Title: Integrability and null canonical gravity

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Abstract: <p>Constraint free initial data can be given for vacuum general relativity on a pair of intersecting null hypersurfaces. Moreover, the Poisson algebra of a set of such free null initial data has been found,but it has an unfamiliar structure, making its quantization difficult. We note that this algebra is essentially a sum of an infinite number of copies of the Poisson algebras of cylindrically symmetric gravity. Using the fact that cylindrically symmetric gravity is integrable we find new free data with an algebra more amenable to quantization.</p>

# Integrability and null canonical gravity

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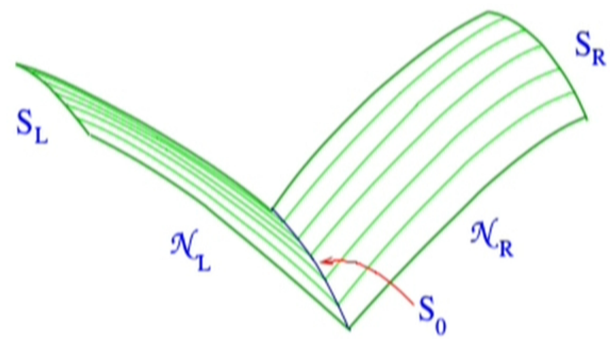
# Plan of talk

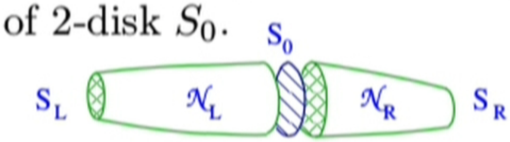
- Outline of canonical vacuum GR in terms of free (unconstrained) initial data on null hypersurfaces.
- The Poisson brackets of free data.
- Poisson brackets in cylindrically symmetric gravity.
  - Full Poisson algebra essentially direct sum of infinitely many copies of cyl. symmetric Poisson algebra.
  - Cyl. symmetric vacuum GR integrable. Quantization known at algebraic level.
- Transformation to new data with known quantization.
- Poisson brackets of new data.
- Quantization of new data.
- General, symmetryless case.
- Things to do.

# Double null sheets as initial data hypersurfaces

- A double null sheet is a pair of intersecting null hypersurfaces (or "lightfronts") - like an open book in spacetime.

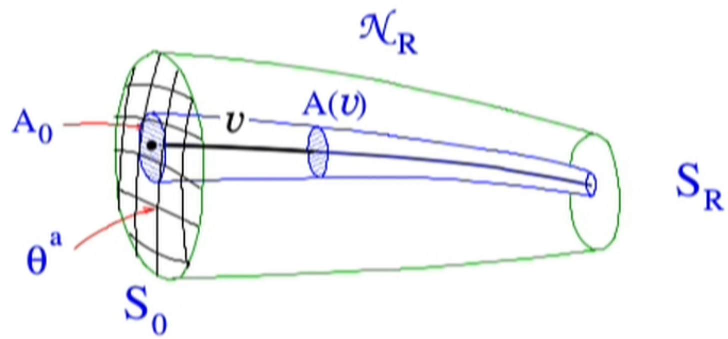



- $\mathcal{N}_R$, $\mathcal{N}_L$ are 3-surfaces swept out by null geodesics emerging normally from the two sides of 2-disk $S_0$.



# The free initial data

Coordinates adapted to $\mathcal{N}$




- $\theta^1, \theta^2$ coordinates on $S_0$. Held constant on generators.
- $v$ is a parameter along each generator defined so that the cross sectional area of an infinitesimal bundle of neighboring generators is

$$A(v) = A_0 v^2$$

where $A_0$ is the cross sectional area at $S_0$. ($\mathcal{N}$ truncated so $A$ monotonic along generators.)

## Data

- "Bulk" data on the 3-manifolds $\mathcal{N}_L$ and $\mathcal{N}_R$. "Surface" data on $S_0$.
- Bulk data = conformal 2-metric $e_{ab}(\theta^1, \theta^2, v)$
  - Induced metric on $\mathcal{N}$ degenerate because $\mathcal{N}$ is null, so

    $$ds^2 = h_{ab} d\theta^a d\theta^b \qquad \text{- no } dv \text{ terms}$$

  - Definition:

    $$e_{ab} = h_{ab}/\sqrt{\det h} \qquad \text{- makes } \det e = 1$$

  - Parametrize $e_{ab}$ by a single complex number valued field $\mu$

    $$ds^2 = h_{ab} d\theta^a d\theta^b = \frac{\rho}{1 - \mu\bar{\mu}}(dz + \mu d\bar{z})(d\bar{z} + \bar{\mu} dz)$$

    with $z = \theta^1 + i\theta^2$ and $\rho = \sqrt{det\, h_{ab}}$
- Surface data on $S_0$: $\rho_0$, $\lambda$, $\tau_a$.

# The Poisson brackets for free data on $\mathcal{N}$ for classical vacuum GR

Brackets not shown vanish.

$$\begin{aligned}\{\mu(\mathbf{1}),\bar\mu(\mathbf{2})\} &= 4\pi G\,\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)\,H(\mathbf{1},\mathbf{2})\Big[\frac{1-\mu\bar\mu}{v_A}\Big]_{\mathbf{1}}\\ &\quad\times\Big[\frac{1-\mu\bar\mu}{v_A}\Big]_{\mathbf{2}}\,e^{\int_{\mathbf{1}}^{\mathbf{2}}(\bar\mu d\mu-\mu d\bar\mu)/(1-\mu\bar\mu)}\end{aligned}$$

for **1**, **2** in the same branch, $\mathcal{N}_A$.

$$\begin{aligned}\{\rho_0(\theta_1),\lambda(\theta_2)\} &= 8\pi G\delta^2(\theta_2-\theta_1)\\ \{\rho_0(\theta),\tau[f]\} &= -8\pi G\pounds_f\rho_0(\theta)\\ \{\lambda(\theta),\tau[f]\} &= -8\pi G\Big[\pounds_f\lambda+\frac{\pounds_f\mu}{(1-\mu\bar\mu)^2}(\partial_{v_R}\bar\mu-\partial_{v_L}\bar\mu)\Big]_\theta\end{aligned}$$

$$\{\tau[f_1],\tau[f_2]\} = -16\pi G\Big[\tau[[f_1,f_2]]-\frac{1}{2}\int_{S_0}\pounds_{[f_1,f_2]}\epsilon$$
$$+\int_{S_0}[\frac{\pounds_{f_1}\mu}{(1-\mu\bar\mu)^2}\{\epsilon\pounds_{f_2}\bar\mu-\frac{1}{2}\pounds_{f_2}\epsilon(\partial_{v_R}\bar\mu+\partial_{v_L}\bar\mu)\}-(1\leftrightarrow 2)]\Big].$$

For **1** in $\mathcal{N}_R - S_0$

$$\{\mu(\mathbf{1}),\lambda(\theta_2)\} = 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)[v_R\partial_{v_R}\mu]_{\mathbf{1}}$$
$$\{\mu(\mathbf{1}),\tau[f]\} = -16\pi G\Big[\pounds_f\mu-\frac{1}{4}\frac{\pounds_f\rho_0}{\rho_0}v_R\partial_{v_R}\mu\Big]_{\mathbf{1}}.$$

For **1** in $S_0$

$$\begin{aligned}\{\mu(\mathbf{1}),\lambda(\mathbf{2})\} &= 0\\ \{\mu(\mathbf{1}),\tau[f]\} &= -8\pi G[\pounds_f\mu]_{\mathbf{1}}.\end{aligned}$$

For **1** in $\mathcal{N}_L - S_0$

$$\begin{aligned}\{\mu(\mathbf{1}),\lambda(\theta_2)\} &= 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)[v_L\partial_{v_L}\mu]_{\mathbf{1}}\\ \{\mu(\mathbf{1}),\tau[f]\} &= -4\pi G\Big[\frac{\pounds_f\rho_0}{\rho_0}v_L\partial_{v_L}\mu\Big]_{\mathbf{1}}.\end{aligned}$$

For $\mathbf{1}\in\mathcal{N}_R$ (including $\mathbf{1}\in S_0$)

$$\begin{aligned}\{\bar\mu(\mathbf{1}),\lambda(\theta_2)\} &= 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)\Big[(v_R\partial_{v_R}\bar\mu)_{\mathbf{1}}\\ &\quad+\Big(\frac{1}{v_R}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar\mu)/(1-\mu\bar\mu)}(\partial_{v_L}\bar\mu)_{\mathbf{1}_0}\Big]\\ \{\bar\mu(\mathbf{1}),\tau[f]\} &= -8\pi G\Big[\Big(2\pounds_f\bar\mu-\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}v_R\partial_{v_R}\bar\mu\Big)_{\mathbf{1}}\\ &\quad-\Big(\pounds_f\bar\mu-\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}\partial_{v_L}\bar\mu\Big)_{\mathbf{1}_0}\Big(\frac{1}{v_R}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar\mu)/(1-\mu\bar\mu)}\Big]\end{aligned}$$

where $\mathbf{1}_0\in S_0$ is the origin of the generator through **1**.
For $\mathbf{1}\in\mathcal{N}_L$

$$\begin{aligned}\{\bar\mu(\mathbf{1}),\lambda(\theta_2)\} &= 4\pi G\frac{1}{\rho_0}\delta^2(\theta_2-\theta_1)\Big[(v_L\partial_{v_L}\bar\mu)_{\mathbf{1}}\\ &\quad+\Big(\frac{1}{v_L}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar\mu)/(1-\mu\bar\mu)}(\partial_{v_R}\bar\mu)_{\mathbf{1}_0}\Big]\\ \{\bar\mu(\mathbf{1}),\tau[f]\} &= -8\pi G\Big[\Big(\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}v_L\partial_{v_L}\bar\mu\Big)_{\mathbf{1}}\\ &\quad+\Big(\pounds_f\bar\mu-\frac{1}{2}\frac{\pounds_f\rho_0}{\rho_0}\partial_{v_R}\bar\mu\Big)_{\mathbf{1}_0}\Big(\frac{1}{v_L}\Big)_{\mathbf{1}}e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}}(\mu d\bar\mu)/(1-\mu\bar\mu)}\Big].\end{aligned}$$

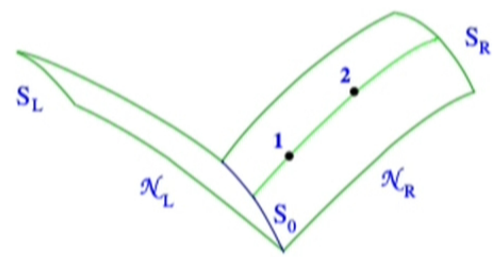
M.R. Phys. Rev. Lett. 101:211101, 2008

## Poisson brackets of the bulk data

$$\{\mu(\mathbf{1}), \mu(\mathbf{2})\} = \{\bar{\mu}(\mathbf{1}), \bar{\mu}(\mathbf{2})\} = 0$$

$$\{\mu(\mathbf{1}), \bar{\mu}(\mathbf{2})\} = 4\pi G \frac{1}{\sqrt{\rho_1 \rho_2}} \delta^2(\theta_2 - \theta_1)\, H(\mathbf{1}, \mathbf{2}) \times [1 - \mu\bar{\mu}]_{\mathbf{1}} [1 - \mu\bar{\mu}]_{\mathbf{2}}\, e^{\int_{\mathbf{1}}^{\mathbf{2}} (\bar{\mu} d\mu - \mu d\bar{\mu})/(1 - \mu\bar{\mu})}.$$

$H(\mathbf{1}, \mathbf{2})$ step function $= 1$ if $\mathbf{2}$ follows $\mathbf{1}$ along the generator, 0 otherwise.




- Only data on same generator have non-zero bracket. Consistent with causality: points on distinct generators spacelike separated.
- Bracket does not quite preserve reality of induced metric on $\mathcal{N}$, but imaginary mode is shock wave that does not enter interior of domain of dependence. Bracket preserves reality of metric there.

# A simpler problem

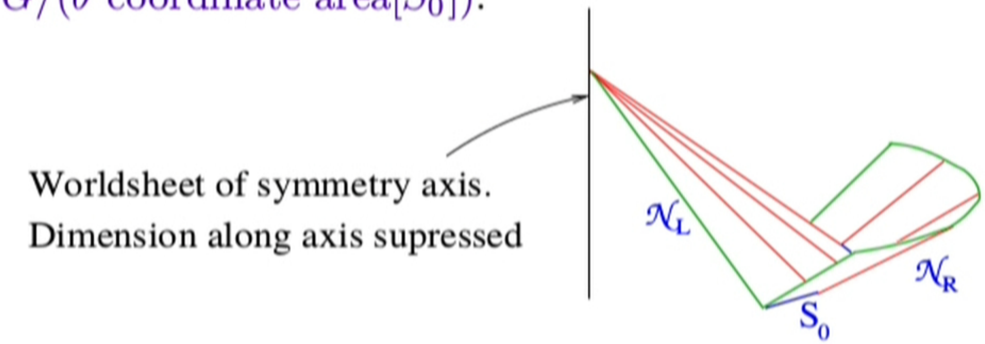
- Step toward quantization: quantize the "one generator algebra"

$$\{\mu(\mathbf{1}), \mu(\mathbf{2})\} = \{\bar{\mu}(\mathbf{1}), \bar{\mu}(\mathbf{2})\} = 0$$

$$\{\mu(\mathbf{1}), \bar{\mu}(\mathbf{2})\} = 4\pi G' \frac{1}{\sqrt{\rho_1 \rho_2}} H(\mathbf{1}, \mathbf{2}) \times [1 - \mu\bar{\mu}]_{\mathbf{1}} [1 - \mu\bar{\mu}]_{\mathbf{2}} \, e^{\int_{\mathbf{1}}^{\mathbf{2}} (\bar{\mu} d\mu - \mu d\bar{\mu})/(1-\mu\bar{\mu})}.$$

Brackets with $\delta^2(\theta_2 - \theta_1)$ removed. $\mu$, $\bar{\mu}$ functions on single line.

- These are brackets in cylindrically symmetric GR on $\mathcal{N}$ swept out by radial light rays from axis, provided $G' = G/(\theta \text{ coordinate area}[S_0])$.




- Obtained as brackets of averages $\langle \mu \rangle$ and $\langle \bar{\mu} \rangle$ over symmetry orbits at symmetric solutions, or from symmetry reduced action.

# Transformation to new variables

- Cylindrically symmetric GR is an integrable system. Quantization exists [Korotkin and Samtleben 1998].
- Transform in steps from $\mu$, $\bar{\mu}$ to variables with known quantization

$$\mu, \bar{\mu} \mapsto \mathcal{V} \mapsto \hat{\mathcal{V}} \mapsto \mathcal{E}$$

- $\mu, \bar{\mu} \mapsto \mathcal{V}$: $\mathcal{V}$ is zweibein for conformal 2-metric $e_{ab}$ on symmetry orbits - $e_{ab} = \mathcal{V}_a{}^i \mathcal{V}_b{}^j \delta_{ij}$. Equivalently $e = \mathcal{V}\mathcal{V}^T$.

$$\mathcal{V} = \frac{1}{\sqrt{1-\mu\bar{\mu}}} \frac{1}{\sqrt{(1-\mu)(1-\bar{\mu})}} \begin{bmatrix} 1-\mu\bar{\mu} & -i(\mu-\bar{\mu}) \\ 0 & (1-\mu)(1-\bar{\mu}) \end{bmatrix}$$

- We treat only the branch $\mathcal{N}_L$ which touches symmetry axis.
- $\mathcal{V}$ assumed regular at axis. (Implies 4-metric not regular there).

- $\mathcal{V} \mapsto \hat{\mathcal{V}}$: Define

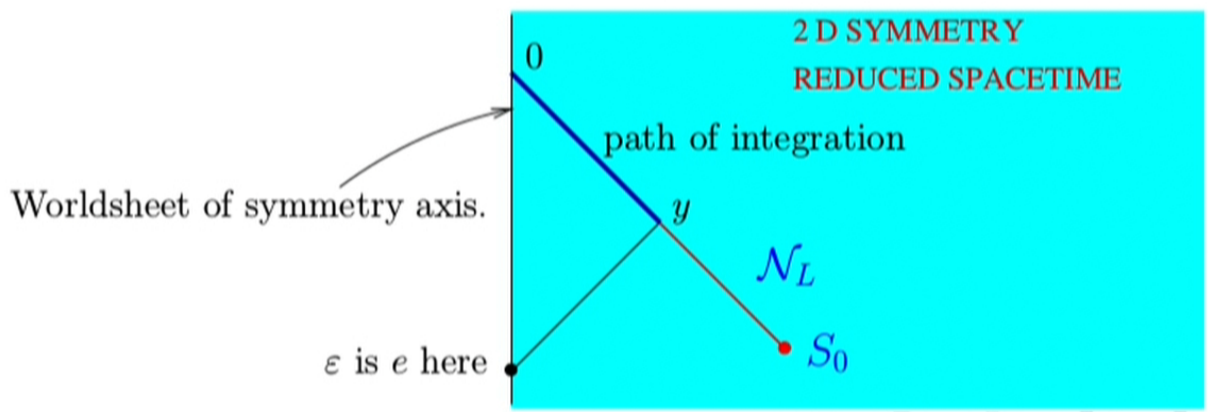
$$J = \mathcal{V}^{-1} d\mathcal{V}$$

so $\mathcal{V}(y) = \mathcal{V}(\mathbf{0})\, \mathcal{P} e^{\int_{\mathbf{0}}^{y} J}$ at $y \in \mathcal{N}_L$, with $\mathbf{0}$ point on $\mathcal{N}_L$ on axis.

$$P = \tfrac{1}{2}(J + J^T) \quad Q = \tfrac{1}{2}(J - J^T)$$

$$\hat{J}(x;y) = Q(x) + \frac{1}{\sqrt{1 - \rho(x)/\rho(y)}} P(x) \quad \hat{\mathcal{V}}(y) = \mathcal{V}(\mathbf{0})\, \mathcal{P} e^{\int_{\mathbf{0}}^{y} \hat{J}(\cdot;y)}$$

- $\hat{\mathcal{V}} \mapsto \mathcal{E}$: "Deformed conformal metric" $\mathcal{E}(y) = \hat{\mathcal{V}}(y) \hat{\mathcal{V}}^T(y)$
  - Map $\mu, \bar{\mu} \mapsto \mathcal{E}$ invertible, mod imaginary shockwave mode.
  - On solutions $\mathcal{E}(y)_{ab}$ is $e_{ab}$ on axis and past lightcone of $y$.



# Poisson algebra of new variables

- A lengthy calculation yields

$$\{\mathcal{E}_{ab}(\mathbf{1}), \mathcal{E}_{cd}(\mathbf{2})\} = p.v.(\frac{64\pi G'}{\rho(\mathbf{1}) - \rho(\mathbf{2})}) \times \mathrm{Sym}_{(ab),(cd)} \left( \mathcal{E}_{ad}(\mathbf{1})\mathcal{E}_{cb}(\mathbf{2}) - \frac{1}{2}\mathcal{E}_{ab}(\mathbf{1})\mathcal{E}_{cd}(\mathbf{2}) \right),$$

- $p.v.(1/x)$ is Cauchy principal value distribution of $1/x$.

## Quantization

- Korotkin and Samtleben 1998 presented associative algebra that quantizes the Poisson algebra of the $\mathcal{E}$s:

$$R(\Delta)\overset{1}{\mathcal{E}}(\mathbf{1})R'(-\Delta+2ia_Q)\overset{2}{\mathcal{E}}(\mathbf{2}) = \overset{2}{\mathcal{E}}(\mathbf{2})\,{}^tR'^t(\Delta+2ia_Q)\overset{1}{\mathcal{E}}(\mathbf{1})\,{}^tR^t(-\Delta)\frac{\Delta-2ia_Q}{\Delta+2ia_Q},$$

$\mathcal{E}_{ab} = \mathcal{E}_{ba}$, *-algebra with $\mathcal{E}^* = \mathcal{E}$.

  - $\Delta = a(\mathbf{1}) - a(\mathbf{2})$ = difference between areas of cylindrical symmetry orbits in $\mathcal{N}_L$ through **1** and **2**.
  - $a_Q = 8\pi G\hbar = 8\pi\times$ Planck area.
  - $R(u)_a{}^b{}_c{}^d = u\delta_a^b\delta_c^d - ia_Q\delta_a^d\delta_c^b$
  - $R'(u)^a{}_{bc}{}^d = (u-ia_Q)\delta_b^a\delta_c^d + ia_Q\delta_c^a\delta_b^d$
  - $R$, $R'$ tensors on product of tangent space of cyl. symmetry orbits at **1** and **2**. $\overset{1}{\mathcal{E}}(\mathbf{1})$ lives in first space, $\overset{2}{\mathcal{E}}(\mathbf{2})$ in the second.
- KS give quantization of $\det\mathcal{E} = 1$ compatible with algebra, but only in asymptotically flat case.
- Algebra of quantum data known, but not suitable representation on Hilbert space. Algebra closely related to $\mathfrak{sl}(2)$ Yangian double.

# General case, without cylindrical symmetry

- In absence of cylindrical symmetry can still apply transformation $\mu, \bar{\mu} \mapsto \mathcal{E}$ to data on each generator. Then

$$\{\mathcal{E}_{ab}(\mathbf{1}), \mathcal{E}_{cd}(\mathbf{2})\} = \delta^2(\theta_1 - \theta_2) p.v.(\frac{64\pi G}{\rho(\mathbf{1}) - \rho(\mathbf{2})}) \times \mathrm{Sym}_{(ab),(cd)} \left( \mathcal{E}_{ad}(\mathbf{1}) \mathcal{E}_{cb}(\mathbf{2}) - \frac{1}{2} \mathcal{E}_{ab}(\mathbf{1}) \mathcal{E}_{cd}(\mathbf{2}) \right),$$

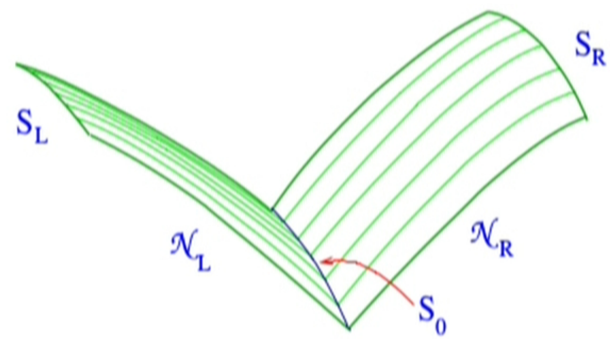
- Can formally generalize quantization, but result is ambiguous - products of $\delta$s etc.

# To do

- Quantize $\det \mathcal{E} = 1$.
- Study the representations of the $\mathcal{E}$ algebra, in general, and in single polarization model [Kuchar 1971].
- $S_0$ data $\lambda$ and $\rho_0$ are present in cylindrically symmetric GR. Here we have ignored them. They should be incorporated into Poisson algebra and quantization. $\tau$ is also present in cylindrically symmetric GR (as "twist constants") but is system integrable if they are non-zero?
- Use the results from cylindrically symmetric GR to quantize data in full GR.

# Double null sheets as initial data hypersurfaces

- A double null sheet is a pair of intersecting null hypersurfaces (or "lightfronts") - like an open book in spacetime.




- $\mathcal{N}_R$, $\mathcal{N}_L$ are 3-surfaces swept out by null geodesics emerging normally from the two sides of 2-disk $S_0$.