Title: Local criticality, diffusion and chaos in generalized Sachdev-Ye-Kitaev models

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Abstract:

Local criticality, diffusion, and chaos in generalized Sachdev-Ye-Kitaev model

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Perimeter Institute, Sep 20 , 2016

Based on [YG, Xiao-Liang Qi, Douglas Stanford, arXiv: 1609.*****]

Outline

- Motivation;
- Brief review of the Sachdev-Ye-Kitaev (SYK) model;
- Generalization to higher dimensions. Example: $(1+1)$-d chain model:
  1. Local criticality and large N thermodynamics;
  2. Emergent conformal symmetry;
  3. OPE and energy transport;
  4. Chaos;
- Discussions.

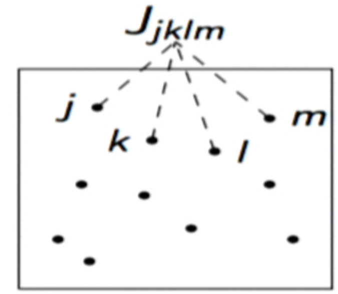
Motivation

- Condensed matter: solvable but non-trivial (lattice) model for strongly interacting system, especially the quantum matters without quasiparticles;
- Quantum chaos and quantum gravity: microscopic model to study the chaos and its propagation;
- Rare known example: the SYK model. $\Rightarrow$ Higher dimensional models with locality.

Sachdev-Ye-Kitaev model

Quantum mechanics (0 + 1-d) of N Majorana fermions:

$$H = \sum_{1 \leqslant j<k<l<m \leqslant N} J_{jklm} \chi_j \chi_k \chi_l \chi_m, \quad \{\chi_j, \chi_k\} = \delta_{jk}$$



All to all random interaction.

Random coefficients $\overline{J_{jklm}} = 0$, $\overline{J_{jklm}^2} = \frac{3! J^2}{N^3}$.
Interesting at large N and strong coupling: $N \gg \beta J \gg 1$. [Sachdev, Ye, 1993; Kitaev 2015; Maldacena, Stanford 2016.]

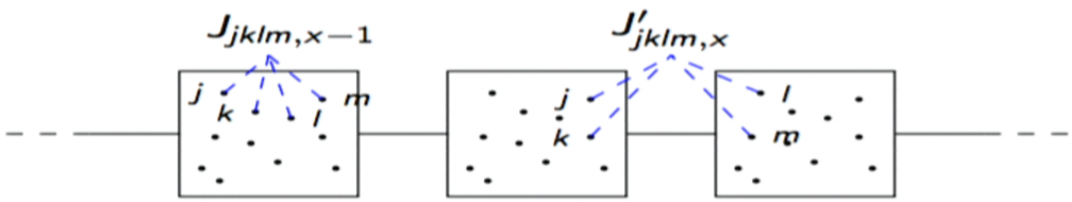
Solvable at large N, but non-trivial (no quasiparticles, strong chaos):

1. Emergent conformal symmetry [Sachdev, Ye; Parcollet, Georges; Kitaev];
2. Extensive zero temperature entropy [Parcollet, Georges, Sachdev; Kitaev];
3. Relevant to AdS_2 gravity [Sachdev; Kitaev; Maldacena, Stanford, Yang].

SYK is a nice model, but no locality.

Generalization with locality

- Generalization to higher dimensions with locality. To be concrete, let us start with a $(1+1)$-d example:



$$H = \sum_{x=1}^{M} \left[\sum_{j<k<l<m} \underbrace{J_{jklm,x}\ \chi_{j,x}\chi_{k,x}\chi_{l,x}\chi_{m,x}}_{\text{SYK term}} + \sum_{j<k;l<m} \underbrace{J'_{jklm,x}\ \chi_{j,x}\chi_{k,x}\chi_{l,x+1}\chi_{m,x+1}}_{\text{Nearest neighbour coupling}} \right]$$

Independent random coefficients:

$$\overline{J^2_{jklm,x}} = \frac{3!J_0^2}{N^3}, \quad \overline{J'^2_{jklm,x}} = \frac{J_1^2}{N^3}$$

- Interesting at $N \gg \beta J \gg 1$, $J := \sqrt{J_0^2 + J_1^2}$.

Plan for the rest of the talk

Analyzing the model via large N effective action:

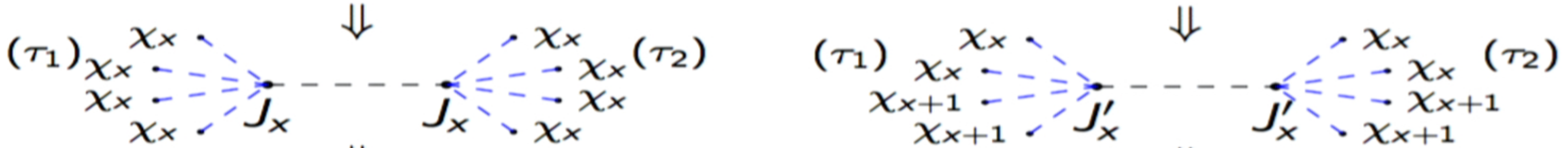
- Saddle point: two point function and thermodynamical quantities;
- Fluctuations around saddle point;
  - Comments on symmetry and pseudo-Goldstone modes;
  - Collective modes and diffusive energy transport;
  - Chaos.

Effective action at large N

Same as 0+1 SYK, our model also self-averages (replicon diagonal):

1. Integrate over $\{J_{jklm}\}$ $\Rightarrow$ disorder averaged partition function;
2. Introduce new fields G and Σ: Σ is Lagrange multiplier enforces $G_x(\tau_1, \tau_2) = \frac{1}{N}\sum_j \chi_{j,x}(\tau_1)\chi_{j,x}(\tau_2)$

$J_{jklm,x}\chi_{j,x}\chi_{k,x}\chi_{l,x}\chi_{m,x}$ $\Downarrow$ $\qquad$ $J'_{jklm,x}\chi_{j,x}\chi_{k,x}\chi_{l,x+1}\chi_{m,x+1}$ $\Downarrow$



$\Downarrow$ $J_0^2 G_x(\tau_1, \tau_2)^4$ $\qquad$ $\Downarrow$ $J_1^2 G_x(\tau_1, \tau_2)^2 G_{x+1}(\tau_1, \tau_2)^2$

Effective action at large N continued

3. Integrate over fermions:

$$\overline{Z} = \int \mathcal{D}G\mathcal{D}\Sigma \exp(-NS_{\text{eff}}[G,\Sigma])$$

$$S_{\text{eff}}[G,\Sigma] = \sum_{x=1}^{M} \Bigg[\underbrace{-\frac{1}{2}\log\det(\partial_\tau - \Sigma_x)}_{\text{From } \frac{1}{2}\chi(\partial_\tau - \Sigma)\chi} + \frac{1}{2}\int_0^\beta d\tau^2 \Bigg(\underbrace{\Sigma_x G_x}_{\text{Lagrange multiplier}}$$

$$\underbrace{-\frac{J_0^2}{4}G_x(\tau_1,\tau_2)^4 - \frac{J_1^2}{4}G_x(\tau_1,\tau_2)^2 G_{x+1}(\tau_1,\tau_2)^2}_{\text{Interaction term}} \Bigg) \Bigg]$$

Large N in front of $S_{\text{eff}} \Rightarrow$ solvable

Two-point function

Large N saddle point analysis:

1. Saddle point equation:

$$G_x^s(i\omega_n) = \frac{1}{-i\omega_n - \Sigma_x^s(i\omega_m)},$$
$$\Sigma_x^s(\tau) = J_0^2 G_x^s(\tau)^3 + \frac{1}{2} J_1^2 G_x(\tau) \left[G_{x-1}^s(\tau)^2 + G_{x+1}^s(\tau)^2\right]$$

2. Using averaged translational symmetry: $G_x^s(\tau_1, \tau_2) = G^s(\tau_1, \tau_2)$

$$G^s(i\omega_n) = \frac{1}{-i\omega_n - \Sigma^s(i\omega_n)}, \quad \Sigma^s(\tau) = J^2 G^s(\tau)^3$$

   $J^2 = J_0^2 + J_1^2$: effective coupling.

3. Saddle point equation same as in the $(0+1)$-d SYK model.

Two-point function continued

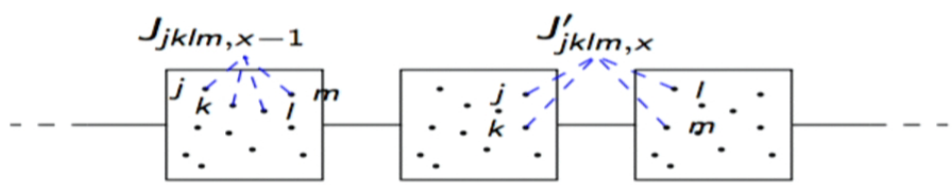
1. Analytic solution at strong coupling $\beta J \gg 1$: [Sachdev,Ye; Parcollet, Georges]

$$G^s(\tau) = \left(\frac{1}{4\pi}\right)^{\Delta} \left(\frac{\beta J}{\pi} \sin \frac{\pi \tau}{\beta}\right)^{-2\Delta}, \; \Delta = \frac{1}{4}$$

2. At $T = 0$, power law correlation for fermion on the same site:

$$\langle \mathcal{T}_\tau \chi_{j,x}(\tau) \chi_{j,x}(0) \rangle \propto \mathrm{sgn}(\tau) |J\tau|^{-2\Delta}$$

3. Fermion correlation functions between different sites strictly zero, due to on-site $\mathbb{Z}_2$ fermion parity symmetry. (In general model, suppressed by large N).



Thermodynamics

Large N thermodynamics: plug G^s and Σ^s back to the effective action:

$$\frac{F}{NM} = \frac{1}{\beta}\left[-\frac{1}{2}\log\det\left(\partial_\tau - \Sigma^s\right) + \frac{1}{2}\int d\tau_1 d\tau_2 \left(\Sigma^s(\tau_1,\tau_2)G^s(\tau_1,\tau_2) - \frac{J^2}{4}G^s(\tau_1,\tau_2)^4\right)\right]$$
$$= U - S_0 T - \frac{\gamma}{2}T^2 + \ldots$$

Same large N free energy density as $0+1$ SYK model:

1. Extensive zero temperature entropy $S_0 = \frac{\mathrm{Catalan}}{2\pi} + \frac{\log 2}{8} = 0.2324\ldots$
2. Specific heat $c_v = \gamma T = \frac{\pi\alpha_K}{16\sqrt{2}\beta J}$, proportional to T. ($\alpha_K \approx 2.852$ numerical constant)

Difference enters at level of quantum fluctuations.

Quantum fluctuations

- Quantum fluctuations around saddle point:

$$G_x = G^s + \delta G_x^s, \quad \Sigma_x = \Sigma^s + \delta \Sigma_x^s$$

- Expands to quadratic order, integrate over $\delta\Sigma \Rightarrow$

$$S_{\rm eff}[G] = S_{\rm eff}[G^s] + \int \delta G_x(\tau_1,\tau_2) Q_{xy}(\tau_1,\tau_2;\tau_3,\tau_4) \delta G_y(\tau_3,\tau_4)$$

- Quadratic form has simple dependence on spatial coordinates

$$Q_{xy}(\tau_1,\tau_2;\tau_3,\tau_4) = \underbrace{K^{-1}(\tau_1,\tau_2;\tau_3,\tau_4)}_{\text{same as 0+1 SYK}} \delta_{xy} - \delta(\tau_{13})\delta(\tau_{24}) \underbrace{S_{xy}}_{\text{was } \delta_{xy}}$$

- K: diagonalizable at $\beta J \gg 1$ [Kitaev 2015; Maldacena, Stanford 2016];
- S: $\delta_{xy} \to c_0 \delta_{xy} + c_1 \delta_{x,y\pm 1}$: "band structure" $s(p) = 1 - c_1 p^2 + \ldots$

Structure of the fluctuations

- Quadratic form $Q = K^{-1} - S$ controls the fluctuations;
- Most of the fluctuations are weak;
- One important subclass of fluctuation dominates (enhenced by βJ);
- Related to symmetry;

Emergent conformal symmetry

$$G^s(i\omega_n) = \frac{1}{\underbrace{-i\omega_n}_{\mathrm{IR},\to 0} - \Sigma^s(i\omega_n)}, \quad \Sigma^s(\tau) = J^2 G^s(\tau)^3$$

1. The model at IR has an emergent conformal symmetry—global time reparametrization: $f \in \mathrm{Diff}(S^1)$:

$$G^s_x(\tau_1, \tau_2) \to (f'(\tau_1) f'(\tau_2))^{\Delta} \, G^s_x(f(\tau_1), f(\tau_2))$$

2. Spontaneous breaking to $\mathrm{PSL}_2(\mathbb{R})$:

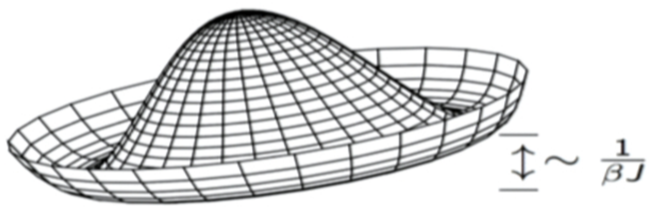
$$(\tau_1 - \tau_2)^{-2\Delta} \to (f'(\tau_1) f'(\tau_2))^{\Delta} (f(\tau_1) - f(\tau_2))^{-2\Delta} = (\tau_1 - \tau_2)^{-2\Delta}$$

$$\forall f(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

3. This symmetry is also explicitly broken by UV terms $-i\omega_n$.

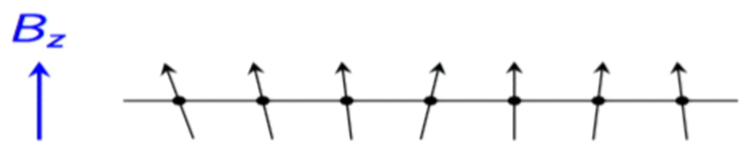
Pseudo-Goldstone mode

- Spontaneous and explicit symmetry breaking $\Rightarrow$ "nearly flat direction": Pseudo-Goldstone mode: $f_x \in \mathrm{Diff}(S^1)/\,\mathrm{PSL}_2(\mathbb{R})$



Tilted Mexican hat

- Analogy: ferromagnetic spin chain with small pining field. Spontaneous breaking SU(2) $\rightarrow U(1)$. Explicit breaking by small B_z.



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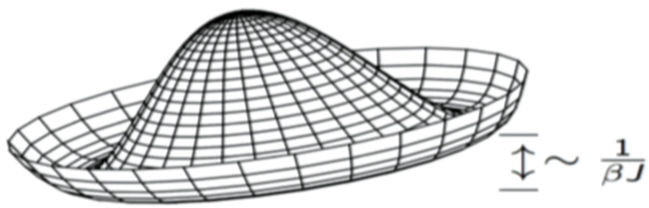

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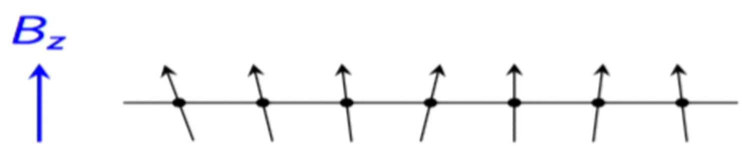


Effective action for pseudo-Goldstone mode

- Effective action for pseudo-Goldstone mode $f_x(\tau) = \tau + \epsilon_x(\tau)$

$$S \simeq \frac{1}{256\pi} \sum_{n,p} \epsilon_{n,p} \left(\underbrace{\frac{\sqrt{2}\alpha_K}{\beta J} n^2(n^2-1)}_{\text{Explicit breaking}} + \underbrace{\frac{J_1^2}{3J^2} p^2 |n| (n^2-1)}_{\text{Kinectic term}} \right) \epsilon_{-n,-p}$$

- Important: determines the energy diffusion and the leading contribution to chaos (will discuss later).

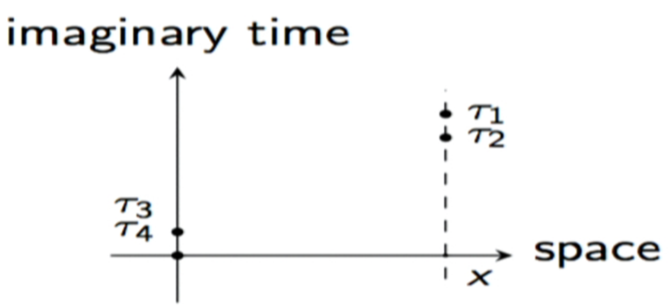
Four-point functions

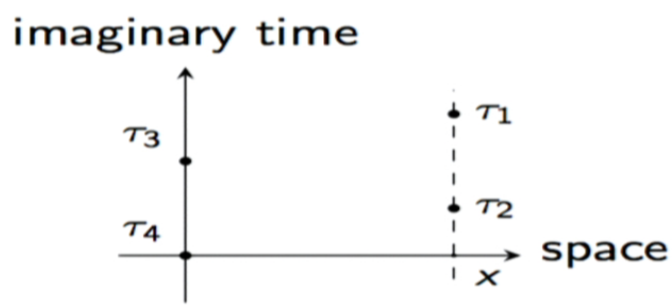
- Connected four-point function determined by quantum fluctuations:

$$\frac{1}{N^2}\sum_{j,k}\langle \chi_{j,x}(\tau_1)\chi_{j,x}(\tau_2)\chi_{k,y}(\tau_3)\chi_{k,y}(\tau_4)\rangle_{\text{conn.}} = \langle G_x(\tau_1,\tau_2)G_y(\tau_3,\tau_4)\rangle - \langle G\rangle\langle G\rangle$$

$$= \langle \delta G_x(\tau_1,\tau_2)\delta G_y(\tau_3,\tau_4)\rangle = \frac{1}{N} Q^{-1}_{xy}(\tau_1,\tau_2;\tau_3,\tau_4)$$

- Next: physical consequence
  1. OPE: collective modes and energy transport;
  2. Out-of-time-ordered correlation function: characterization of chaos.

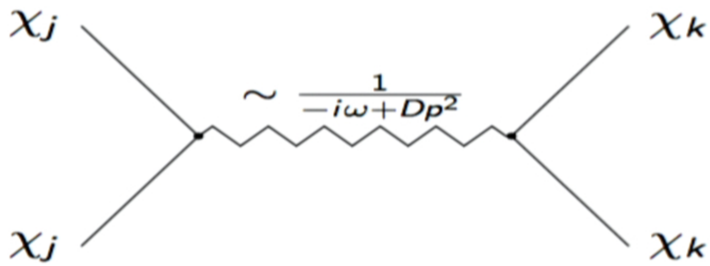




OPE and energy diffusion

OPE region: $\tau_1 \approx \tau_2 \gg \tau_3 \approx \tau_4$: the four-point function $\sim$ two-point function of collective modes.

1. Leading contribution: energy momentum tensor.



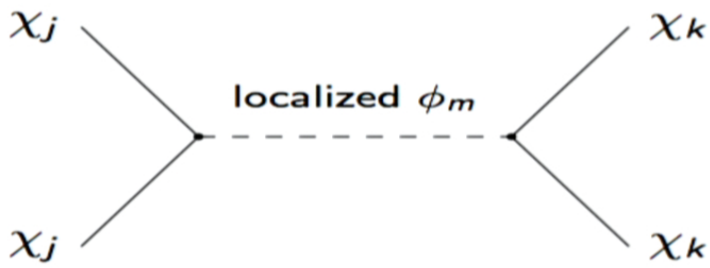
2. A diffusion pole $\frac{1}{-i\omega + Dp^2}$ with diffusion constant:

$$D = \frac{2\pi J_1^2}{3\sqrt{2}\alpha_K J}, \quad \alpha_K \approx 2.852 \quad \text{(numerical constant)}$$

J_1: coupling between neighbor sites; J: effective on-site coupling.

OPE and "stringy" modes

1. Subleading contributions: an infinite family of collective local critical modes ϕ_m, $m = 1, 2, \ldots$, localization length $\sim$ lattice scale;



2. Corresponds to "low-tension strings" in $0+1$ SYK; [Maldacena, Stanford 2016]
3. Responsible to the correction of chaos exponents (later slides).

Chaos region of four-point function

- A special time-order responsible for chaos in many-body system: out-of-time-order (OTO)

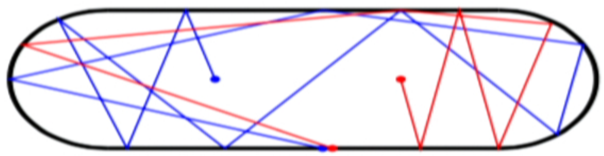
$$\langle \chi_{j,x}(t)\chi_{k,y}(0)\chi_{j,x}(t)\chi_{k,y}(0)\rangle_{\text{conn.}}$$

[Kitaev; Maldacena, Stanford, Shenker; ...]

- Next:
  1. briefly explain why OTO is relevant to chaos.
  2. show the results in our model.

From classical to quantum

Chaos: high sensitivity on initial condition.



Particles in a stadium

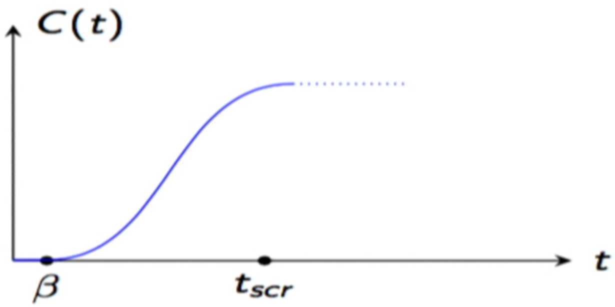
- Classical chaos: Poisson bracket $\{q(t), p(0)\}_{\mathrm{PB}} = \frac{\partial q(t)}{\partial q(0)} \sim e^{\lambda_L t}$,
  λ_L: Lyapunov exponent.

- From classical to quantum:
  $\{q(t), p(0)\}_{\mathrm{PB}} \to [\widehat{q}(t), \widehat{p}(0)] \to [W(t), V(0)]$
  [Larkin, Ovchinnikov 1969].

- $W(t)$, $V(0)$ -generic (hermitian) Heisenberg operators in many-body system

Quantum chaos in many-body system

- Diagnostics: $C(t) = -\,\mathrm{Tr}\left([W(t), V(0)]^2 \rho_\beta\right) = -\langle [W(t), V(0)]^2 \rangle_\beta$ (operator 2-norm on thermal ensemble).
- Example: 0+1 SYK model.



$C(t) \sim \frac{1}{N} e^{\lambda_L t}$. $t_{scr} \sim \frac{1}{\lambda_L} \log N$: time scale of "memory" ;

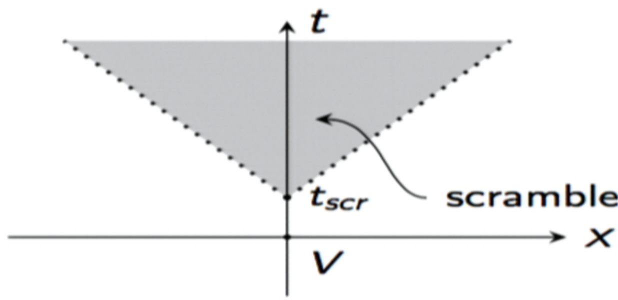
- λ_L: measure how fast chaos develops.
- Maldacena-Shenker-Stanford bound $\lambda_L \leqslant \frac{2\pi}{\beta}$.

Butterfly velocity

- QM $\Rightarrow$ QFT with spatial locality:

$$C(x,t) := -\langle [W(x,t), V(0,0)]^2 \rangle_\beta$$

- Extra delay of the scrambling time: e.g. $t_{scr}(x) = t_{scr} + |x|/v$, $v = v_B$: butterfly velocity.

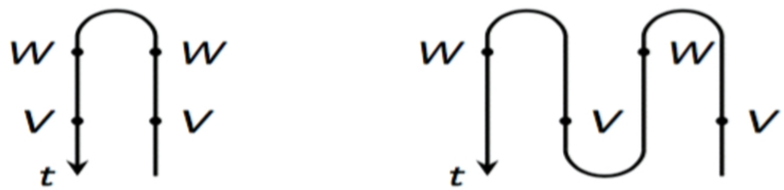


- Butterfly velocity: the speed of chaos propagation. [Roberts, Stanford, Susskind 2014]

$C(t)$ and OTO correlation function

- Growing of $C(t)$ is a measure of chaos. $\Rightarrow$ OTO?
- $C(t) = -\langle [W(t), V(0)]^2 \rangle_\beta$ contains four terms.

$$C(t) = \langle \underbrace{V(0)W(t)W(t)V(0) + W(t)V(0)V(0)W(t)}_{\text{"Accessible correlators"}} \underbrace{- W(t)V(0)W(t)V(0) - V(0)W(t)V(0)W(t)}_{\text{"Out-of-time-ordered correlators"}} \rangle_\beta$$



- OTO correlator: non-trivial exponential growing.

Chaos in generalized SYK

OTO correlation function is responsible for chaos.

- In our $(1+1)$-d SYK chain model:

$$\frac{\langle \chi_{j,x}(t)\chi_{k,0}(0)\chi_{j,x}(t)\chi_{k,0}(0)\rangle_{\text{conn.}}}{\langle \chi_{j,x}(t)\chi_{k,0}(0)\chi_{j,x}(t)\chi_{k,0}(0)\rangle_{\text{disc.}}} \sim \frac{(\beta J)^{1/2}}{N} \exp \frac{2\pi}{\beta}(t - |x|/v_B)$$

- $\lambda_L = \frac{2\pi}{\beta}$ true at least to $\frac{1}{(\beta J)^2}$.
  Correction vanishes at $\frac{1}{\beta J}$ order: localization of the "stringy" modes;

- Butterfly velocity:

$$v_B = \frac{2\pi J_1}{(3\sqrt{2}\alpha_K \beta J)^{1/2}}, \quad \alpha_K \approx 2.852 \text{ (numerical constant)}$$

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Comments on the v_B

- Interesting relation to diffusion:

$$v_B^2 = 2\pi TD$$

agree with holographic calculation on incoherent black hole [Blake 2016]. Relevant to incoherent metal [Hartnoll 2014].

- Compared with holographic calculation [Roberts, Swingle 2016] $v_B \propto T^{1-1/z} \Rightarrow z = 2$;

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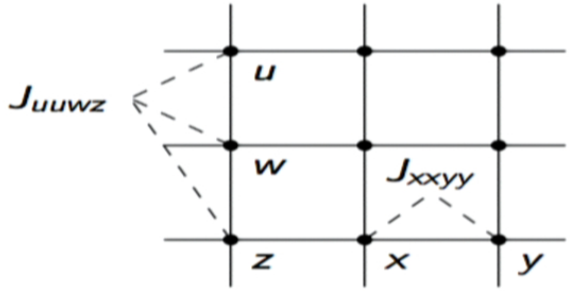
Brief discussion on general construction

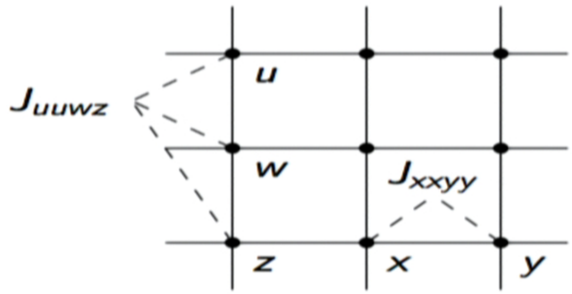
- Our model can be defined on arbitrary lattice Γ in any dimensions.

$$H = \sum_{x,y,z,w \in \Gamma} \sum_{j,l,k,m=1}^{N} J_{jklm,xyzw} \chi_{j,x} \chi_{k,y} \chi_{l,z} \chi_{m,w}$$



- Independent random numbers $\overline{J^2_{jklm,xyzw}} = \frac{J^2_{xyzw}}{N^3}$, solvable at large N.
- Locality: J_{xyzw} are "local functions" of $xyzw$.

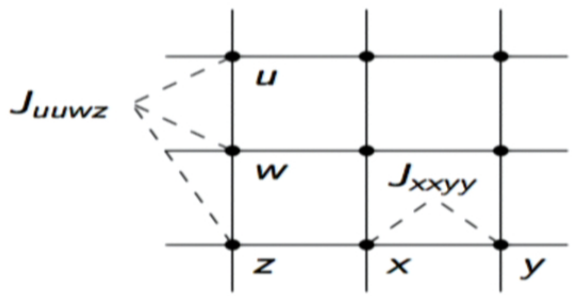
Brief discussion on general construction



1. Emergent conformal symmetry at strong coupling, local criticality for fermions, maximal chaos;
2. Diffusive energy transport and butterfly velocity: $v_{B,j}^2 = 2\pi TD_j$ remains for all directions x_j, $j = 1, 2 \ldots, d$.

Further generalization: add global symmetries.

Brief discussion on general construction



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Summary and discussion

Summary:

1. A generalized construction of Sachdev-Ye-Kitaev model at higher dimensions with locality;
2. Solvable at large N, local criticality, emergent symmetry at strong coupling $N \gg \beta J \gg 1$;
3. OPE and diffusive energy transport;
4. Maximal chaos. Butterfly velocity, related to diffusion constant by $v_B^2 = 2\pi TD$

Discussions:

1. A platform to study properties of strongly correlated system **exactly**;
2. Is our model holographic?
    - The relation suggests a dual to "incoherent black hole".
    - One single family of modes (reparametrizations) dominate, dual to "gravitational field"?