Title: Moonshine, topological modular forms, and 576 fermions.

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Abstract: I will report on progress understanding the 576-fold periodicity in TMF in terms of conformal field theoretic constructions. Sporadic finite groups and their cohomology will play a role.

576 fermions

Motivating theorem [Stolz–Teichner]

Let X be a manifold.

Consider a cat

morphisms: "$\mathcal{N}=1$ SUSY 1-manifold in X"

1-manifolds locally $\mathbb{R}^{1|1}$ w/ local geometry

576 fermions

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$M(X \times [0,1])$

↙ restrict to 0 ↘ restrict to 1

$M(X)$ $M(X)$

$M(X) = Fun_{sym\otimes}(EBord^{1|1}(X), SVect_{\mathbb{R}})$

$=$ "$\mathcal{N}=1$ SUSY 1-D QFTs over X"

this space is a cocycle model for $KO(X)$.

By this I mean that

$M(X)$ / "cobordism of field theories" $= KO^0(X)$

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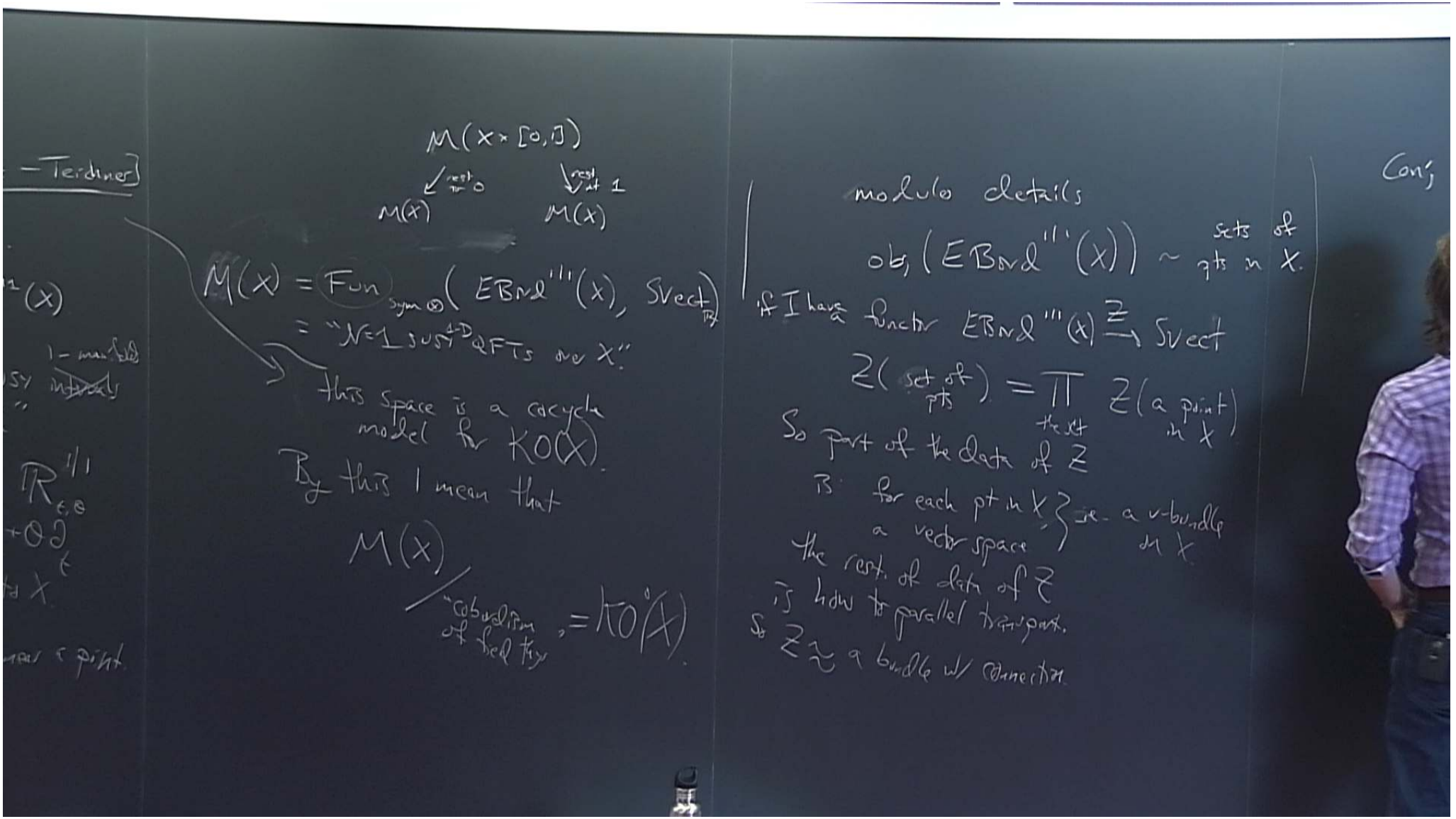
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modulo details
$\mathrm{ob}(EBrd'''(X)) \sim$ sets of pts in X.
if I have a functor $EBrd'''(X) \xrightarrow{Z} \mathrm{Svect}$
$Z(\text{set of pts}) = \prod_{\text{the set}} Z(\text{a point in } X)$
So part of the data of Z
is: for each pt in X, } i.e. a v-bundle on X
a vector space
the rest of data of Z
is how to parallel transport.
So $Z \cong$ a bundle w/ connection
Con;

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Conj [Segal - Stolz - Teichner]

There is an analogy

$\mathcal{N}=1$	1-D QFT	KO
$\mathcal{N}=1$	2-D QFT	TMF

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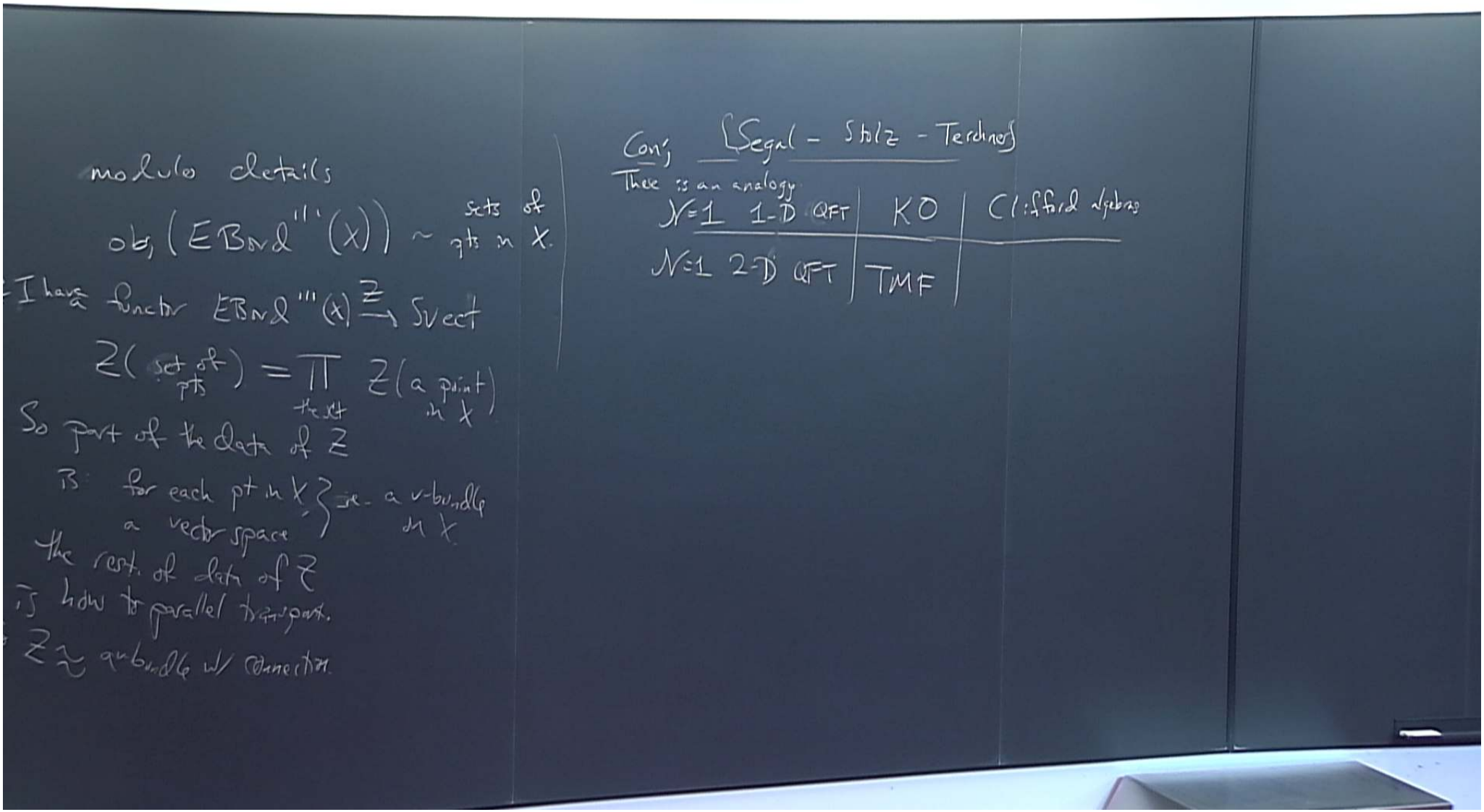
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N=1 1-D QFT | KO | Clifford algebras
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$M^n(X)$ = space of sym monoidal (∞,n)-cat trans

$EBord_2(X)$ ⇓ SAlgs bimods intertwiners

trivial functor

the 1-TFT determined by $Cliff(n)$.

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$$\frac{M(X)}{\text{"cobordism of field thys"}} = KO^0(X)$$

modulo det

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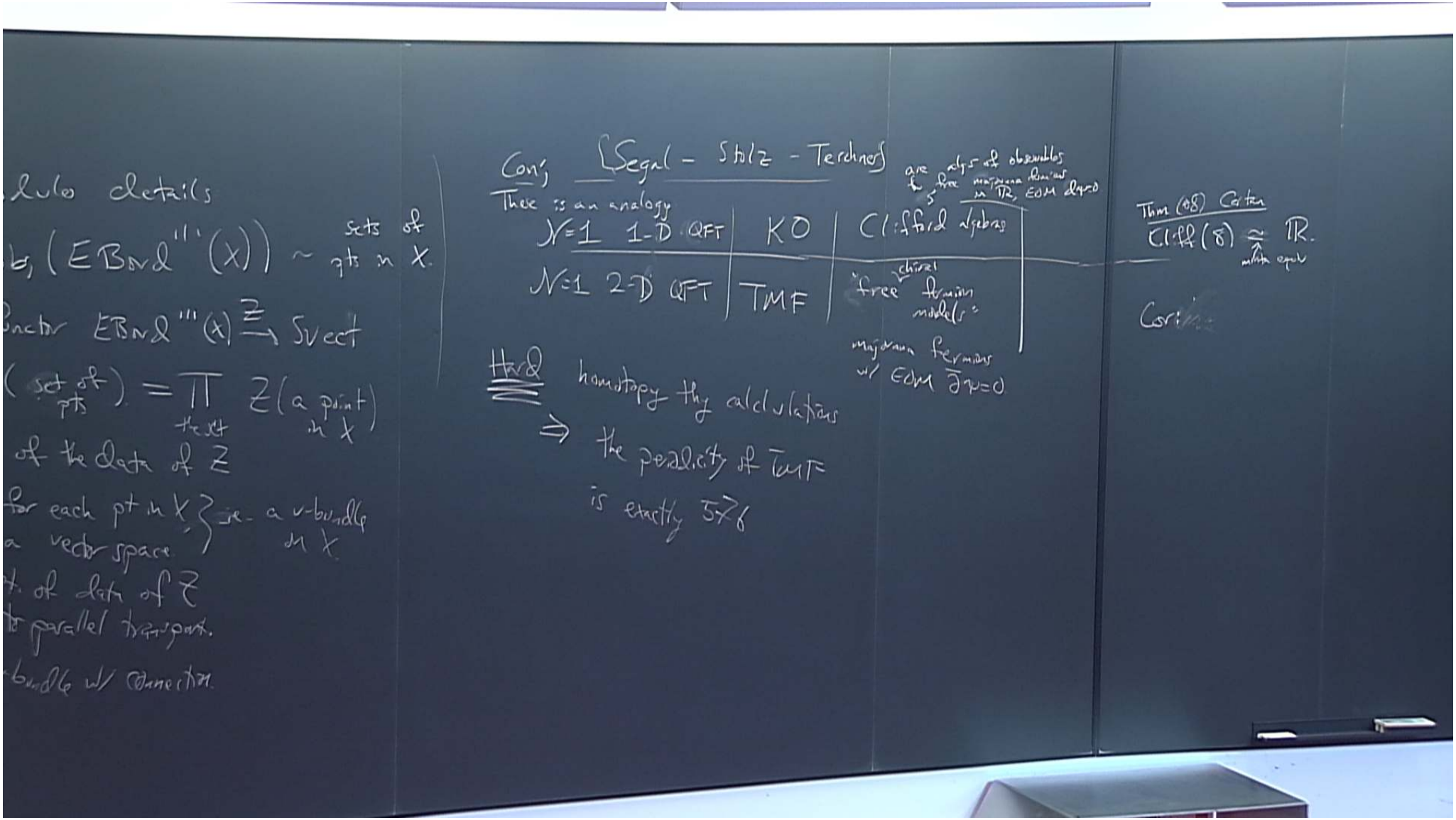
Cons; [Segal - Stolz - Teichner]
are the algs of observables for 5 free majorana fermions in TR, EOM dyed
Thm (08) Cotton
Cliff(8) ≅ TR.
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N=1 1-D QFT | KO | Clifford algebras
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majorana fermions w/ EOM ∂̄ψ=0
Hard homotopy thy calculations
⇒ the periodicity of TMF is exactly 576
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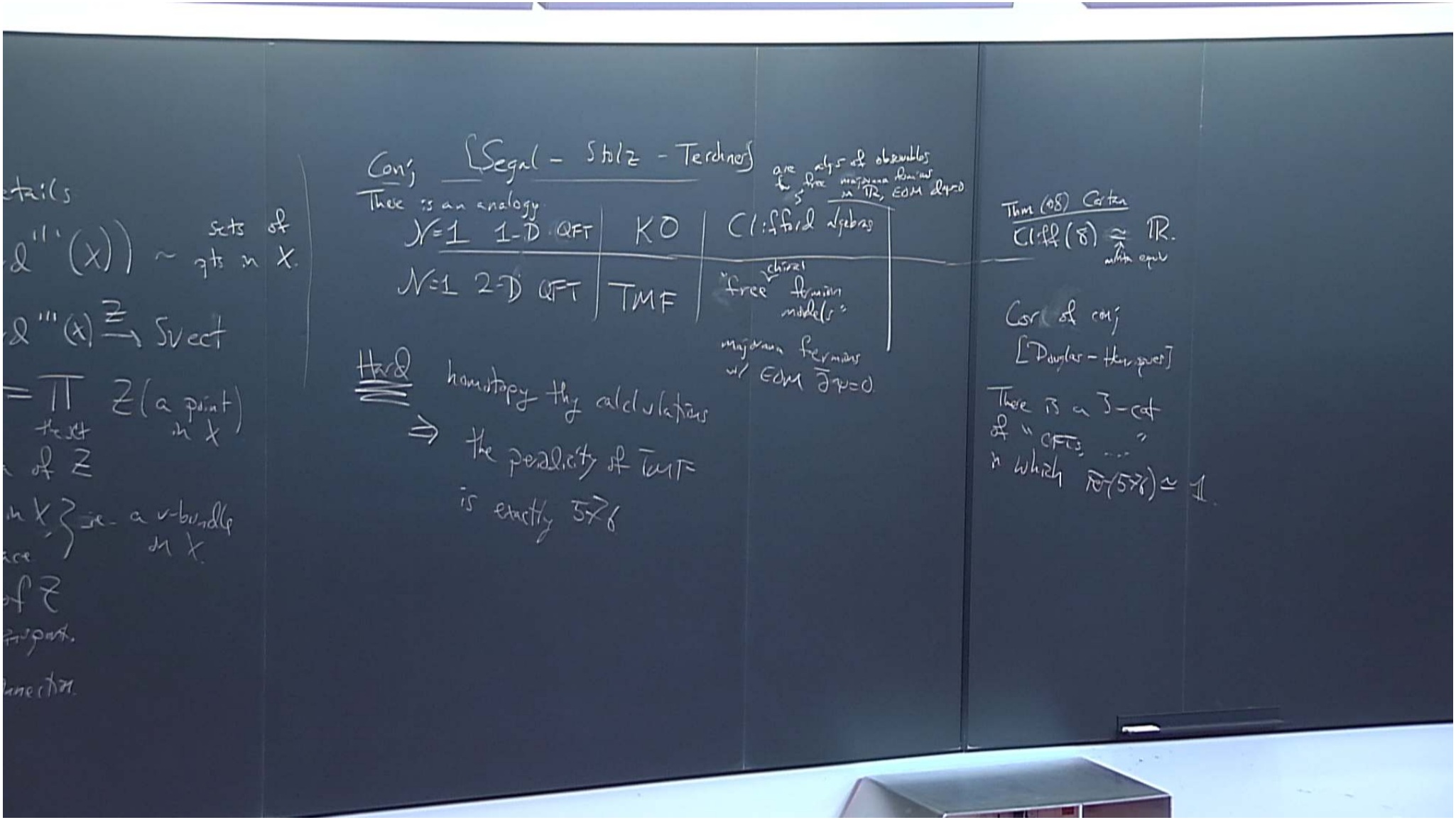
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Cliff(8) ≅ 1R.
unital equiv
Cor of conj
[Douglas - Henriques]
There is a 3-cat of "CFTs, ..." in which Ω⁷(S⁷) ≅ 1
sets of pts in X.
Z(a point) in X
a v-bundle on X.

How to find Morita equivs?

Suppose $G \curvearrowright A$ (top. gp. acting on alg.)

a Hamiltonian for this action is a homomorphism $\mu: G \to A^\times$ (multiplicative map)

s.t. action of $g \in G$ on $a \in A$ is $a \mapsto \mu(g)\, a\, \mu(g)^{-1}$

$M^n(X)$ = space of sym mon (ax not trans)

$EBord_2(X)$ ⇓ trivial functor $\Rightarrow$ $SAlg_\infty$ (bimod? Grothendieck)

the 1-2-FT determined by $Cliff(n)$.

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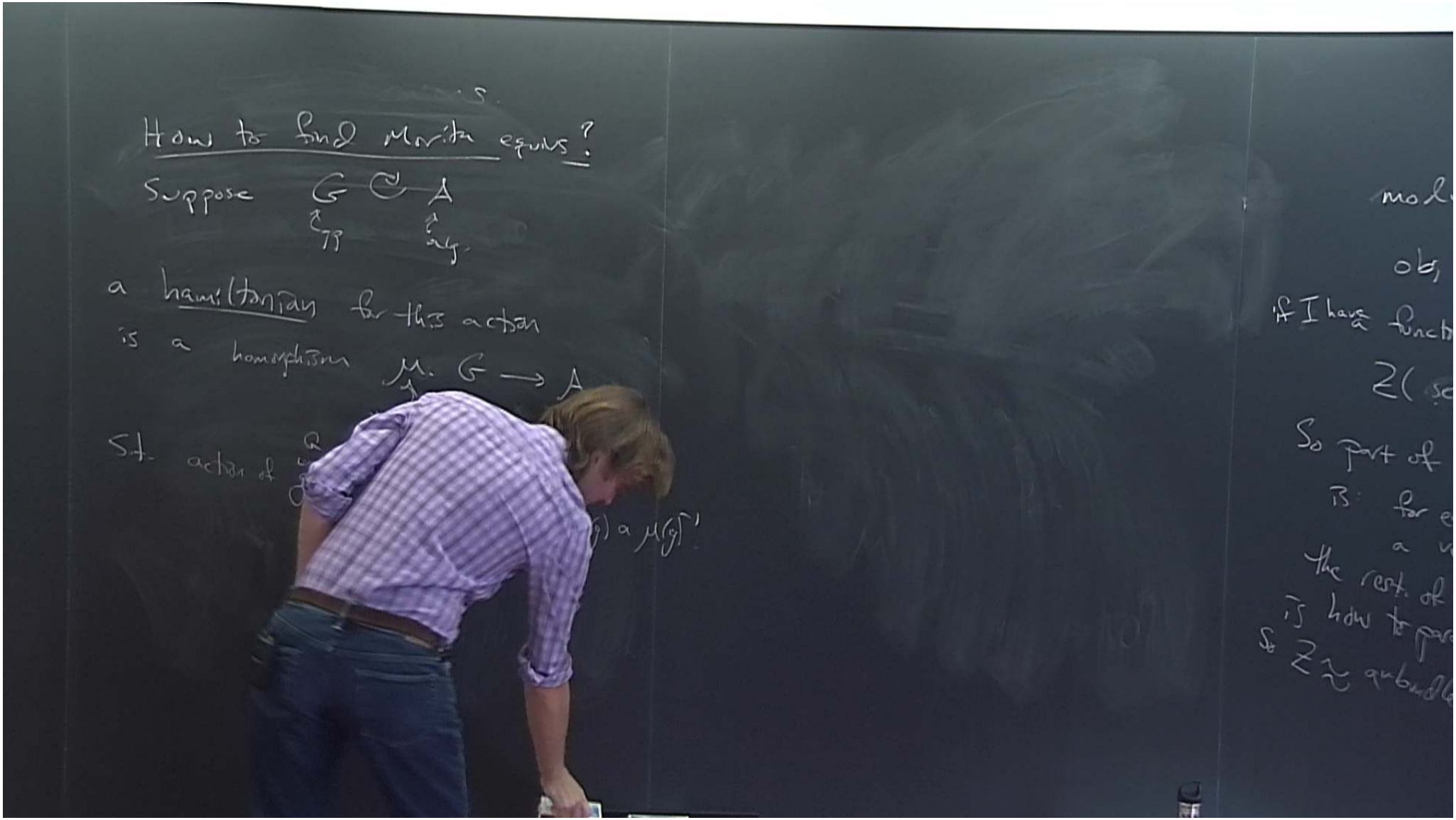
How to find Morita equivs?
Suppose G ↷ A
a hamiltonian for this action
is a homomorphism μ: G → A
s.t. action of
mod
ob
if I have a funct
Z(
So part of
B: for e
a
the rest of
is how to
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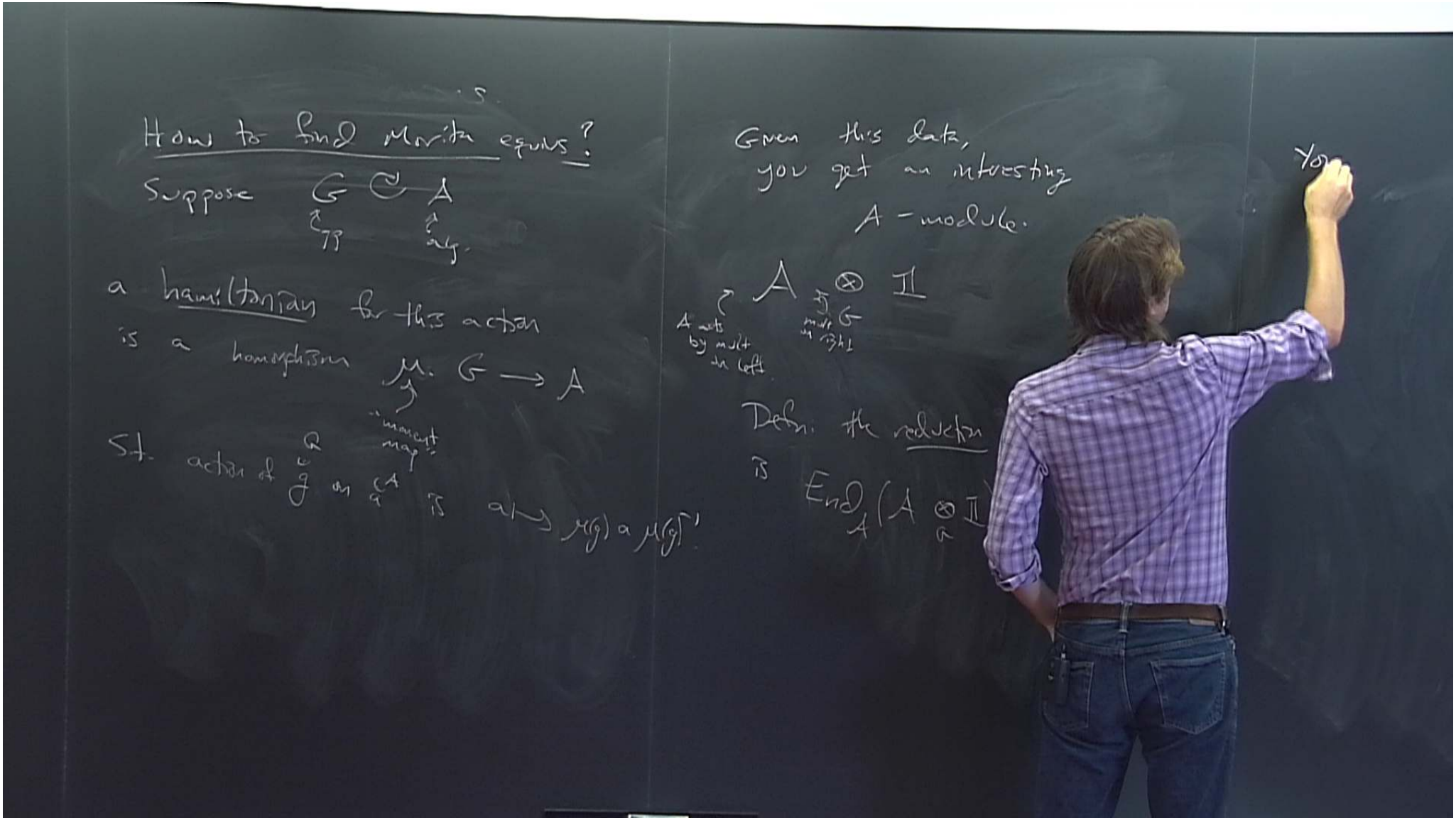
How to find Morita equivs?
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moment map
s.t. action of G on A is a ↦ μ(g) a μ(g)'
Given this data,
you get an interesting
A-module.
A ⊗ Π
A acts by mult on left
G acts on right
Defn. the reduction
is End_A(A ⊗ Π)

Given this data, you get an interesting A-module.

$A \otimes_G \Pi$

A acts by mult on left; G mult on right

Defn. the reduction $A/\!/G$ is $\mathrm{End}_A(A \otimes_G \Pi)$.

) a $\mu(g)$'

You can hope that the bimodule

$A \xrightarrow{A \otimes_G \Pi} A/\!/G$

is a Morita equiv.

If $G \subset A$ ~~reductive~~ nice enough, then $A/\!/G = (A \otimes_G \Pi)^G$

$= A^G /\!/ \ldots$

Conj. [illegible]

There is an ana

$N=1$ 1

$N=1$ 2-

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E.g.

If G

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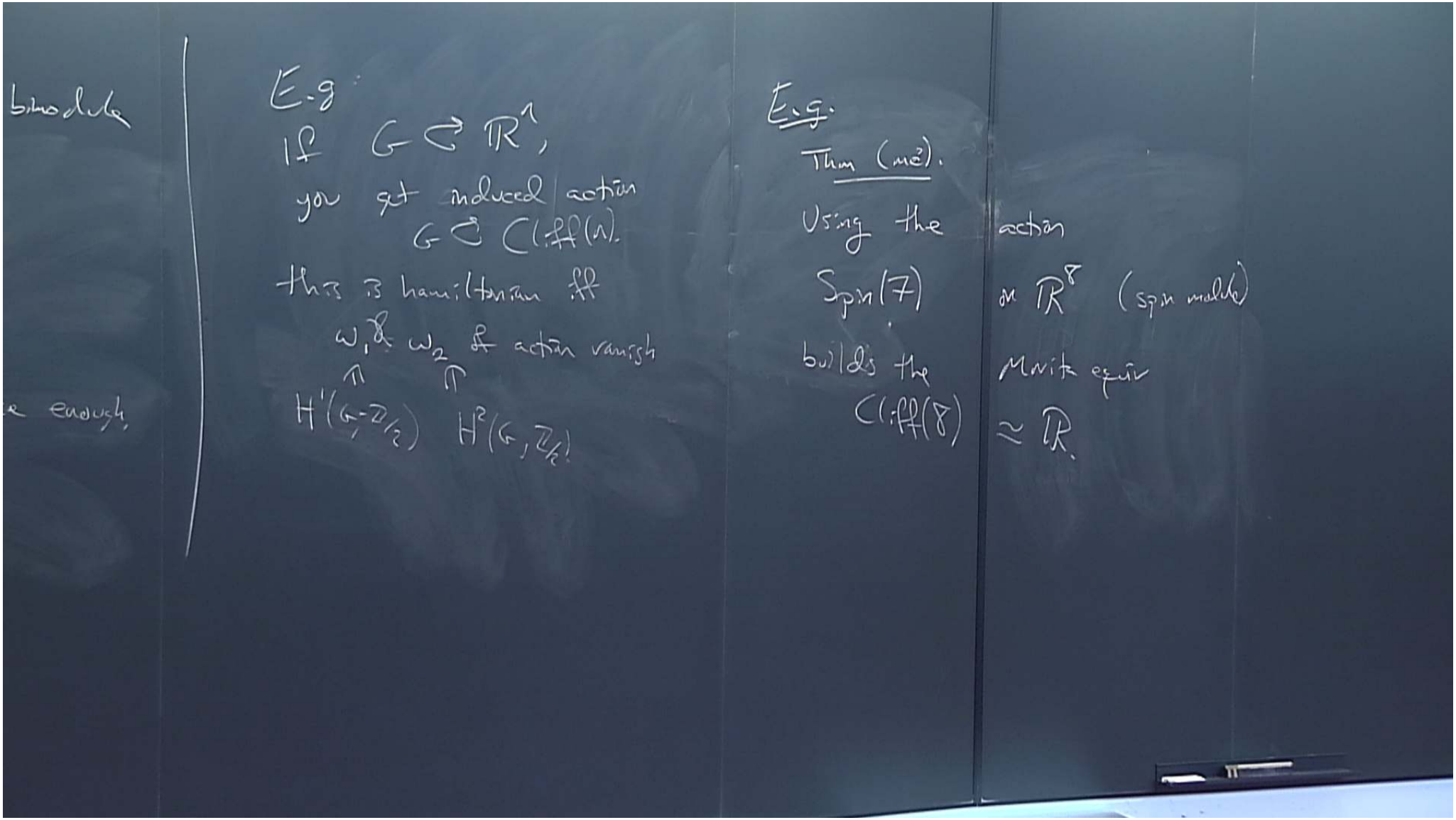
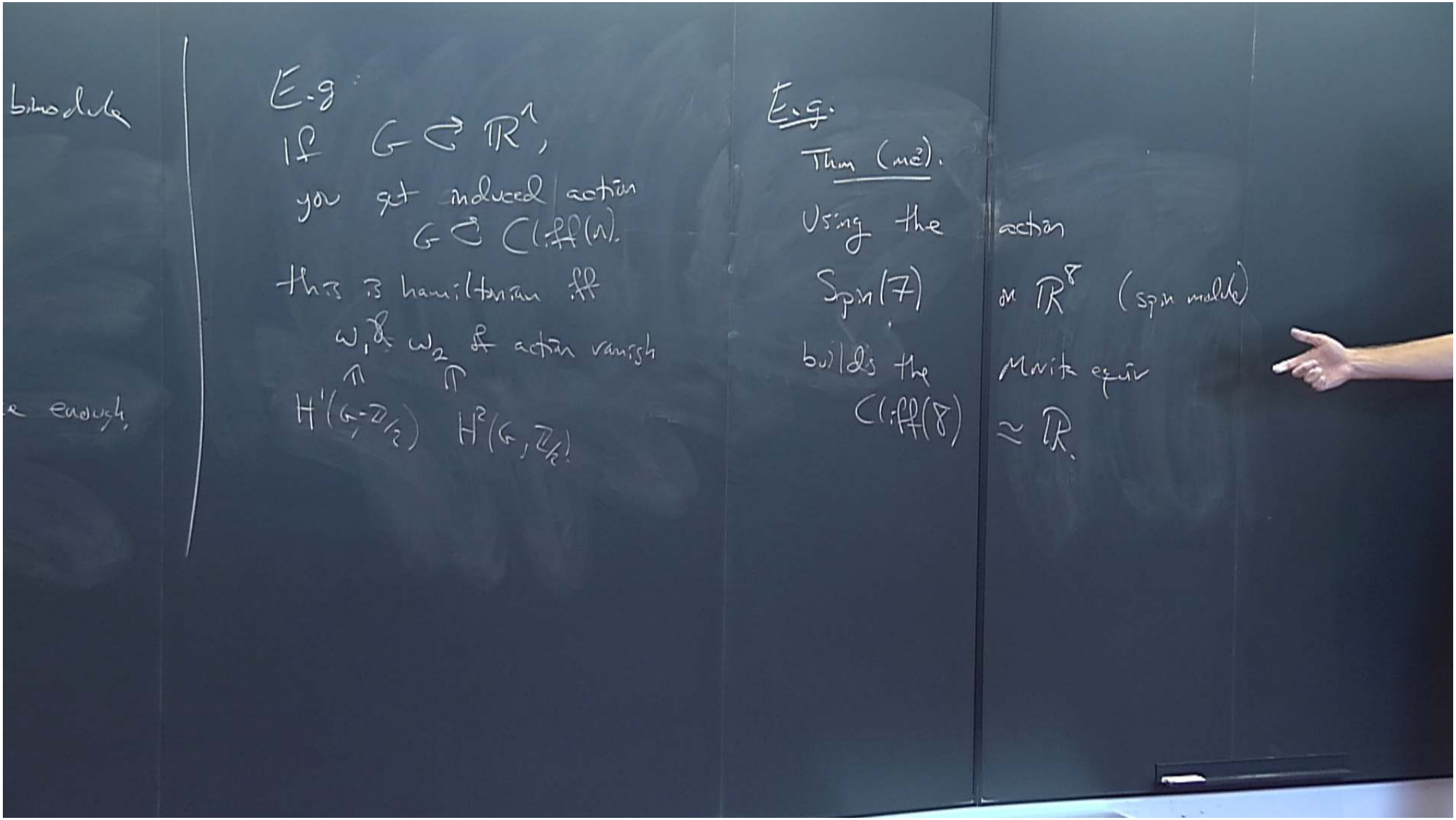

E.g.
If $G \curvearrowright \mathbb{R}^1$,
you get induced action
$G \curvearrowright Diff(n)$.
this is hamiltonian iff
ω_1 & ω_2 of action vanish
$H^1(G; \mathbb{Z}/2)$ $H^2(G, \mathbb{Z}/2)$
E.g.
Thm (me).
Using the action
Spin(7)
on $\mathbb{R}^8$ (spin module)
builds the
Minkowski equiv
$Diff(8)$
$\simeq \mathbb{R}$.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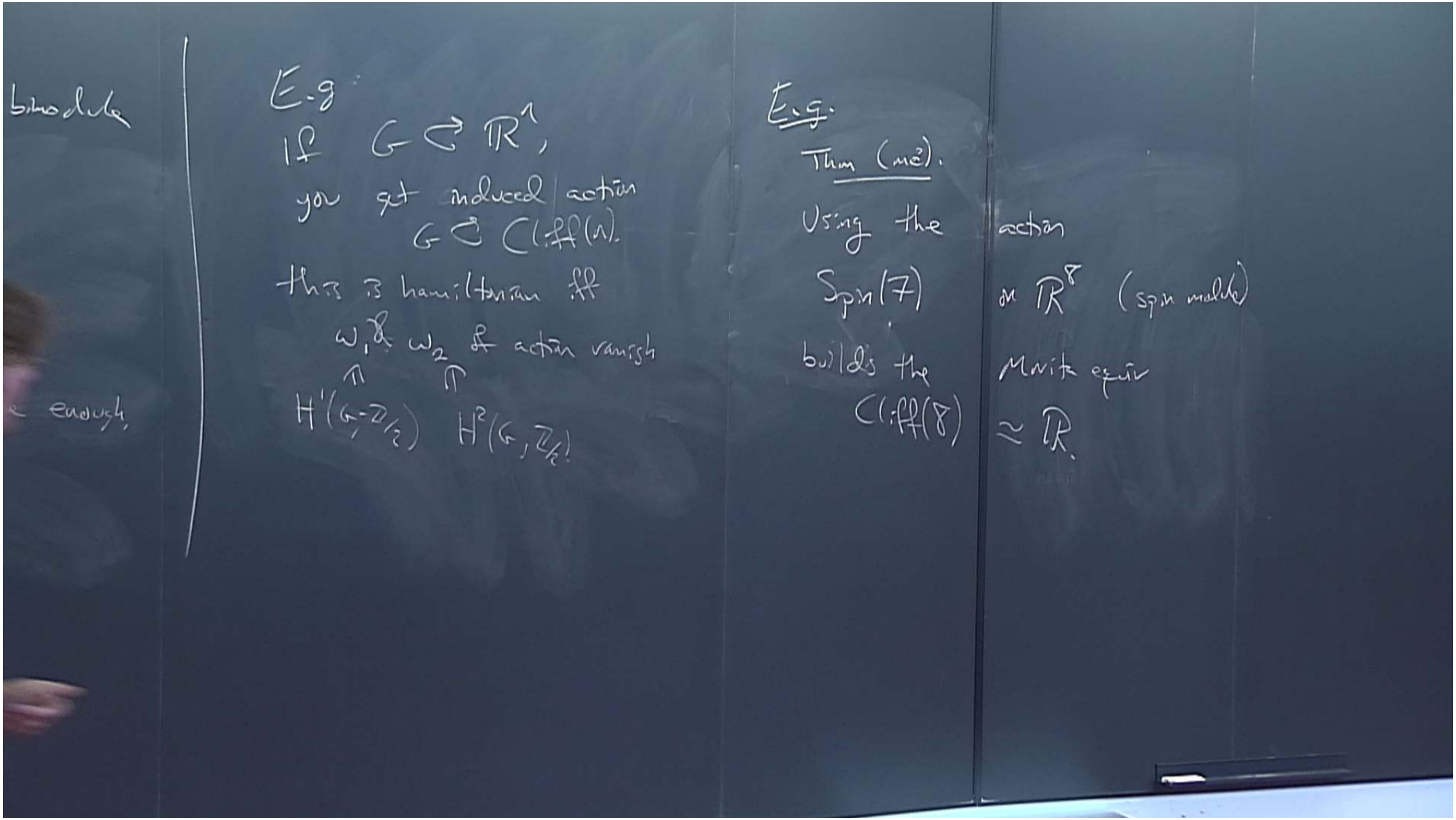
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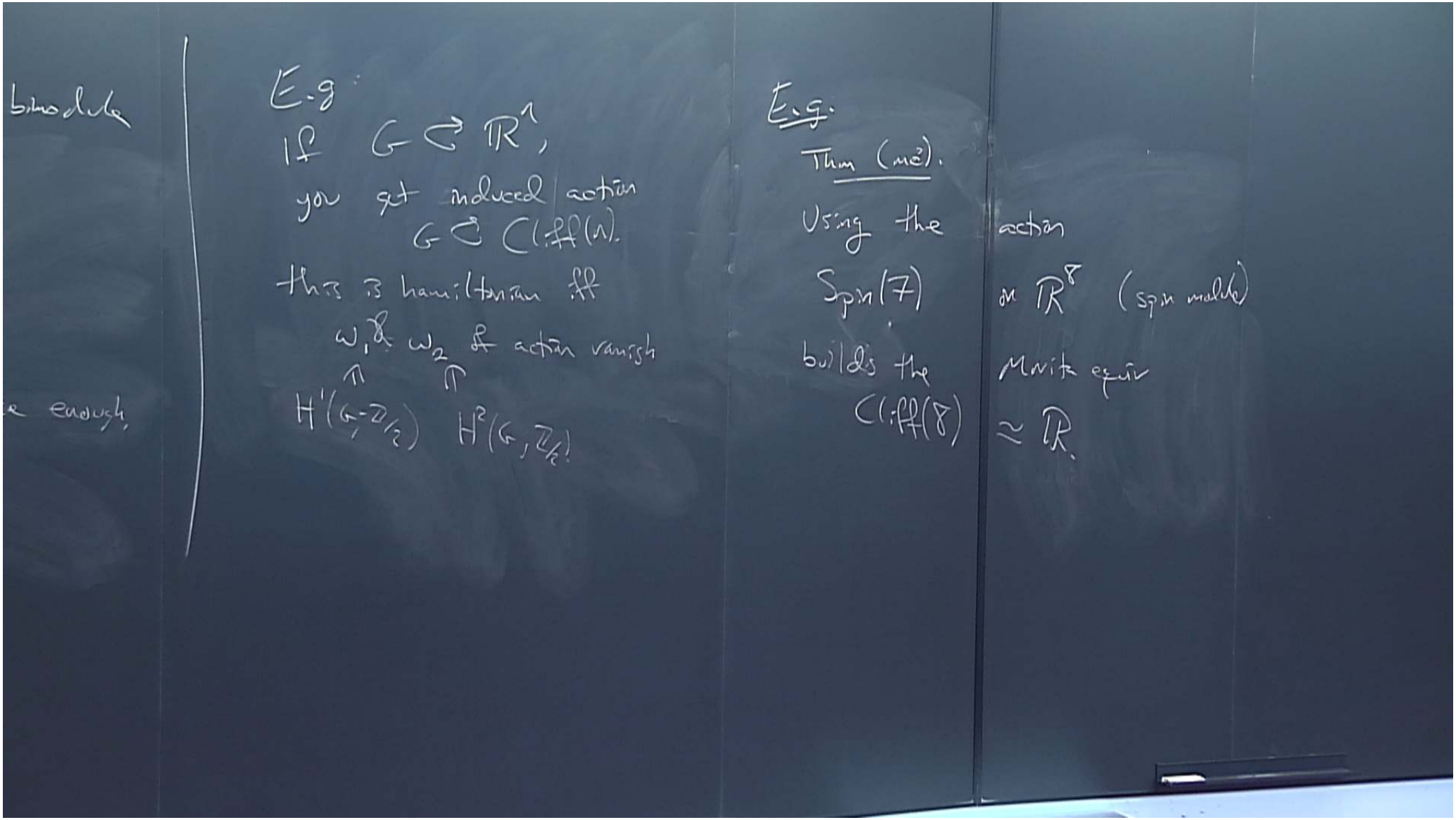
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How to find Morita equivs?

Suppose $G \curvearrowright A$

G: ?? A: ? monoidal cat

play the same game.

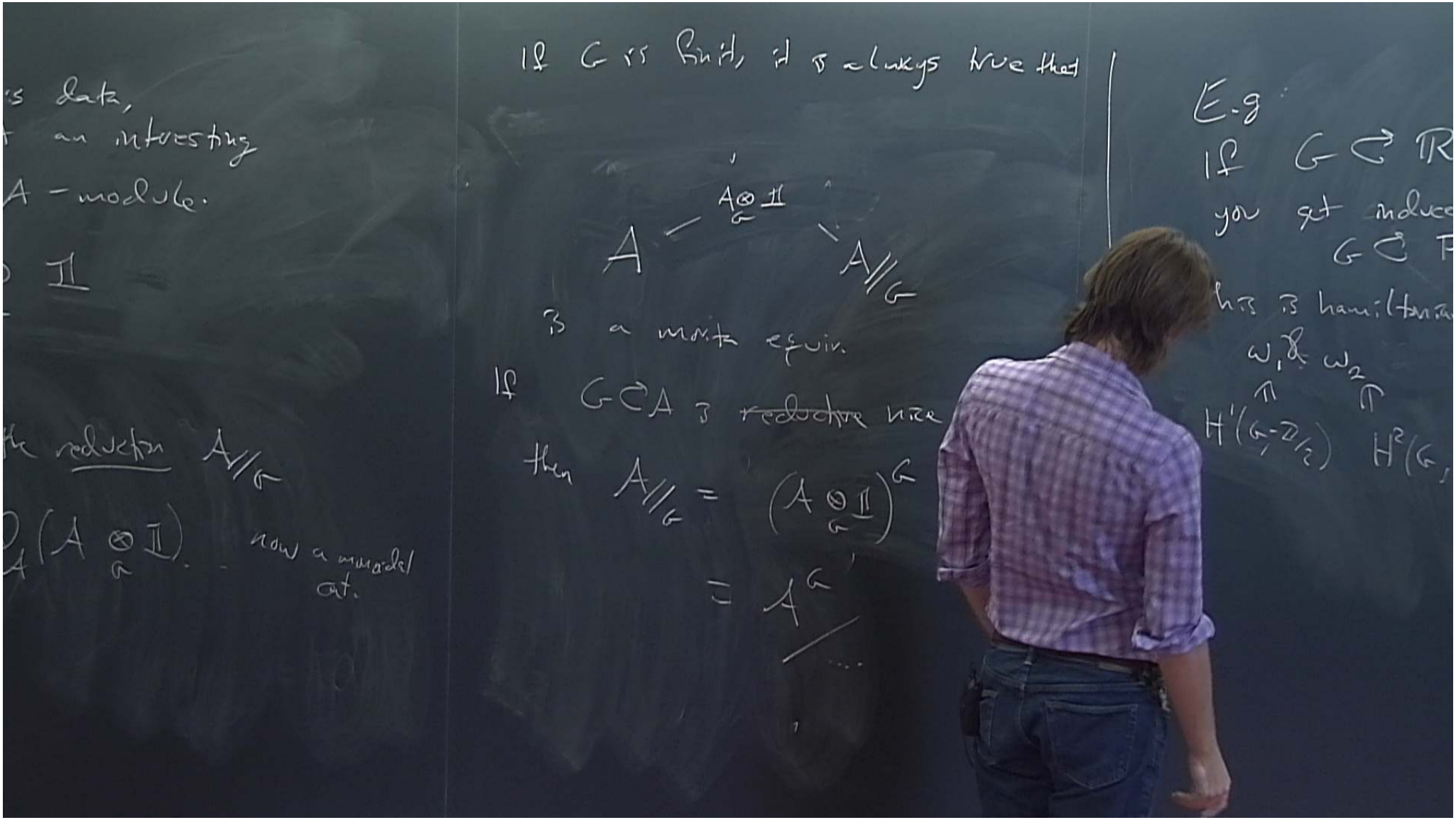
i.e. realize the action as conj via a monoidal functor $G \to A$.

Given this data, you get an interesting A-module.

$A \otimes \mathbb{1}$

(A acts by mult on left; G mult on right)

Defn. the reduction $A/\!/G$ is $\mathrm{End}_A(A \otimes \mathbb{1})^G$. now a monoidal cat.


is data,
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II
the reduction A//G
(A ⊗ II)
now a manifold
If G is finite, it is always true that
A ⊗ II
A
A//G
is a morita equiv.
If G ⊂ A is reductive nice
then A//G = (A ⊗ II)^G
= A^G
E.g
If G ⊂ R
you get induce
G ⊂ F
his is hamiltonia
ω1 & ω2
H'(G, Z/2)
H²(G,

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$\uparrow$ $\uparrow$

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E.g.

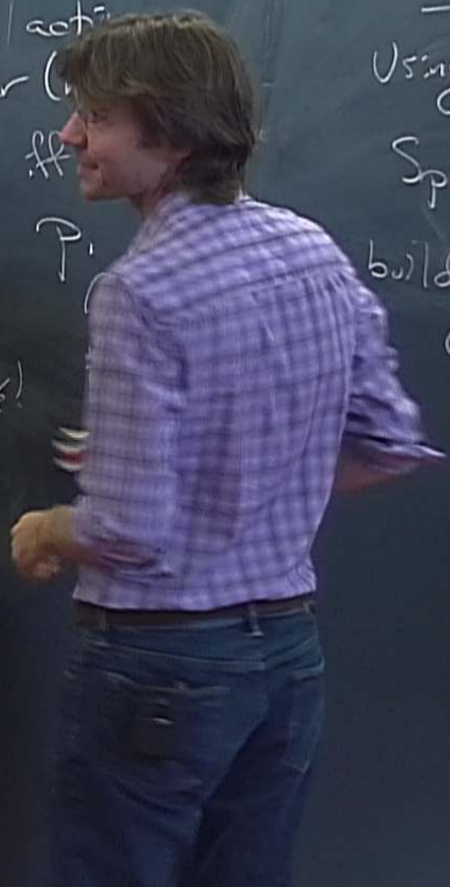
Thm (

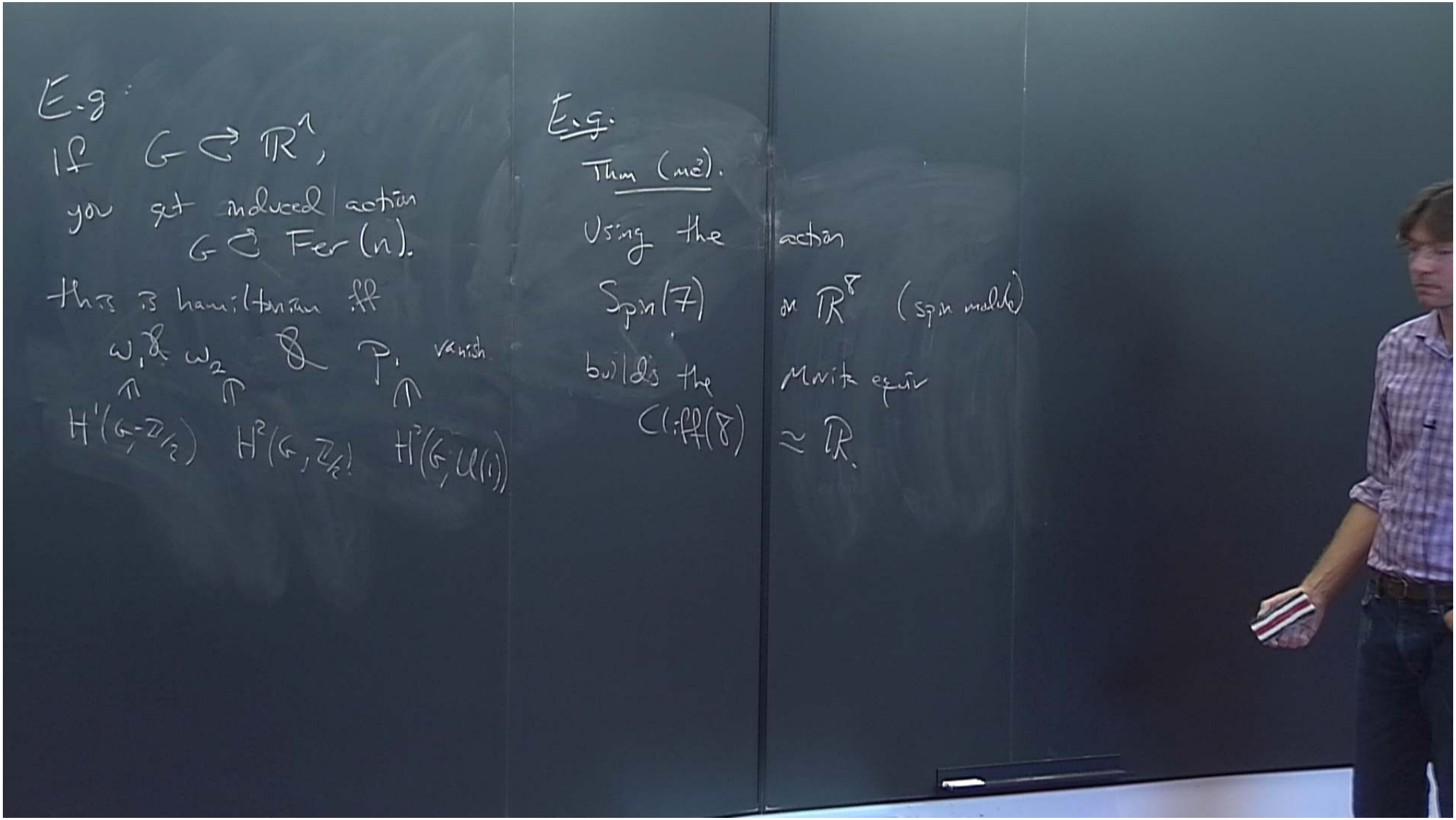
Using t

$Spin(7$

builds t

C(iff




E.g:
If $G \curvearrowright \mathbb{R}^1$,
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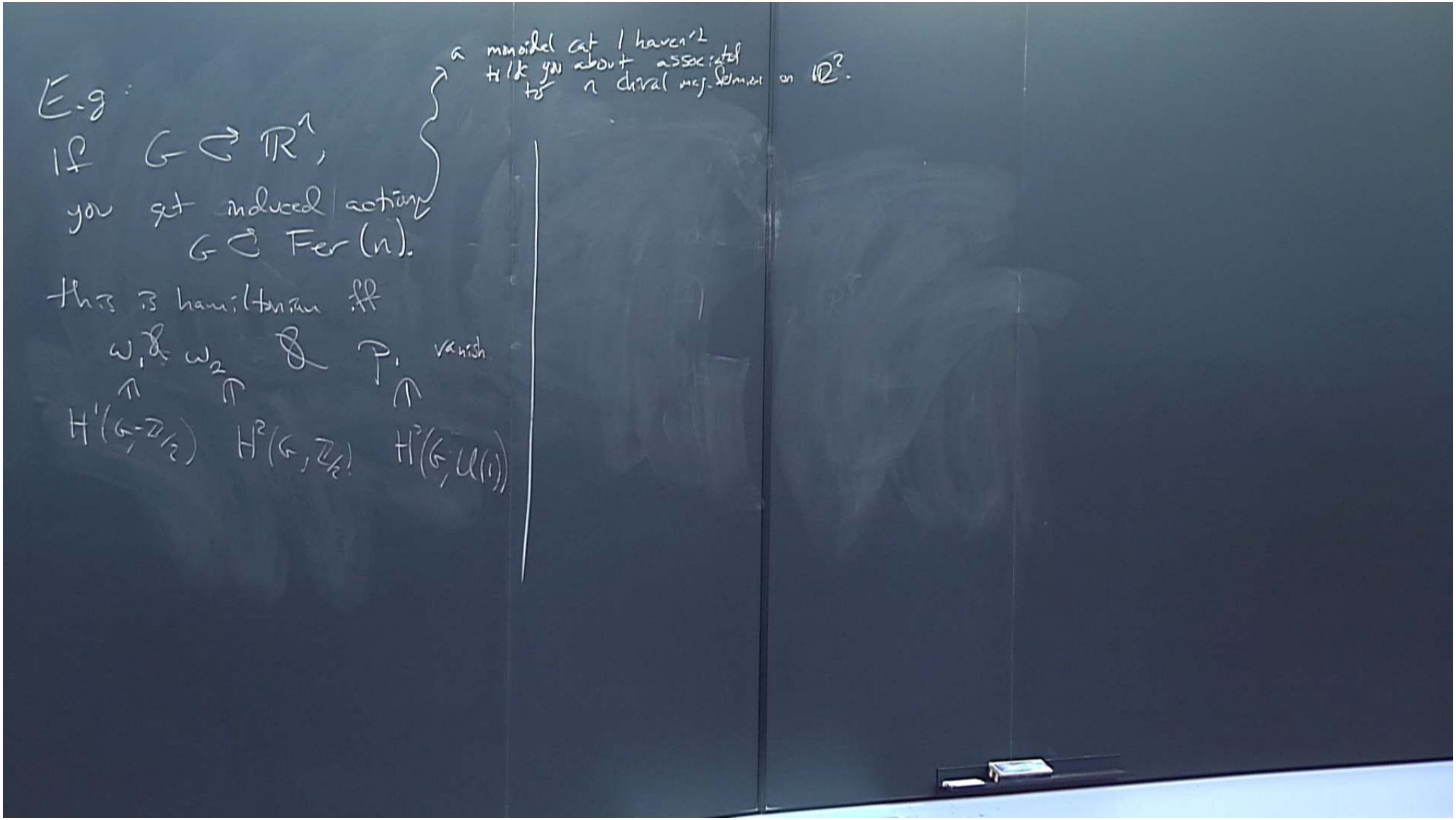
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a monoidal cat I haven't
told you about associated
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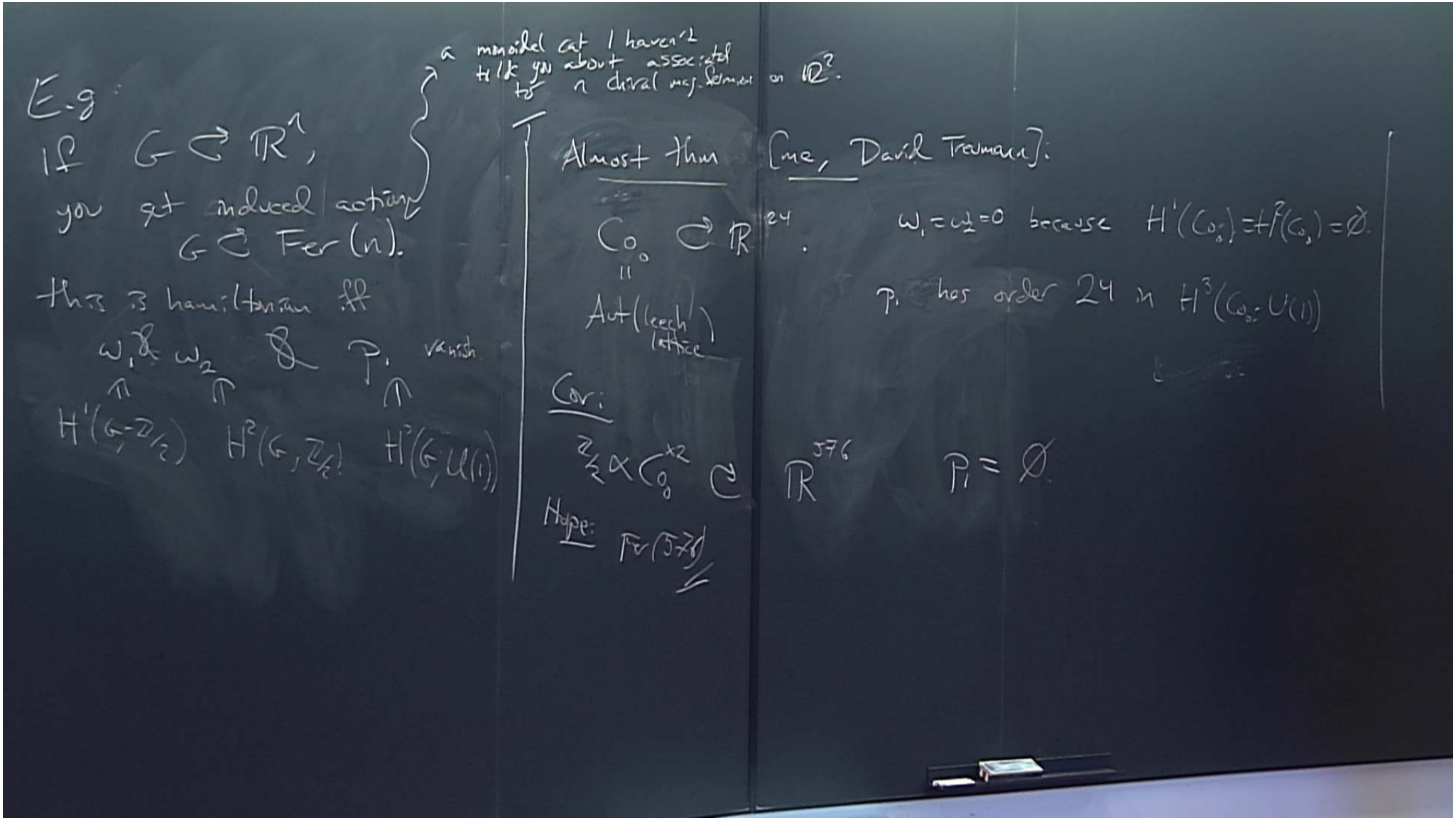
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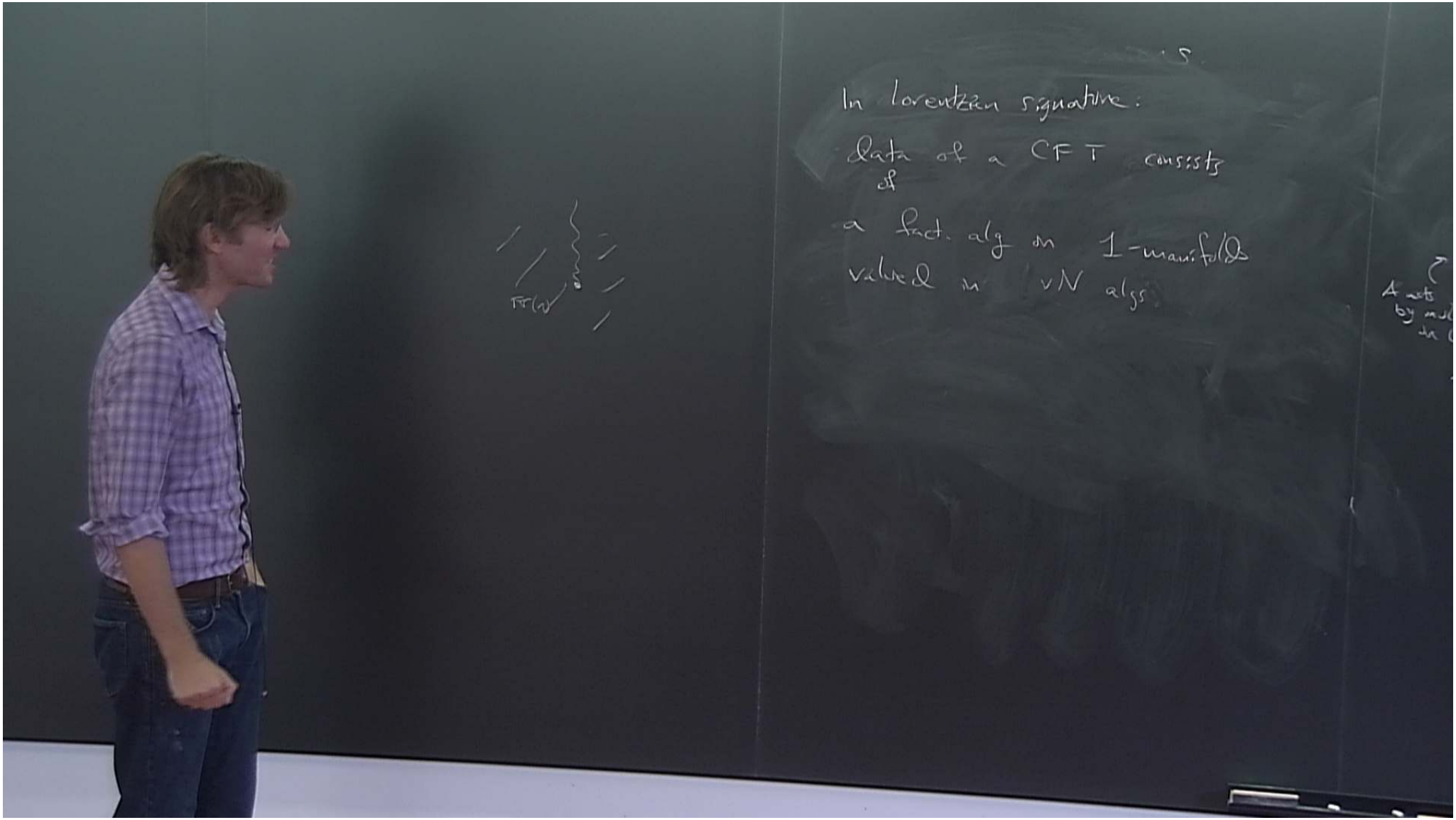
Almost thm [me, David Treumann]:
$Co_0 \curvearrowright \mathbb{R}^{24}$.
$Co_0 = Aut(\text{Leech lattice})$
$\omega_1 = \omega_2 = 0$ because $H^1(Co_0) = H^2(Co_0) = \emptyset$.
P_1 has order 24 in $H^3(Co_0; U(1))$
Cor:
$\mathbb{Z}/2 \ltimes Co_0^{\times 2} \curvearrowright \mathbb{R}^{576}$ $P_1 = \emptyset$
Hope: $Fer(576)$


In lorentzian signature:
data of a CFT consists of
a fact. alg on 1-manifolds
valued in vN algs

In Lorentzian signature:

data of a CFT consists of

a fact. alg on spin 1-manifolds valued in
- vN algs
- cats

A
|
Mod(A)

↑ "many object vN algs"

Eg.

1-manifold

Fer(n)(I) = Cliff(Γ_{L^2}($\not{\partial}$)$^{\otimes n}$)

spinor bundle

vN complete
"side standard
Fock module

⇝ Type III fact

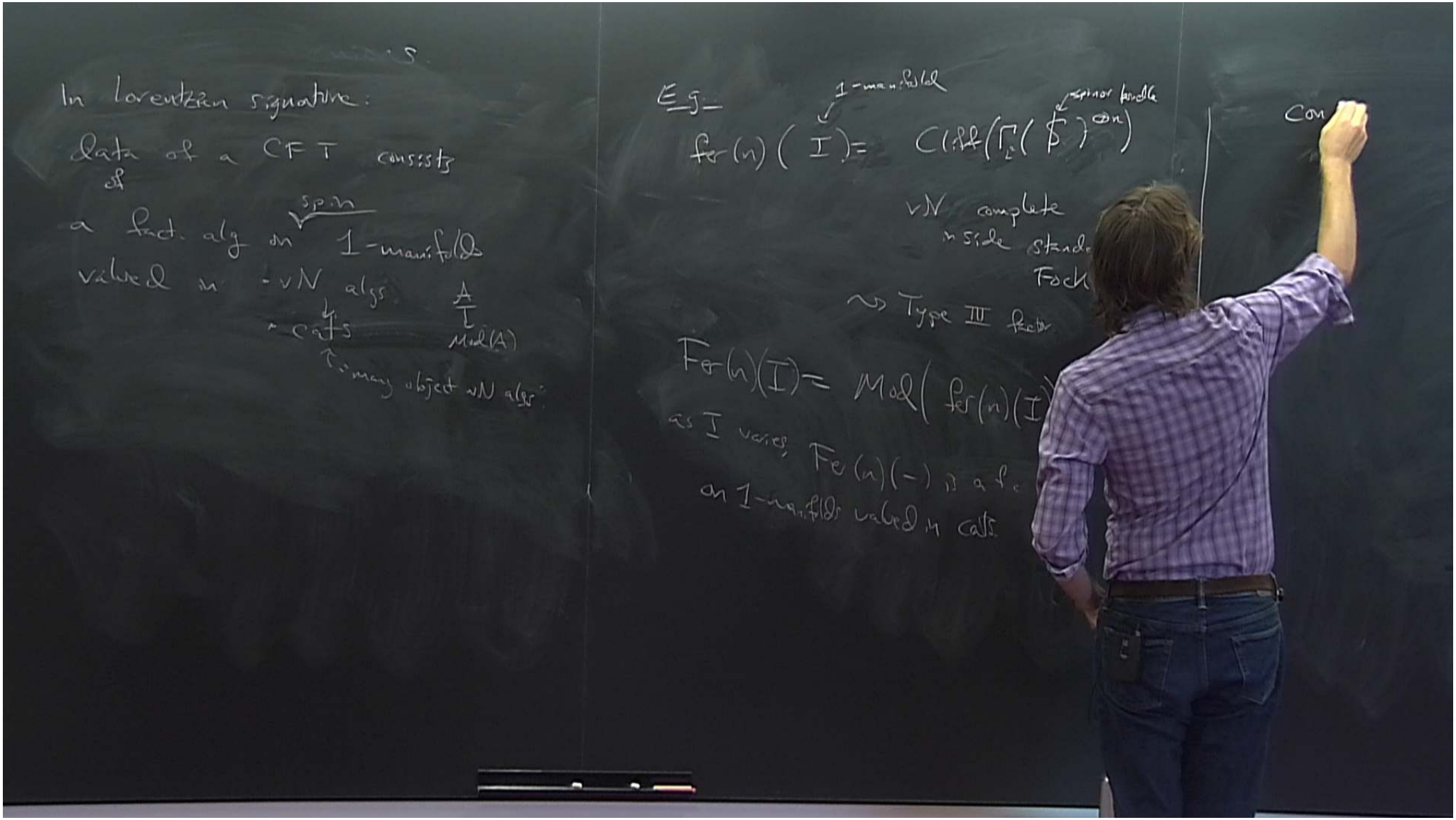
If G ⟳ fin,

A — A⊗G

∃ a mor

If G⊂A ∃

then A//G =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Mod(A)
many object vN algs
Eg:
1-manifold
spinor bundle
vN complete
Fock
Type III
as I varies
on 1-manifolds valued in cats

$\mathrm{Cliff}(\Gamma_L(\not{D}^{\text{spinor bundle}})^{\otimes n})$

vN complete
inside standard
Fock module

Type III factor

$\mathcal{A}(\mathrm{fer}(n)(I))$

$(-)$ is a f.
ed in cats

there is an action of

$\mathrm{Diff}_+(I) \curvearrowright \mathrm{Fer}(n)(I)$

↑ diffeos trivial "near" ∂I
↑ enough to be trivial on a 1-jet.

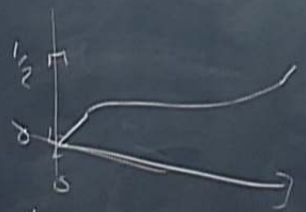
"conformal of central charge c"
means this action is the action

$\mathrm{Diff}_+(I) \curvearrowright \mathrm{Vect}) \otimes \mathrm{Fer}(n)(I)$

Choose some diffeo

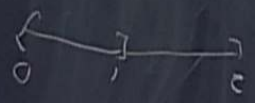
$[0,1] \xrightarrow{\sim} [0,\tfrac{1}{2}]$

✓ Derivative 1 at 0.



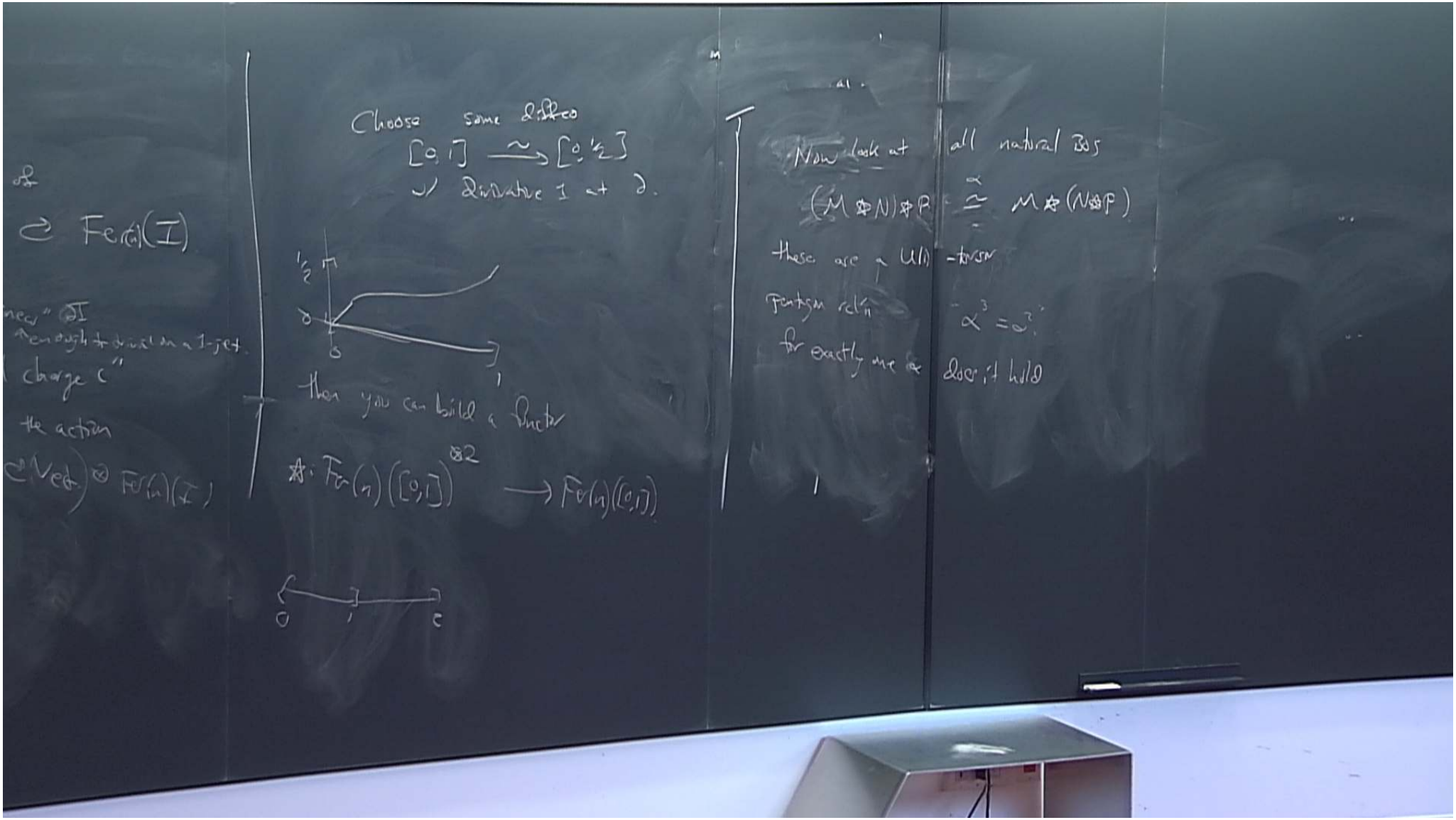
then you can build a functor

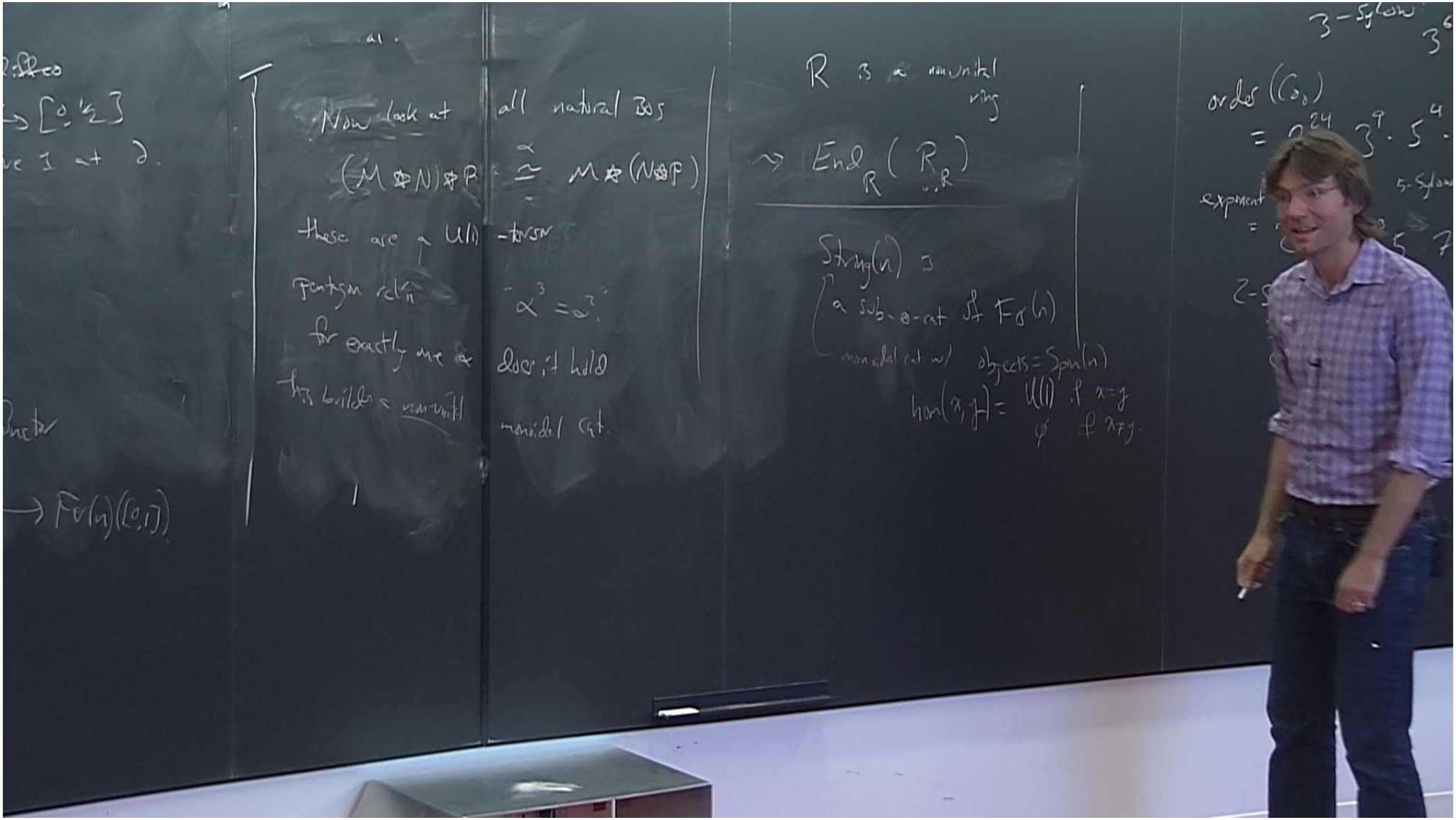
$\mathrm{Fer}(n)([0,1])^{\otimes 2} \longrightarrow \mathrm{Fer}(n)([0,1])$

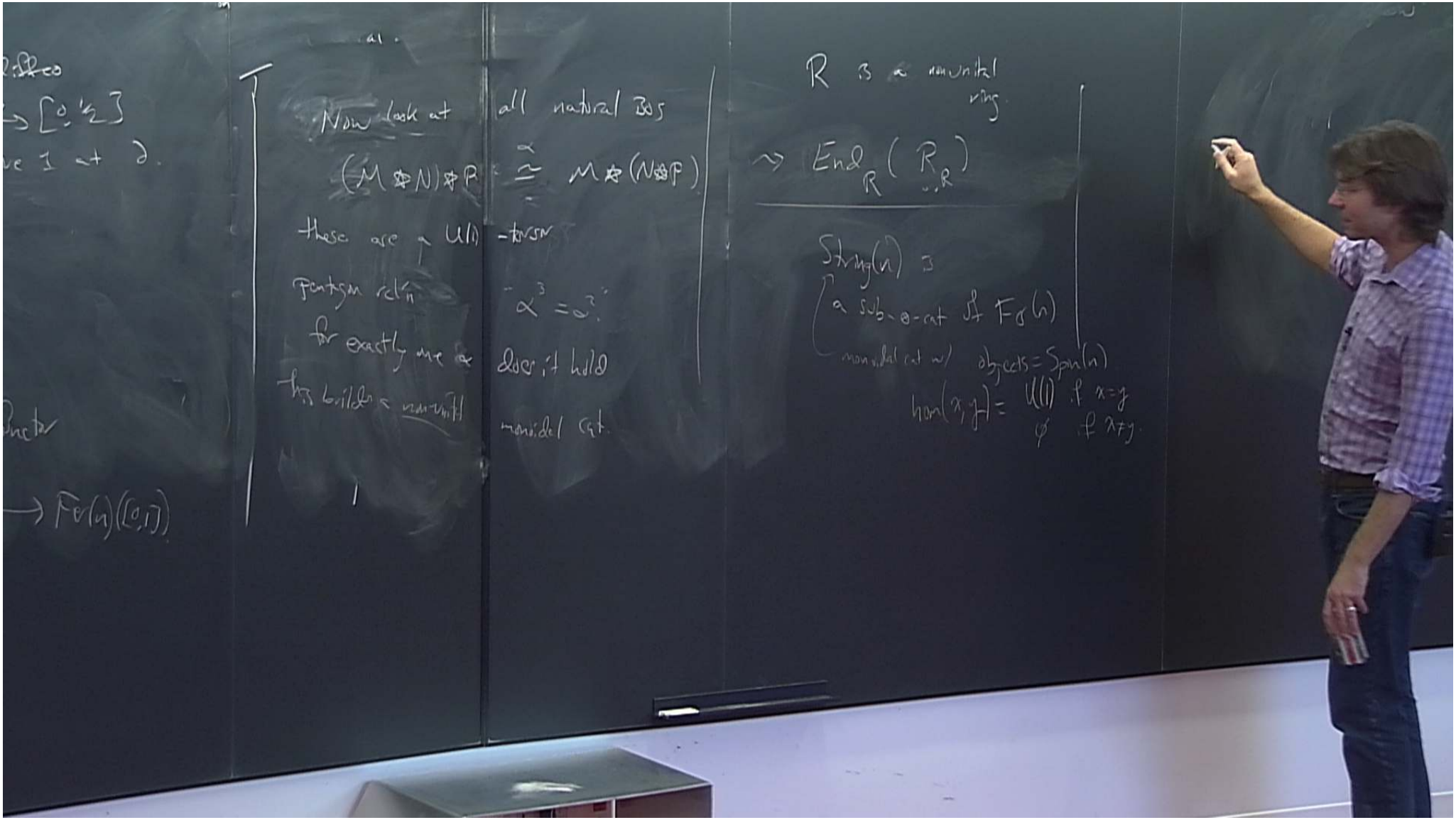


monoidal cat I h
tell you about
to n dim

Almost


Choose some diffeo
$[0,1] \xrightarrow{\sim} [0,\tfrac{1}{2}]$
✓ derivative 1 at 0.
$\tfrac{1}{2}$
1
0
0
1
then you can build a functor
$\otimes^2$
$\mathrm{Fer}(n)([0,1]) \to \mathrm{Fer}(n)([0,1])$
0
1
c
Now look at all natural iso's
$(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$
these are a Uhlenbeck-tensor
pentagon relation
$\alpha^3 = \alpha^2$
for exactly one α does it hold
$\mathrm{Fer}(n)(I)$
$\mathrm{Vect}) \otimes \mathrm{Fer}(n)(I)$
charge c"
the action


Now look at
all natural Bos
these are a U(1)-torsor
for exactly one of
this holds
monoidal cat.
R is a non-unital ring
String(n) is
a sub-2-cat of Fd(n)
monoidal cat w/ objects = Spin(n)
hom(x,y) = U(1) if x=y
∅ if x≠y.
order (G0)
exponent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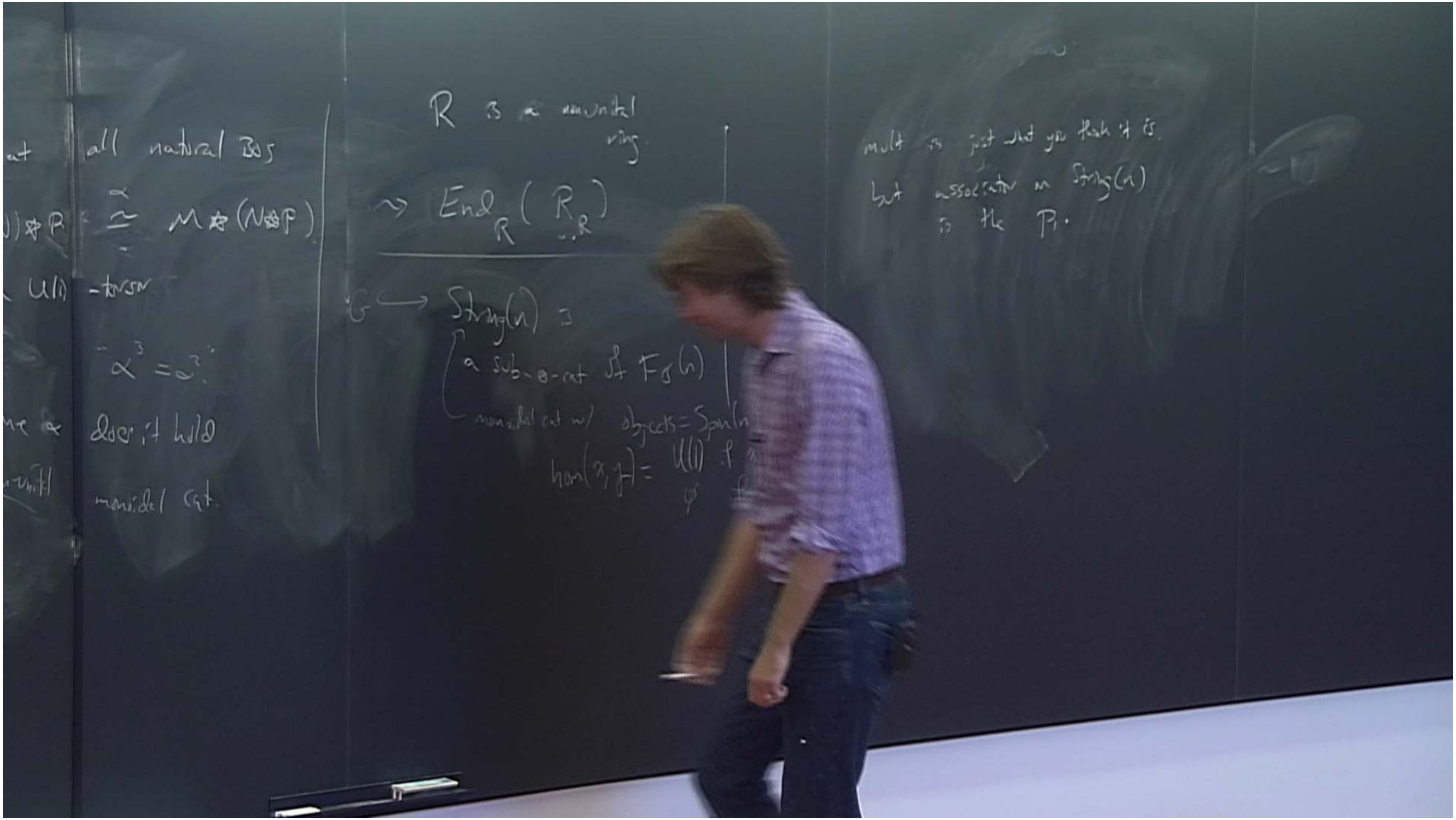
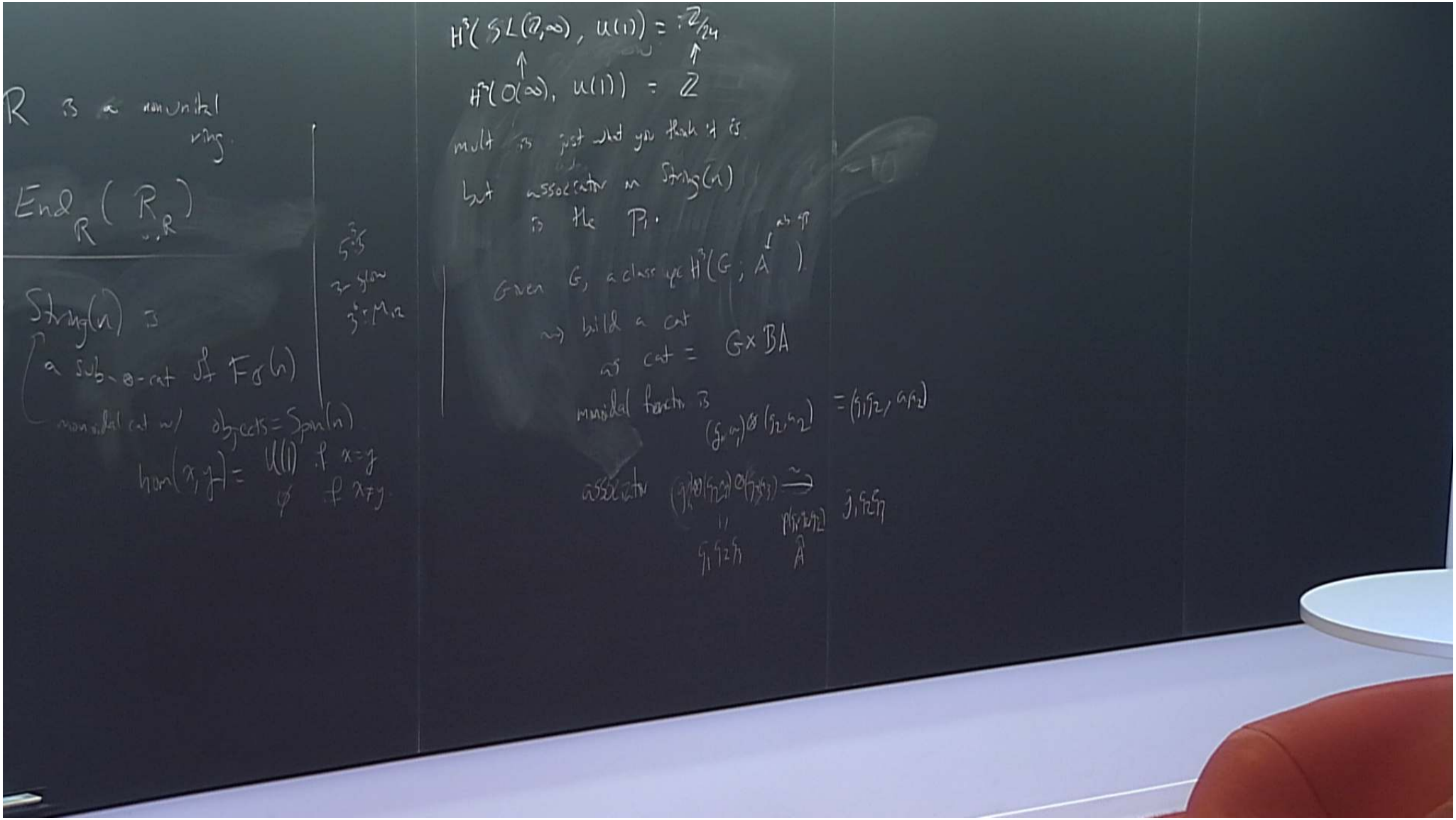

all natural Bos
does it hold
monoidal cat.
R is a non-unital ring.
End_R(R_R)
String(n) :
a sub-∞-cat of Fg(n)
monoidal cat w/ objects = Span(n
hom(x, y) = U(1)
mult is just what you think it is.
but associator in String(n)
is the P_i.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monoidal cat w/ objects = Span(n)
hom(x,y) = U(1) if x=y
∅ if x≠y
S^3
3-sphere
3-Man
H^3(SL(2,∞), U(1)) = Z/24
H^3(O(∞), U(1)) = Z
mult is just what you think it is
but associator in String(n)
is the P_1.
Given G, a class in H^3(G, A^{abs pt})
↝ build a cat
as cat = G × BA
monoidal functor is
associator