Title: Moonshine, topological modular forms, and 576 fermions.

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URL: http://pirsa.org/16090051

Abstract: I will report on progress understanding the 576-fold periodicity in TMF in terms of conformal field theoretic constructions. Sporadic finite groups and their cohomology will play a role.

576 fermions

Motivating theorem [Stolz–Teichner]

Let X be a manifold.

Consider a cat

morphisms: "$\mathcal{N}=1$ SUSY 1-manifold in X"

1-manifolds locally $\mathbb{R}^{1|1}$ w/ local geometry

576 fermions

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- objects are sums of these things over a point

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- objects are germs of these things near a point

$M(X \times [0,1])$

$\swarrow$ restrict to 0 $\qquad$ $\searrow$ restrict to 1

$M(X) \qquad\qquad M(X)$

$M(X) = \mathrm{Fun}_{\mathrm{sym}^\otimes}(EBord^{1|1}(X), SVect_{\mathbb{R}})$

$=$ "$\mathcal{N}=1$ SUSY 1-D QFTs on X"

this space is a cocycle model for $KO(X)$.

By this I mean that

$$\frac{M(X)}{\text{"cobordism of field theories"}} = KO^0(X)$$

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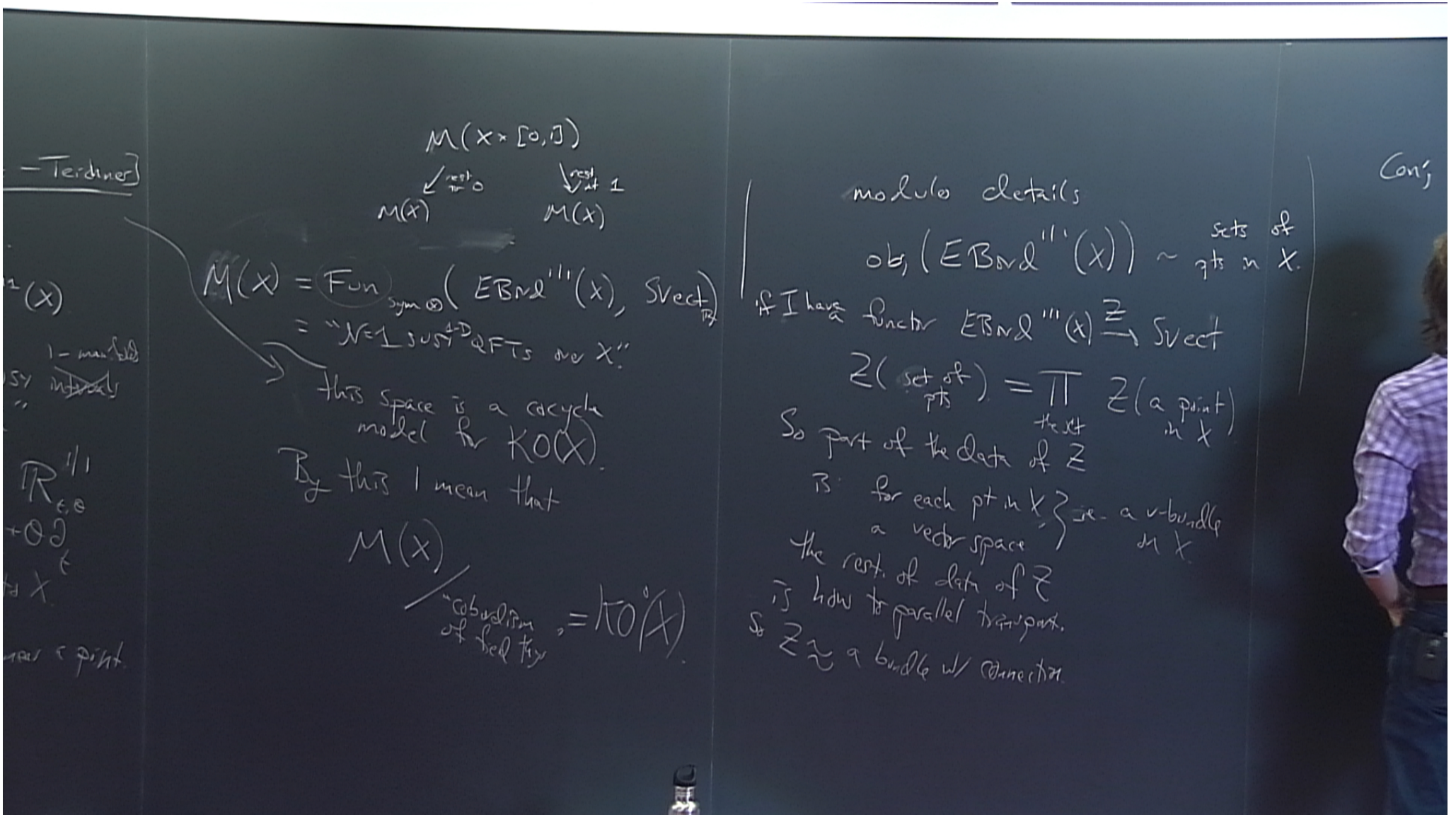
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modulo details
ob(EBrd'''(X)) ~ sets of pts in X.
if I have a functor EBrd'''(X) → Z sVect
Z(set of pts) = ∏ over the set Z(a point in X)
So part of the data of Z is: for each pt in X, a vector space } i.e. a v-bundle on X.
the rest of data of Z is how to parallel transport.
So Z ≅ a bundle w/ connection
Con;

modulo details:

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I have a functor $EBord^{1,1}(X) \xrightarrow{Z} Vect$

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$Z \approx$ a bundle w/ connection.

Conj [Segal - Stolz - Teichner]

There is an analogy

$\mathcal{N}=1$	1-D QFT	KO
$\mathcal{N}=1$	2-D QFT	TMF

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$\mathcal{N}=1$	1-D QFT	KO	Clifford algebras
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576 fermions

Motivating theorem [Stolz–Teichner]

Let X be a manifold.

Consider a cat $EBord^{1/2}(X)$

- morphisms: "$\mathcal{N}=1$ SUSY 1-manifolds in X"

1/2-manifolds locally $\mathbb{R}^{1|1}_{t,\theta}$ w/ local geometry $\partial_\theta + \theta\partial_t$ together w/ a smooth map to X

- objects are sums of these things over 1 point

$M^n(X)$ = space of sym mon (ax not triv)

this space is a cocycle model for $KO(X)$.

By this I mean that

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$EBord_2(X)$ ⇓

trivial functor → SAlgs bimods intertwiners

the 1-TFT determined by $Cliff(n)$.

modulo det

$ob_1(EBord$

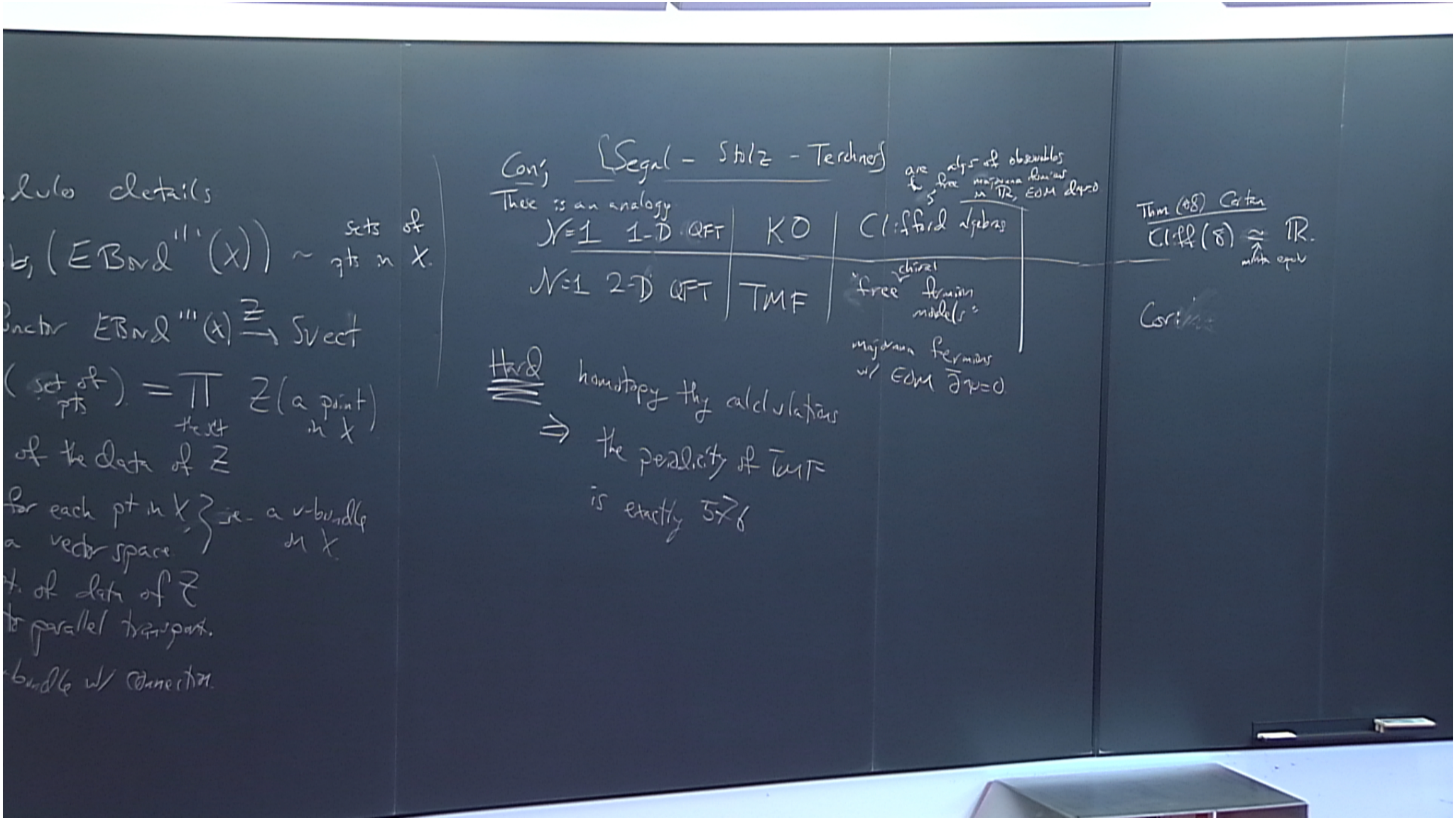
if I have a functor $EBord$

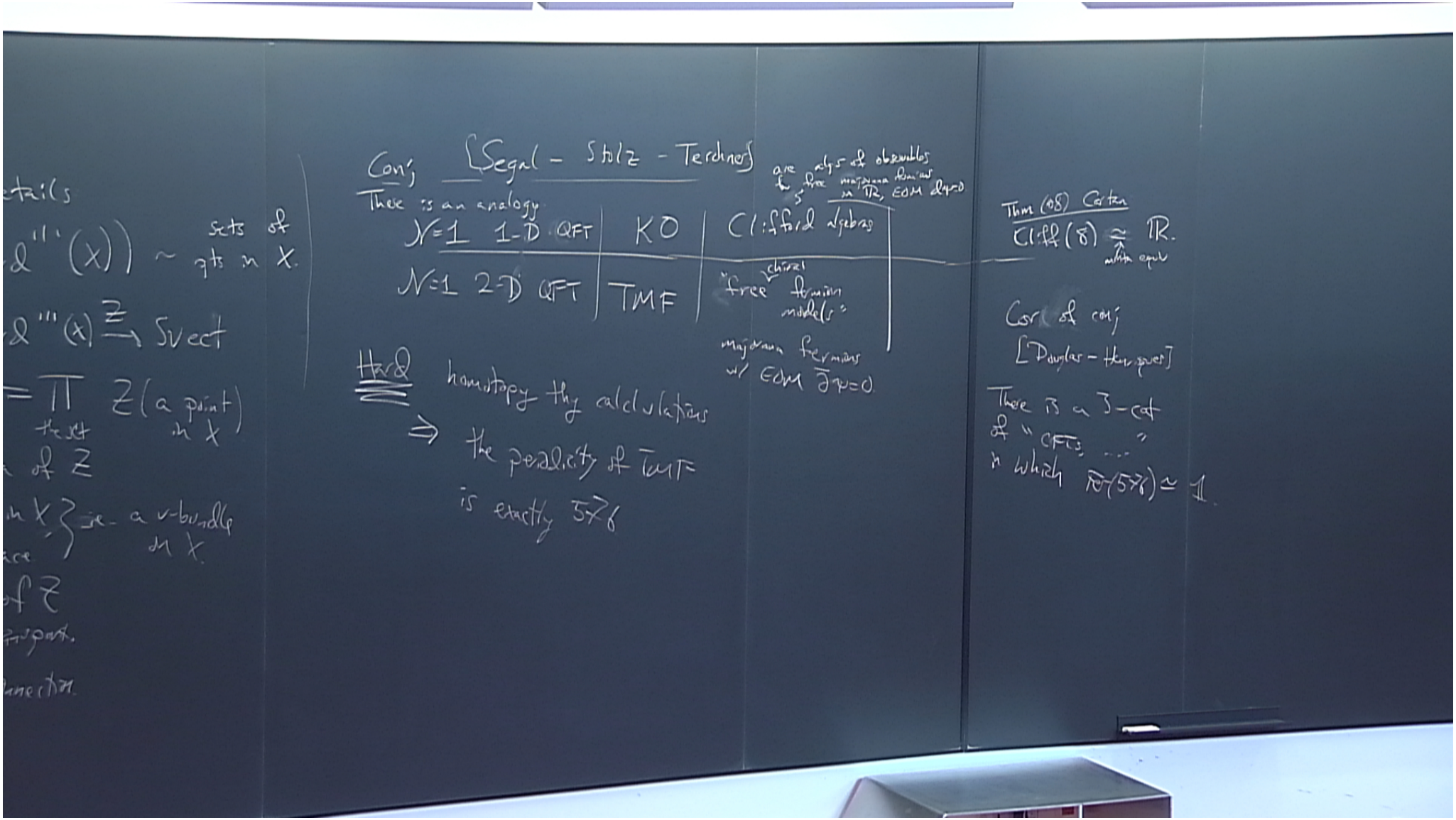
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Conj [Segal - Stolz - Teichner]
There is an analogy
N=1 1-D QFT | KO | Clifford algebras
N=1 2-D QFT | TMF | free chiral fermion models
Thm (08) Cotton
Cliff(8) ≅ ℝ
Hard
⇒ the periodicity of TMF is exactly 576


Conj [Segal - Stolz - Teichner]
There is an analogy
N=1 1-D QFT | KO | Clifford algebras
N=1 2-D QFT | TMF | "free chiral fermion models"
are algs of observables in free majorana fermions in 1D, EOM $\partial_t \psi = 0$
majorana fermions w/ EOM $\partial \psi = 0$
Hard homotopy thy calculations
⇒ the periodicity of TMF is exactly 576
Thm (08) Cartan
$Cliff(8) \cong \mathbb{R}$.
with equiv
Cor of conj [Douglas - Henriques]
There is a 3-cat of "CFTs, ..." in which $\pi_0(576) \cong 1$.

How to find Morita equivs?

Suppose $G \curvearrowright A$ (top. gp. acting on alg.)

a Hamiltonian for this action is a homomorphism $\mu: G \to A^{\times}$ (multiplicative map)

s.t. action of $g \in G$ on $a \in A$ is $a \mapsto \mu(g)\, a\, \mu(g)^{-1}$.

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$EBord_2(X)$ ⇓ SAlgs bimod: Cinftwise

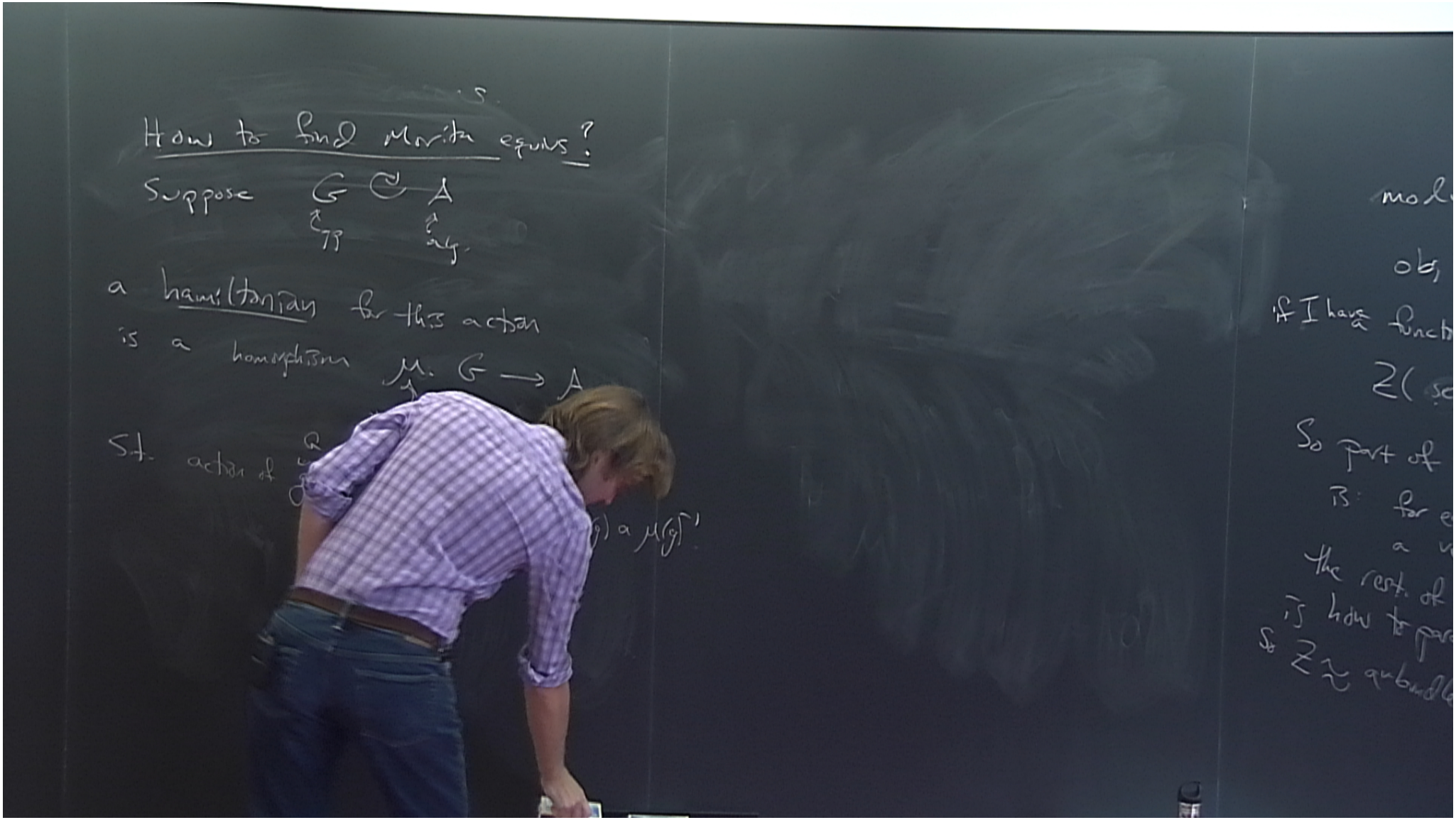
trivial functor

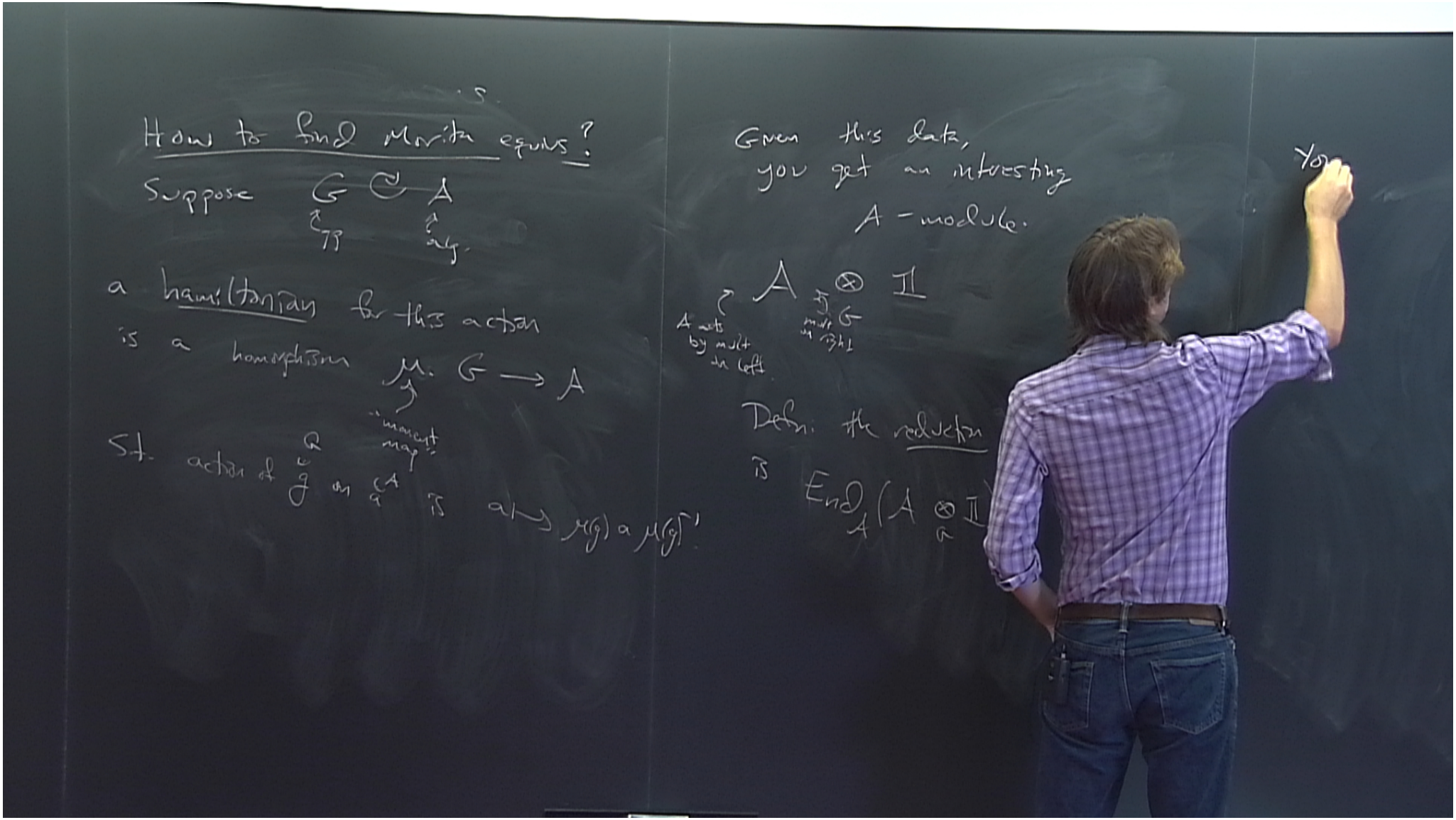
the 1-2 FT determined by $Cliff(n)$.

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Suppose
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How to find Morita equivs?
Suppose G ↷ A
a hamiltonian for this action
is a homomorphism μ: G → A
moment map
s.t. action of G on A is a ↦ μ(g) a μ(g)^{-1}
Given this data,
you get an interesting
A-module.
A ⊗ Π
A acts by mult on left
G acts on right
Defn: the reduction
is End_A(A ⊗ Π)
G

Given this data,
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$$A \underset{G}{\otimes} \mathbb{I}$$

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Defn. the reduction $A/\!/G$ is $\mathrm{End}_A(A \underset{G}{\otimes} \mathbb{I})$.

) a $\mu(g)$'

You can hope that the bimodule

$$A - \overset{A \underset{G}{\otimes} \mathbb{I}}{-} - A/\!/G$$

is a morita equiv.

If $G \subset A$ is ~~reductive~~ nice enough,

then $A/\!/G = \left(A \underset{G}{\otimes} \mathbb{I}\right)^G$

$= A^G$ /....

Cor; [illegible]

There is an ana

$N=1$ 1

$N=1$ 2-

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$A \otimes_G \Pi$

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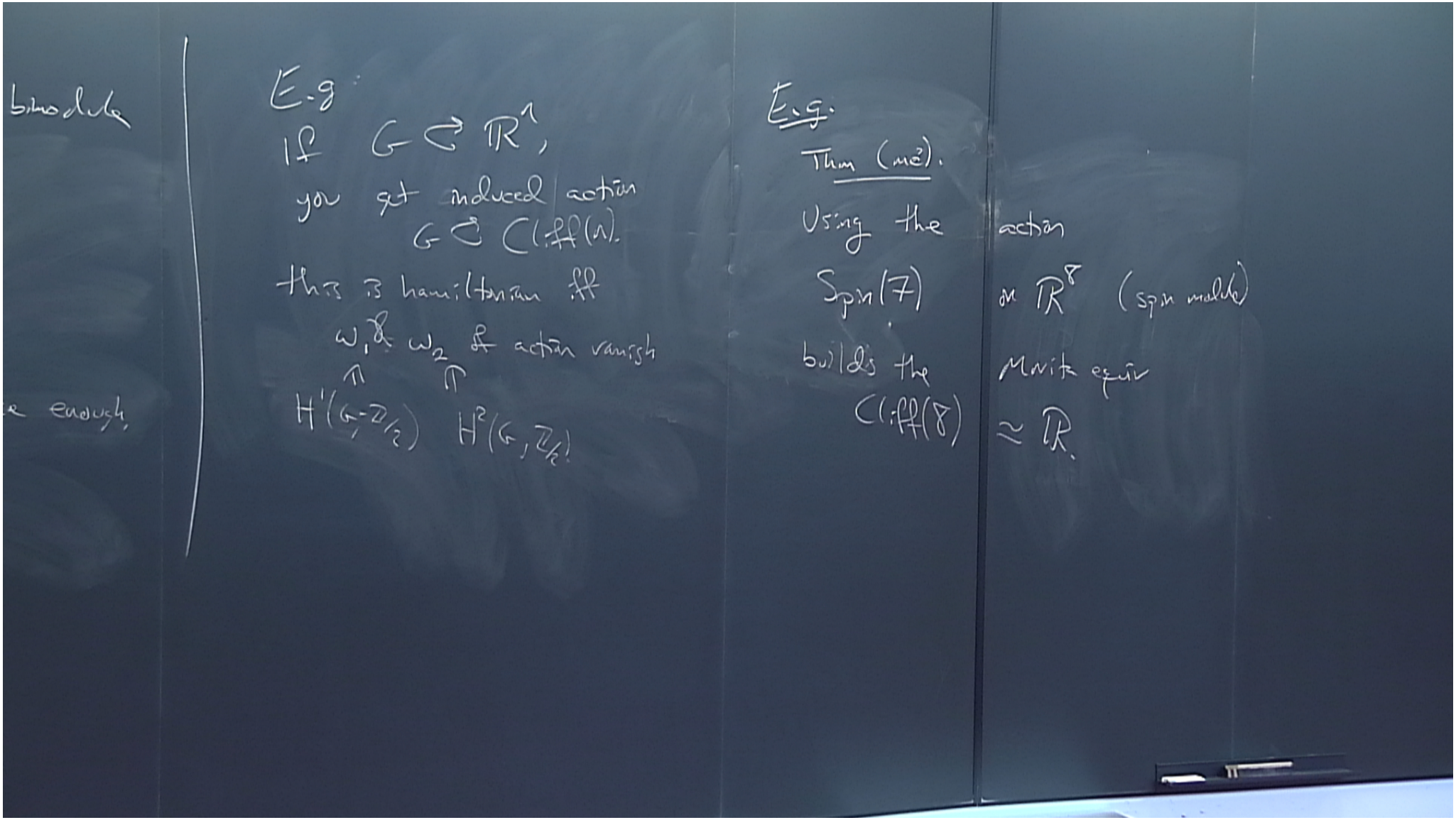
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E.g.

If G

you g

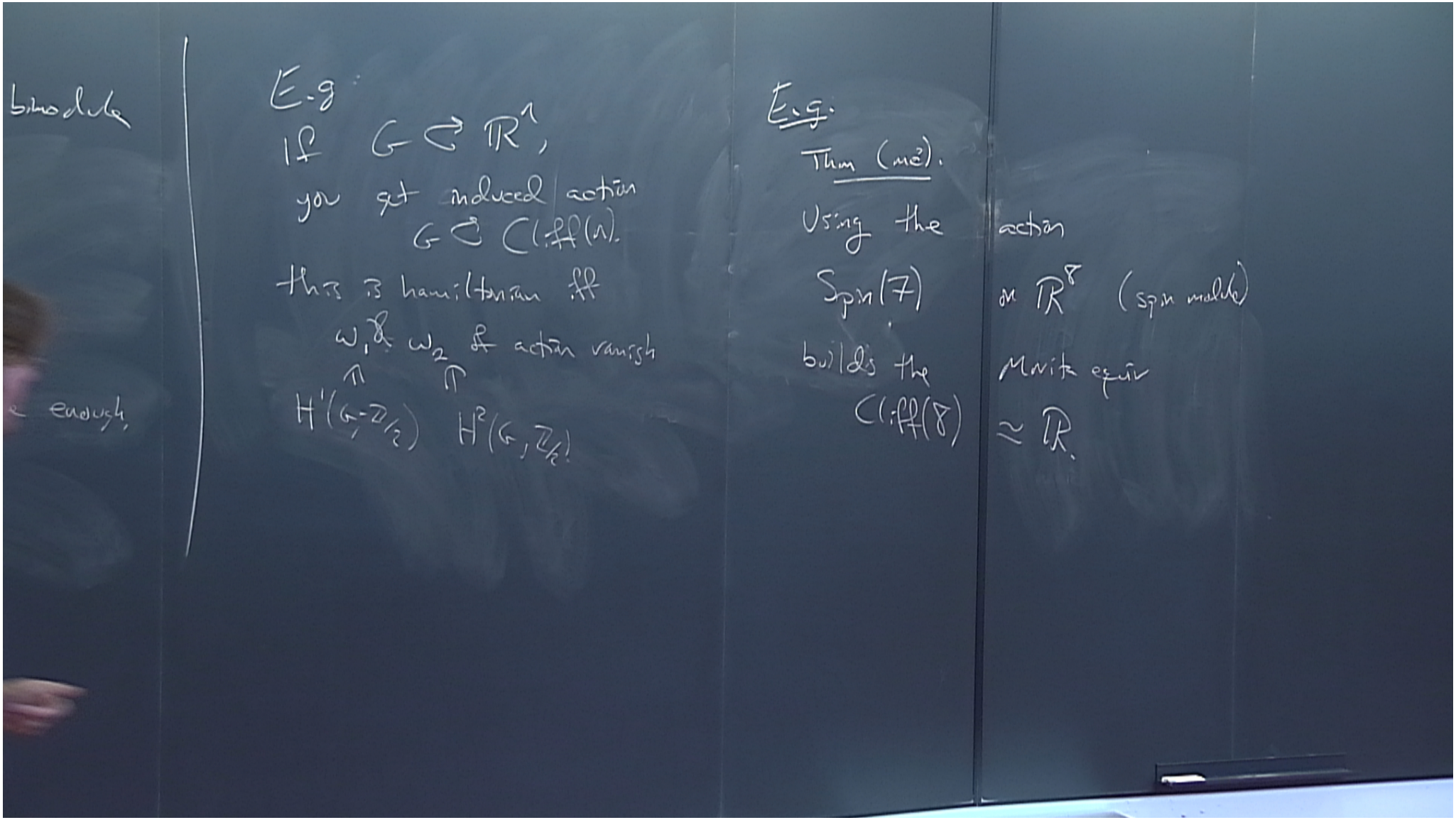

E.g.
If $G \curvearrowright \mathbb{R}^1$,
you get induced action
$G \curvearrowright Diff(n)$.
this is hamiltonian iff
ω_1 & ω_2 of action vanish
$H^1(G; \mathbb{Z}/2)$ $H^2(G, \mathbb{Z}/2)$
E.g.
Thm (mc²).
Using the
Spin(7)
builds the
Diff(8)
action
on $\mathbb{R}^8$ (spin manifolds)
Morita equiv
$\simeq \mathbb{R}$.

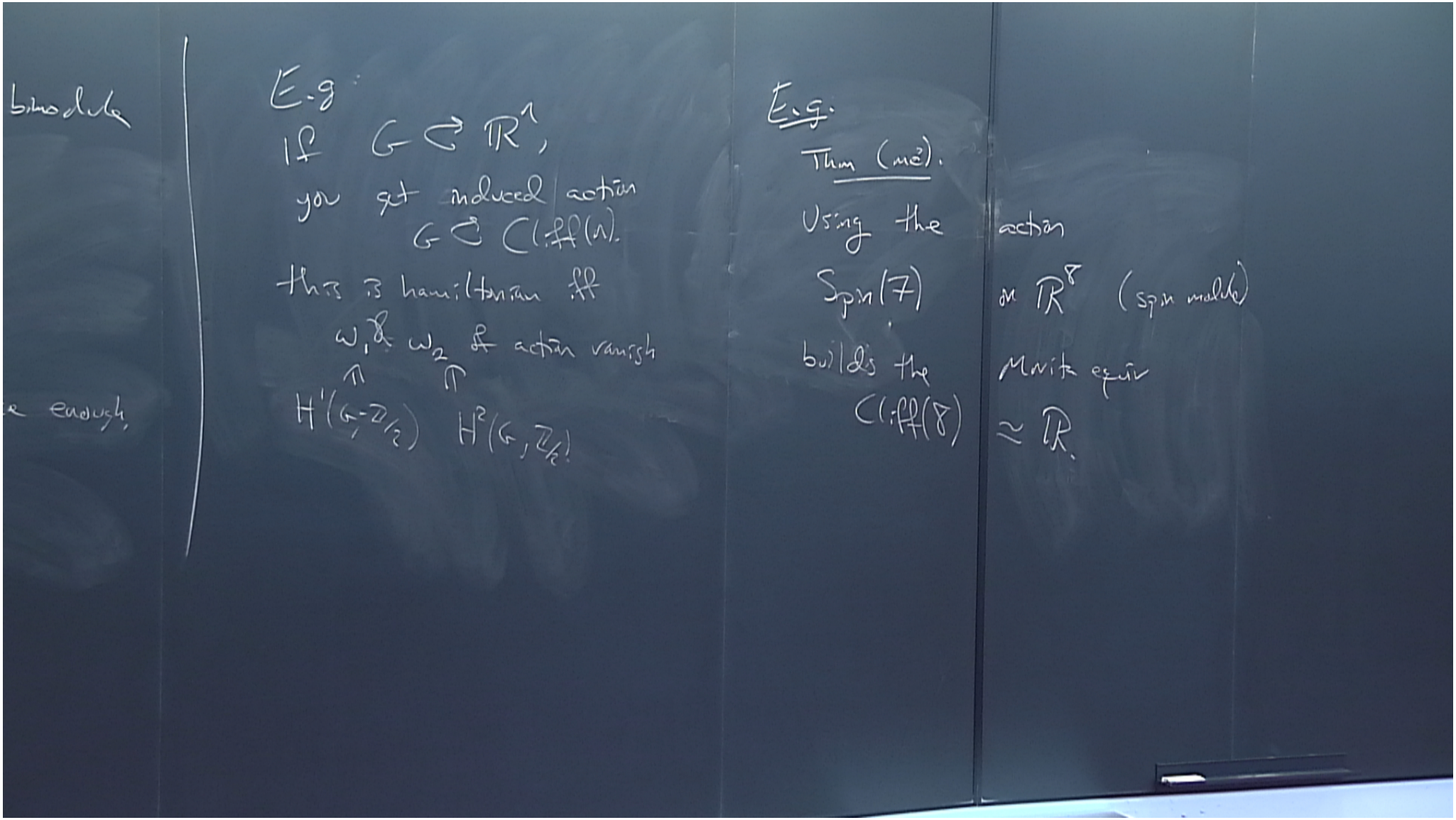
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Suppose $G \curvearrowright A$

G: ?? A: ? monoidal cat

play the same game.

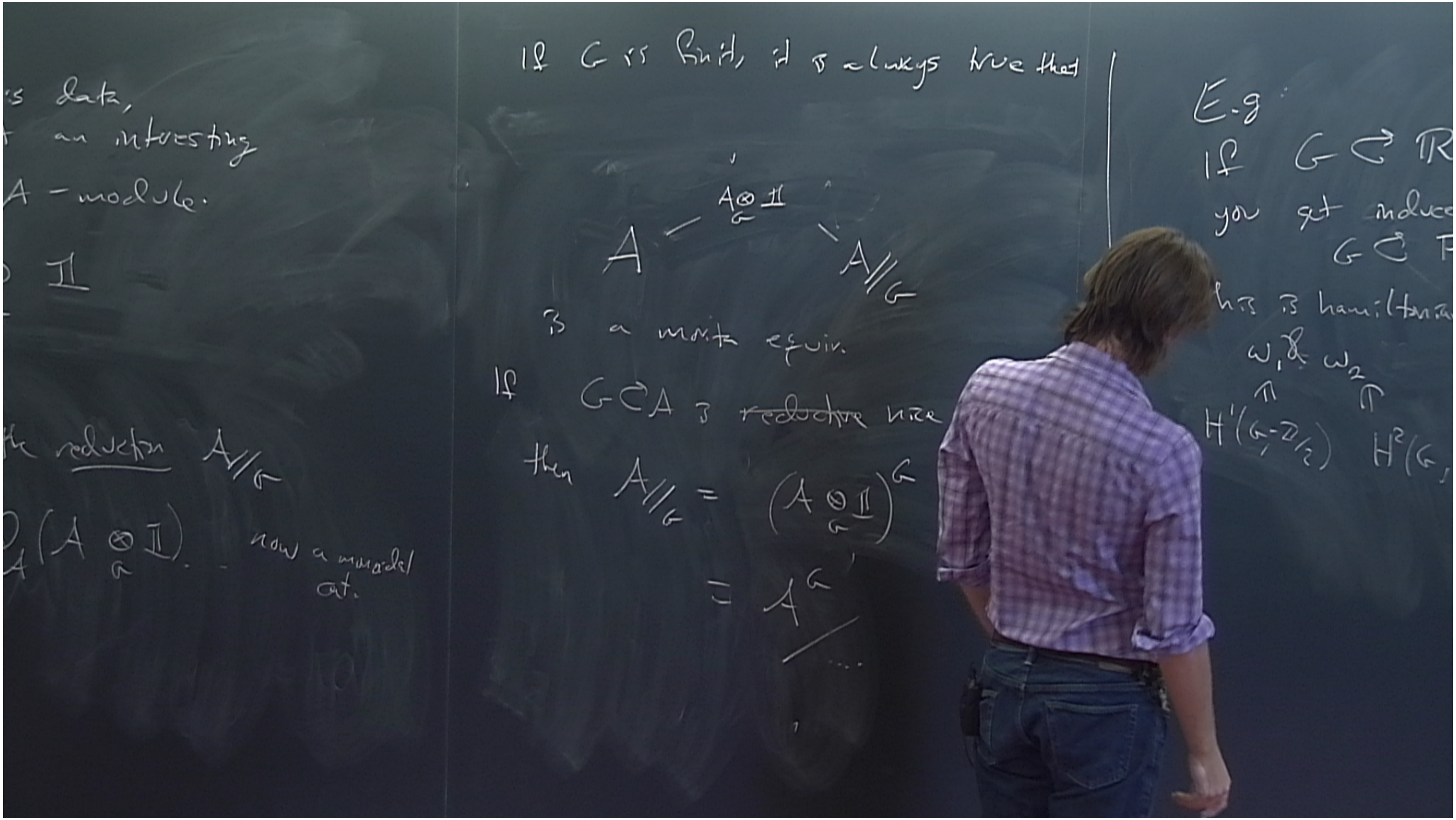
i.e. realize the action as conj via a monoidal functor $G \to A$.

Given this data, you get an interesting A-module.

$A \otimes_G \mathbb{1}$

(A acts by mult on left; G acts by mult on right)

Defn. the reduction $A/\!/G$ is $\mathrm{End}_A(A \otimes_G \mathbb{1})$. now a monoidal cat.


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If G is finite, it is always true that
A
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A//G
is a monita equiv.
If G ⊂ A is reductive nice
then A//G = (A ⊗_G II)^G
= A^G
E.g
If G ⊂ R
you get induce
G ⊂ F
his is hamiltonia
ω1 & ω2
H¹(G,
H²(G,

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E.g.

If $G \curvearrowright \mathbb{R}^1$,

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$G \subset \mathrm{Fer}($

this is hamiltonian iff

w_1 & w_2 & P

$H^1(G, \mathbb{Z}/2)$ $H^2(G, \mathbb{Z}/2$

E.g.

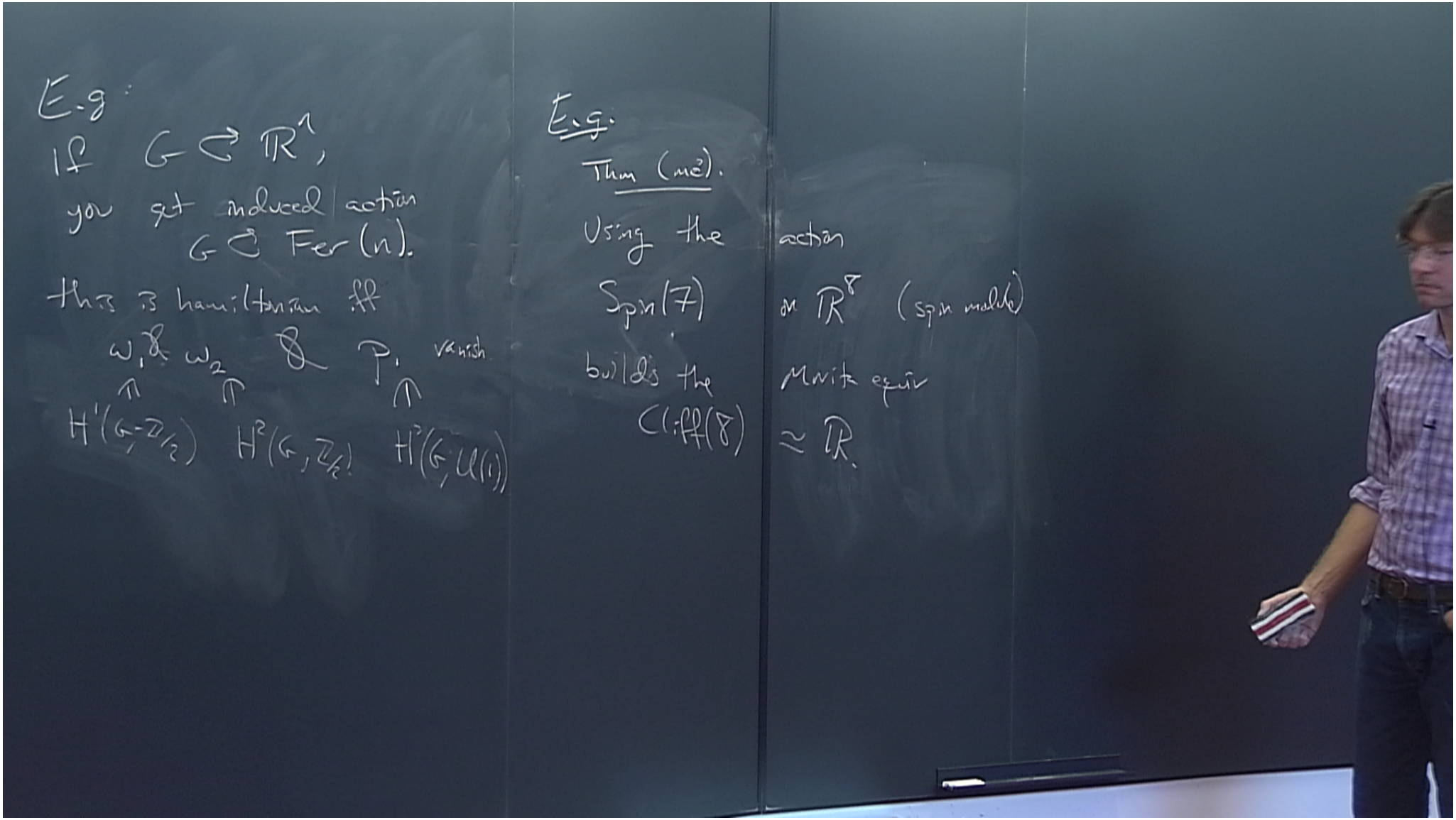
Thm (

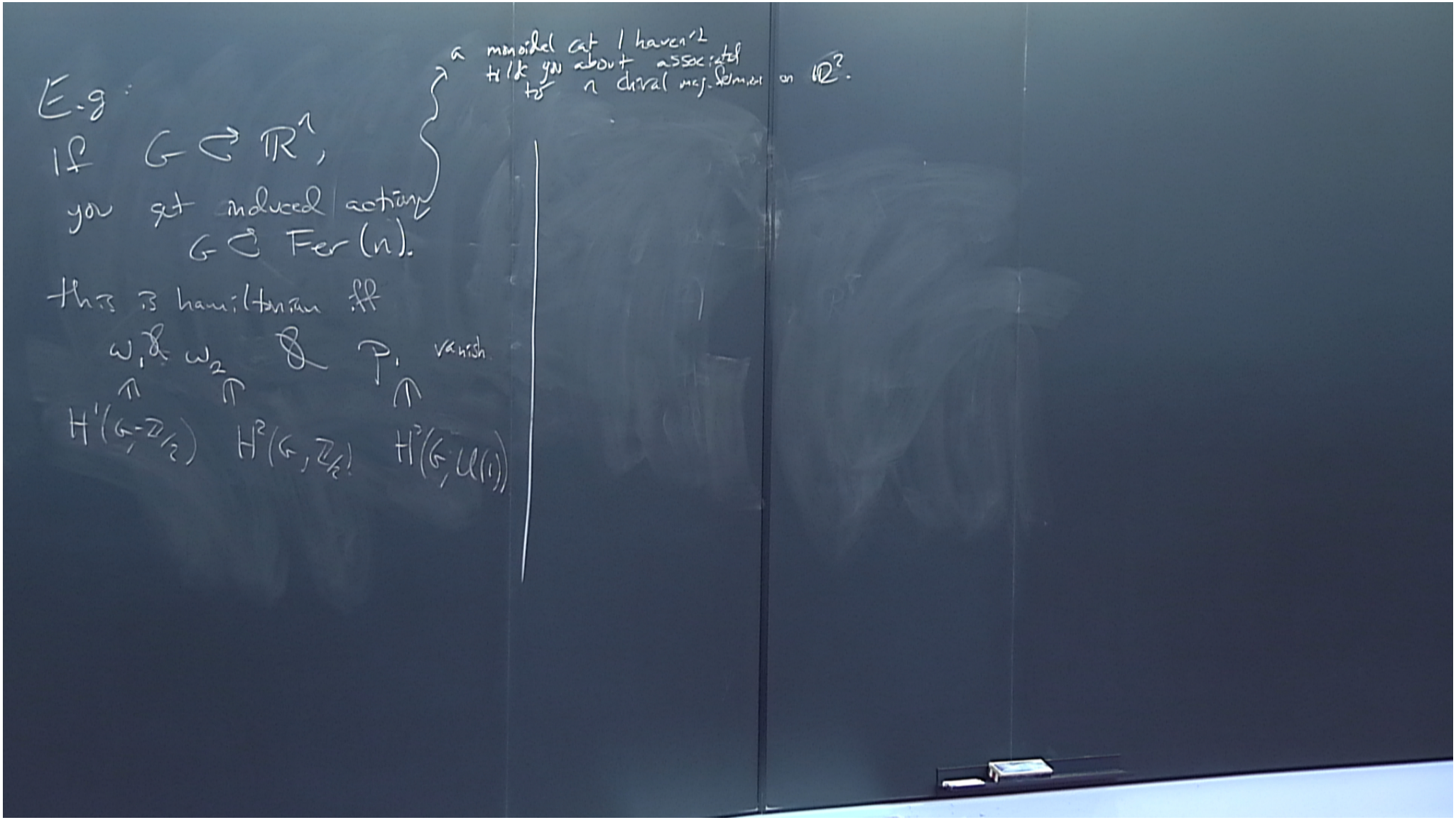
Using t

Spin(7

builds t

C(iff


E.g:
If $G \curvearrowright \mathbb{R}^1$,
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This is hamiltonian iff ω_1 & ω_2 & p_1 vanish

$\omega_1 \in H^1(G; \mathbb{Z}/2)$ $\omega_2 \in H^2(G; \mathbb{Z}/2)$ $p_1 \in H^3(G; U(1))$

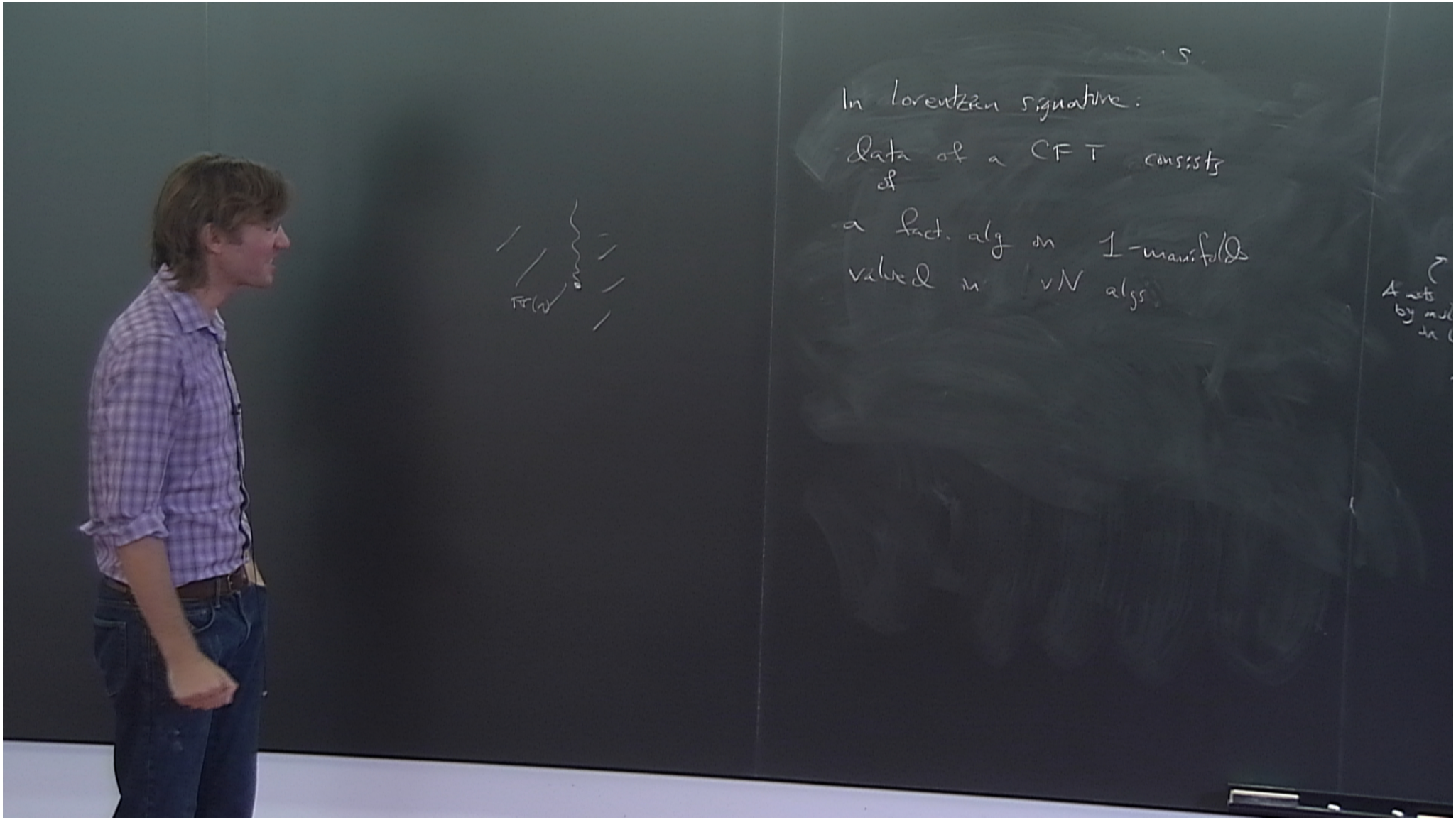
Almost thm [me, David Treumann]:

$Co_0 = \mathrm{Aut}(\text{Leech lattice}) \curvearrowright \mathbb{R}^{24}$. $\omega_1 = \omega_2 = 0$ because $H^1(Co_0) = H^2(Co_0) = \emptyset$.

p_1 has order 24 in $H^3(Co_0; U(1))$

Cor: $\mathbb{Z}/2 \ltimes Co_0^{\times 2} \curvearrowright \mathbb{R}^{576}$ $p_1 = \emptyset$.

Hope: $\mathrm{Fer}(576)$


In lorentzian signature:
data of a CFT consists of
a fact. alg on 1-manifolds
valued in vN algs

In Lorentzian signature:

Data of a CFT consists of

a fact. alg on spin 1-manifolds

valued in - vN algs

- cats

↑ - many object vN algs

A

|

Mod(A)

Eg -

1-manifold

spinor bundle

$\mathrm{Fer}(n)(I) = \mathrm{Cliff}(\Gamma_{L^2}(\not{D})^{\otimes n})$

vN complete

inside standard Fock module

⇝ Type III fact

If G ⟳ fin,

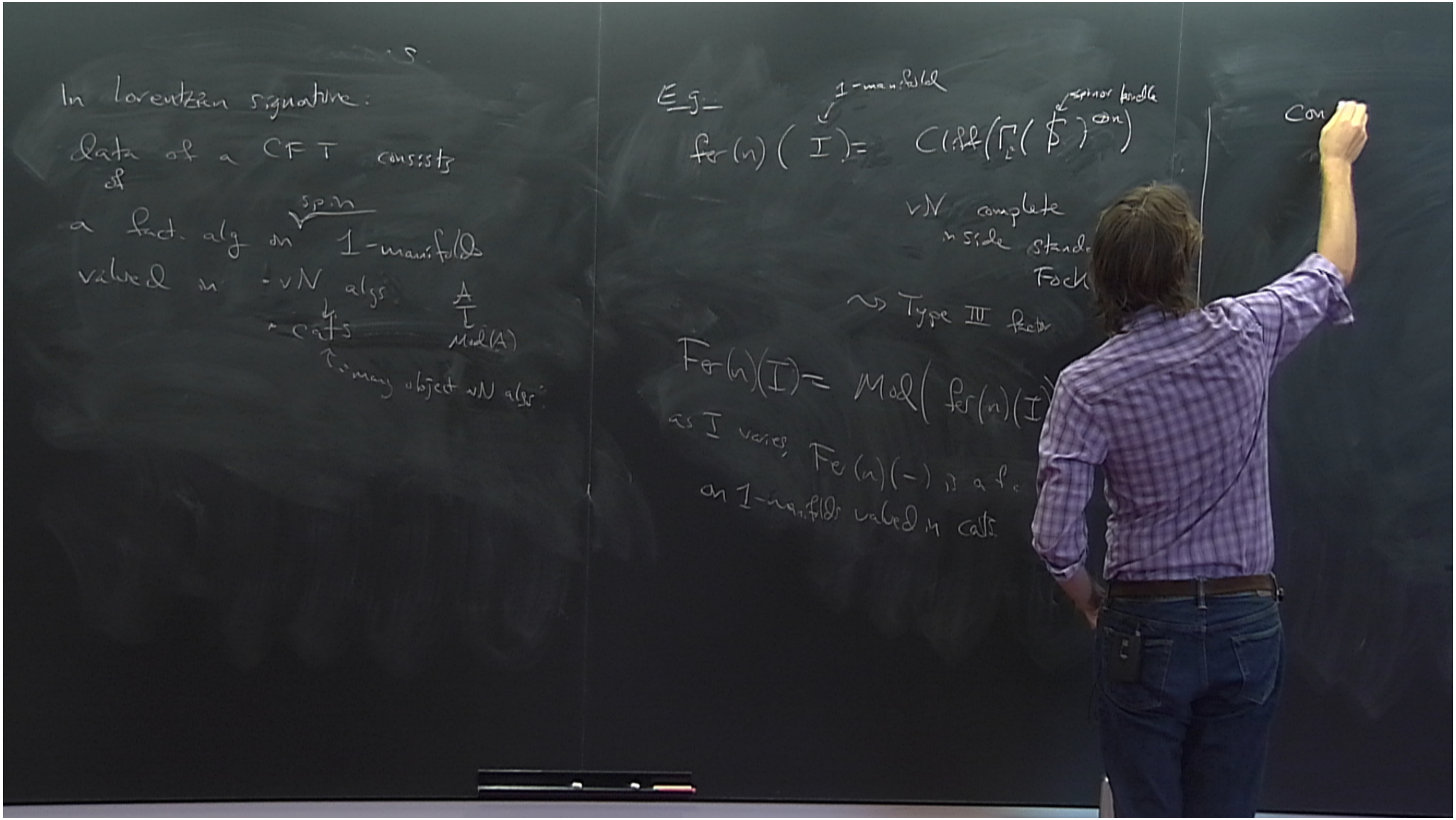
A — A⊗G

3 a mate

If G ⊂ A ⟳

then A//G =

=


In Lorentzian signature:
data of a CFT consists of
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valued in vN algs
cats
Mod(A)
Eg:
1-manifold
spinor bundle
vN complete
Type III
as I varies,
on 1-manifolds valued in cats

spinor bundle

$C(\mathrm{Aff}(\Gamma(S)^{\otimes n}))$

vN complete
inside standard
Fock module

Type III factor

$\mathcal{R}(\mathrm{Fer}(n)(I))$

$(-)$ is a f_c

ed in cats

there is an action of

$\mathrm{Diff}_+(I) \curvearrowright \mathrm{Fer}(n)(I)$

diffeos trivial "near" ∂I
(enough to be trivial on a 1-jet.)

"conformal of central charge c"

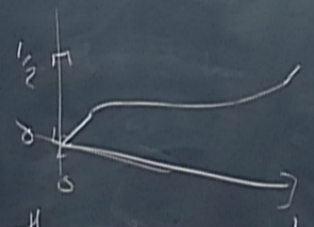
means this action is the action

$\mathrm{Diff}_+(I) \curvearrowright (\mathrm{Vect}) \otimes \mathrm{Fer}(n)(I)$

Choose some diffeo

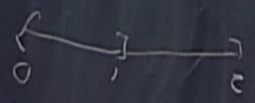
$[0,1] \xrightarrow{\sim} [0, \tfrac{1}{2}]$

ν derivative 1 at ∂.



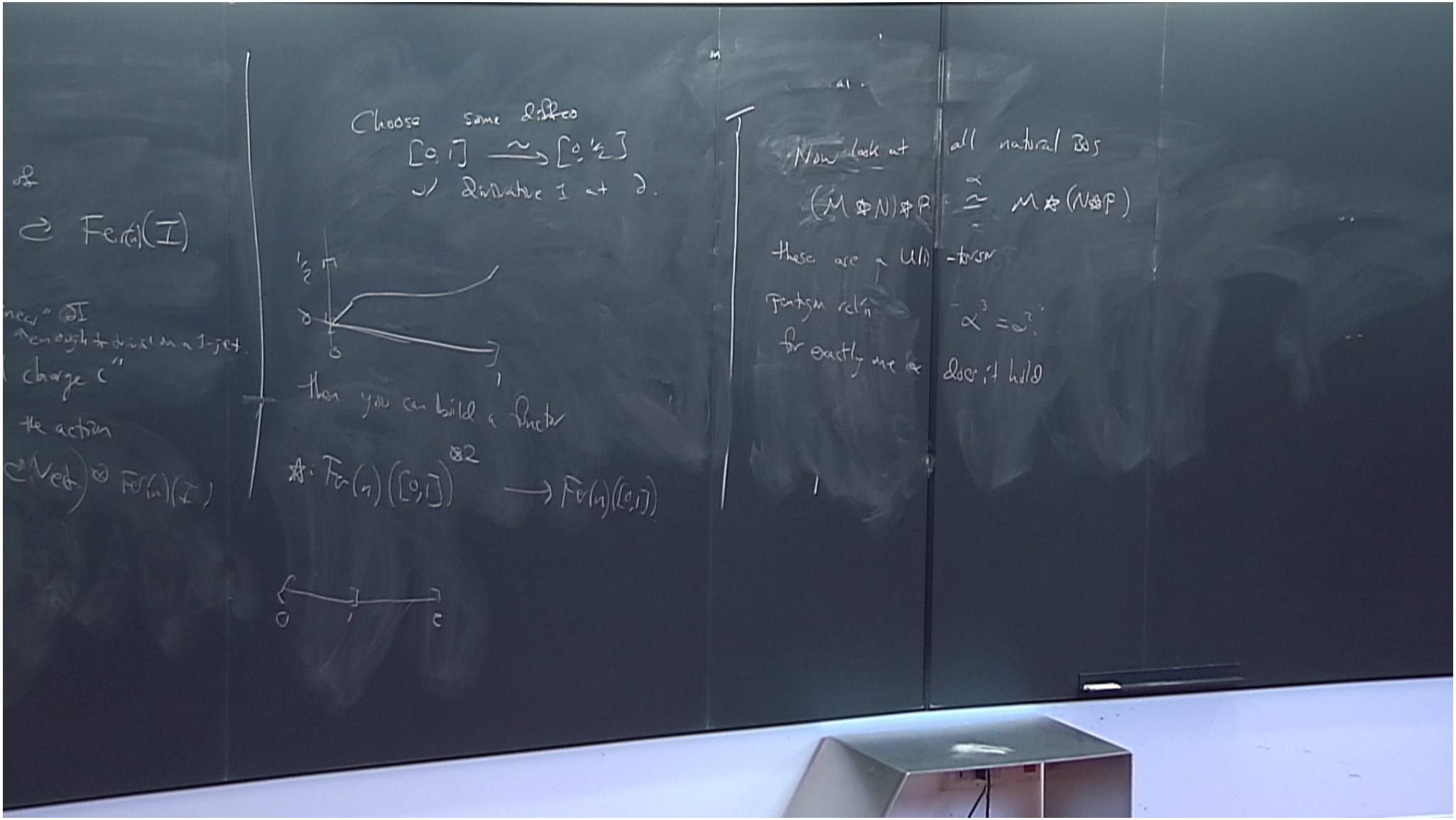
then you can build a functor

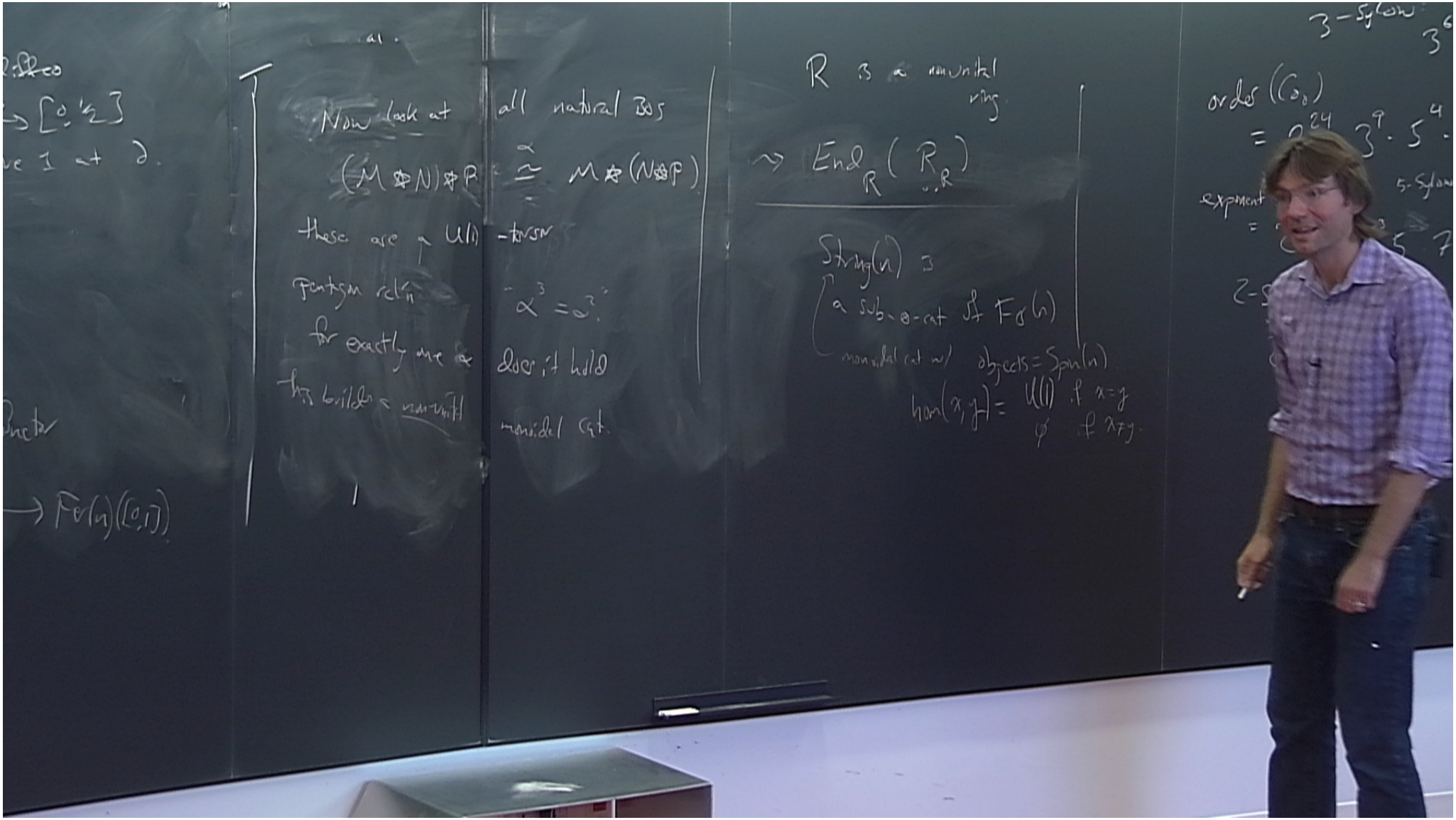
$\mathrm{Fer}(n)([0,1])^{\otimes 2} \longrightarrow \mathrm{Fer}(n)([0,1])$

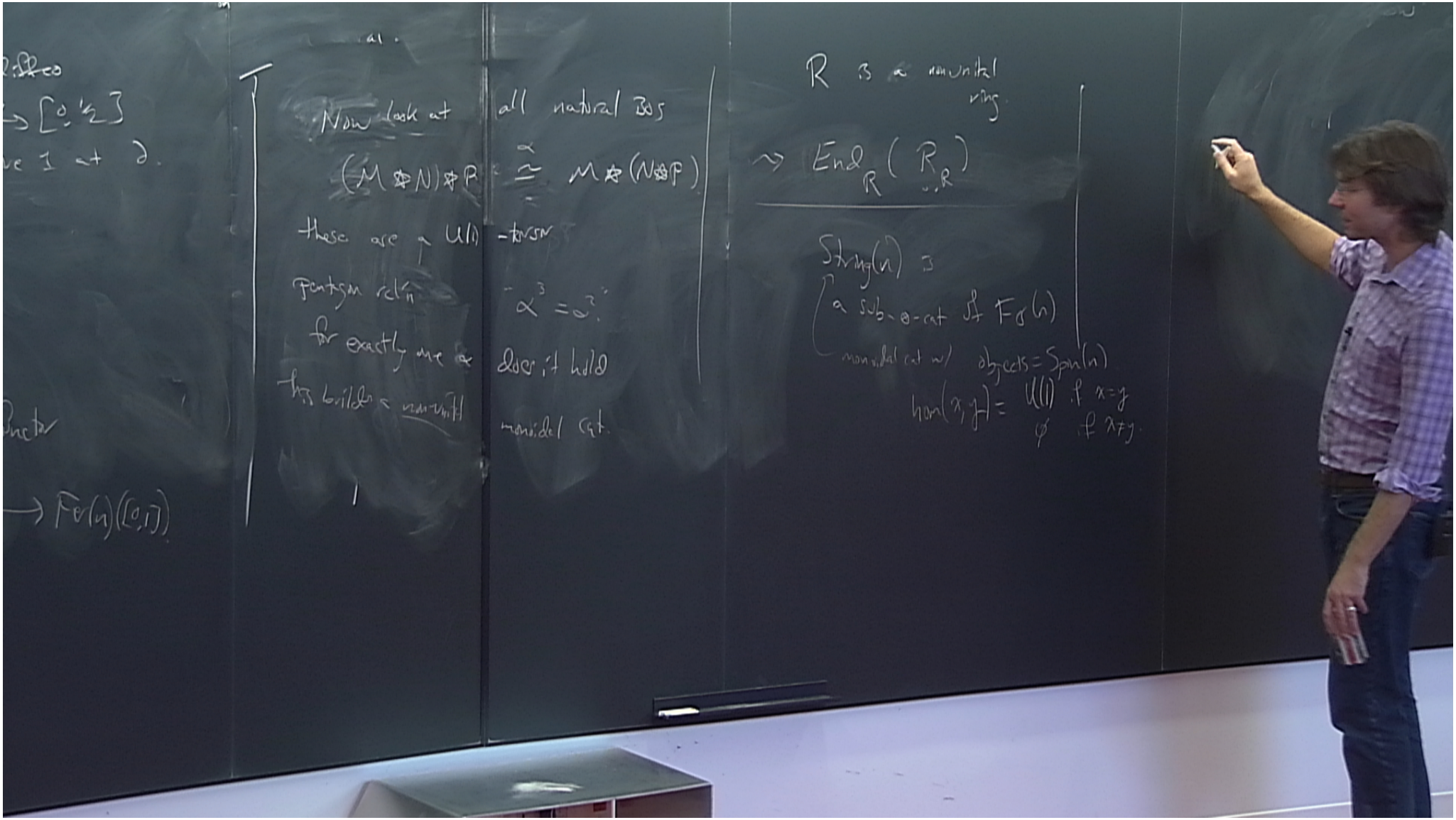


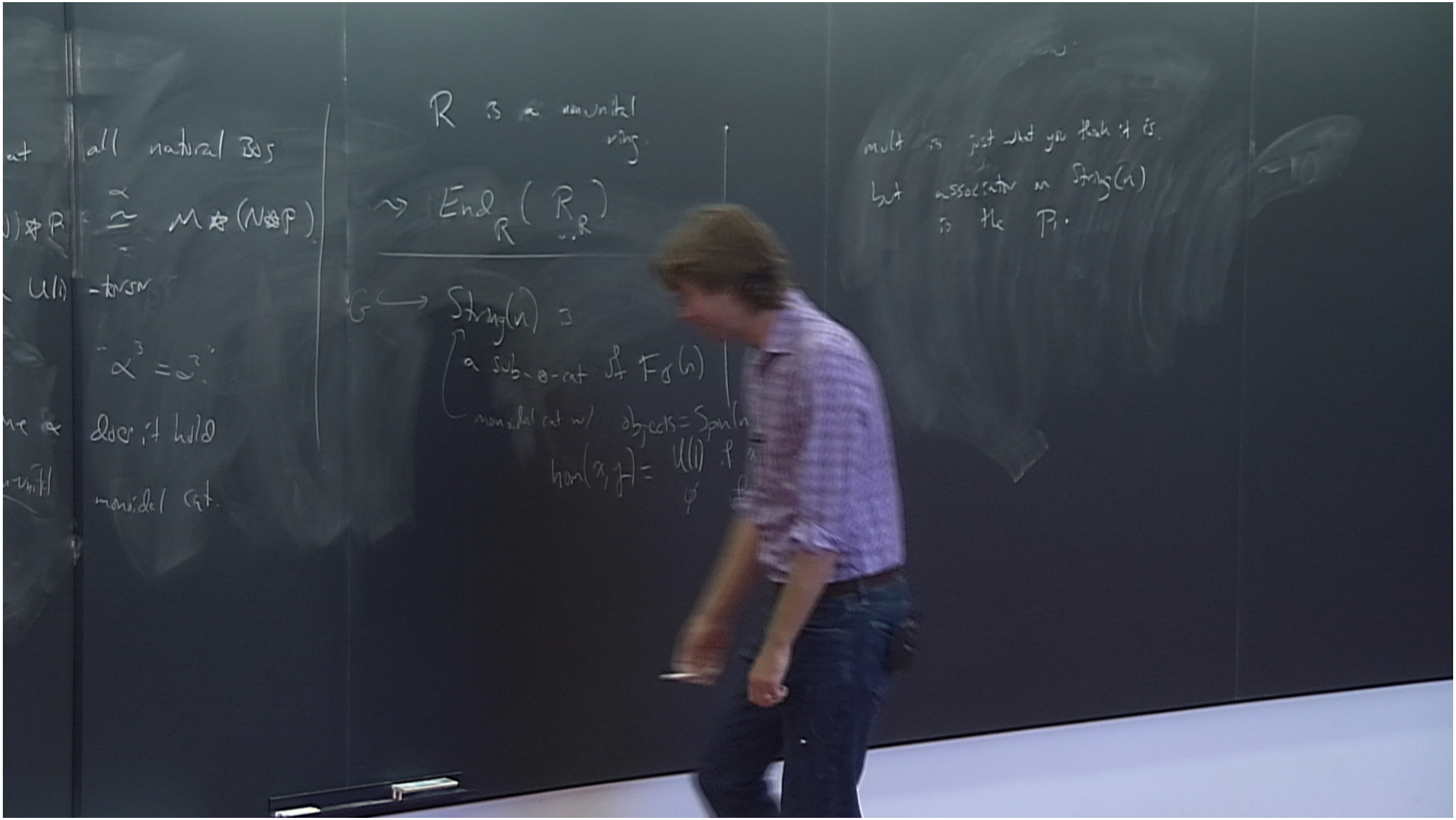
monoidal cat I h
talk you about
to n dim

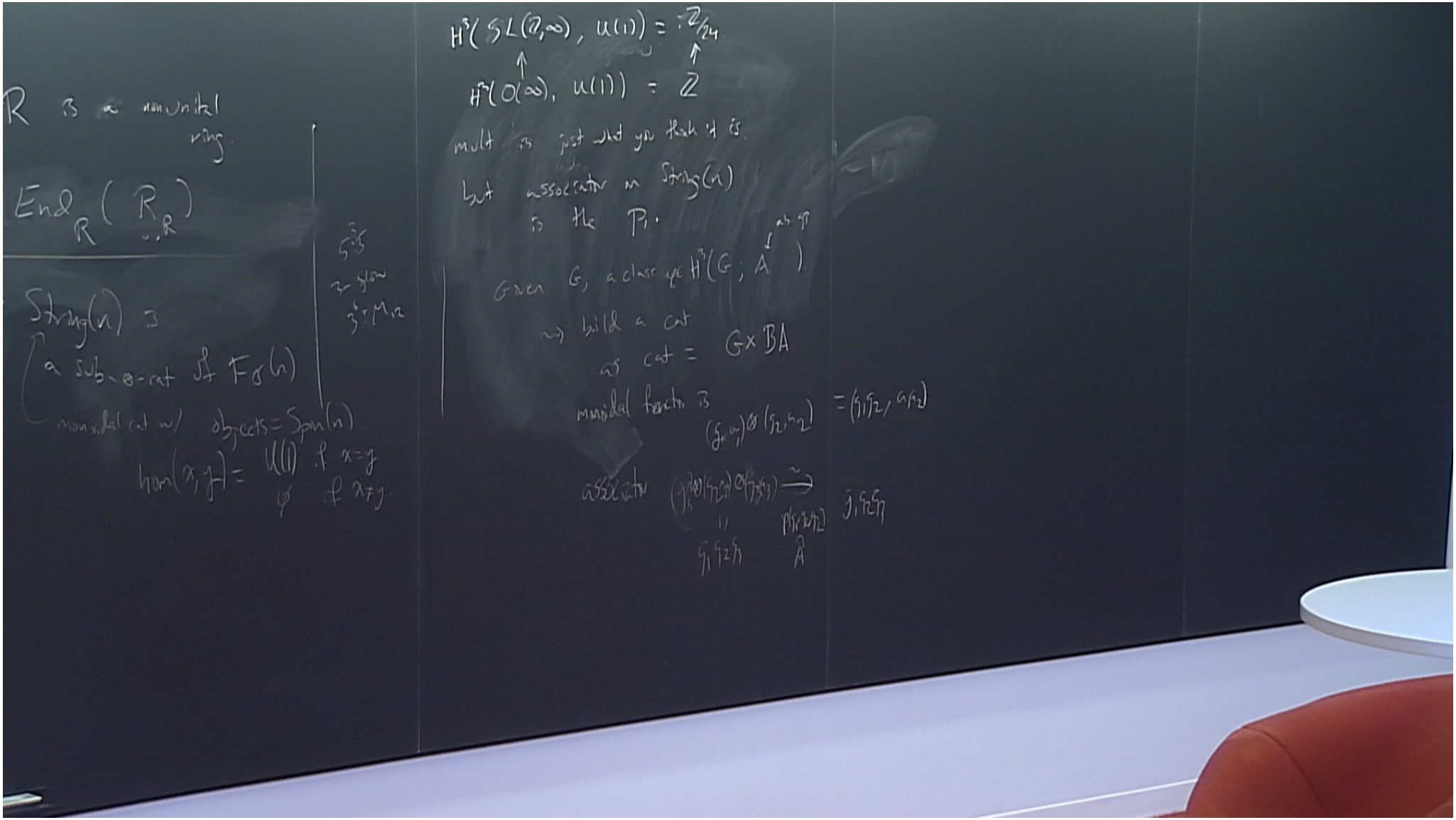
Almost


Choose some diffeo
[0,1] ≃ [0,½]
✓ derivative 1 at 0.
½
1
0
0
1
then you can build a functor
⊗: Fer(n)([0,1])^⊗2 → Fer(n)([0,1])
0
1
c
Now look at all natural ISOs
(M⊗N)⊗P ≃ M⊗(N⊗P)
these are a U(1)-torsor
pentagon relation
for exactly one α does it hold
α³ = α²
Fer(n)(I)
Vect ⊗ Fer(n)(I)


Now look at
all natural isos
these are a
for exactly one of
does it hold
monoidal cat.
R is a non-unital ring
order
Pirsa: 16090051
Page 34/37


Now look at
all natural iso's
these are
for exactly one
this builds
does it hold
monoidal cat.
R is a non-unital ring.
String(n) :=
a sub-2-cat of F_σ(n)
objects = Span(n)
if x=y
if x≠y.


all natural Bos
does it hold
monoidal cat.
R is a non-unital ring.
End_R(R_R)
String(n) is
a sub-⊗-cat of Fs(n)
monoidal cat w/ objects = Span(n
mult is just what you think it is.
but associator in String(n)
is the P_i.


R is a non-unital ring.
End_R(R_R)
String(n) is
a sub-∞-cat of Fg(n)
monoidal cat w/ objects = Span(n)
hom(x,y) = U(1) if x=y
∅ if x≠y
S^3 S^3
3-glow
3^3-Mix
H^3(SL(∞,∞), U(1)) = Z/24
H^3(O(∞), U(1)) = Z
mult is just what you think it is
but associator in String(n)
is the P_1.
Given G, a class in H^3(G; A^{ab gp})
~> build a cat
as cat = G × BA
monoidal functr is
(g_1,a_1) ⊗ (g_2,a_2) = (g_1g_2, a_1 g_1(a_2))
associator
g_1g_2g_3
A