Title: Supersymmetric Field Theories for Mathematicians

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Abstract:

Spinors + SUSY algebras

In n dimensions

$\mathrm{Spin}(n, \mathbb{C})$

$V = \mathbb{C}^n$ vector representation

Suppose S is a spin rep. (complex)

And $\Gamma : S \otimes S \longrightarrow V$

is a map of reps. which is symmetric.

Then, we can construct a super-Lie algebra called super-translation

$$T = \Pi S \oplus V$$

Lie bracket: $v \in V$, $\psi \in S$

$$[v, -] = 0$$

$$[\psi, \psi'] = \Gamma(\psi \otimes \psi') \in V$$

The SUSY algebra is the semi-direct product

$$\mathfrak{so}(n,\mathbb{C}) \ltimes T$$

i.e. it is

$$\begin{array}{ccc} -1 & & 0 \\ S & & V \oplus \mathfrak{so}(n,\mathbb{C}) \\ \psi \in & & v \in \qquad A \in \end{array}$$

$[A, -]$ is $\mathfrak{so}(n,\mathbb{C})$ action on spinor/vector

$$[\psi, \psi'] = \Gamma(\psi \otimes \psi')$$

$$[v, \psi] = 0$$

$$[v, v'] = 0$$

Clifford Algebra

n dimensions, $V = \mathbb{C}^n$

$e_1 \ldots e_n$ an orthonormal basis

$\langle e_i, e_j \rangle = \delta_{ij}$

$$Cl_n = \mathbb{C}\langle e_1, \ldots, e_n \rangle / (e_i e_j + e_j e_i = \delta_{ij})$$

This is an associative algebra. A basis is $e_{i_1} \cdots e_{i_k}$

$i_1 < i_2 \cdots < i_k$

$i_1 < i_2 \cdots < i_k$

$\dim Cl_n = 2^n$, an element is even/odd if it's written with even/odd number of generators.

There's a copy of $so(n,\mathbb{C}) \subseteq Cl_n$

If E_{ij} is elementary matrix

$so(n,\mathbb{C}) \ni E_{ij} - E_{ji} \longmapsto e_i \cdot e_j$

$[e_i e_j, e_j e_k] = e_i e_k$

$\exp so(n,\mathbb{C}) \subseteq Cl_n$

is the subgroup of elements of the form

$$e^{\alpha} = 1 + \alpha + \frac{\alpha^2}{2} \cdots$$

$$\alpha \in so(n,\mathbb{C}) \subseteq Cl_n$$

<u>Fact</u> This is $Spin(n,\mathbb{C})$

Representations of Cl_n

Suppose $n = 2m$

Then, we can let

$$f_i^+ = e_{2i-1} + i e_{2i}$$

$$f_i^- = e_{2i-1} - i e_{2i}$$

Since these are null

$$[f_i^\pm, f_j^\pm] = 0$$

$$\parallel$$

$$f_i^\pm f_j^\pm + f_j^\pm f_i^\pm$$

$$[f_i^+, f_j^-] = 2\delta_{ij}$$

$$[f_i^{\pm}, f_j^{\pm}] = 0$$

$$\parallel$$

$$f_i^{\pm} f_j^{\pm} + f_j^{\pm} f_i^{\pm}$$

$$[f_i^{+}, f_j^{-}] = 2\delta_{ij}$$

There's a rep. on

$\mathbb{C}[f_i^-]$ in which $\to 2^n$ dimensional v. space

f_i^- acts by mult.

f_i^+ by $2\frac{\partial}{\partial f_i^-}$

$i_1 < i_2 \dots < i_k$

Call this representation S_{Cl}. This is the only irreducible rep up to a change of parity.

$$Cl_n = \mathrm{End}\, S = gl(2^{m-1}|2^{m-1}) \qquad 2m = n$$

$$\left(\begin{array}{c|c} \text{ev} & \text{odd} \\ \hline \text{odd} & \text{ev} \end{array}\right)$$

$\mathrm{Spin}(n, \mathbb{C}) \subseteq Cl_n(\mathbb{C})$

$n = 2m$

S_{Cl} has even + odd pieces, which are preserved by $\mathrm{Spin}(n, \mathbb{C})$

$S^+ =$ even part
$S^- =$ odd part

these are 2 irreducible spin representations of dimension $2^{n/2 - 1}$

n odd, $n = 2m+1$

$f_i^{\pm} = e_{2i-1} \pm i e_{2i} \quad i = 1, \ldots, m$

e_{2m+1}

There's a rep $\mathbb{C}[f_i^-, \varepsilon]$ where

f_i^- acts by mult

f_i^+ acts by $2\frac{\partial}{\partial f_i^-}$

e_{2m+1} acts by $\varepsilon + \frac{\partial}{\partial \varepsilon}$

This is of dimension 2^{m+1}

S_{Cl} = this rep as a rep of $Spin(n, \mathbb{C})$

S^+, S^- reps of $Spin(2m+1, \mathbb{C})$

These are isomorphic

The element

$$\gamma = e_1 e_2 \dots e_{2m+1} \in Cl_{2m+1}(\mathbb{C})$$

commutes with $Spin(2m+1, \mathbb{C})$

and maps S^+ to S^-

And, $\gamma^2 = \pm 1$.

$\mathrm{Spin}(n, \mathbb{C}) \subseteq Cl_n(\mathbb{C})$

$n = 2m$

$S_{\mathbb{C}}$ has even + odd pieces, which are preserved by $\mathrm{Spin}(n, \mathbb{C})$

$S^+ =$ even part
$S^- =$ odd part

these are 2 irreducible spin representation of dimension $2^{n/2 - 1}$

$n = 2m+1$

S, one irreducible spin rep. of dimension $2^m = 2^{(n-1)/2}$

n even

$V \subseteq (S_+ \oplus S_-)^{\otimes 2}$

Possibilities

1) $V \subseteq S_+ \otimes S_-$ $\quad 4$
2) $V \subseteq S^2 S_+$ $\quad 2$ (and $V \subseteq S^2 S_-$)
3) $V \subseteq \Lambda^2 S_+$ $\quad 6$ (and $V \subseteq \Lambda^2 S_-$)

Representations of Cl_n $\quad [f_i^{\pm}, f_j^{\pm}] = 0$

n odd

$V \subseteq S \otimes S$

a) $V \subseteq S^2 S$ $(\mathrm{dim}^n\ 3$

b) $V \subseteq \Lambda^2 S$ $(\mathrm{dim}^n\ 5)$

Spin reps in Low dimensions

$n=2$ $\quad SO(2,\mathbb{C}) = \mathbb{C}^\times$

$$\mathrm{Spin}(2,\mathbb{C}) = \mathbb{C}^\times \xrightarrow{z^2} SO(2,\mathbb{C})$$

$V = \mathbb{C}^2$ is a sum of reps $V^{1,0}, V^{0,1}$ of weights $1, -1$ under $SO(2,\mathbb{C})$

$V^{1,0}$ spanned by ∂_z $\quad V^{0,1}$ by $\partial_{\bar{z}}$

$S_+ = \mathbb{C}$, weight $\frac{1}{2}$ under action of $\mathbb{C}^\times = SO(2,\mathbb{C})$

$S_- = \mathbb{C}$ weight $-\frac{1}{2}$

$$\mathrm{Sym}^2 S^+ = V^{1,0}$$

$$\mathrm{Sym}^2 S^- = V^{0,1}$$

We can have the (n,m) susy alg

$$S_{n,m} = S^+ \otimes \mathbb{C}^n \oplus S^- \otimes \mathbb{C}^m$$

$\mathbb{C}^n$, $\mathbb{C}^m$ are auxiliary spaces. We need a symmetric inner product on them, $\langle \, , \, \rangle$

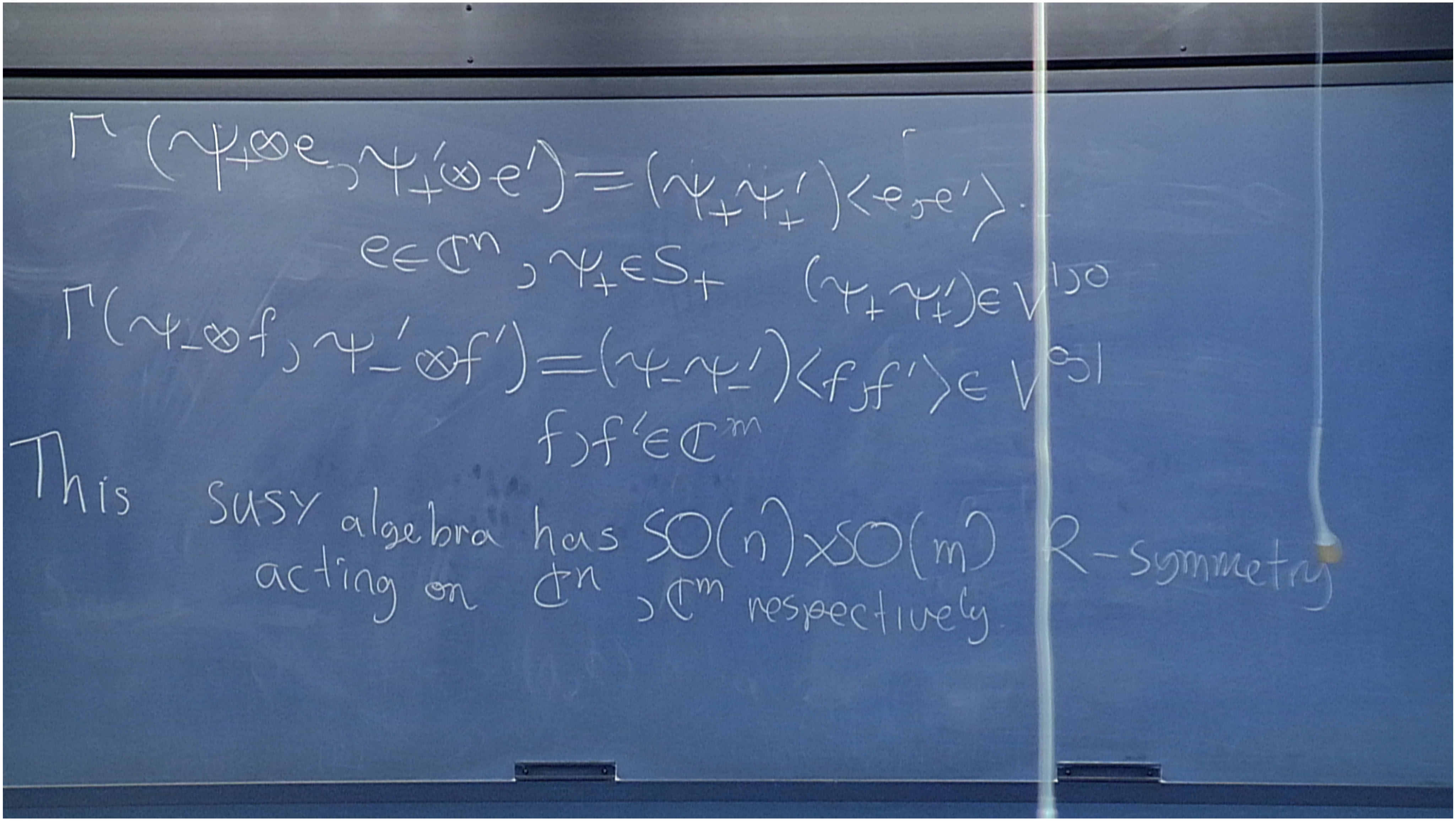
$\Gamma : \mathrm{Sym}^2 S_{n,m} \longrightarrow V$ is

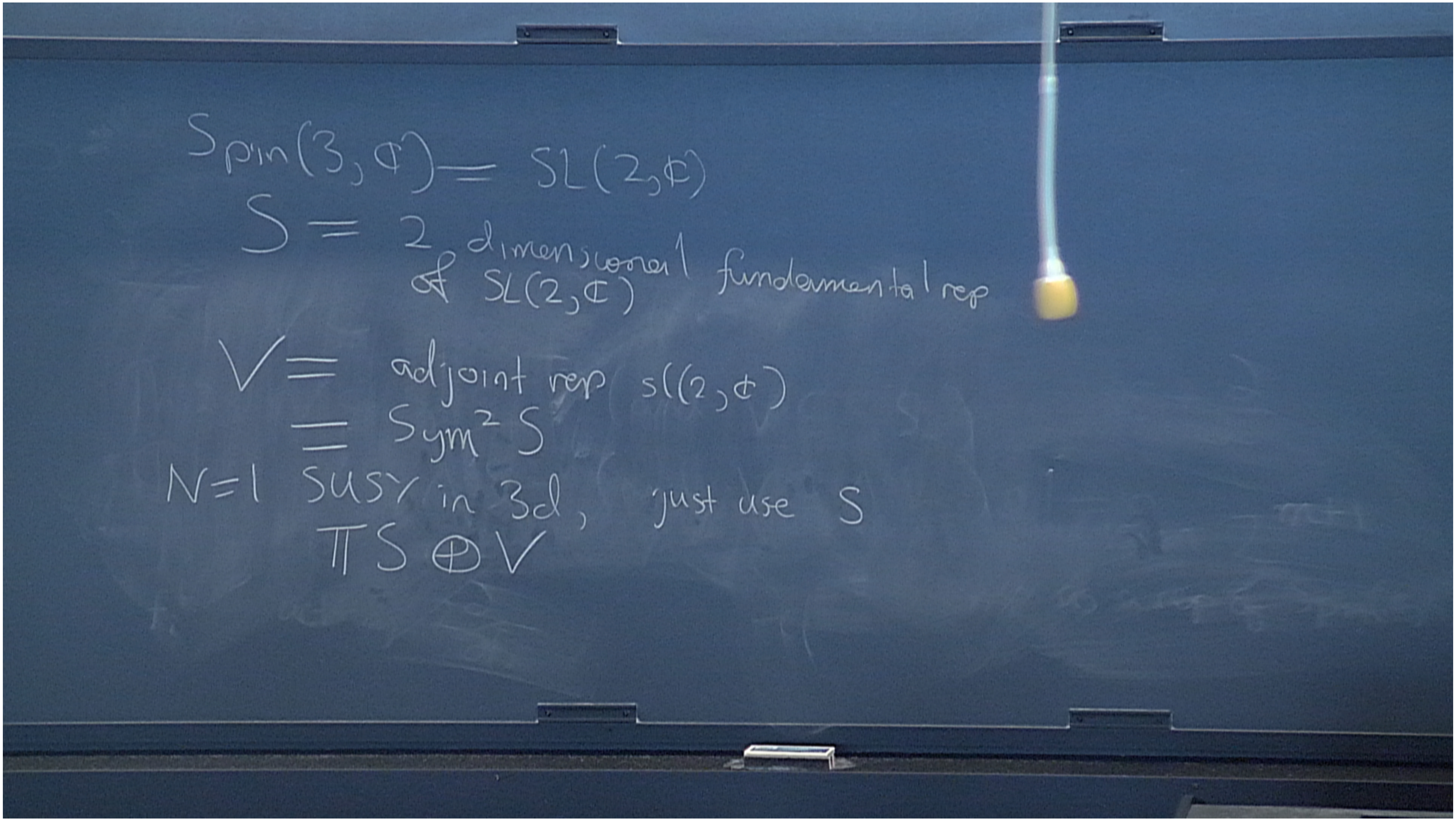
$$\Gamma(\psi_+ \otimes e, \psi_+' \otimes e') = (\psi_+, \psi_+')\langle e, e' \rangle$$

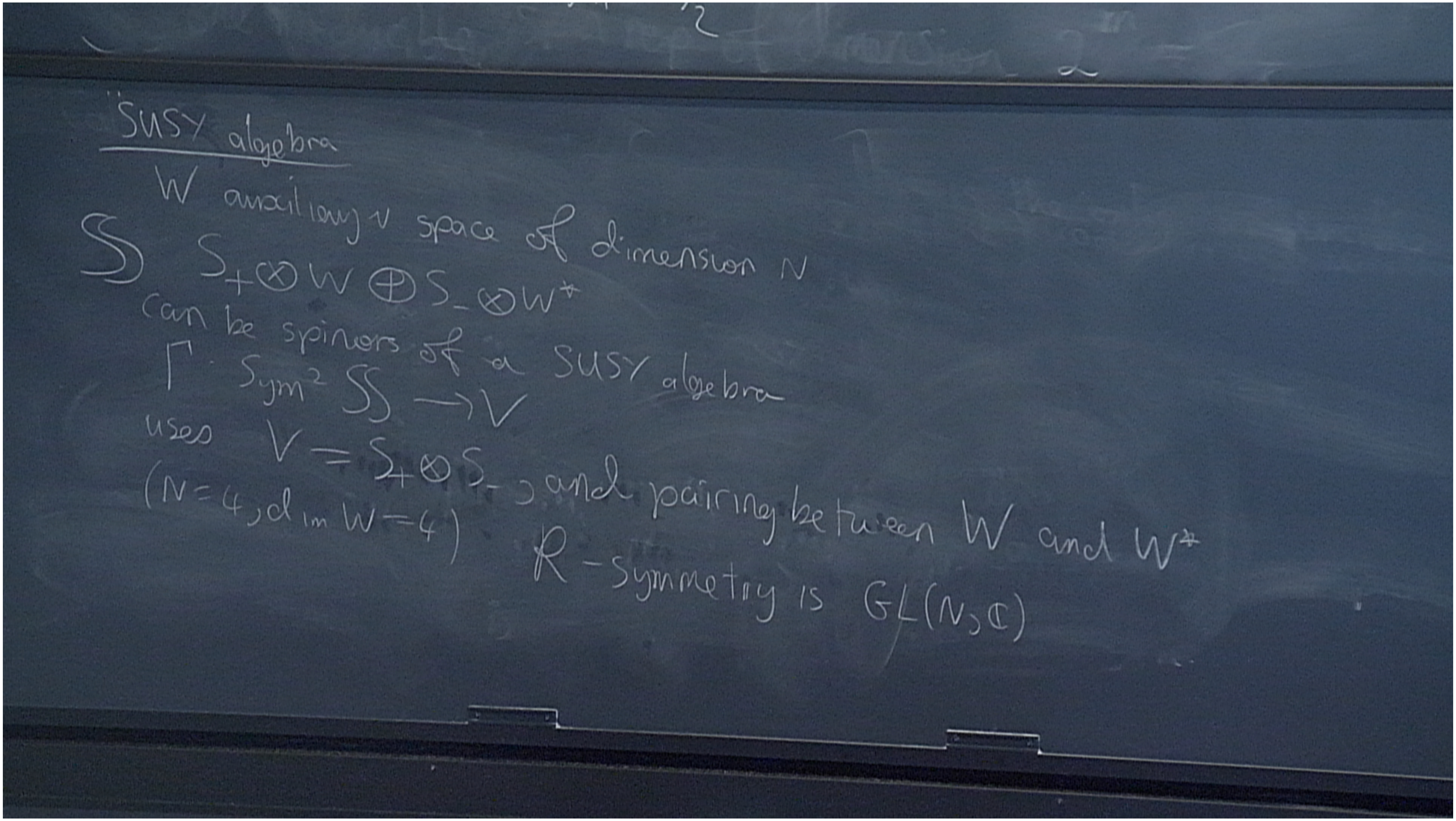
$$e \in \mathbb{C}^n, \quad \psi_+ \in S_+ \qquad (\psi_+, \psi_+') \in V^{1,0}$$

$$\Gamma(\psi_- \otimes f, \psi_-' \otimes f') = (\psi_-, \psi_-')\langle f, f' \rangle \in V^{0,1}$$

$$f, f' \in \mathbb{C}^m$$


$\Gamma(\psi_+\otimes e, \psi_+'\otimes e') = (\psi_+\psi_+')\langle e, e'\rangle$
$e \in \mathbb{C}^n, \psi_+ \in S_+$
$(\psi_+\psi_+') \in V^{1,0}$
$\Gamma(\psi_-\otimes f, \psi_-'\otimes f') = (\psi_-\psi_-')\langle f, f'\rangle \in V^{0,1}$
$f, f' \in \mathbb{C}^m$
This SUSY algebra has $SO(n)\times SO(m)$ R-symmetry
acting on $\mathbb{C}^n$, $\mathbb{C}^m$ respectively.


Spin(3, ℂ) = SL(2, ℂ)
S = 2 dimensional fundamental rep
of SL(2, ℂ)
V = adjoint rep sl(2, ℂ)
= Sym² S
N=1 SUSY in 3d, just use S
ΠS ⊕ V


SUSY algebra
W auxiliary space of dimension N
$\mathbb{S}$ $S_+ \otimes W \oplus S_- \otimes W^*$
can be spinors of a SUSY algebra
$\Gamma: \mathrm{Sym}^2 \mathbb{S} \longrightarrow V$
uses $V = S_+ \otimes S_-$ and pairing between W and W^*
$(N=4, \dim W = 4)$
R-symmetry is $GL(N, \mathbb{C})$