Title: Shifted Yangians, loop groups, and (co)products

Date: Sep 29, 2016 02:00 PM

URL: http://pirsa.org/16090061

Abstract: <p>I'll describe a family of algebras called shifted Yangians, which arise as deformation quantizations of certain spaces related to loop groups. I'll also describe coproducts for these algebras, which are related to multiplication in the loop group. Physically, this fits into the story of Coulomb branches associated to 3d N=4 quiver gauge theories, where the above multiplication maps arises by taking products of scattering matrices.</p>

Shifted Yangians,
loop groups,
and (co) products

Joint w/ Finkelberg
Kamnitzer
Pham
Rybnikov

Plan: 1) Prelim.
2) Yangians
3) Shifted Yangians
4) Products
5) Connection to physics

- algebra
- loop groups
- moduli spaces of bundles

1) Preliminaries

- Always over $\mathbb{C}$
- $\mathfrak{g}$ simple, $\mathfrak{h} \subset \mathfrak{g}$ Cartan,

  $\{\alpha_i\}_{i \in I}$ simple roots
- G connected alg gp

  $B^\pm$ Borel

  $\cup$ $\qquad T$ torus

  $U^\pm$ Unipotents

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T torus

- Let z be the coordinate on $\mathbb{C}$

Have loop groups (algebraic)

$G[z] = G(\mathbb{C}[z]) = \mathrm{Maps}(\mathbb{C}, G)$

$= \mathrm{Maps}(\mathbb{P}^1 \setminus \{\infty\}, G)$

$G[z^{-1}] = \mathrm{Maps}(\mathbb{P}^1 \setminus \{0\}, G)$

$\cup$

$G_1[z^{-1}] =$ those maps taking $\infty \mapsto 1 \in G$

ex: $G = SL_2$

$$G_1[z^{-1}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l} a,b,c,d \in \mathbb{C}[z^{-1}] \\ \det = 1 \\ a,d \in 1 + z^{-1} \cdots \\ b,c \in z^{-1}\mathbb{C}[z^{-1}] \end{array} \right\}$$

Rmk: Really want $G_1[[z^{-1}]]$

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Rmk: Really want $G_1[[z^{-1}]]$

$G_1[[z^{-1}]]$ is a scheme

$G[z^{-1}]$ is an ind-scheme

2) Yangians

The $\underline{\text{Yangian}}$ $Y = Y(\mathfrak{g})$

is an assoc $\mathbb{C}$-alg.

$\underline{\text{gens}}$: $E_i^{(r)}, H_i^{(r)}, F_i^{(r)}$ $\quad i \in I$, $r \geq 1$

$\underline{\text{rels}}$: later!

- Can choose "root vectors" $E_\alpha^{(r)}, F_\alpha^{(r)}$ $\quad r \geq 1$

$$Y \underset{\substack{\uparrow \\ \text{vector space}}}{\cong} \mathbb{C}[E_\alpha^{(r)}, H_i^{(r)}, F_\alpha^{(r)}]$$

$\underline{\text{Thm}}$:

- Can choose "root vectors" $E_\alpha^{(r)}, F_\alpha^{(r)} \quad r \geq 1$

$$Y \underset{\substack{\uparrow \\ \text{vector space}}}{\cong} \mathbb{C}[E_\alpha^{(r)}, H_i^{(r)}, F_\alpha^{(r)}]$$

- Y is a Hopf alg

<u>Thm:</u> (Kamnitzer–Webster–W.–Yacobi)
There is a filtration on Y

$$\deg X^{(r)} = r \;, \quad X = E_\alpha, H_i, F_\alpha$$

and $\operatorname{gr} Y \cong \mathcal{O}(G_1[z^{-1}])$

In fact, this an iso. of graded Poisson–Hopf algebras.

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$\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$

$, \mathfrak{g}[z], z^{-1}\mathfrak{g}[z^{-1}]$

bialgebras

$G[z], G_1[z^{-1}]$

on-Lie groups

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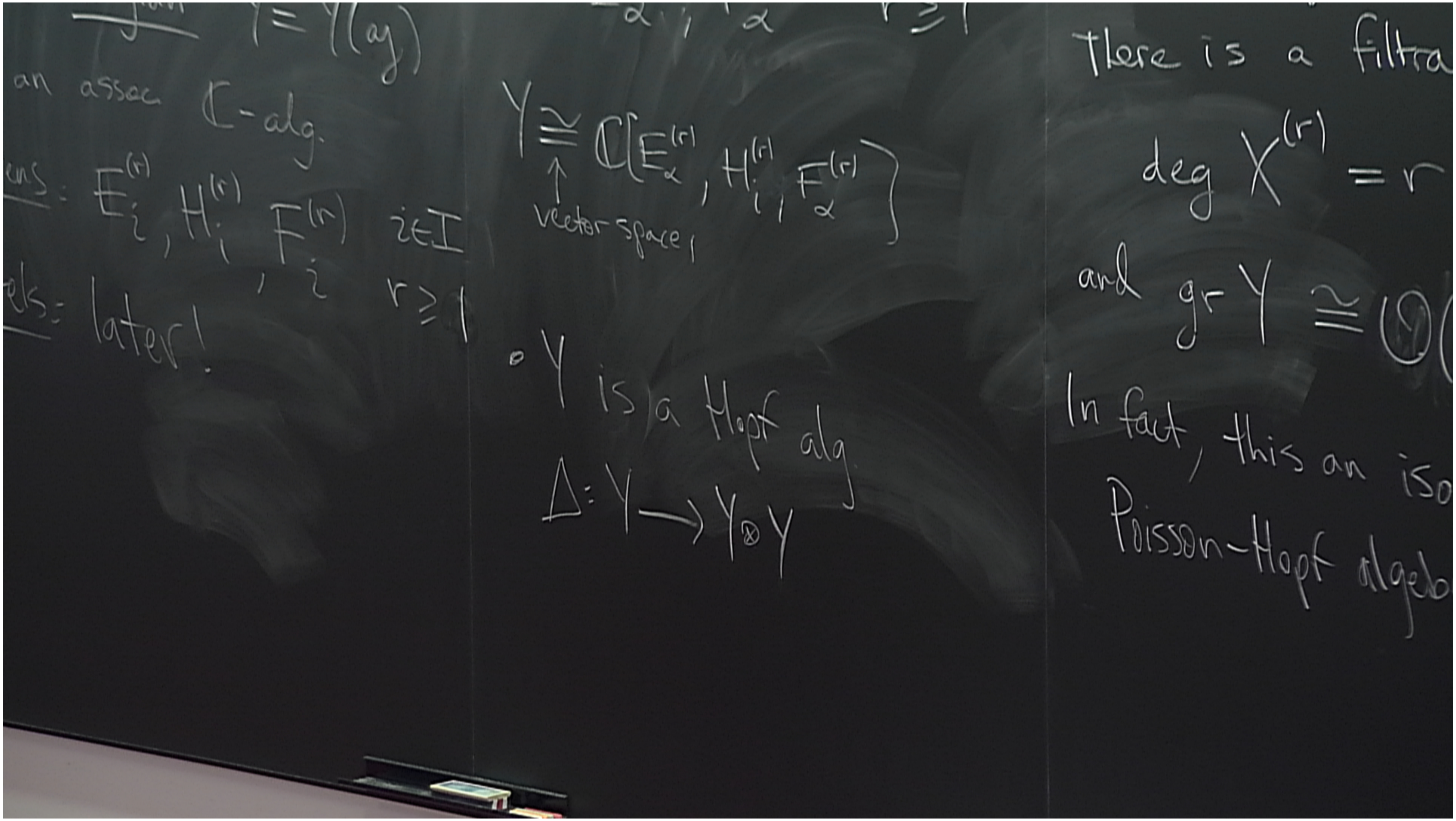

$Y = Y(\mathfrak{g})$
an assoc $\mathbb{C}$-alg.
gens: $E_i^{(r)}, H_i^{(r)}, F_i^{(r)}$ $i \in I$ $r \geq 1$
rels: later!
$Y \cong \mathbb{C}[E_\alpha^{(r)}, H_i^{(r)}, F_\alpha^{(r)}]$
vector space
• Y is a Hopf alg
$\Delta : Y \to Y \otimes Y$
There is a filtra
$\deg X^{(r)} = r$
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In fact, this an iso
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$,z^{-1}]$

$), z^{-1}\mathfrak{g}[z^{-1}]$

ebras

group

2) There is a second filtration on Y, which yields classical limit $U(\mathfrak{g}[z])$

$$\begin{tikzcd} & A \arrow[dl, squiggly, "\text{Filtration } 1"'] \arrow[dr, squiggly, "\text{Filtration } 2"] & \\ U(\mathfrak{g}) & & \mathcal{O}(G^*) \end{tikzcd}$$

group with Lie algebra $\mathfrak{g}^*$

Quantum duality principle

ex: $G = SL_2$

$$G_1[z^{-1}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l} a,b,c,d \in \mathbb{C}[z^{-1}] \\ \det = 1 \\ a,d \in 1 + z^{-1} \ldots \\ b,c \in z^{-1}\mathbb{C}[z^{-1}] \end{array} \right\}$$

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$b,c \in z^{-1}\mathbb{C}[z^{-1}]$

$G_1[[z^{-1}]]$

scheme

-scheme

e,h,f

$$U(sl_2)\otimes\mathbb{C}[\hbar] = A$$

$$\cup$$

$$\langle E = \hbar e,\ H = \hbar h,\ F = \hbar f \rangle = A'$$

$[e,f] = h$

$[E,F] = \hbar H$

$G^* = (sl_2^*, +)$

ns

...ian $Y = Y(\mathfrak{g})$

... $\mathbb{C}$-alg.

$\dots, H_i^{(r)}, F_i^{(r)}$ $i \in I$, $r \geq 1$

...ter!

<u>ex:</u> $G = PGL_2$

Any $g \in G_1[z^{-1}]$ can be <u>Gauss decomp'd</u>

$$\begin{pmatrix} 1 & e \\ & 1 \end{pmatrix}\begin{pmatrix} h & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ f & 1 \end{pmatrix}$$

$e = e^{(1)} z^{-1} + e^{(2)} z^{-2} + \dots$

$h = 1 + h^{(1)} z^{-1} + \dots$

$E^{(r)} \mapsto e^{(r)}$, etc.

<u>Thm:</u> (Kamnitzer-Webster-W.-Yacobi)

There is a filtration on Y

$$\deg X^{(r)} = r, \quad X = E_\alpha, H_i, F_\alpha$$

and $\operatorname{gr} Y \cong \mathcal{O}(G_1[z^{-1}])$

In fact, this an iso. of graded Poisson-Hopf algebras.

<u>Remark.</u>

$\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$

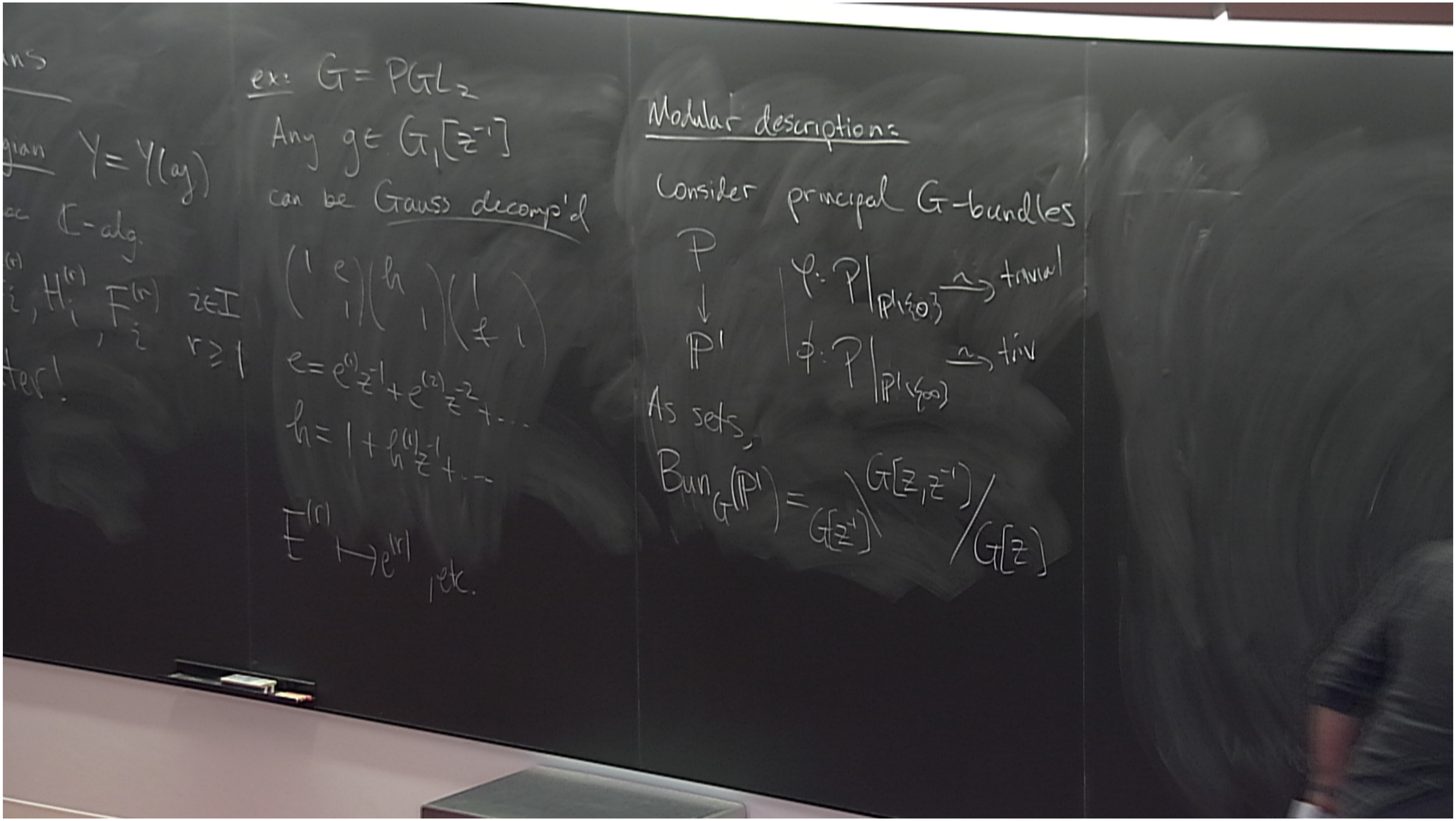
1) $\mathfrak{g}[z, z^{-1}]$, $\mathfrak{g}[z]$, $z^{-1}\dots$

are Lie bialgebras

$\Updownarrow$

$G[z, z^{-1}]$, $G[z]$, $G_1[\dots$

are Poisson-Lie groups.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$e = e^{(1)}z^{-1} + e^{(2)}z^{-2} + \dots$
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$E^{(r)} \mapsto e^{(r)}$, etc.
Modular description:
Consider principal G-bundles
$P \to \mathbb{P}^1$
$\varphi : P|_{\mathbb{P}^1 \setminus \{0\}} \xrightarrow{\sim}$ trivial
$\phi : P|_{\mathbb{P}^1 \setminus \{\infty\}} \xrightarrow{\sim}$ triv
As sets,
$\mathrm{Bun}_G(\mathbb{P}^1) = {}_{G[z^{-1}]}\backslash G[z, z^{-1}] / G[z]$

z

$[z^{-1}]$

decomp'd

$z^{-2}+\dots$

$+\dots$

etc.

Modular description:

- Consider principal G-bundles

$$\begin{array}{c} P \\ \downarrow \\ \mathbb{P}^1 \end{array} \quad \begin{array}{l} \varphi : P|_{\mathbb{P}^1 \setminus \{0\}} \xrightarrow{\sim} \text{trivial} \\ \phi : P|_{\mathbb{P}^1 \setminus \{\infty\}} \xrightarrow{\sim} \text{triv} \end{array}$$

As sets,

$$\mathrm{Bun}_G(\mathbb{P}^1) = G[z^{-1}] \backslash G[z,z^{-1}] / G[z]$$

Then over $\mathbb{C}^*$,

$$g = \varphi \circ \phi^{-1} : P_{\mathrm{triv}}|_{\mathbb{C}^*} \xrightarrow{\sim} P_{\mathrm{triv}}|_{\mathbb{C}^*}$$

$$\rightsquigarrow g \in G[z,z^{-1}]$$

- The affine Grassmannian Gr_G

$$Gr = \{ (P, \varphi) : \text{---}''\text{---} \}$$

$$\cong G[z,z^{-1}] / G[z]$$

2) There is a
on Y, whi
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$$\begin{array}{ccc} & A & \\ \text{Filtration 1} \swarrow & & \searrow \text{Filtration 2} \\ U(\mathfrak{g}) & & \mathcal{O}(G^* \end{array}$$

group with
algebra $\mathfrak{g}$

Open subset:

$$Gr_0 = \{(P, \varphi) : P \text{ is trivial}\}$$

$$= G[z^{-1}]G[[z]] / G[[z]]$$

$$\cong G_1[z^{-1}]$$

e, h, f

$U(\mathfrak{sl}_2) \otimes \mathbb{C}[\hbar$

$$\left\langle \begin{array}{l} E = \hbar e \\ H = \hbar h \\ F = \hbar f \end{array} \right\rangle$$

$$G^* = (\mathfrak{sl}_2^*, +)$$

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Q: How does Poisson structure appear?

E
H
F

3) Shifted Yangians

Def: The Cartan doubled Yangian Y_∞

gens: $E_i^{(r)}, F_i^{(r)}$ $r \geq 1$

$H_i^{(s)}$ $s \in \mathbb{Z}$

rels: $[H_i^{(r)}, H_j^{(s)}] = 0$

$[E_i^{(r)}, F_j^{(s)}] = \delta_{ij} H_i^{(r+s-1)}$

$[H_i^{(s+1)}, E_j^{(s)}] - [H_i^{(r)}, E_j^{(s+1)}] = \frac{(\alpha_i, \alpha_j)}{2}(H_i^{(r)} E_j^{(s)} + E_j^{(s)} H_i^{(r)})$

$[E_i^{(r+1)}, E_j^{(s)}] - [E_i^{(r)}, E_j^{(s+1)}] = \ldots$

+ replace E's with F's

Then over

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+ replace E's with F's

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- The ordinary Yangian contains a copy of $U(\mathfrak{g})$

$\langle E_i^{(1)}, H_i^{(1)}, F_i^{(1)} \rangle$

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• Choose a coweight μ for G

let $\mu_i = \langle \mu, \alpha_i \rangle$

(i.e. integer μ_i $\forall i \in I$)

re affine

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...gian contains

$U(\mathfrak{g})$

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...se a coweight

...or G

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$\forall i \in I)$

Def: The <u>Shifted Yangian</u>

$$Y_\mu = Y_\infty \Big/ \left\langle \begin{matrix} H_i^{(r)} = 0, & r < -\mu_i \\ H_i^{(-\mu_i)} = 1 & \end{matrix} \right\rangle$$

Open subset:

$$Gr_0 = \{(P, \varphi) : P \text{ is trivia...}$$

$$= G[z^{-1}]G[z] \Big/ G[z]$$

$$\cong G_1[z^{-1}]$$

Q: How does Poisson structure appear?

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Properties:

a) Y_0 is the ordinary Yangian

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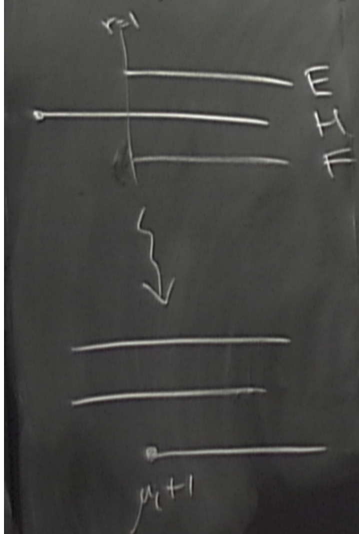
Then $Y_\mu \hookrightarrow Y_0$



c) If μ is non-dominant

i.e. some $\mu_i < 0$

$\Rightarrow [E_i^{(1)}, F_i^{(-\mu_i)}] = 1$

$\Rightarrow$ contain Weyl algebra / CCR

$\Rightarrow$ no f.d. reps for Y_μ.

Thm: (FKPRW)

1) The PBW theorem holds for Y_μ

2) Y_μ is filtered, and quantizes

$$W_\mu := U_1^{+}[z^{-1}]\, T_1[z^{-1}]\, z^{\mu}\, U_1^{-}[z^{-1}]$$

$$\subseteq G[z, z^{-1}]$$

: (FKPRW)

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$= U_1^+[z^{-1}]\, T_1[z^{-1}]\, z^\mu\, U_1^-[z^{-1}]$

$\subseteq G[z, z^{-1}]$

Guess: $Y_\infty \rightsquigarrow$

$U_1^+[z^{-1}]\, T[z, z^{-1}]\, U_1^-[z^{-1}]$

Question: How does the corresponding Poisson structure come from?

Conjecture: It is "induced" from $G[z, z^{-1}]$.

ordinary Yangian contains

copy of $U(\mathfrak{g})$

$\langle E_i^{(1)}, H_i^{(1)}, F_i^{(1)} \rangle$

- Choose a coweight μ for G

$\mu_i = \langle \mu, \alpha_i \rangle$

integer μ_i $\forall i \in I$)

$$\begin{array}{ccc} M & \supset & N \\ \cup & & \cup \\ S & & S \cap N \end{array}$$

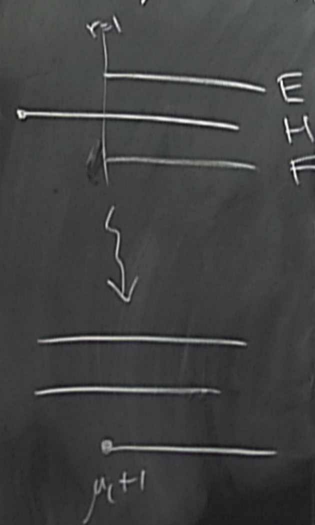
<u>Properties</u>:

a) Y_0 is the ordinary Yangian

b) If μ is <u>dominant</u>

<u>i.e.</u> all $\mu_i \geq 0$

Then $Y_\mu \hookrightarrow Y_0$



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$\langle E_i^{(1)}, H_i^{(1)}, F_i^{(1)} \rangle$

$E_j^{(1)} + E_j^{(1)} H_i^{(1)}$

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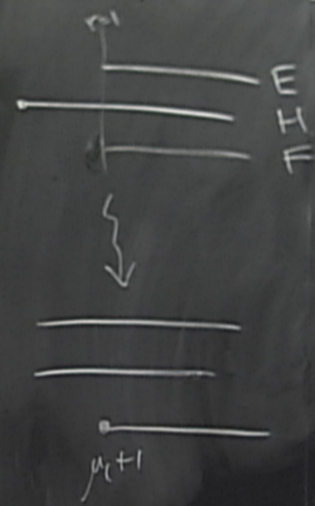
$$\begin{matrix} M & \supset & N \\ \cup & & \cup \\ S & & S \cap N \end{matrix}$$

Want: To define symplectic leaves of N as $S \cap N$

Need: $S \cap N \subset S$ is symplectic submanifold.

Then $Y_\mu \hookrightarrow Y_0$



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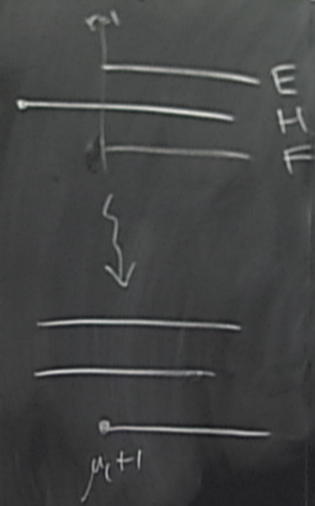
Need: $S \cap N \subset S$ is symplectic submanifold.

Rmk: $N \subset M$ is Poisson

$N = \bigcup S$

Then $Y_\mu \hookrightarrow Y_0$



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Modular description:

Kamnitzer

Thm: (Braverman–Finkelberg – Nakajima)

W_μ is isomorphic to

$\{(P, \varphi, P_B):$ a) $P \downarrow \mathbb{P}^1$, φ trivialization away from 0; b) P_B is a B-reduction of P; c) P_B has type $w_0\mu$ and transports to B^- at ∞ under $\varphi\}$

rels: $[H_i^{(r)}, H_j^{(s)}] = 0$

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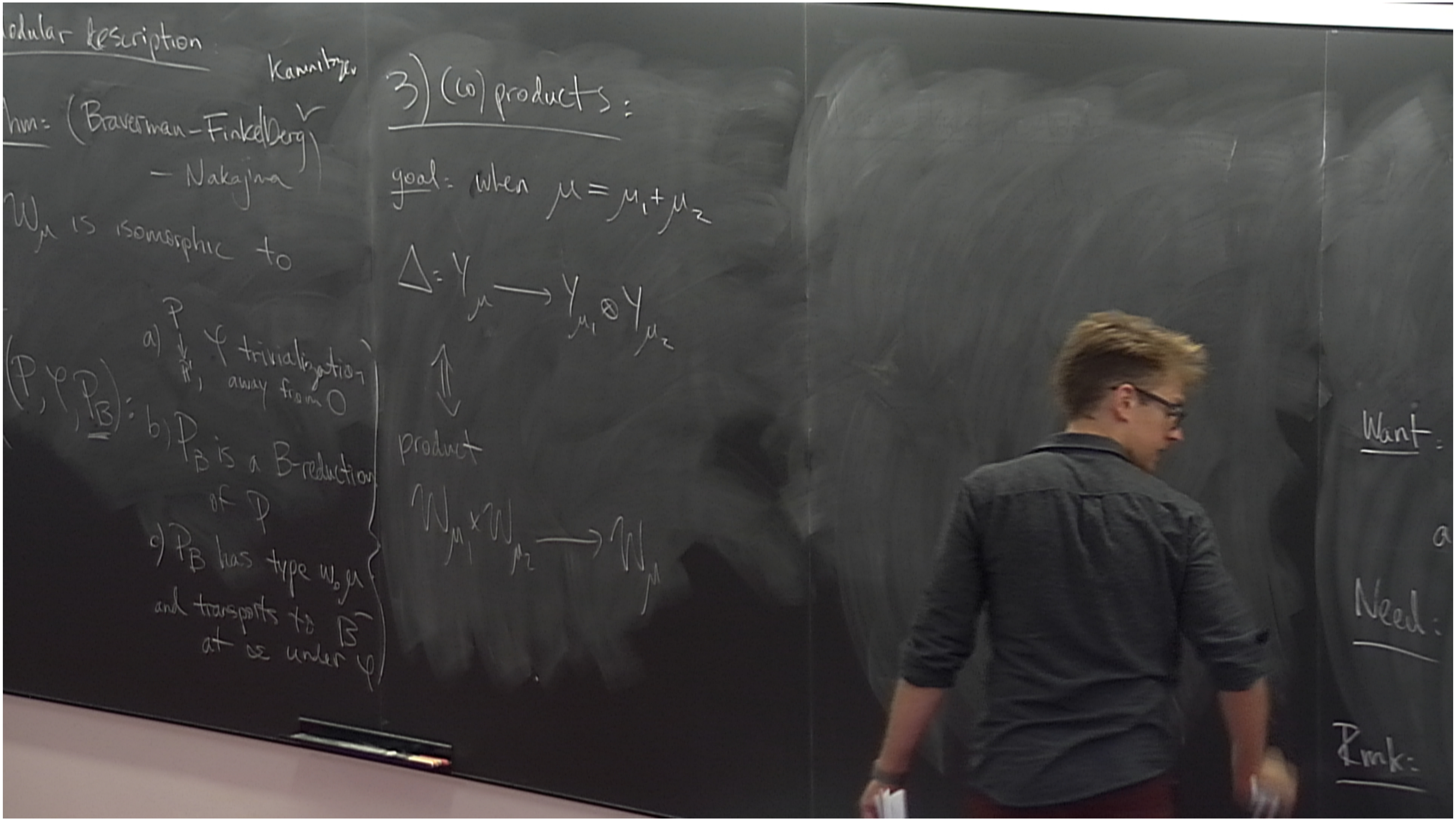
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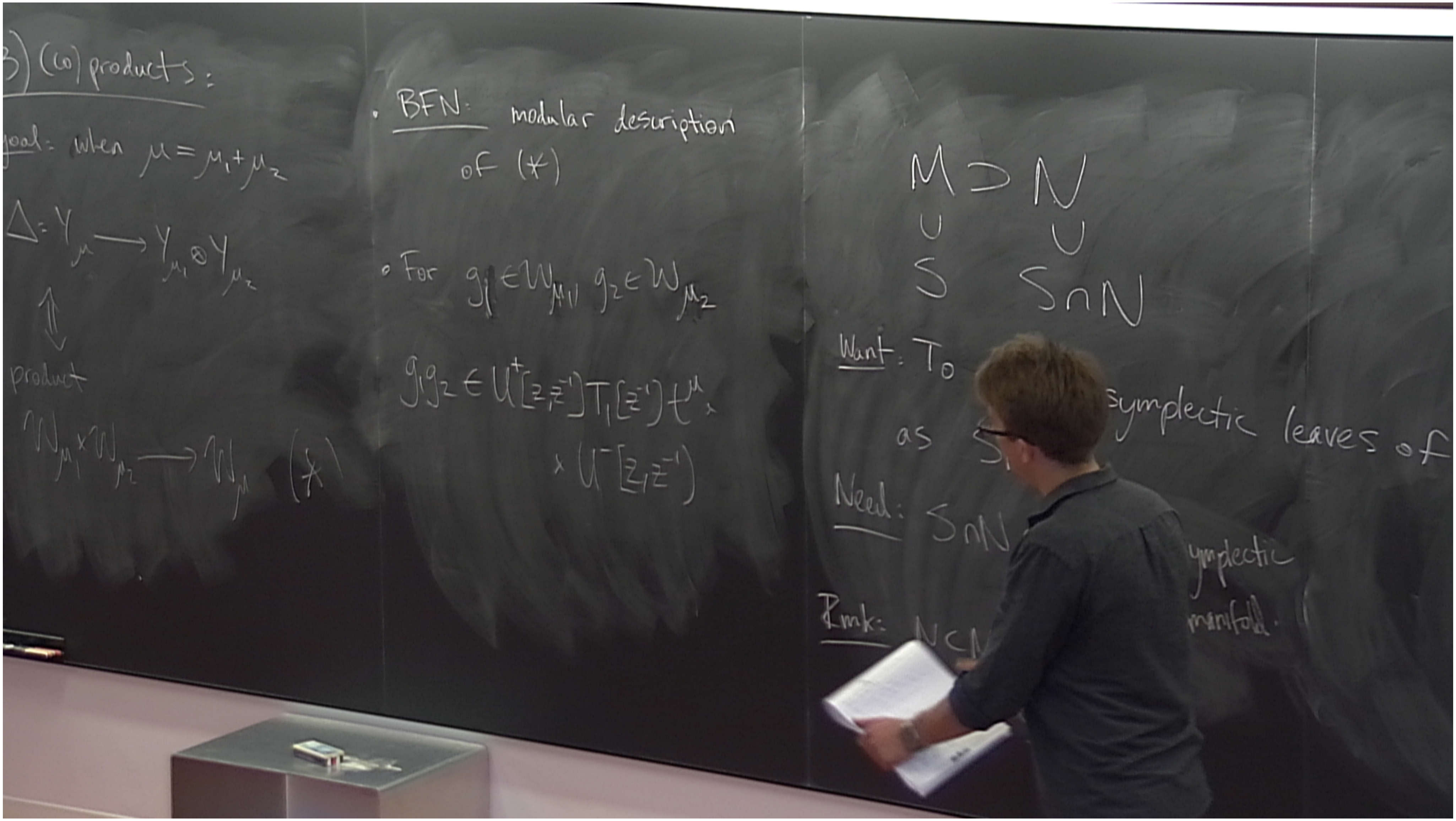
W_μ is isomorphic to
(P, φ, P_B):
a) φ trivialization away from 0
b) P_B is a B-reduction of P
c) P_B has type $w_0\mu$ and transports to B^- at ∞ under φ
3) (co) products:
goal: when $\mu = \mu_1 + \mu_2$
$\Delta : Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$
product
$W_{\mu_1} \times W_{\mu_2} \longrightarrow W_\mu$
Want:
Need:
Rmk: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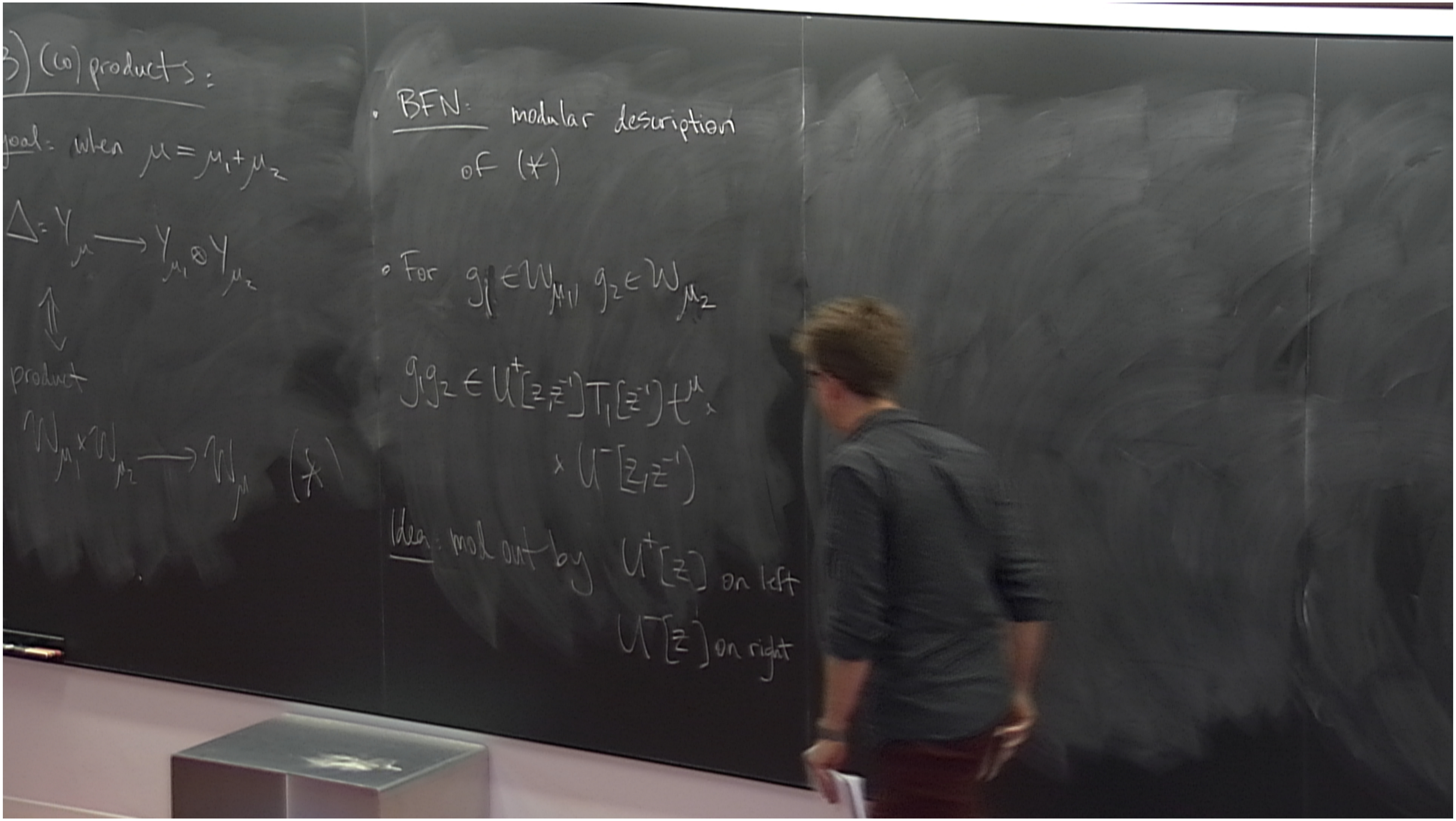
$\Delta : Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$
product
"$W_{\mu_1} \times W_{\mu_2} \longrightarrow W_\mu$" $(*)$
BFN: modular description of $(*)$
For $g_1 \in W_{\mu_1}$, $g_2 \in W_{\mu_2}$
$g_1 g_2 \in U^+[z, z^{-1}] T_1[z^{-1}] t^\mu \times U^-[z, z^{-1}]$
$M \supset N$
S
$S \cap N$
Want: T_0
symplectic leaves of
as
Need: $S \cap N$
symplectic
manifold.
Rmk:


3) (co)products:
goal: when $\mu = \mu_1 + \mu_2$
$\Delta: Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$
$\Updownarrow$
product
$W_{\mu_1} \times W_{\mu_2} \longrightarrow W_\mu \quad (*)$
• BFN: modular description of $(*)$
• For $g_1 \in W_{\mu_1}$, $g_2 \in W_{\mu_2}$
$g_1 g_2 \in U^+[z,z^{-1}]\, T_1[z^{-1}]\, t^\mu \times U^-[z,z^{-1}]$
Idea: mod out by $U^+[z]$ on left
$U^-[z^{-1}]$ on right

• **BFN:** modular description of $(*)$

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Idea: mod out by $U^+[z]$ on left, $U^-[z]$ on right

Note: $U^+[z,z^{-1}] = U^+[z]\,U_1^+[z^{-1}]$

e.g. $\begin{pmatrix}1 & \mathbb{C}[z,z^{-1}]\\ & 1\end{pmatrix} = \begin{pmatrix}1 & \mathbb{C}[z]\\ & 1\end{pmatrix}\begin{pmatrix}1 & z^{-1}\mathbb{C}[z^{-1}]\\ & 1\end{pmatrix}$

Then Y_μ

- **BFN:** modular description of (*)

- For $g_1 \in \mathcal{W}_{\mu_1}$, $g_2 \in \mathcal{W}_{\mu_2}$

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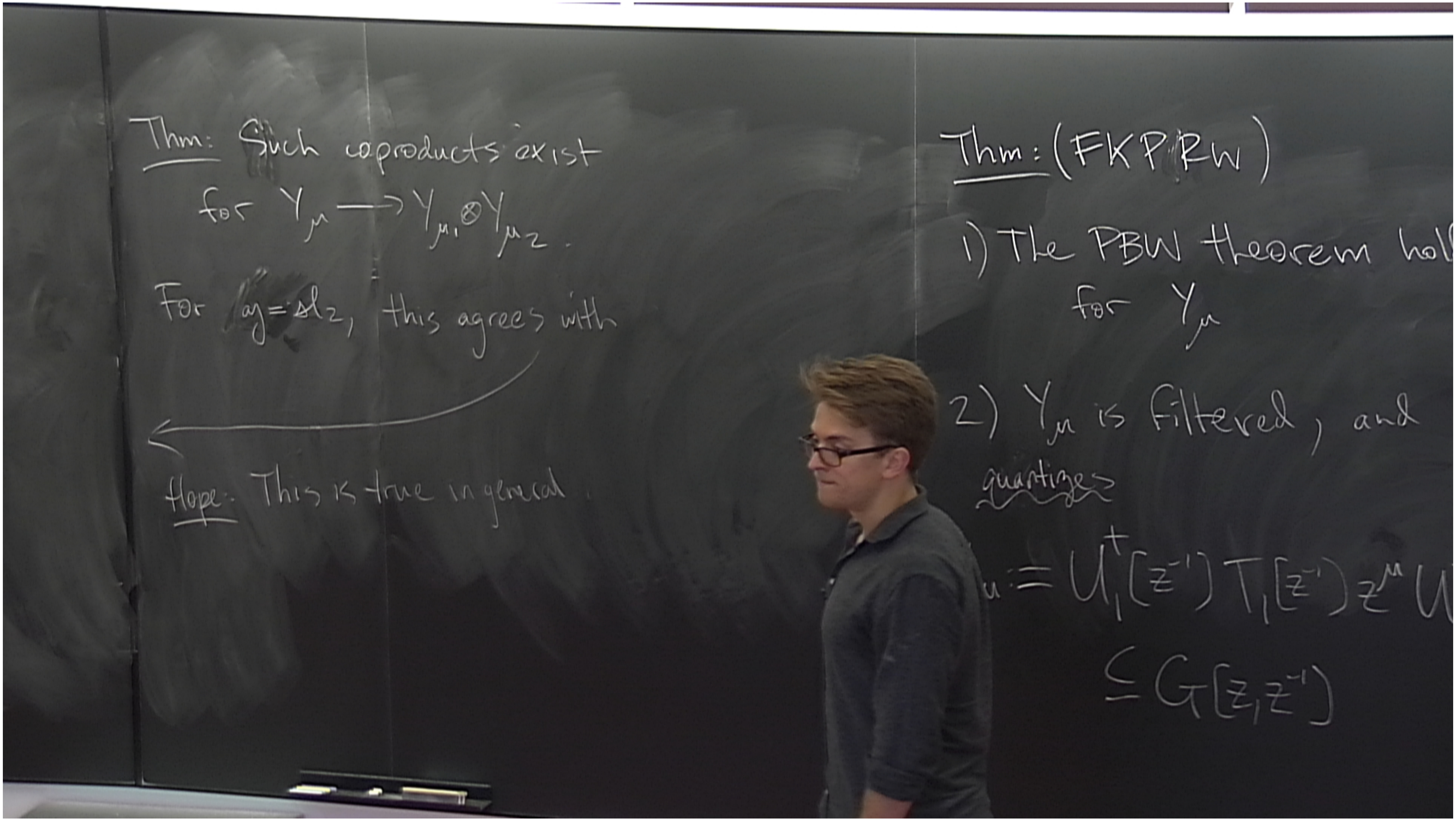
$\Longrightarrow$ can define

$$\mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} \longrightarrow \mathcal{W}_{\mu}$$

as this reduction


Thm: Such coproducts exist
for $Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$.
For $\mathfrak{g} = \mathfrak{sl}_2$, this agrees with
Hope: This is true in general.
Thm: (FKPRW)
1) The PBW theorem hol
for Y_μ
2) Y_μ is filtered, and
quantizes
$\mu := U_1^+[z^{-1}]\, T_1[z^{-1}]\, z^\mu\, U$
$\subseteq G[z, z^{-1}]$

4) Connections to Physics

- Recent work describing Coulomb branches in 3d $N=4$ SUSY gauge theory

(see Bullimore–Dimofte–Gaiotto, BFN)

Thm: (FKPRW)

1) The PBW theorem hol

for Y_μ

2 ... s filtered, and

W_μ ... $) T_1[z^{-1}) z^\mu W$

... $[z, z^{-1}]$

4) Connections to Physics

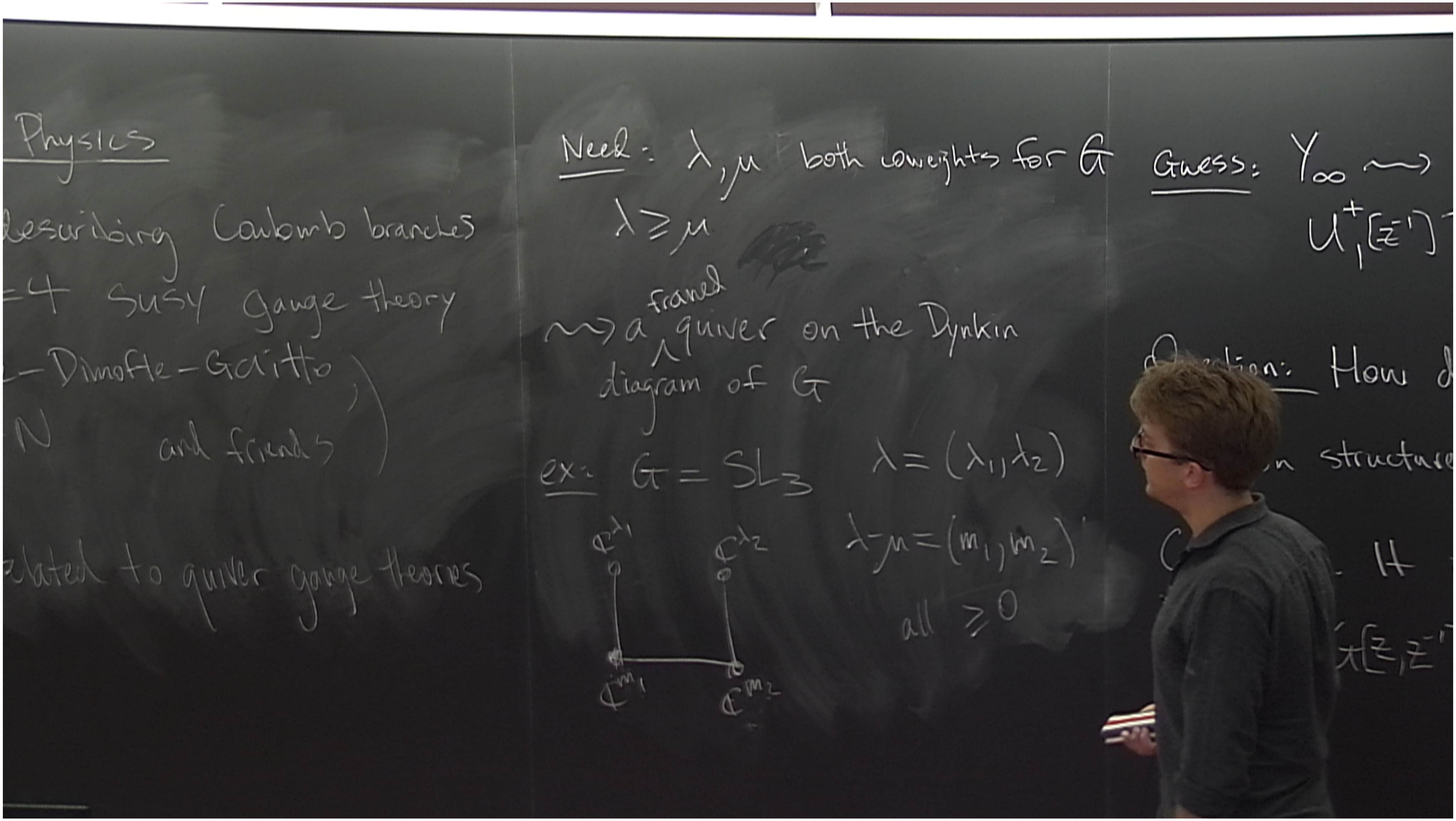
- Recent work describing Coulomb branches in 3d $N=4$ SUSY gauge theory (see Bullimore–Dimofte–Gaiotto, BFN and friends)

- W_μ and Y_μ related to quiver gauge theories

Y_μ is filtered, quantizes

$:= U_1^+[z^{-1}]\, T_1[z$

$\subseteq G[z, z$


Physics
describing Coulomb branches
SUSY gauge theory
Dimofte-Gaitto
and friends
related to quiver gauge theories
Need: λ, μ both coweights for G
λ ≥ μ
⇝ a framed quiver on the Dynkin diagram of G
ex: G = SL3
λ = (λ1, λ2)
λ−μ = (m1, m2)
all ≥ 0
Guess: Y∞ ⇝
How d
structure

<u>Need</u>: λ, μ both coweights for G

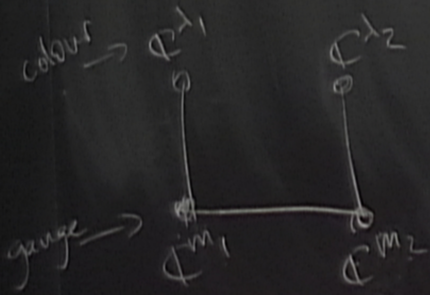
$\lambda \geq \mu$

$\rightsquigarrow$ a framed quiver on the Dynkin diagram of G

<u>ex</u>: $G = SL_3$ $\quad \lambda = (\lambda_1, \lambda_2)$

$\lambda - \mu = (m_1, m_2)$

all ≥ 0

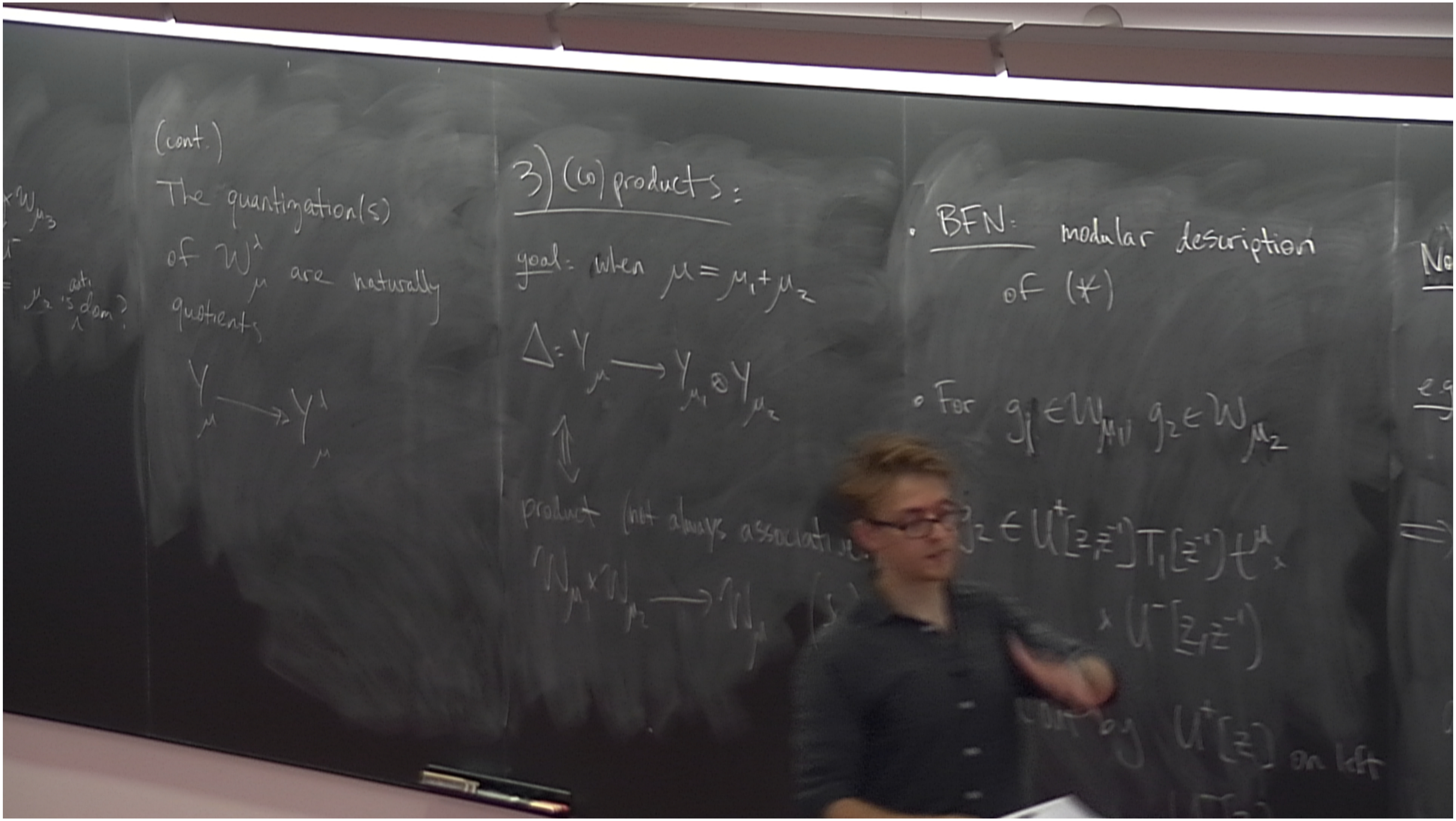


<u>Thm</u>: Coulomb branch associated to Quiver(λ, μ) is a subvariety

$$\mathcal{W}^{\lambda}_{\mu} \subseteq \mathcal{W}_{\mu}$$

$$\|$$

$$g \in U_1^+[z^{-1}]\, T_1[z^{-1}] \cdots$$


(cont.)
The quantization(s)
of W^λ_μ are naturally
quotients
$Y_\mu \twoheadrightarrow Y^\lambda_\mu$
3) (co)products:
goal: when $\mu = \mu_1 + \mu_2$
$\Delta: Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$
product (not always associative)
$W_{\mu_1} \times W_{\mu_2} \longrightarrow W_\mu$
• BFN: modular description
of (*)
• For $g_1 \in W_{\mu_1}$, $g_2 \in W_{\mu_2}$
on left

$\times W_{\mu_3}$

μ_2 's dom?

(cont.)

The quantization(s) of W^λ_μ are naturally quotients

$$Y_\mu \longrightarrow Y^\lambda_\mu$$

• BDG: "multiplication" arises naturally in this context

3) (co)products:

goal: when $\mu = \mu_1 + \mu_2$

$$\Delta : Y_\mu \longrightarrow Y_{\mu_1} \otimes Y_{\mu_2}$$

$$\Updownarrow$$

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