Title: The commutative limit of non-commutative geometries.

Date: Sep 01, 2016 02:30 PM

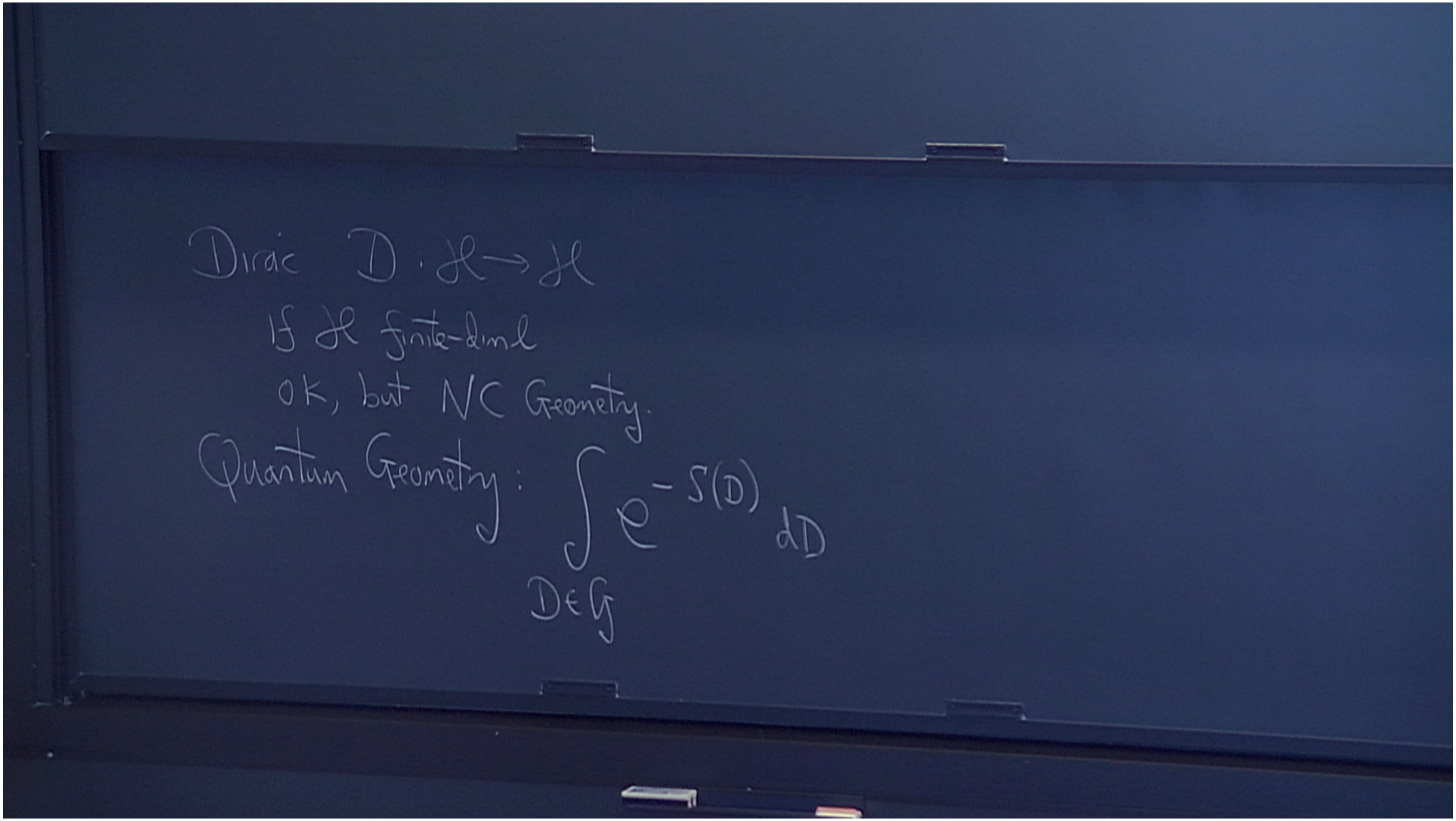
URL: http://pirsa.org/16090038

Abstract: <p>A formulation of a limit of a sequence of finite non-commutative spectral triples is presented. Examples of the commutative limits are the coadjoint orbits of semisimple Lie groups.</p>

Dirac $D : \mathcal{H} \to \mathcal{H}$

If $\mathcal{H}$ finite-dim'l

OK, but NC Geometry.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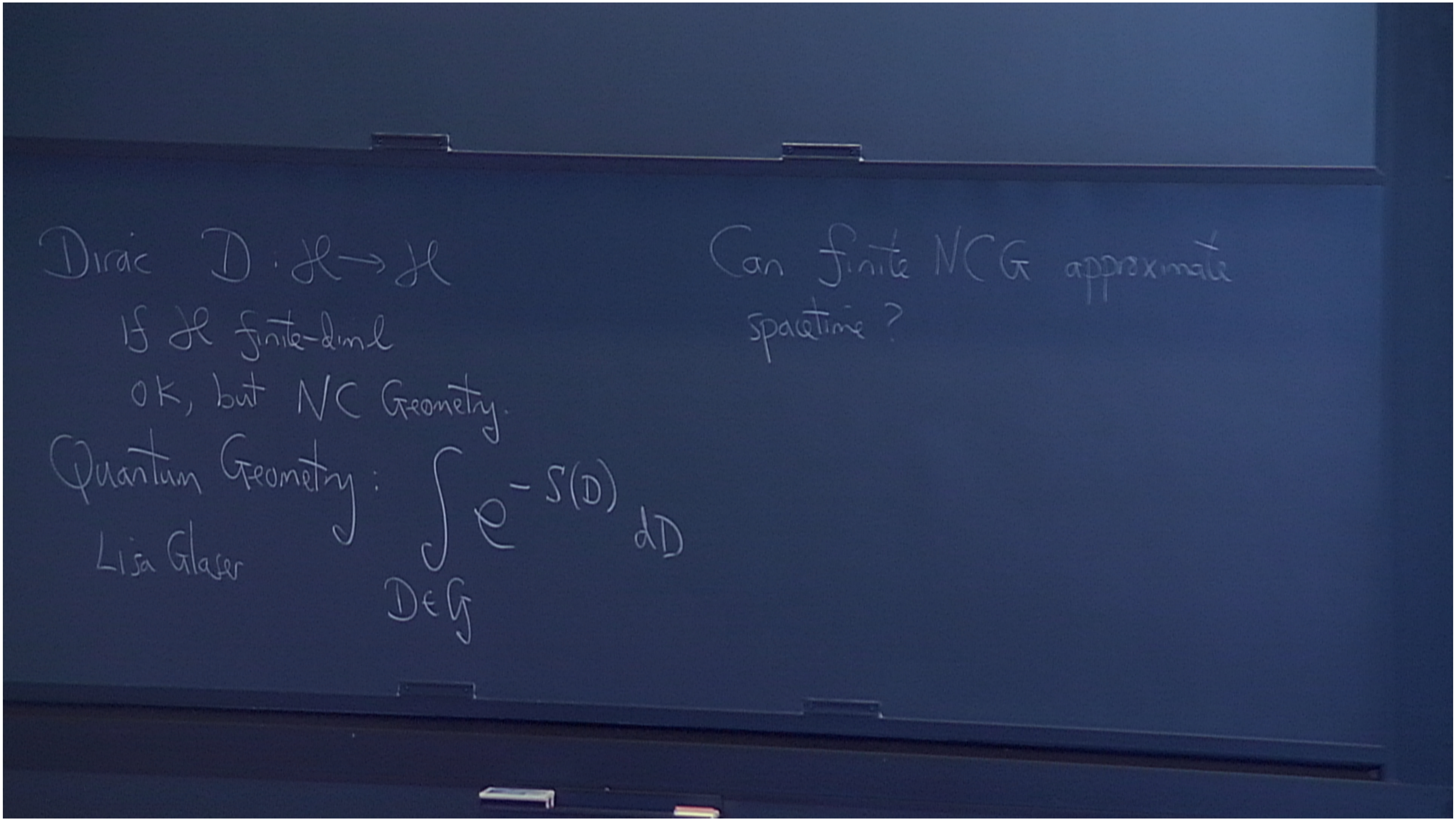
Quantum Geometry: $\int_{D \in \mathcal{G}} e^{-S(D)} \, dD$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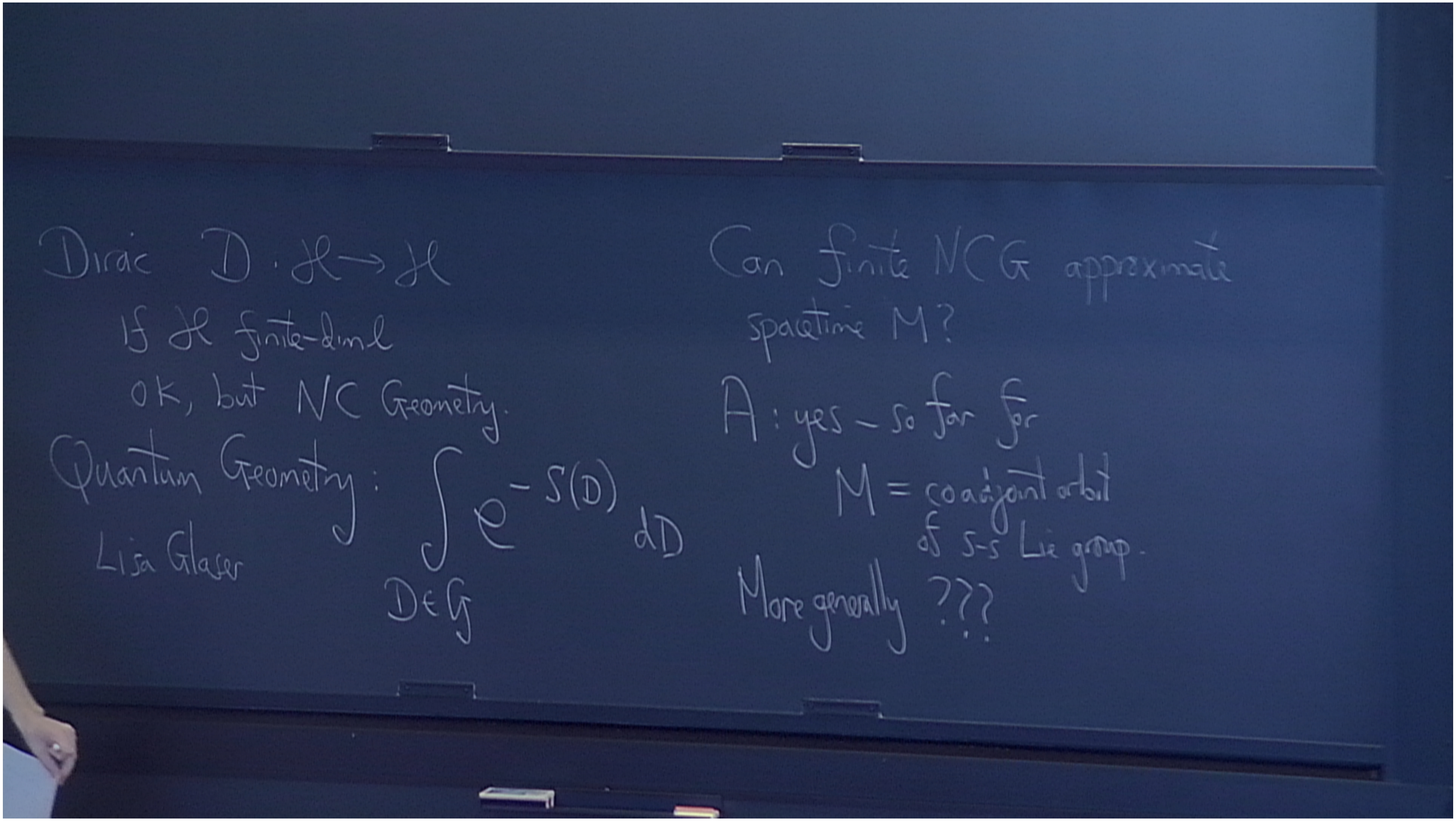
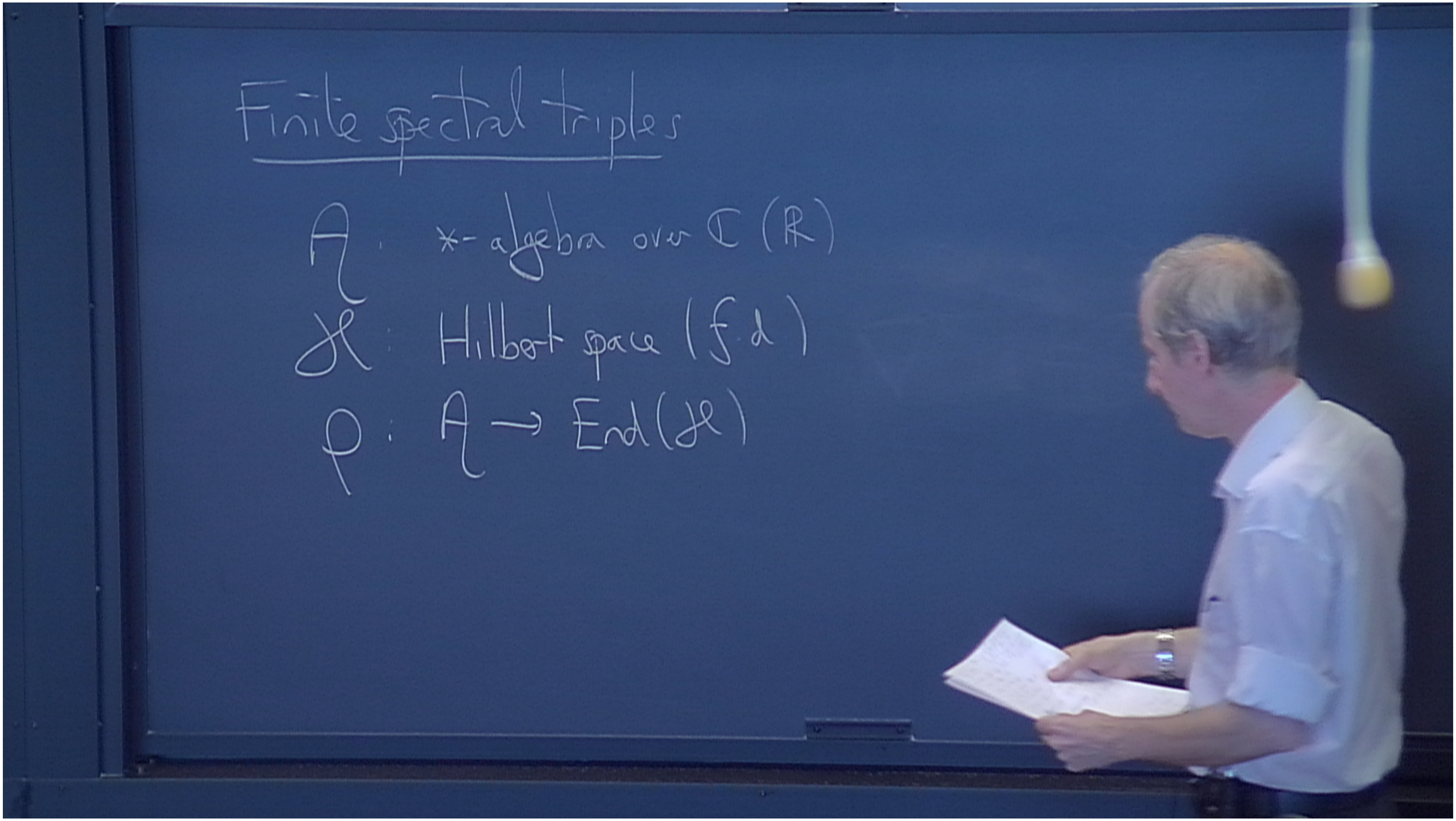
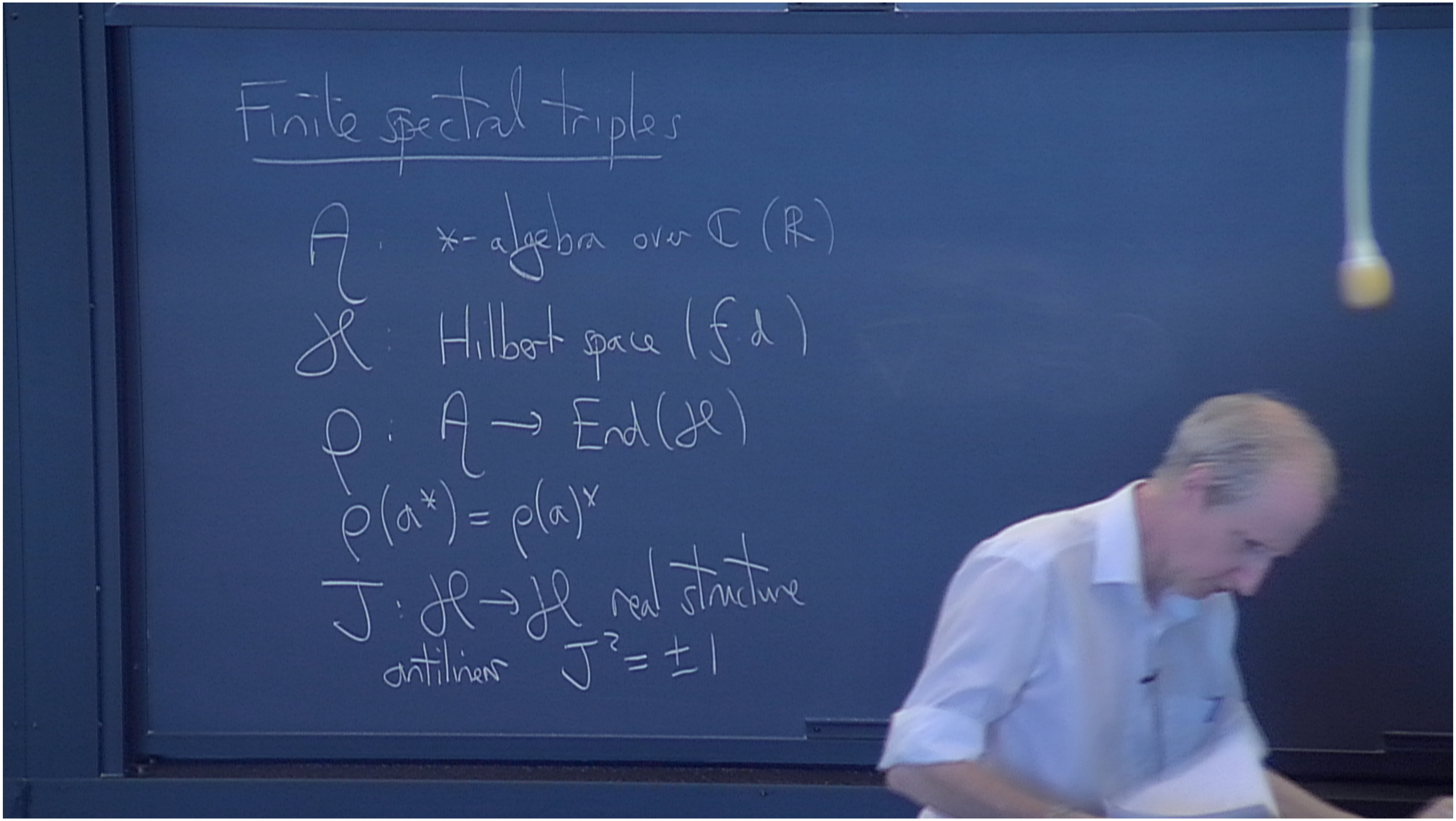
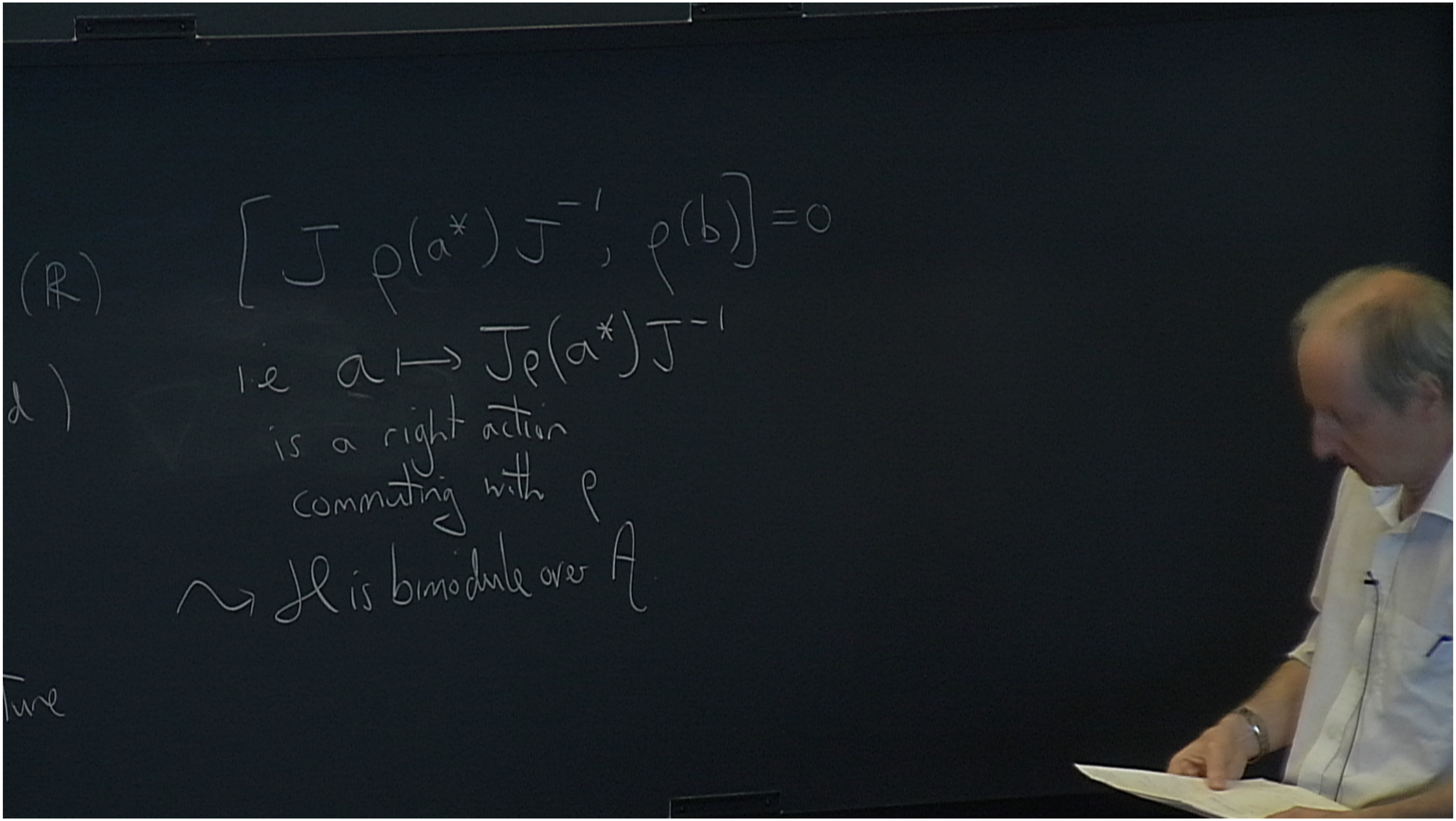
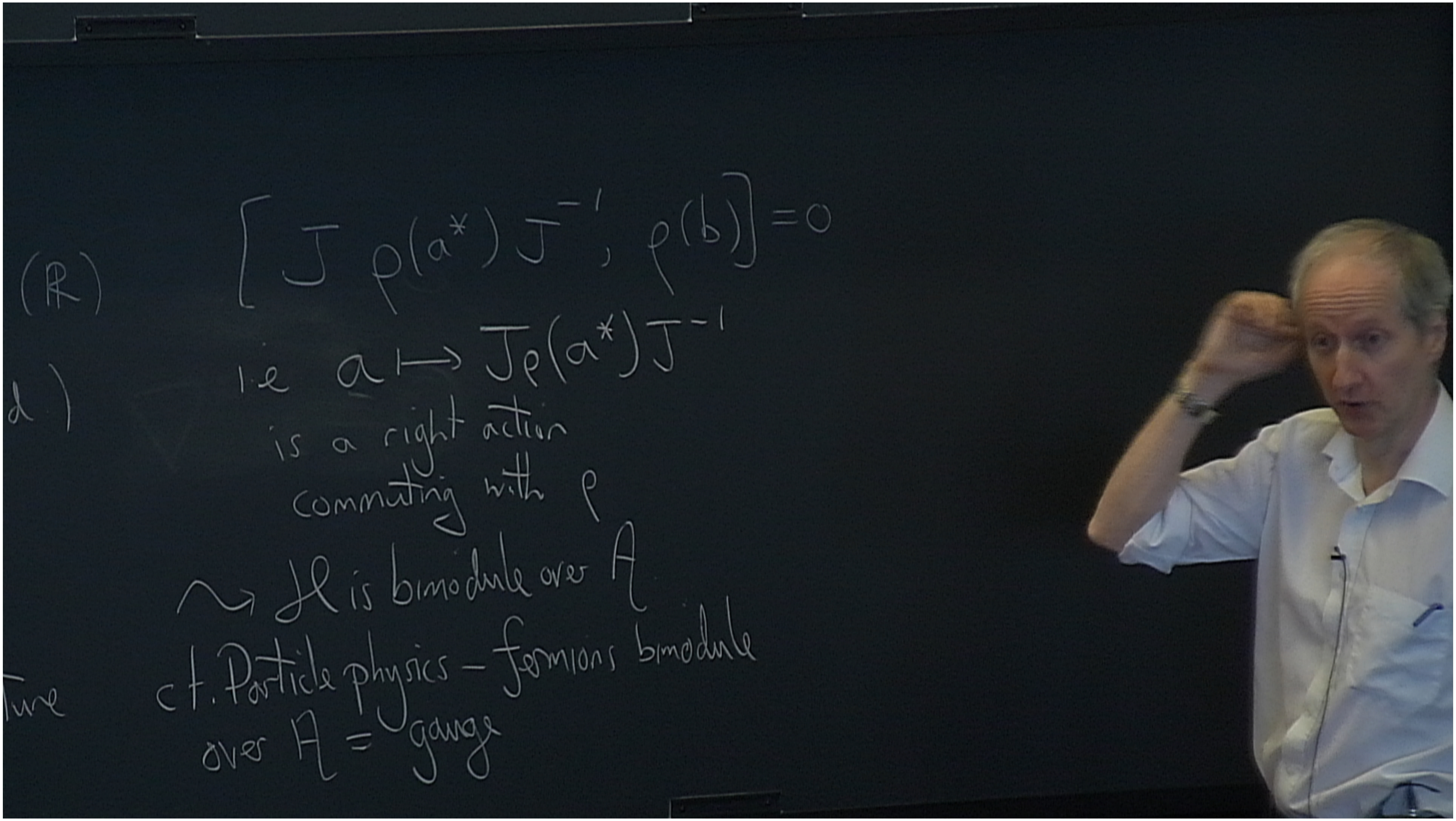
Lisa Glaser
Can finite NCG approximate spacetime?


Dirac D: ℋ → ℋ
If ℋ finite-dim'l
OK, but NC Geometry.
Quantum Geometry: ∫ e^{-S(D)} dD
D∈G
Lisa Glaser
Can finite NCG approximate
spacetime M?
A: yes - so far for
M = coadjoint orbit
of s-s Lie group.
More generally ???


Finite spectral triples
A : *-algebra over ℂ (ℝ)
H : Hilbert space (f.d)
ρ : A → End(H)


Finite spectral triples
A : *-algebra over ℂ (ℝ)
ℋ : Hilbert space (f.d)
ρ : A → End(ℋ)
ρ(a*) = ρ(a)*
J : ℋ → ℋ real structure
antilinear J² = ±1


[J ρ(a*) J⁻¹, ρ(b)] = 0
i.e a ↦ Jρ(a*)J⁻¹
is a right action
commuting with ρ
⇝ H is bimodule over A


(R)
d)
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⇝ H is bimodule over A
cf. Particle physics – fermions bimodule
over A = gauge

$$[J \rho(a^*) J^{-1}, \rho(b)] = 0$$

i.e. $a \mapsto J\rho(a^*)J^{-1}$

is a right action

commuting with ρ

$\rightsquigarrow \mathcal{H}$ is bimodule over A

cf. Particle physics – fermions bimodule over A = gauge

$$J\rho(a^*)J^{-1}\psi \overset{\text{def}}{=} \psi\rho(a)$$

$$[J\rho(a^*)J^{-1}, \rho(b)] = 0$$

i.e. $a \mapsto J\rho(a^*)J^{-1}$
is a right action
commuting with ρ

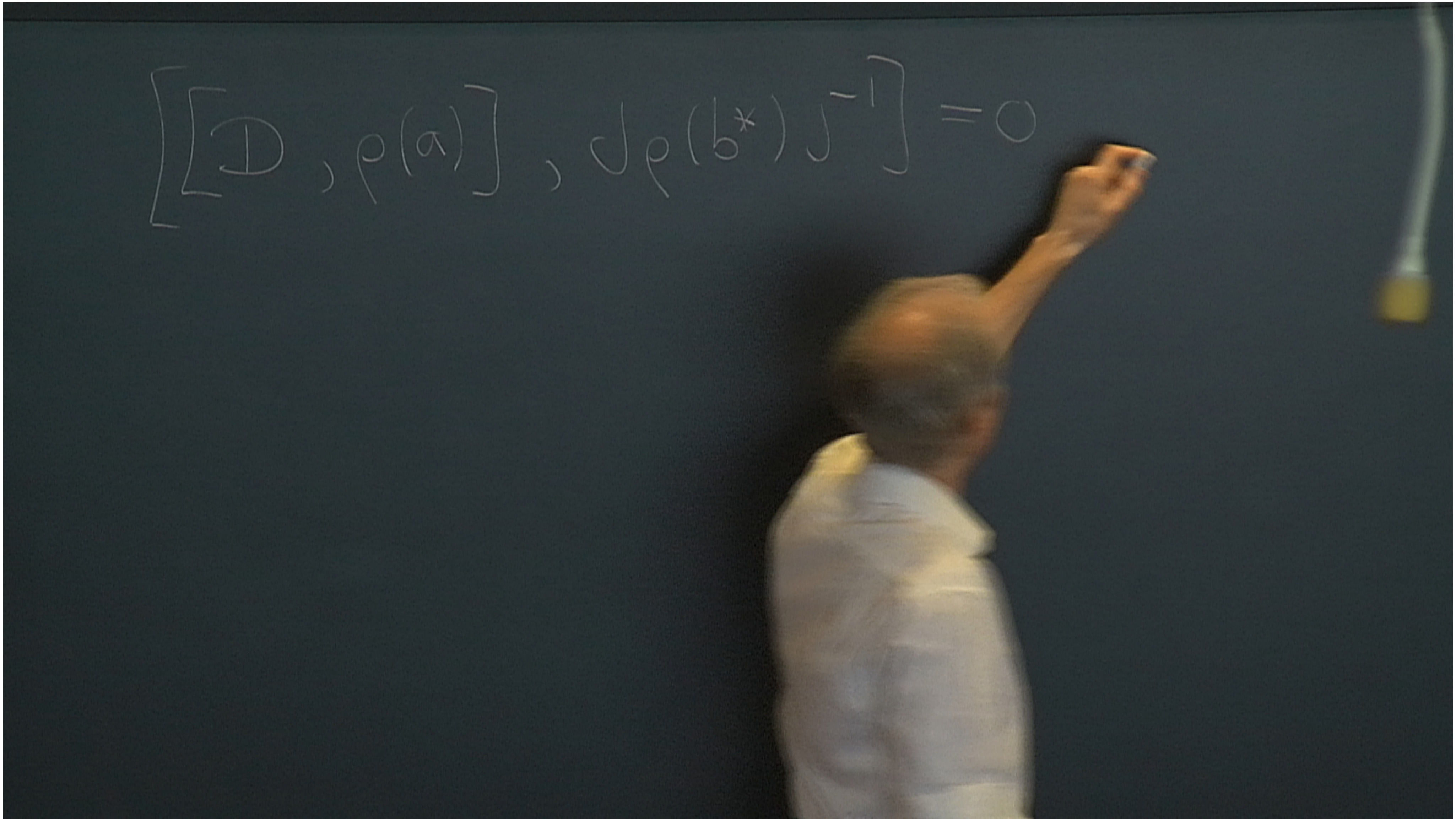
$\rightsquigarrow$ $\mathcal{H}$ is bimodule over $\mathcal{A}$.

cf. Particle physics – fermions bimodule over $\mathcal{A}$ = gauge

$$J\rho(a^*)J^{-1}\psi \overset{\text{def}}{=} \psi\rho(a)$$

$$D = D^*$$

$$DJ = JD.$$


$[[D, \rho(a)], J\rho(b^*)J^{-1}] = 0$

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"first-order condition"

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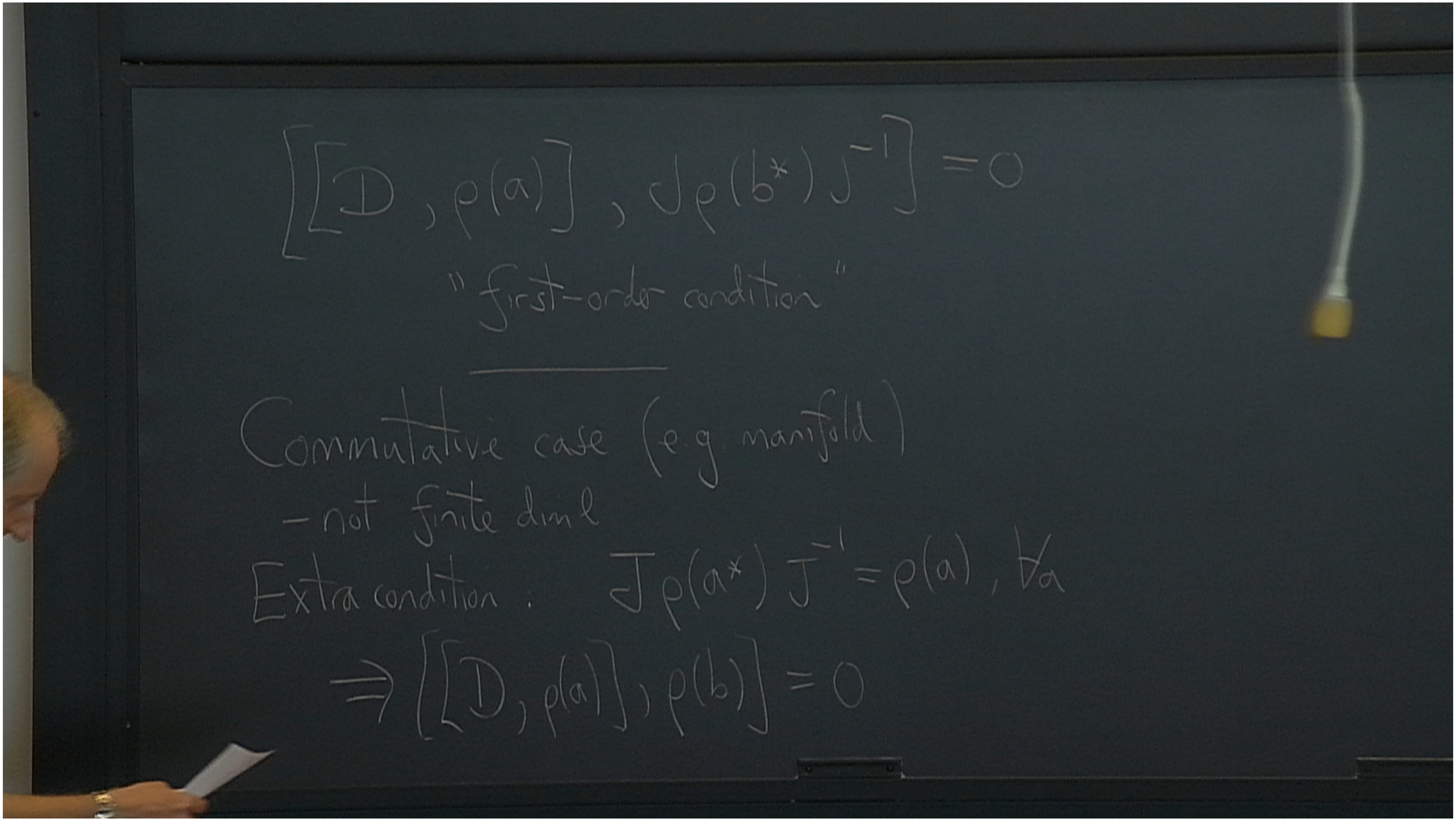
"first-order condition"

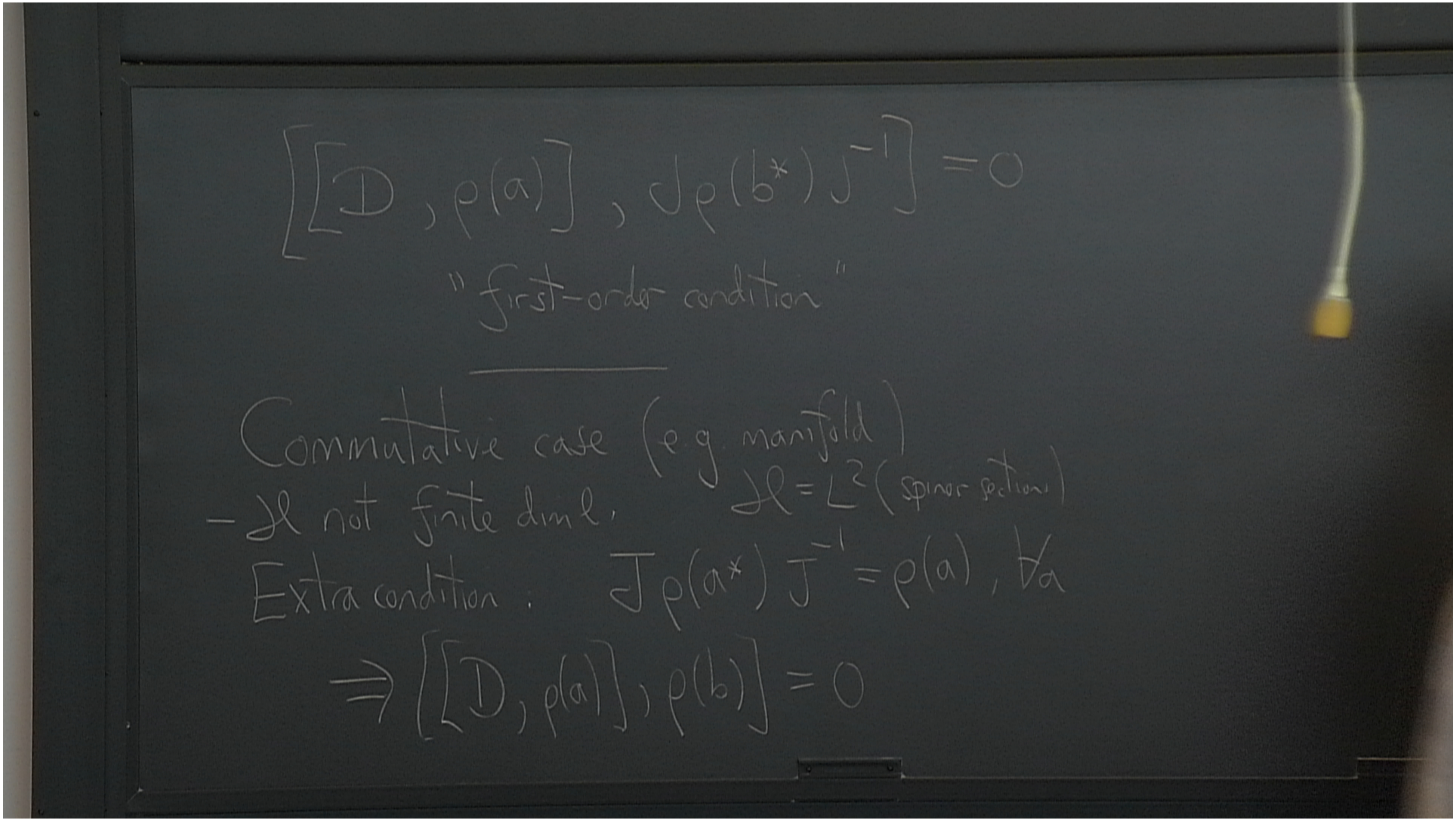
Commutative case (e.g. manifold)

- not finite dim'l

Extra condition: $J\rho(a^*)J^{-1} = \rho(a), \forall a$



$[[D, \rho(a)], J\rho(b^*)J^{-1}] = 0$
"first-order condition"
Commutative case (e.g. manifold)
- not finite dim'l
Extra condition: $J\rho(a^*)J^{-1} = \rho(a), \forall a$
$\Rightarrow [[D, \rho(a)], \rho(b)] = 0$


[[D, ρ(a)], Jρ(b*)J^-1] = 0
"first-order condition"
Commutative case (e.g. manifold)
- H not finite dim'l, H = L^2(spinor sections)
Extra condition: Jρ(a*)J^-1 = ρ(a), ∀a
⇒ [[D, ρ(a)], ρ(b)] = 0

$) J^{-1}] = 0$

(…nifold)

$= L^2(\text{spinor sections})$

$J^{-1} = \rho(a), \ \forall a$

$= 0$

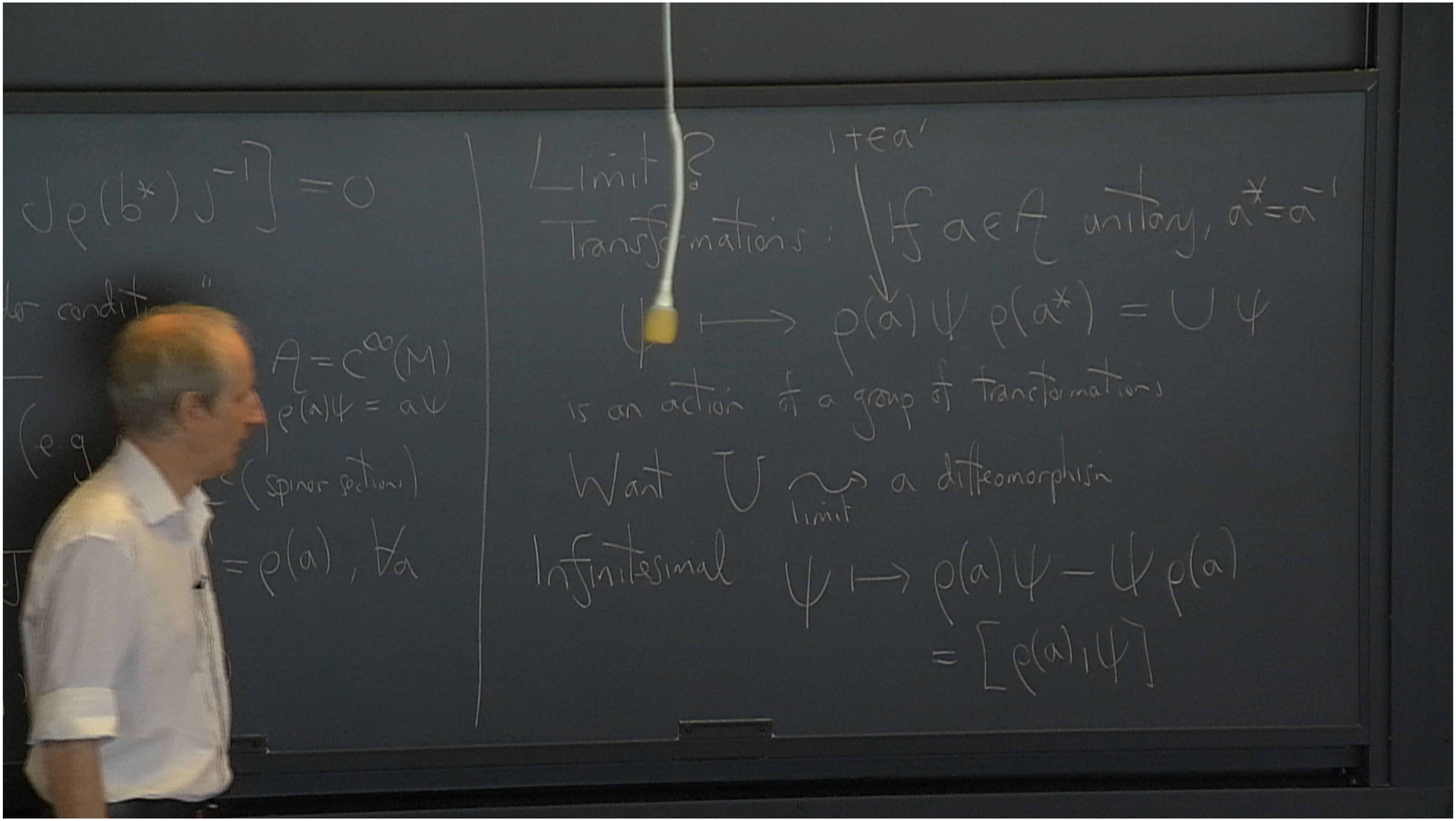
Limit?

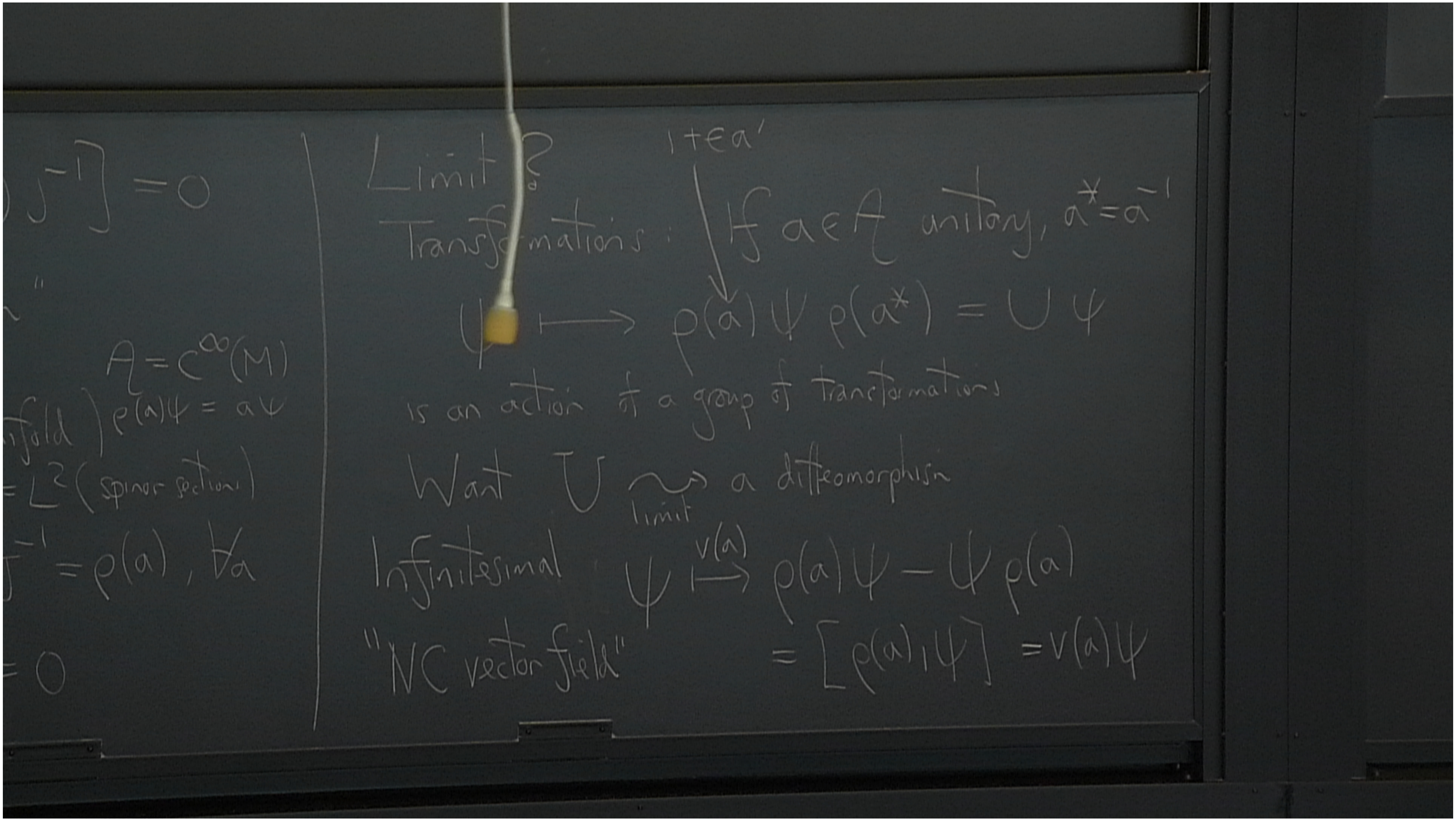
Transformations: $\forall a \in A$ unitary, $a^* = a^{-1}$

$$\psi \longmapsto \rho(a)\,\psi\,\rho(a^*) = U\psi$$

is an action of a group of transformations

Want $U \underset{\text{limit}}{\rightsquigarrow}$ a diffeomorphism


Jρ(b*) J^{-1}] = 0
A = C^∞(M)
ρ(a)ψ = aψ
(spinor sections)
= ρ(a), ∀a
Limit?
1+εa'
Transformations: If a∈A unitary, a* = a^{-1}
ψ ⟼ ρ(a) ψ ρ(a*) = U ψ
is an action of a group of transformations
Want U ⇝ a diffeomorphism
limit
Infinitesimal ψ ↦ ρ(a)ψ − ψρ(a)
= [ρ(a), ψ]

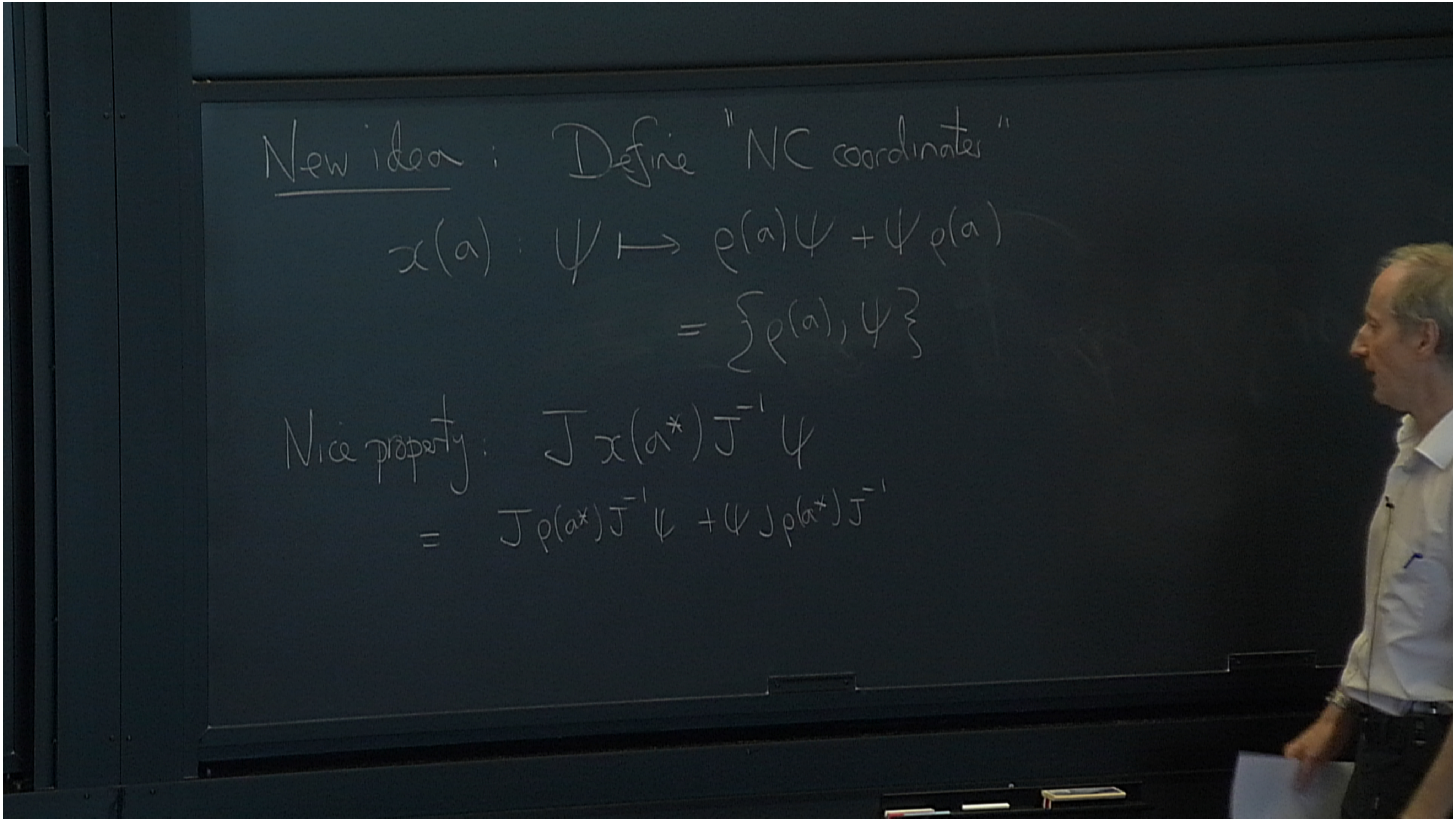


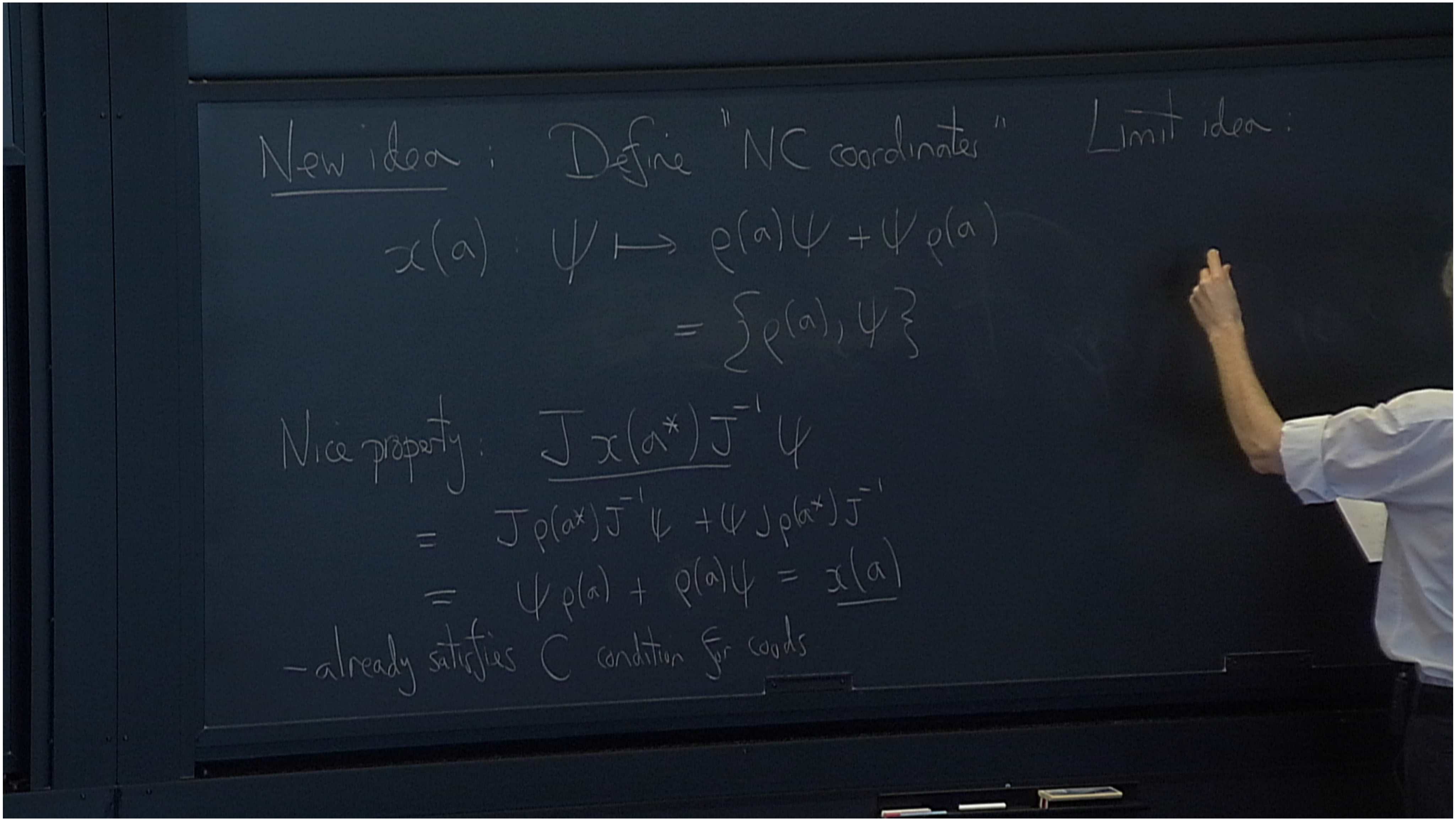
Limit?
1+εa'
Transformations: If a∈A unitary, a*=a⁻¹
ψ ⟼ ρ(a) ψ ρ(a*) = U ψ
is an action of a group of transformations
Want U ⤳ a diffeomorphism
limit
Infinitesimal ψ ⟼ ρ(a)ψ − ψρ(a)
v(a)
"NC vector field" = [ρ(a), ψ] = v(a)ψ
A = C∞(M)
ρ(a)ψ = aψ
=L²(spinor sections)
=ρ(a), ∀a
=0

New idea: Define "NC coordinates"

$$x(a) : \psi \longmapsto \rho(a)\psi + \psi\rho(a)$$

$$= \{\rho(a), \psi\}$$


New idea: Define "NC coordinates"
x(a): ψ ↦ ρ(a)ψ + ψρ(a)
= {ρ(a), ψ}
Nice property: J x(a*) J⁻¹ ψ
= J ρ(a*) J⁻¹ ψ + ψ J ρ(a*) J⁻¹


New idea: Define "NC coordinates"
Limit idea:
x(a) : ψ ↦ ρ(a)ψ + ψρ(a)
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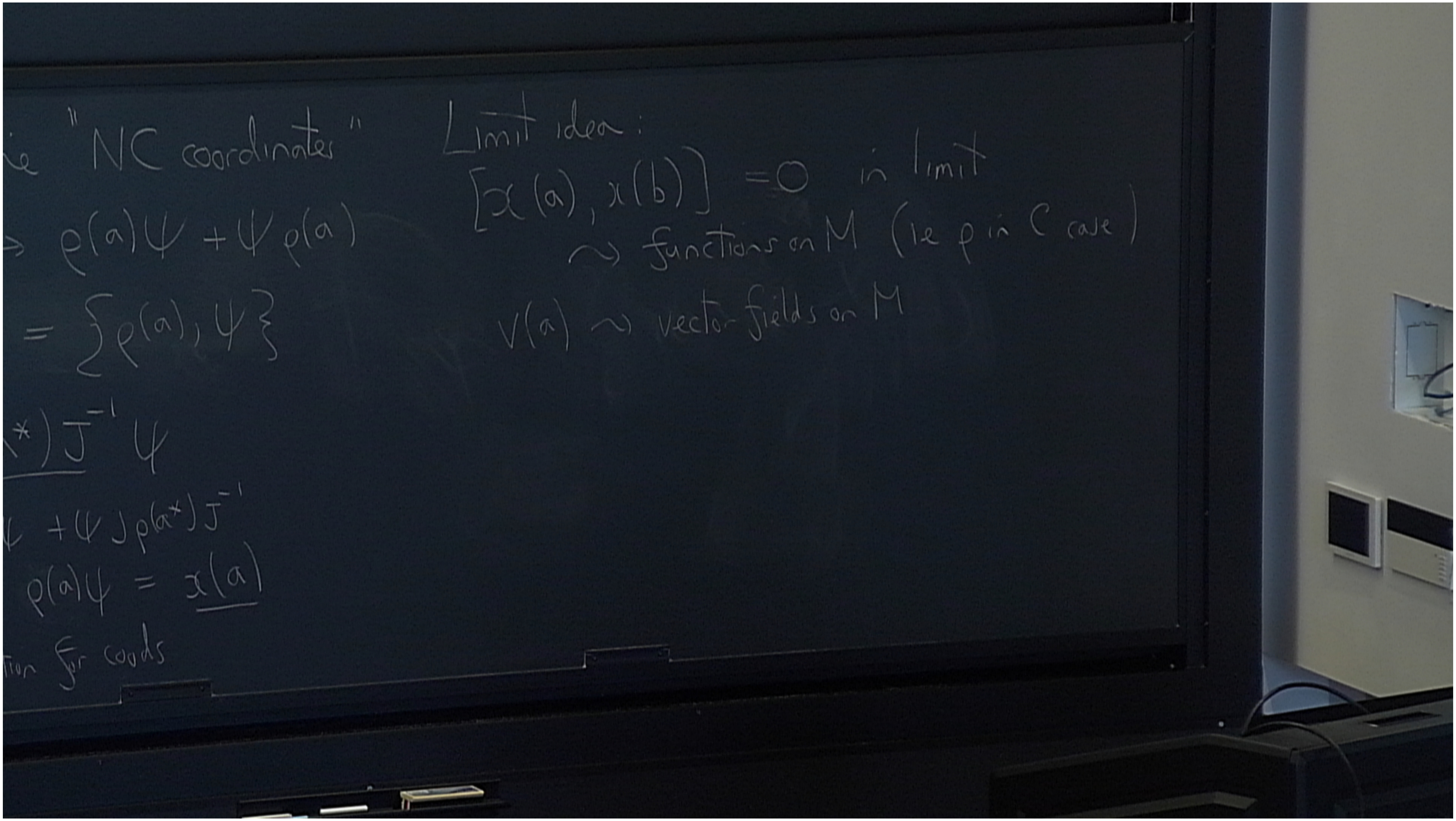
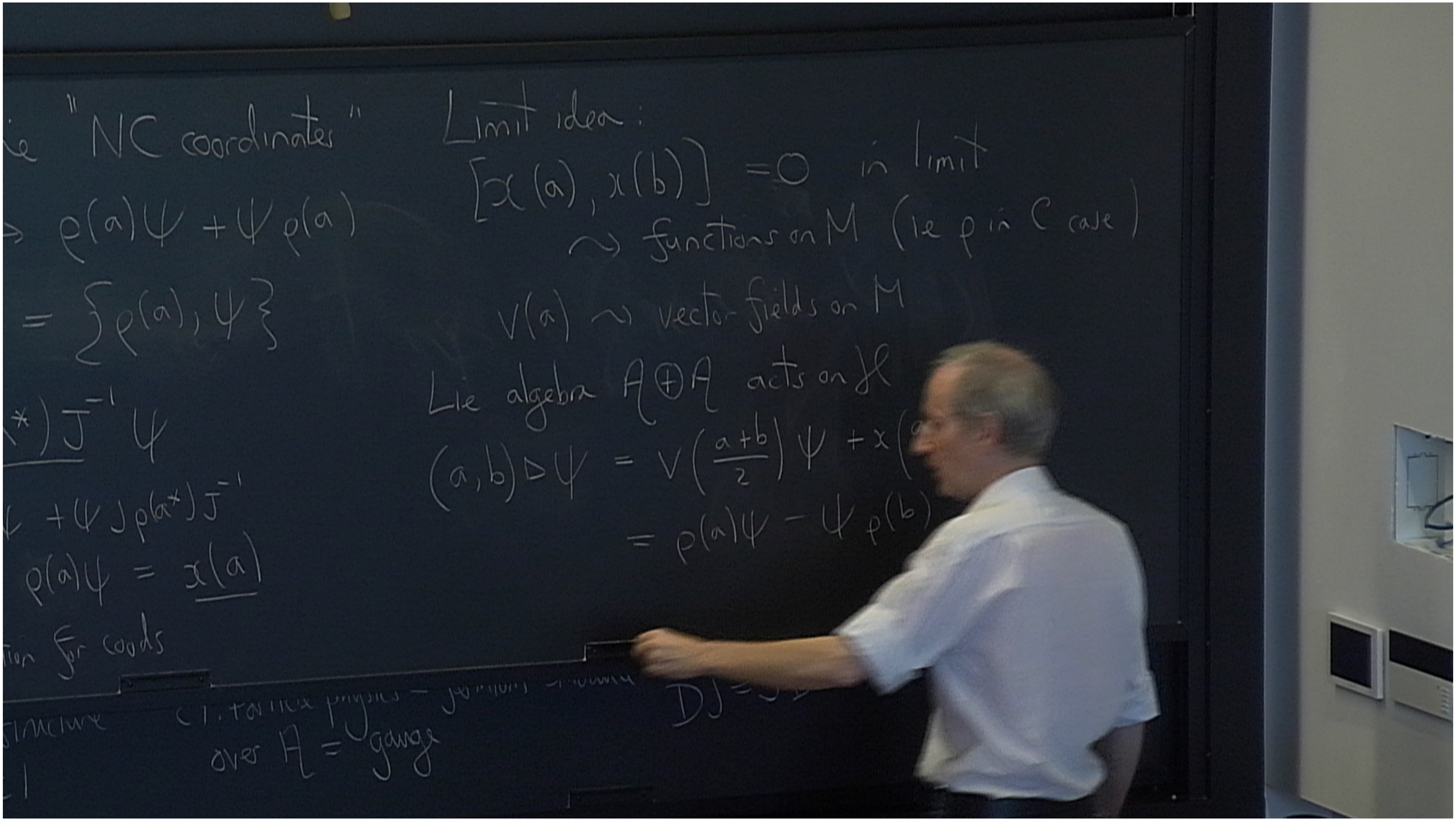
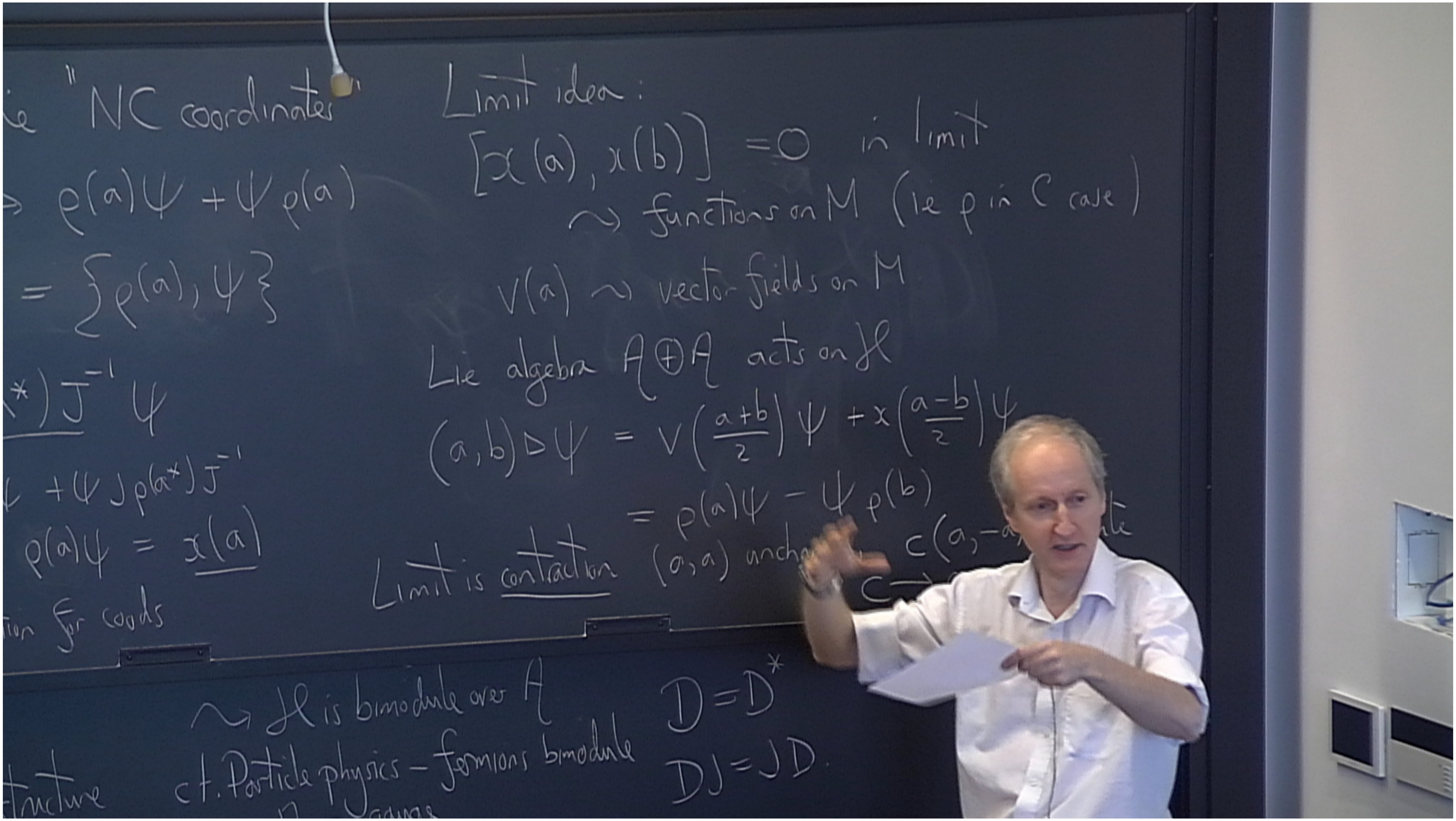
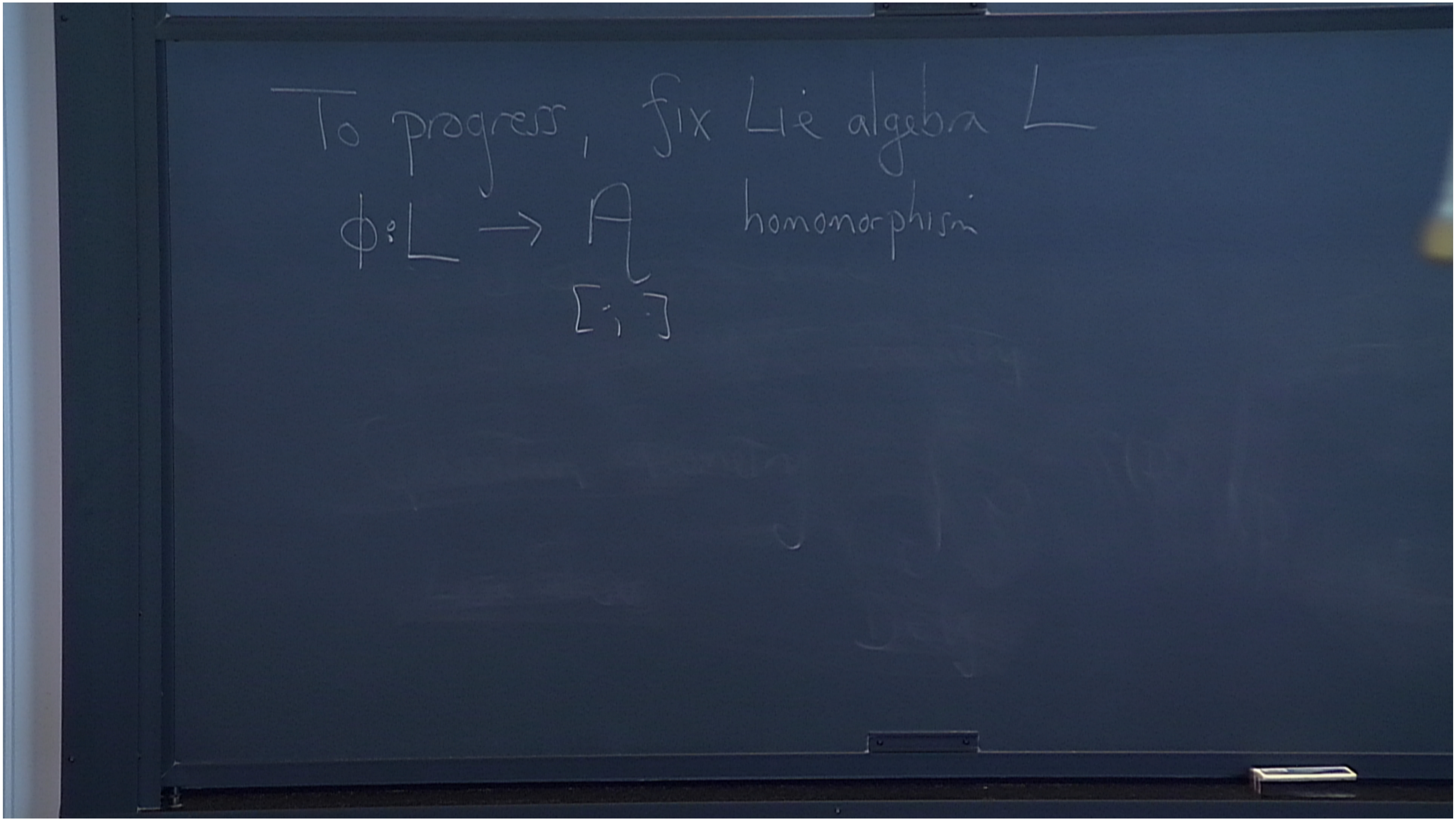
Nice property: J x(a*) J⁻¹ ψ
= J ρ(a*) J⁻¹ ψ + ψ J ρ(a*) J⁻¹
= ψ ρ(a) + ρ(a) ψ = x(a)
- already satisfies C condition for coords

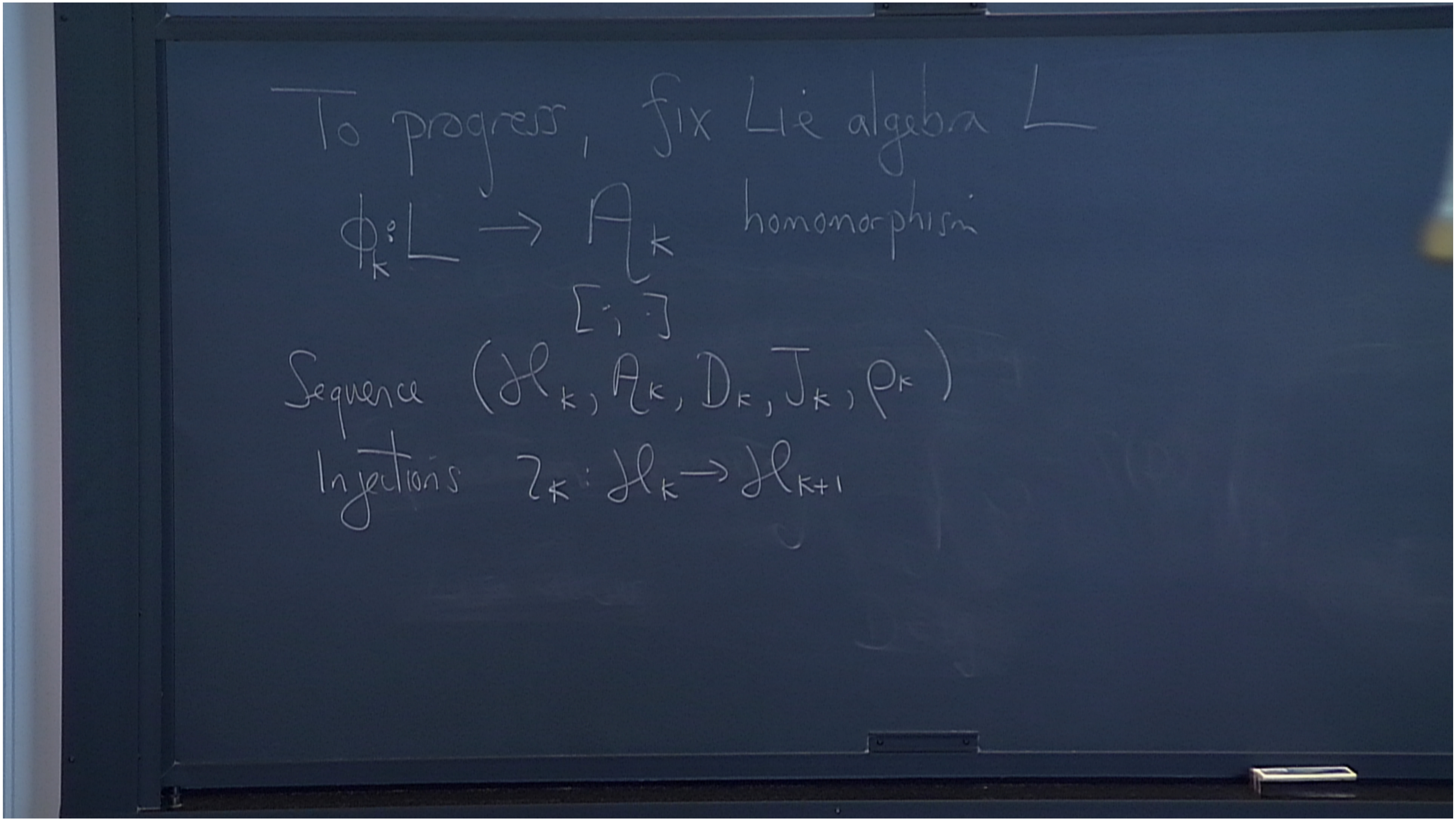

ie "NC coordinates"
⇒ ρ(a)ψ + ψρ(a)
= {ρ(a), ψ}
*) J^-1 ψ
ψ + ψ J ρ(a*) J^-1
ρ(a)ψ = x(a)
tion for coords
Limit idea:
[x(a), x(b)] = 0 in limit
~ functions on M (ie ρ in C case)
v(a) ~ vector fields on M

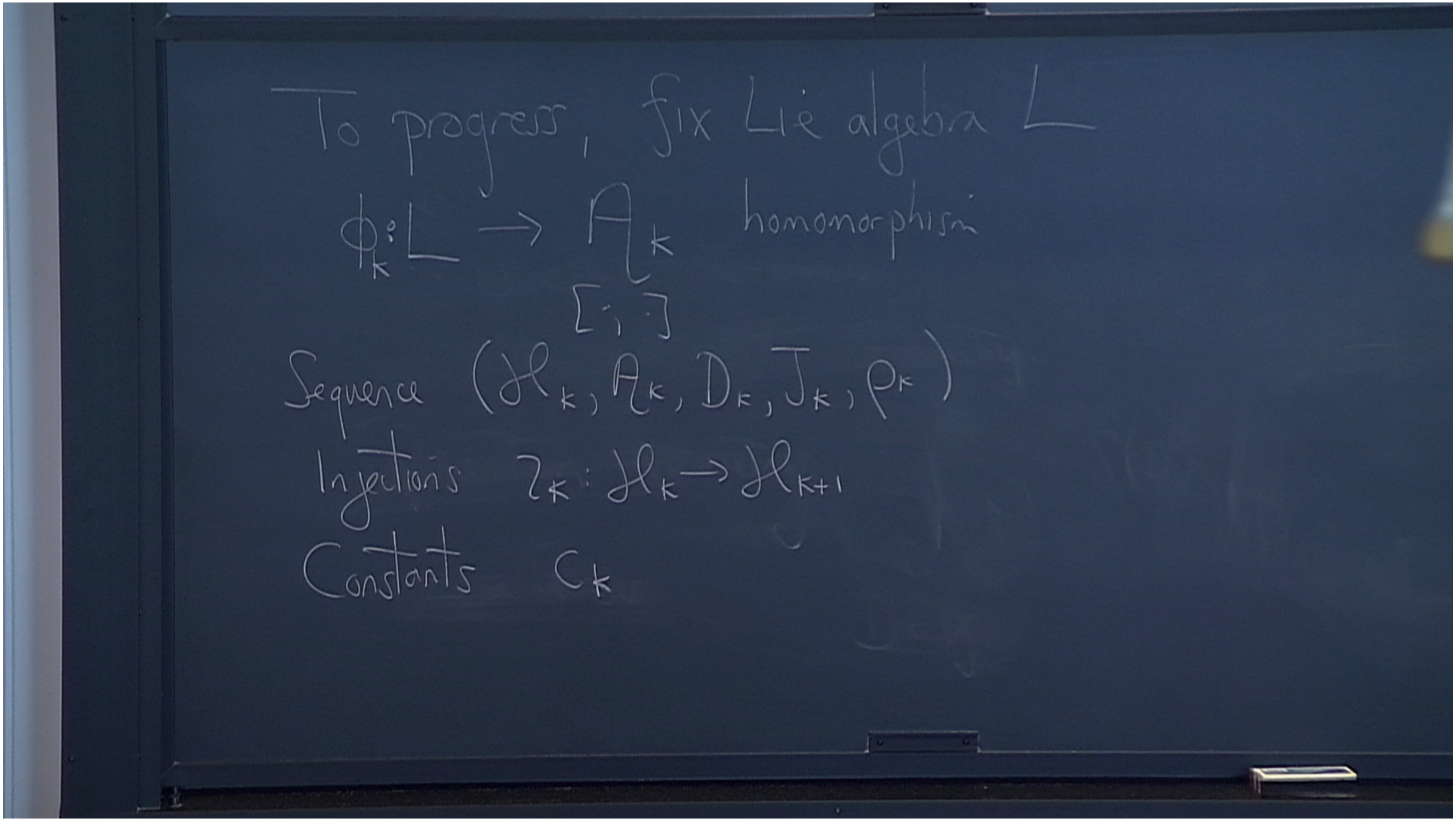

"NC coordinates"
Limit idea:
[x(a), x(b)] = 0 in limit
functions on M (ie ρ in C case)
= {ρ(a), ψ}
V(a) ~ vector fields on M
Lie algebra A ⊕ A acts on H
(a,b) ▷ ψ = V((a+b)/2) ψ + x(
= ρ(a)ψ − ψ ρ(b)
ρ(a)ψ = x(a)
over A = gauge

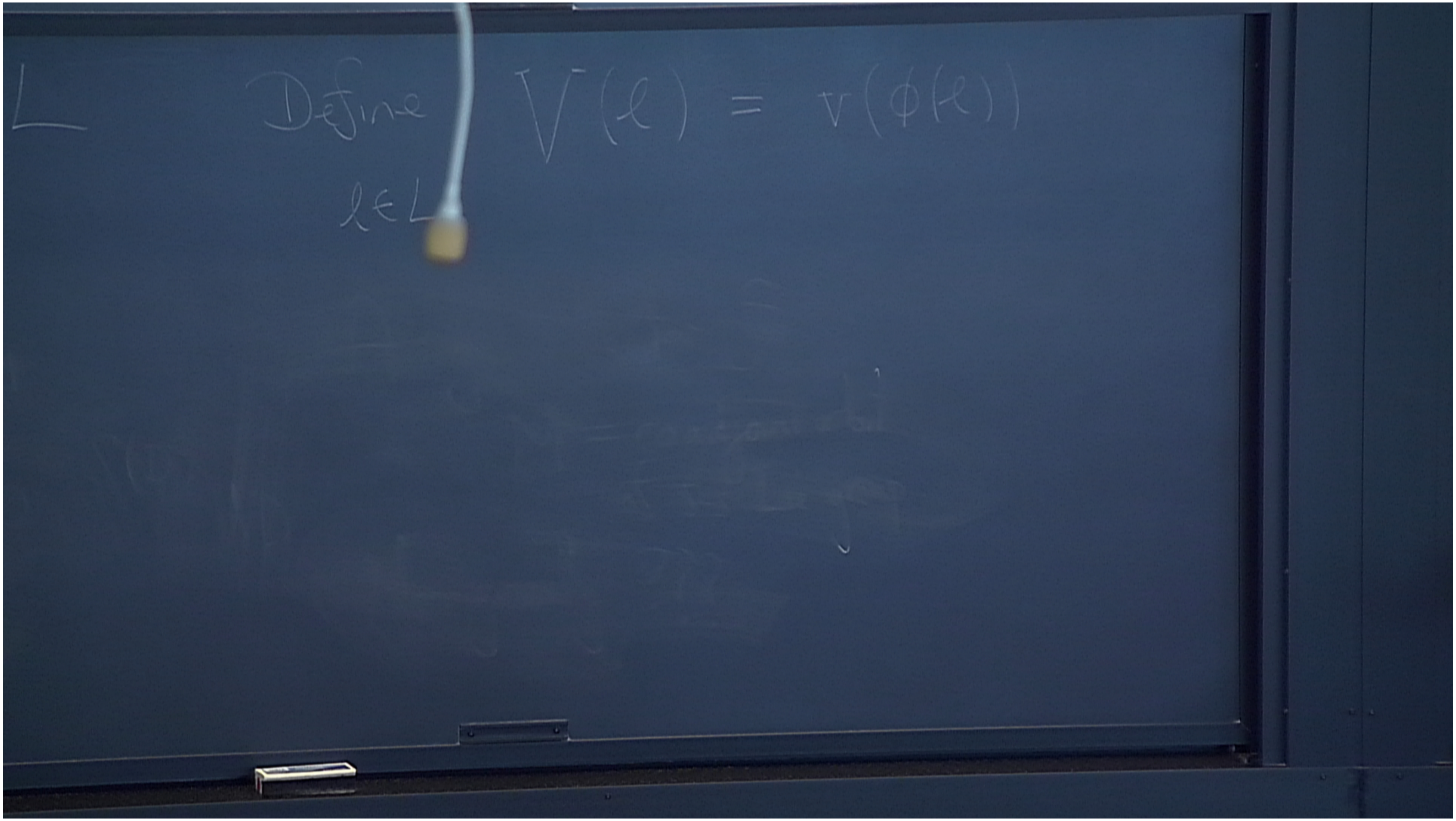

"NC coordinates"
Limit idea:
[x(a), x(b)] = 0 in limit
~> functions on M (ie ρ in C case)
V(a) ~> vector fields on M
Lie algebra A ⊕ A acts on H
(a,b) ▷ Ψ = V((a+b)/2) Ψ + x((a−b)/2) Ψ
= ρ(a)Ψ − Ψ ρ(b)
Limit is contraction
ρ(a)Ψ + Ψ ρ(a)
= {ρ(a), Ψ}
ρ(a)Ψ = x(a)
~> H is bimodule over A
cf. Particle physics – fermions bimodule
D = D*
DJ = JD



To progress, fix Lie algebra L
φ: L → A homomorphism
[·,·]


To progress, fix Lie algebra L
$\phi_k : L \to A_k$ homomorphism
$[\cdot,\cdot]$
Sequence $(\mathcal{H}_k, A_k, D_k, J_k, \rho_k)$
Injections $\iota_k : \mathcal{H}_k \to \mathcal{H}_{k+1}$


To progress, fix Lie algebra L
φk : L → Ak homomorphism
[·,·]
Sequence (Hk, Ak, Dk, Jk, ρk)
Injections ιk : Hk → Hk+1
Constants ck


L
Define $V(\ell) = v(\phi(\ell))$
$\ell \in L$

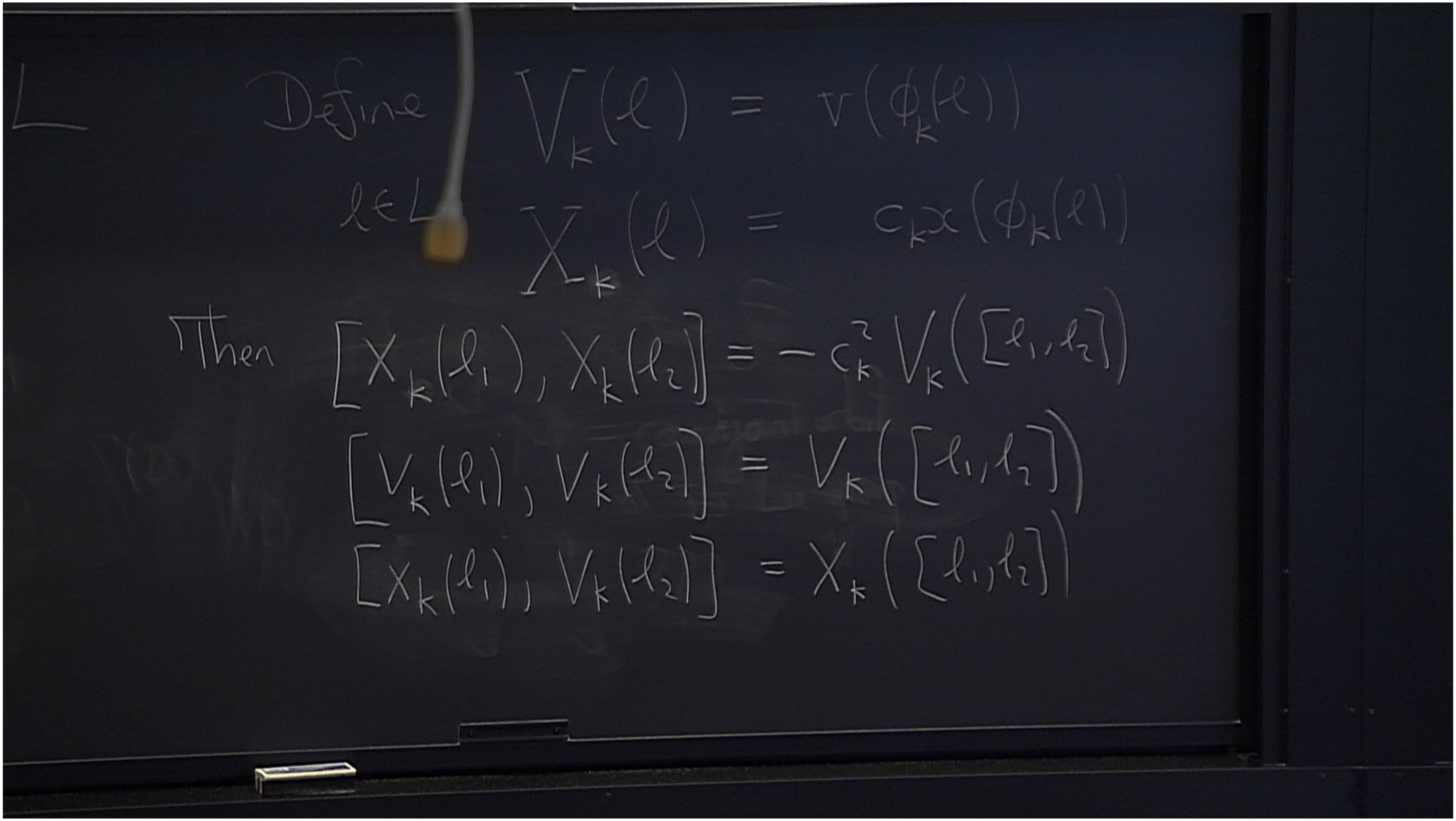
L

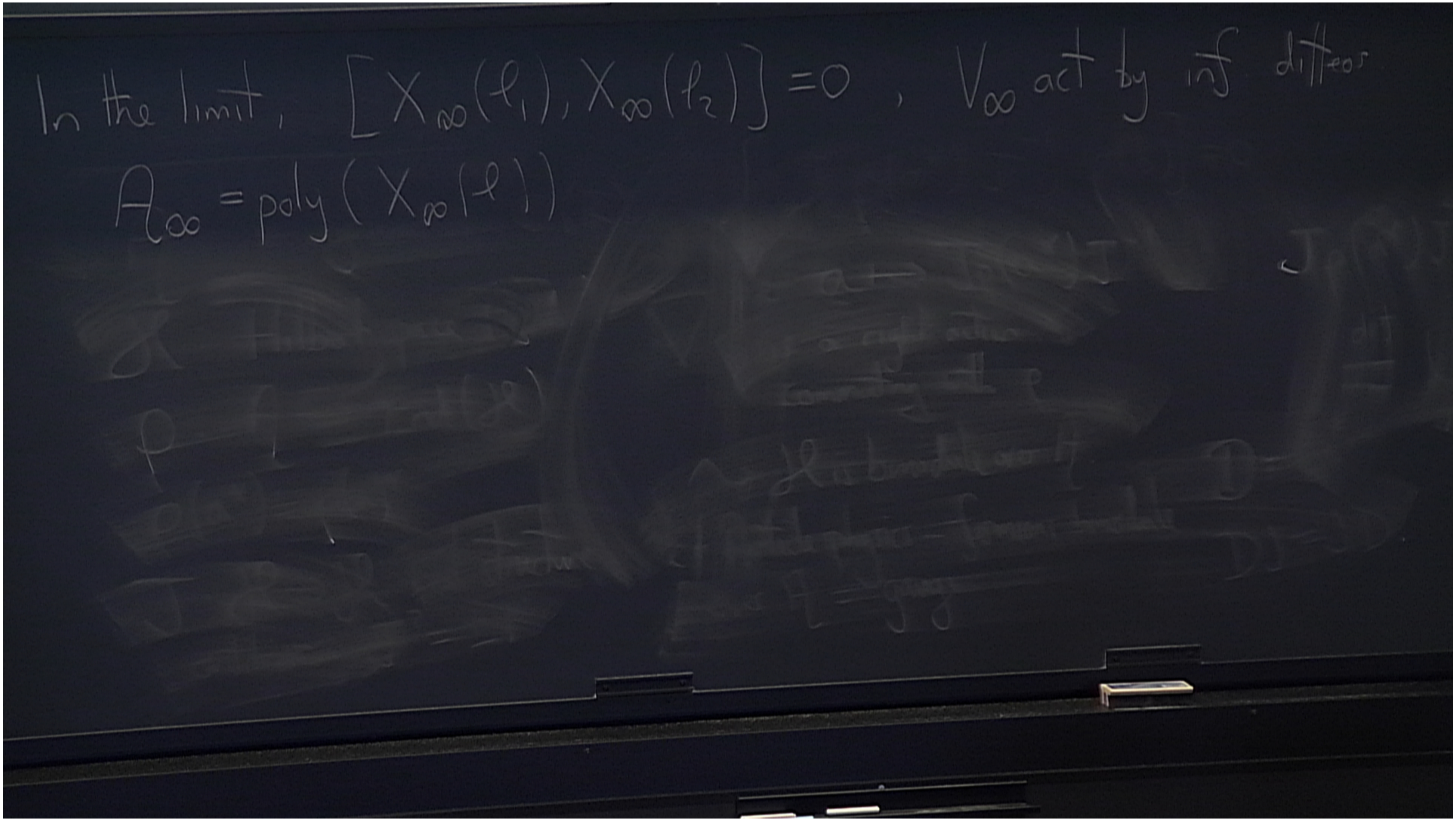
Define $V_k(\ell) = v(\phi_k(\ell))$

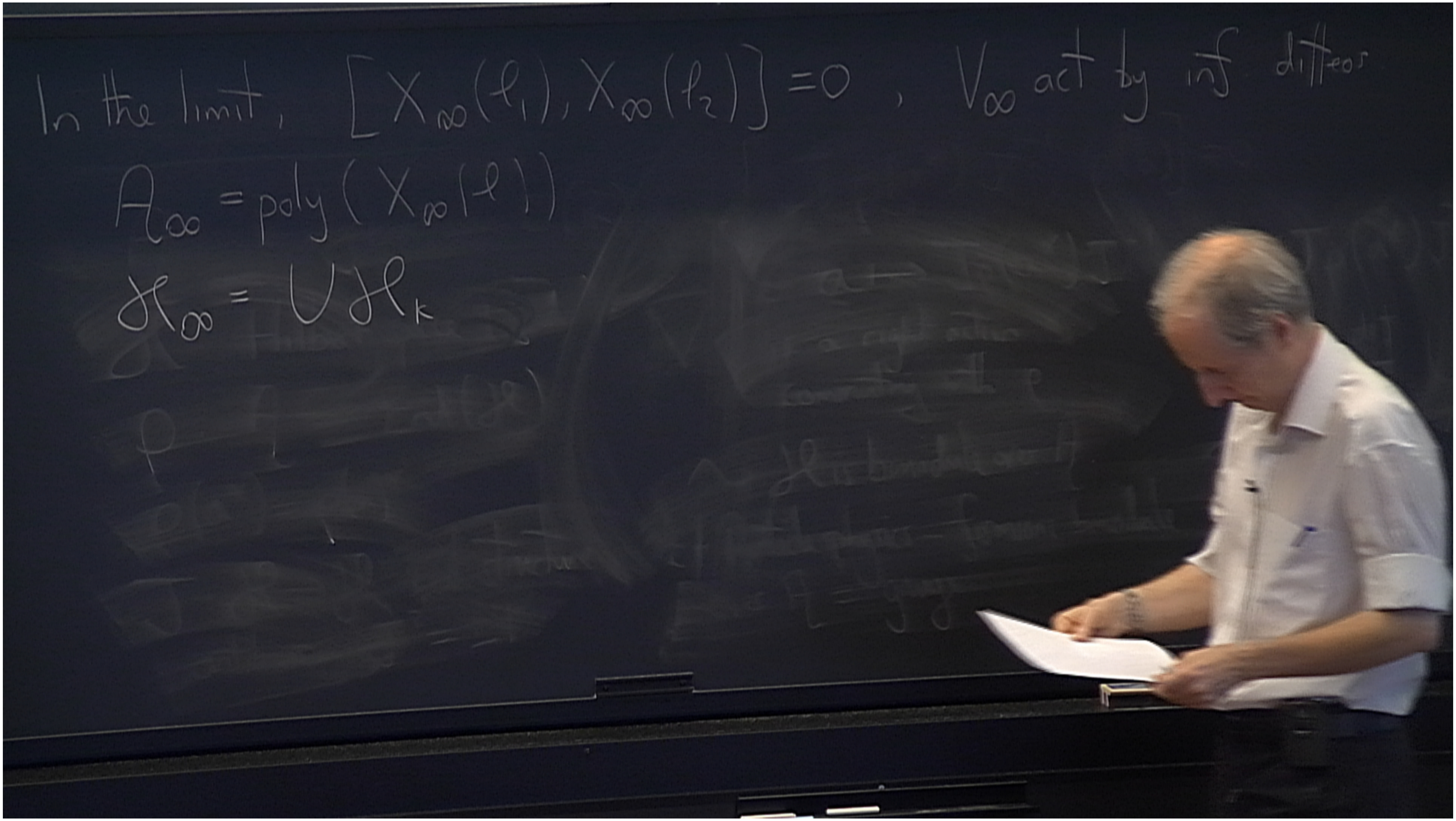
$\ell \in L$ $\quad X_k(\ell) = c_k x(\phi_k(\ell))$

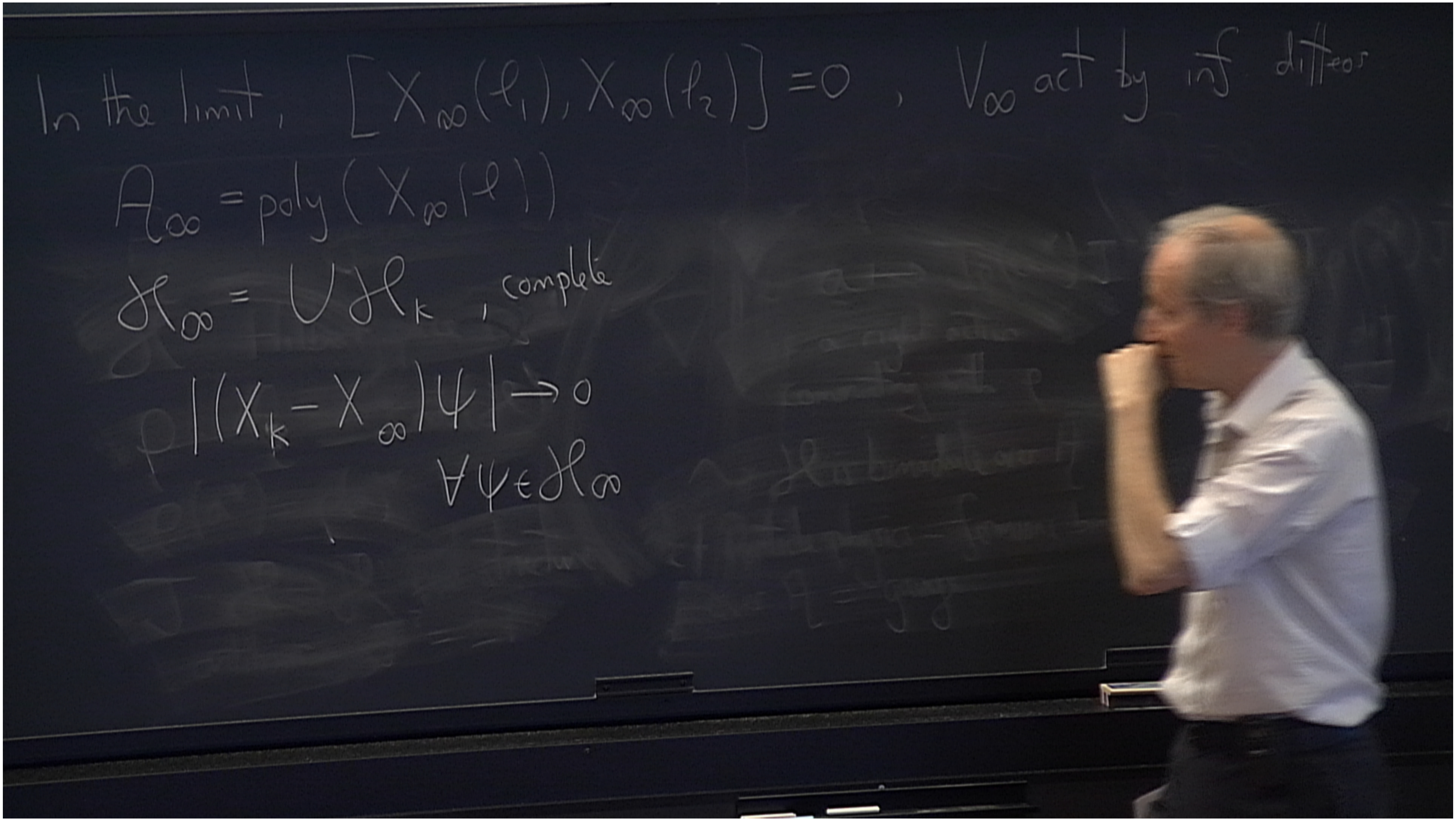
Then $[X_k(\ell_1), X_k(\ell_2)] = -c_k^2 V_k([\ell_1, \ell_2])$

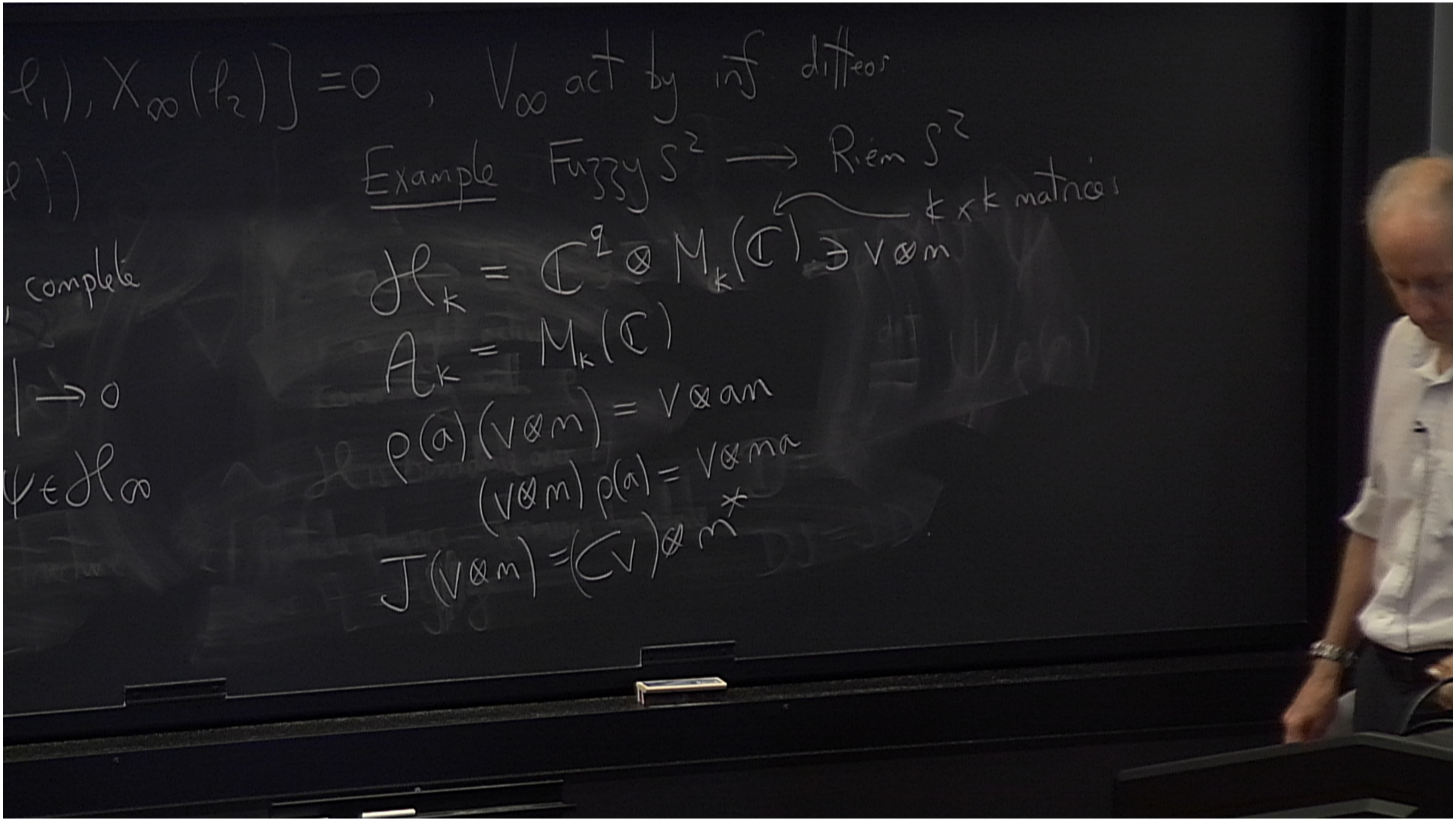
$[V_k(\ell_1), V$

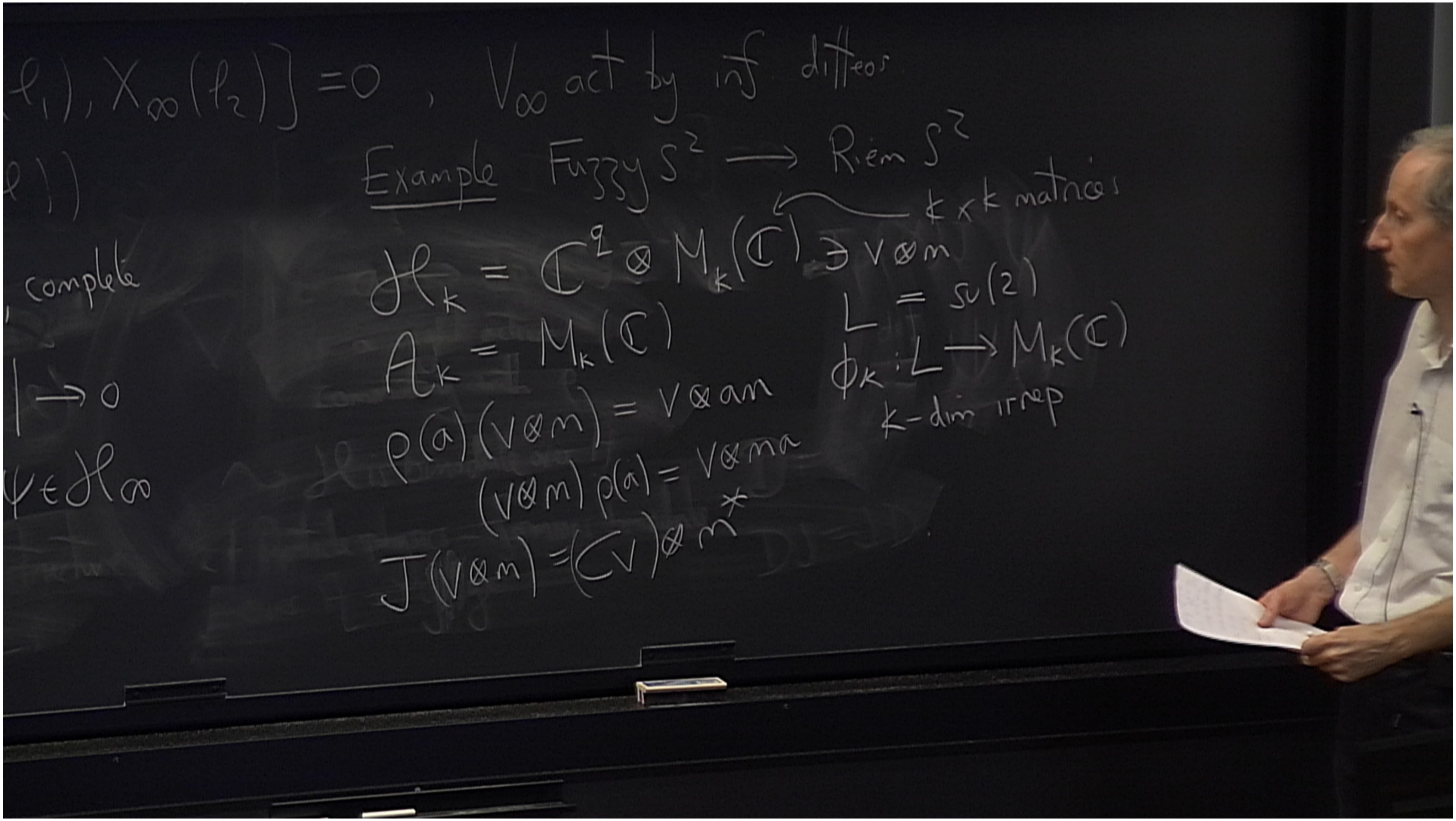

Define $\bar{V}_k(\ell) = v(\phi_k(\ell))$
$\ell \in L$ $\bar{X}_k(\ell) = c_k x(\phi_k(\ell))$
Then $[X_k(\ell_1), X_k(\ell_2)] = -c_k^2 V_k([\ell_1, \ell_2])$
$[V_k(\ell_1), V_k(\ell_2)] = V_k([\ell_1, \ell_2])$
$[X_k(\ell_1), V_k(\ell_2)] = X_k([\ell_1, \ell_2])$


In the limit, $[X_\infty(\ell_1), X_\infty(\ell_2)] = 0$, V_∞ act by inf diffeos
$A_\infty = \text{poly}(X_\infty(\ell))$


In the limit, $[X_\infty(\ell_1), X_\infty(\ell_2)] = 0$, V_∞ act by inf diffeos
$A_\infty = \mathrm{poly}(X_\infty(\ell))$
$\mathcal{H}_\infty = \bigcup \mathcal{H}_k$


In the limit, $[X_\infty(\ell_1), X_\infty(\ell_2)] = 0$, V_∞ act by inf diffeos
$A_\infty = \mathrm{poly}(X_\infty(\ell))$
$\mathcal{H}_\infty = \bigcup \mathcal{H}_k$, complete
$\|(X_k - X_\infty)\psi\| \to 0$
$\forall \psi \in \mathcal{H}_\infty$

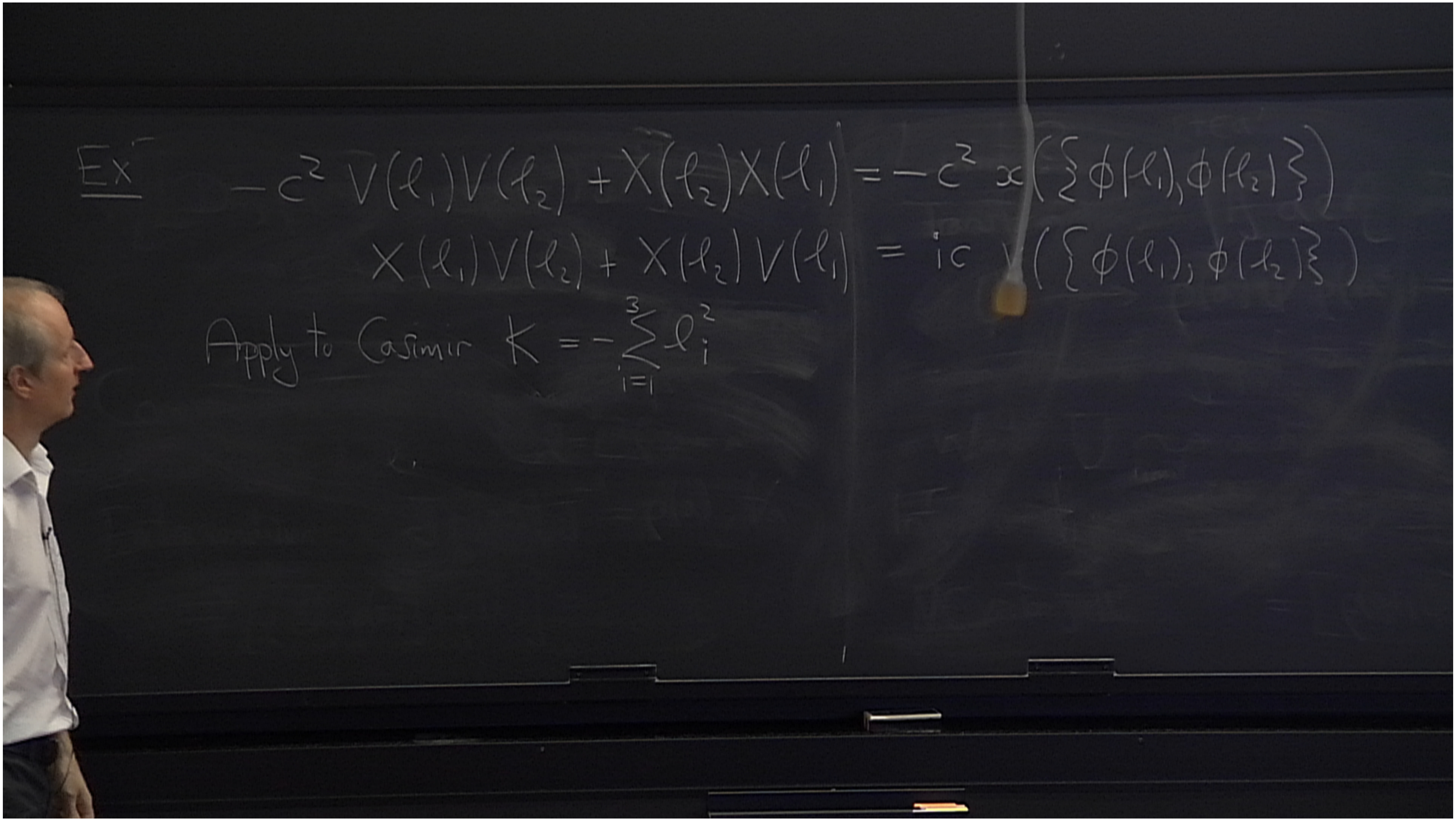

V∞ act by inf diffeos
Example Fuzzy S² ⟶ Riem S²
k×k matrices
H_k = ℂ² ⊗ M_k(ℂ) ∋ v⊗m
A_k = M_k(ℂ)
ρ(a)(v⊗m) = v⊗am
(v⊗m)ρ(a) = v⊗ma
J(v⊗m) = (Cv)⊗m*
complete
ψ ∈ H∞

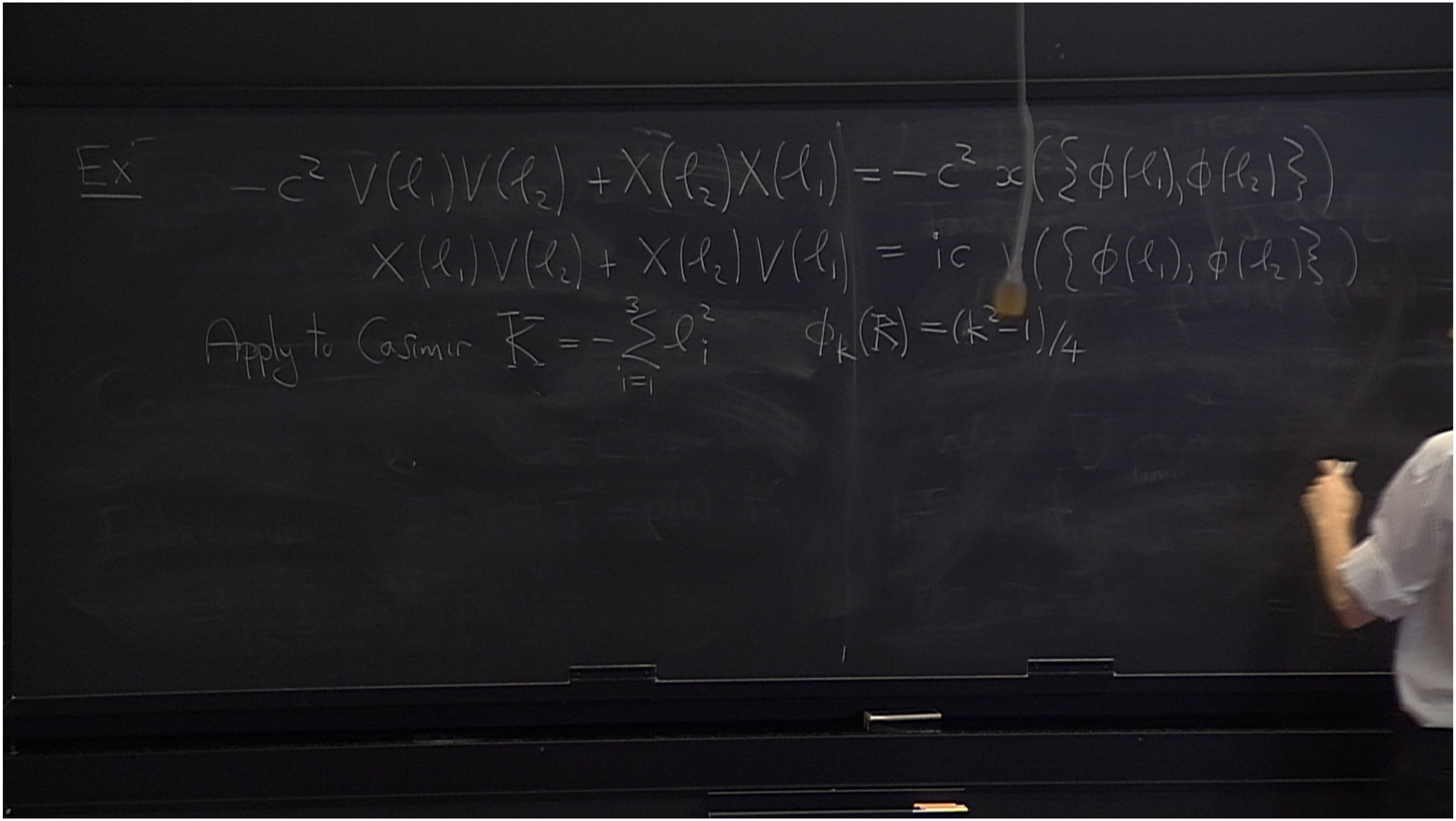

ℓ₁), X∞(ℓ₂)] = 0 , V∞ act by inf diffeos
Example Fuzzy S² ⟶ Riem S²
k×k matrices
ℋ_k = ℂ² ⊗ M_k(ℂ) ∋ v⊗m
L = su(2)
A_k = M_k(ℂ)
φ_k : L ⟶ M_k(ℂ)
ρ(a)(v⊗m) = v⊗am
k-dim irrep
(v⊗m) ρ(a) = v⊗ma
J(v⊗m) = (Cv)⊗m*
complete
| → 0
ψ ∈ ℋ∞

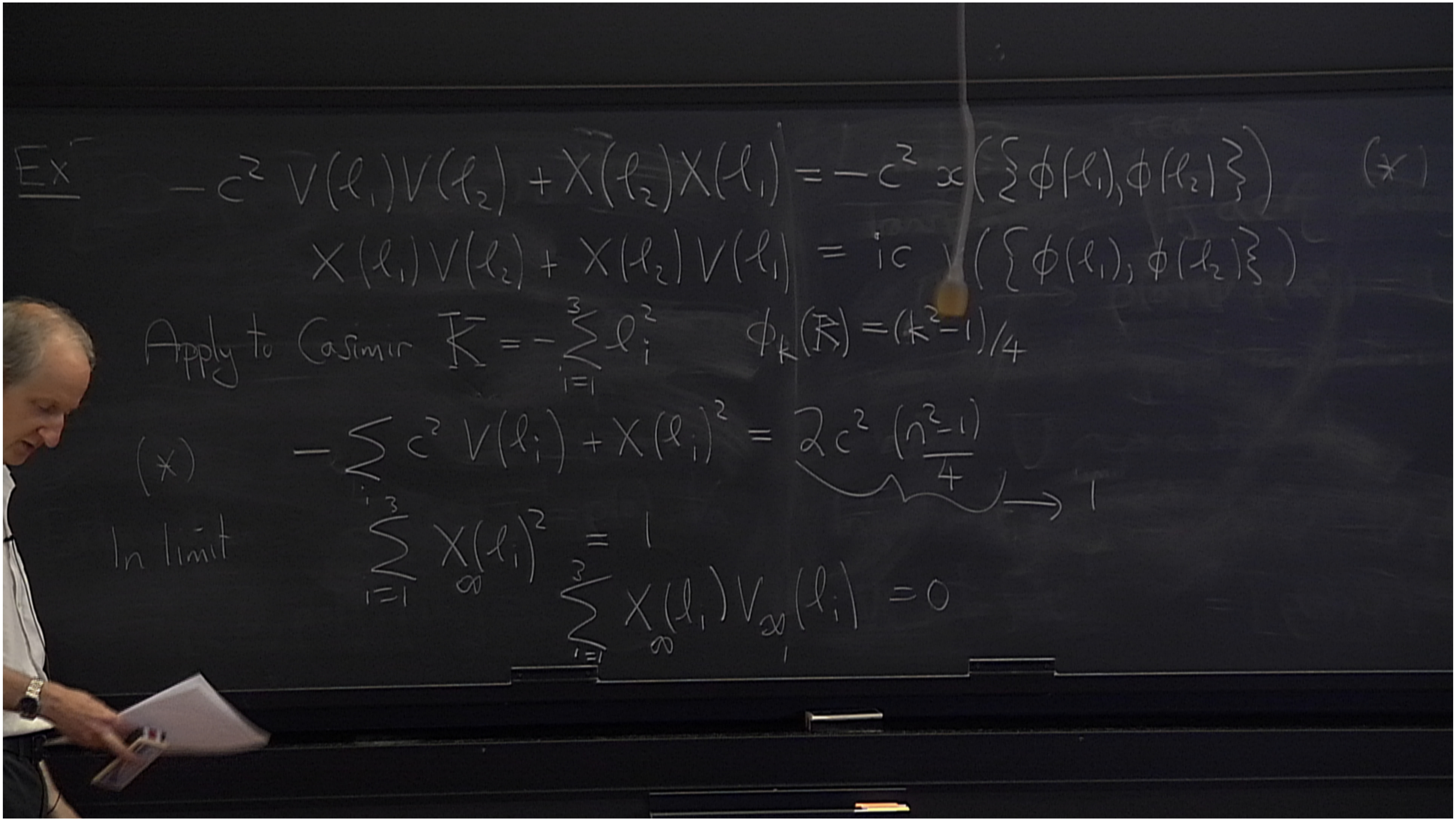
Ex

$$-c^2 V(\ell_1)V(\ell_2) + X(\ell_2)X(\ell_1) = -c^2\, x(\{\phi(\ell_1),\phi(\ell_2)\})$$

$$X(\ell_1)V(\ell_2) + X(\ell_2)V(\ell_1) = ic\, x(\{\phi(\ell_1),\phi(\ell_2)\})$$


Ex
$-c^2 V(\ell_1)V(\ell_2) + X(\ell_2)X(\ell_1) = -c^2 x(\{\phi(\ell_1), \phi(\ell_2)\})$
$X(\ell_1)V(\ell_2) + X(\ell_2)V(\ell_1) = ic\, V(\{\phi(\ell_1), \phi(\ell_2)\})$
Apply to Casimir $K = -\sum_{i=1}^{3} \ell_i^2$


Ex
$-c^2 V(\ell_1)V(\ell_2) + X(\ell_2)X(\ell_1) = -c^2 x(\{\phi(\ell_1),\phi(\ell_2)\})$
$X(\ell_1)V(\ell_2) + X(\ell_2)V(\ell_1) = ic\ (\{\phi(\ell_1),\phi(\ell_2)\})$
Apply to Casimir $K = -\sum_{i=1}^{3} \ell_i^2$
$\phi_k(K) = (k^2-1)/4$


Ex
$-c^2 V(\ell_1)V(\ell_2) + X(\ell_2)X(\ell_1) = -c^2 x(\{\phi(\ell_1),\phi(\ell_2)\})$ (*)
$X(\ell_1)V(\ell_2) + X(\ell_2)V(\ell_1) = ic\, v(\{\phi(\ell_1),\phi(\ell_2)\})$
Apply to Casimir $\bar{K} = -\sum_{i=1}^{3} \ell_i^2$ $\phi_k(\bar{K}) = -(k^2-1)/4$
(*) $-\sum c^2 V(\ell_i) + X(\ell_i)^2 = 2c^2 \frac{(n^2-1)}{4} \longrightarrow 1$
In limit $\sum_{i=1}^{3} X_\infty(\ell_i)^2 = 1$
$\sum_{i=1}^{3} X_\infty(\ell_i)V_\infty(\ell_i) = 0$

Ex

$$-c^2 V(\ell_1)V(\ell_2) + X(\ell_2)X(\ell_1) = -c^2 x\left(\{\phi(\ell_1),\phi(\ell_2)\}\right) \qquad (*)$$

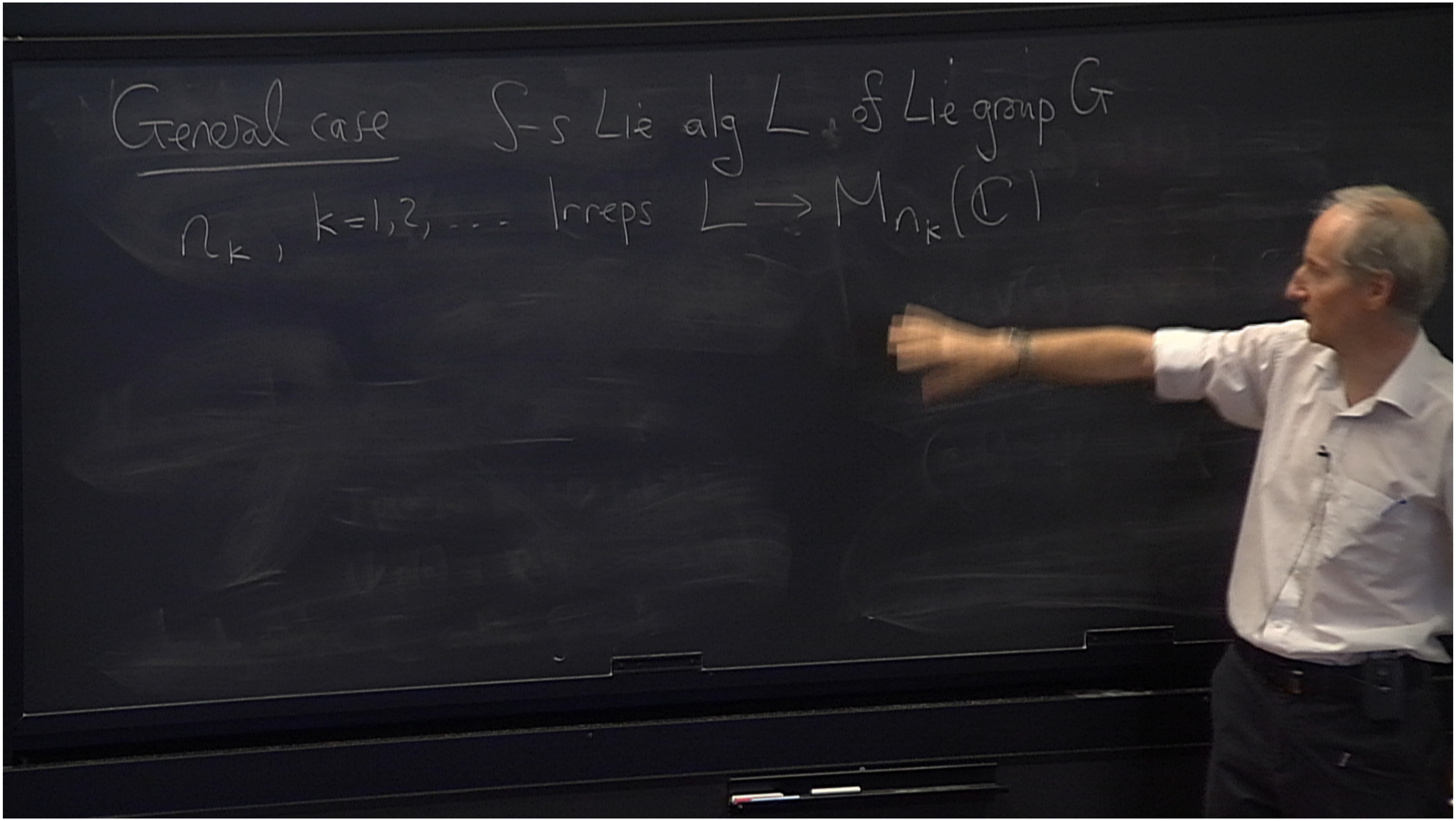
$$X(\ell_1)V(\ell_2) + X(\ell_2)V(\ell_1) = ic\, V\left(\{\phi(\ell_1),\phi(\ell_2)\}\right)$$

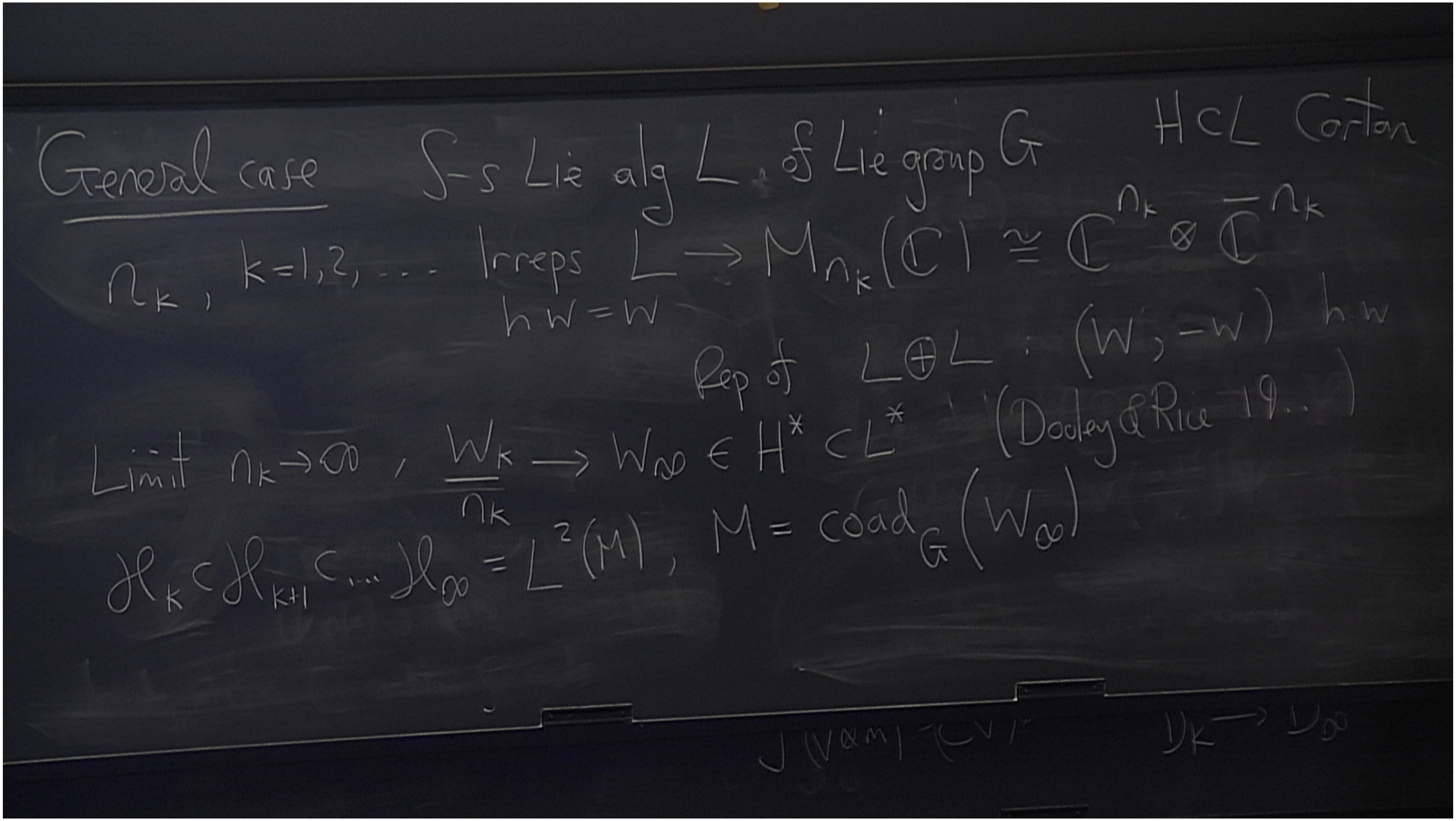
Apply to Casimir $\bar{K} = -\sum_{i=1}^{3} \ell_i^2 \qquad \phi_k(\bar{K}) = (k^2-1)/4$

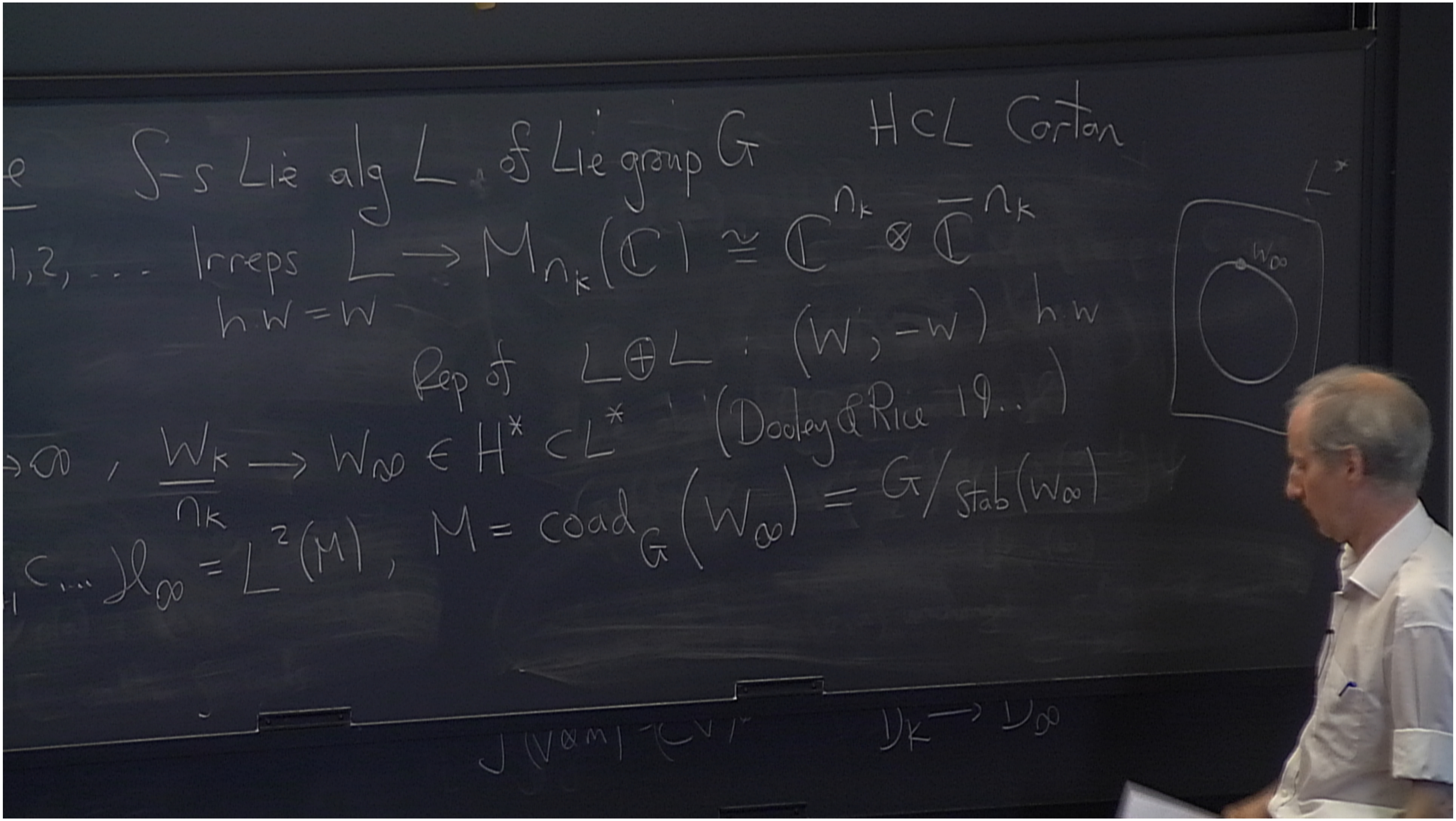
$$(*) \qquad -\sum_i c^2 V(\ell_i)^2 + X(\ell_i)^2 = 2c^2 \frac{(n^2-1)}{4} \longrightarrow 1$$

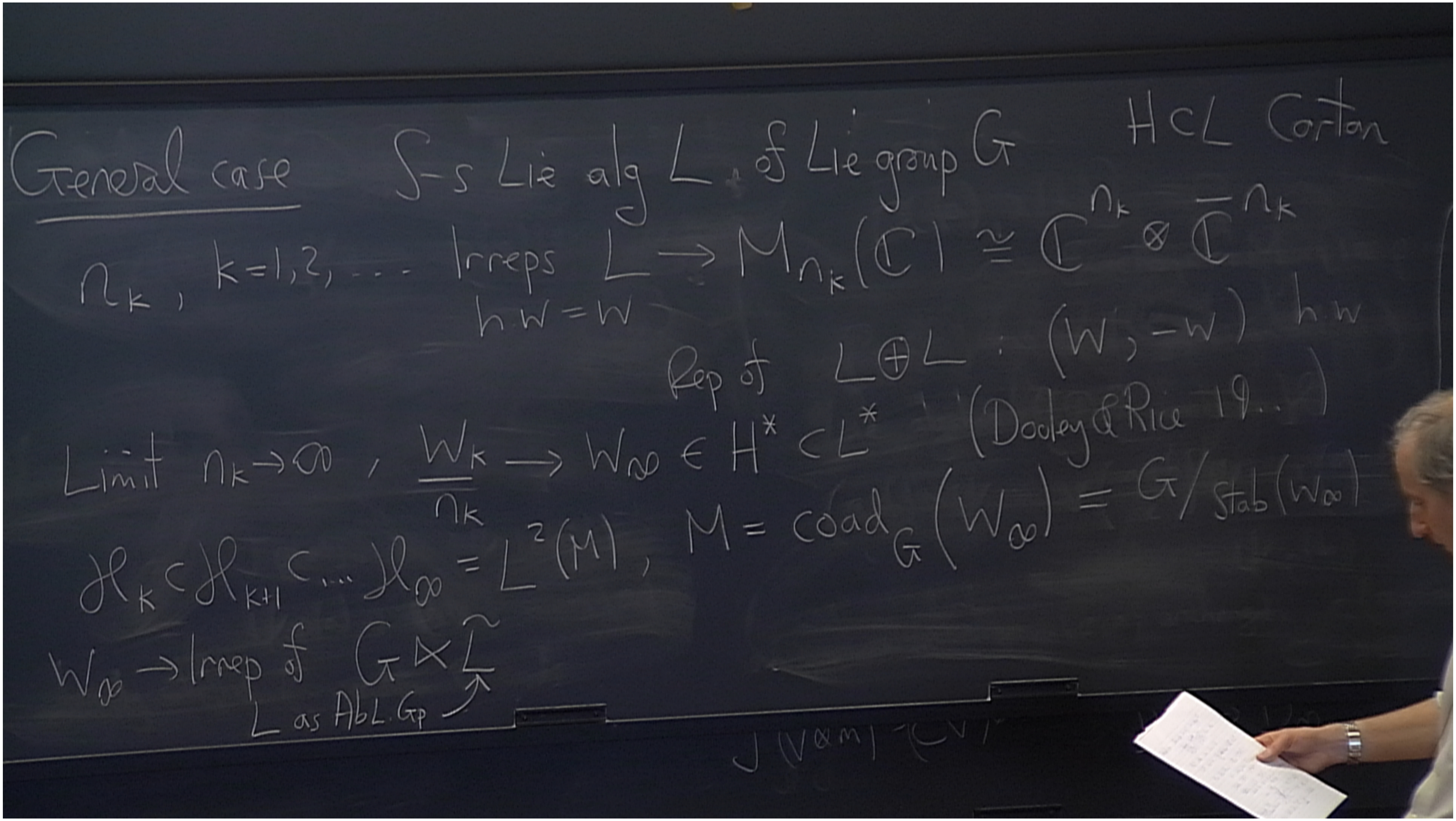
In limit
$$\sum_{i=1}^{3} X_\infty(\ell_i)^2 = 1$$

$$\sum_{i=1}^{3} X_\infty(\ell_i)\, V_\infty(\ell_i) = 0 \qquad \epsilon_{ijk}\, x_k \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right) = 0$$


General case   S-s Lie alg L of Lie group G
n_k, k=1,2,... Irreps L → M_{n_k}(ℂ)


General case
S-s Lie alg L of Lie group G
H ⊂ L Cartan
n_k, k = 1, 2, ... Irreps L → M_{n_k}(ℂ) ≅ ℂ^{n_k} ⊗ ℂ̄^{n_k}
h.w = w
Rep of L ⊕ L : (w, −w) h.w
(Dooley & Rice 19..)
Limit n_k → ∞ , w_k / n_k → w_∞ ∈ H* ⊂ L*
ℋ_k ⊂ ℋ_{k+1} ⊂ ... ℋ_∞ = L²(M), M = coad_G(w_∞)
D_k → D_∞


S-s Lie alg L of Lie group G
H ⊂ L Cartan
1, 2, ... Irreps L → $M_{n_k}(\mathbb{C}) \cong \mathbb{C}^{n_k} \otimes \bar{\mathbb{C}}^{n_k}$
h.w = W
Rep of L ⊕ L : (W, −W) h.w
(Dooley & Rice 19..)
$\frac{W_k}{n_k} \to W_\infty \in H^* \subset L^*$
$M = \text{coad}_G(W_\infty) = G/\text{stab}(W_\infty)$
$\mathcal{H}_\infty = L^2(M)$
L^*
W_∞
$\mathcal{D}_k \to \mathcal{D}_\infty$


General case
S-s Lie alg L of Lie group G
H ⊂ L Cartan
n_k , k = 1, 2, . . . Irreps L → M_{n_k}(ℂ) ≅ ℂ^{n_k} ⊗ ℂ̄^{n_k}
h.w = w
Rep of L ⊕ L : (w, −w) h.w
(Dooley & Rice 19..)
Limit n_k → ∞ , w_k / n_k → w_∞ ∈ H* ⊂ L*
ℋ_k ⊂ ℋ_{k+1} ⊂ ... ℋ_∞ = L²(M) , M = coad_G (w_∞) = G / stab(w_∞)
w_∞ → Irrep of G ⋉ L̃
L as Abl. Gp

General case

S-s Lie alg L of Lie group G $\quad$ $H \subset L$ Cartan

n_k, $k = 1, 2, \dots$ Irreps $L \to M_{n_k}(\mathbb{C}) \cong \mathbb{C}^{n_k} \otimes \bar{\mathbb{C}}^{n_k}$

$h.w = w$

Rep of $L \oplus L$: $(w, -w)$ h.w

(Dooley & Rice 19..)

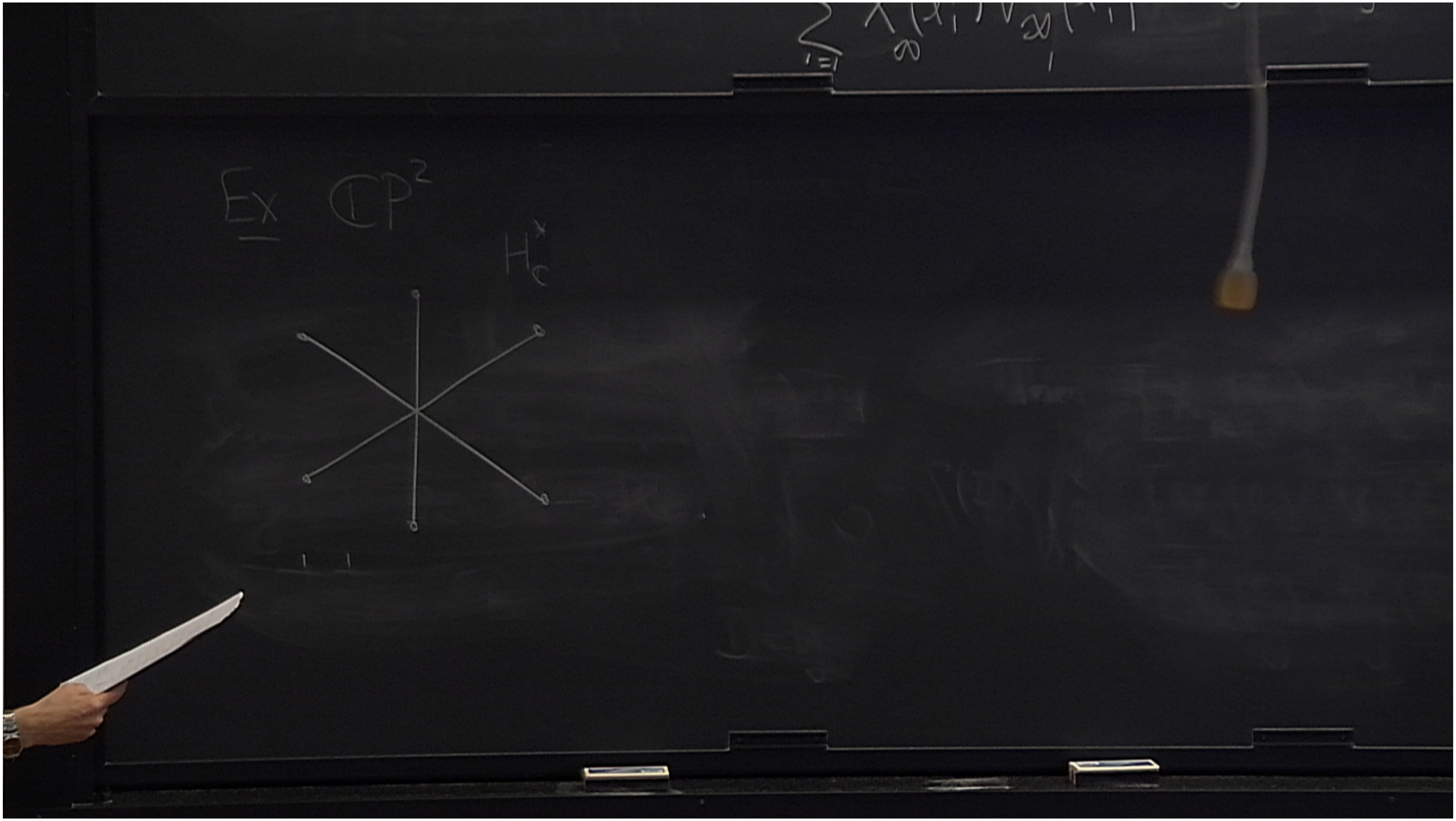
Limit $n_k \to \infty$, $\frac{w_k}{n_k} \to w_\infty \in H^* \subset L^*$

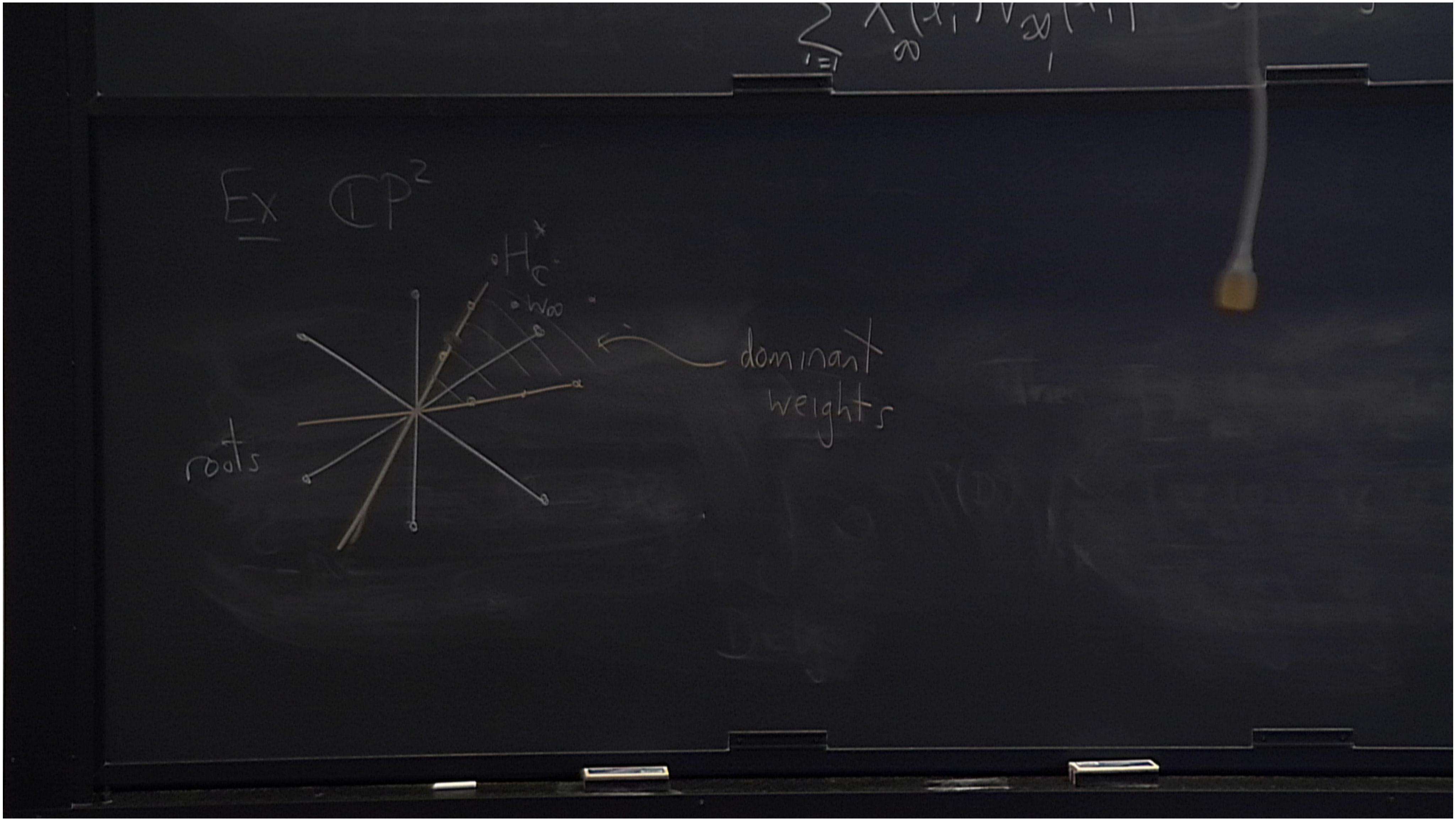
$\mathcal{H}_k \subset \mathcal{H}_{k+1} \subset \dots \mathcal{H}_\infty = L^2(M)$, $M = \mathrm{coad}_G(w_\infty) = G/\mathrm{Stab}(w_\infty)$

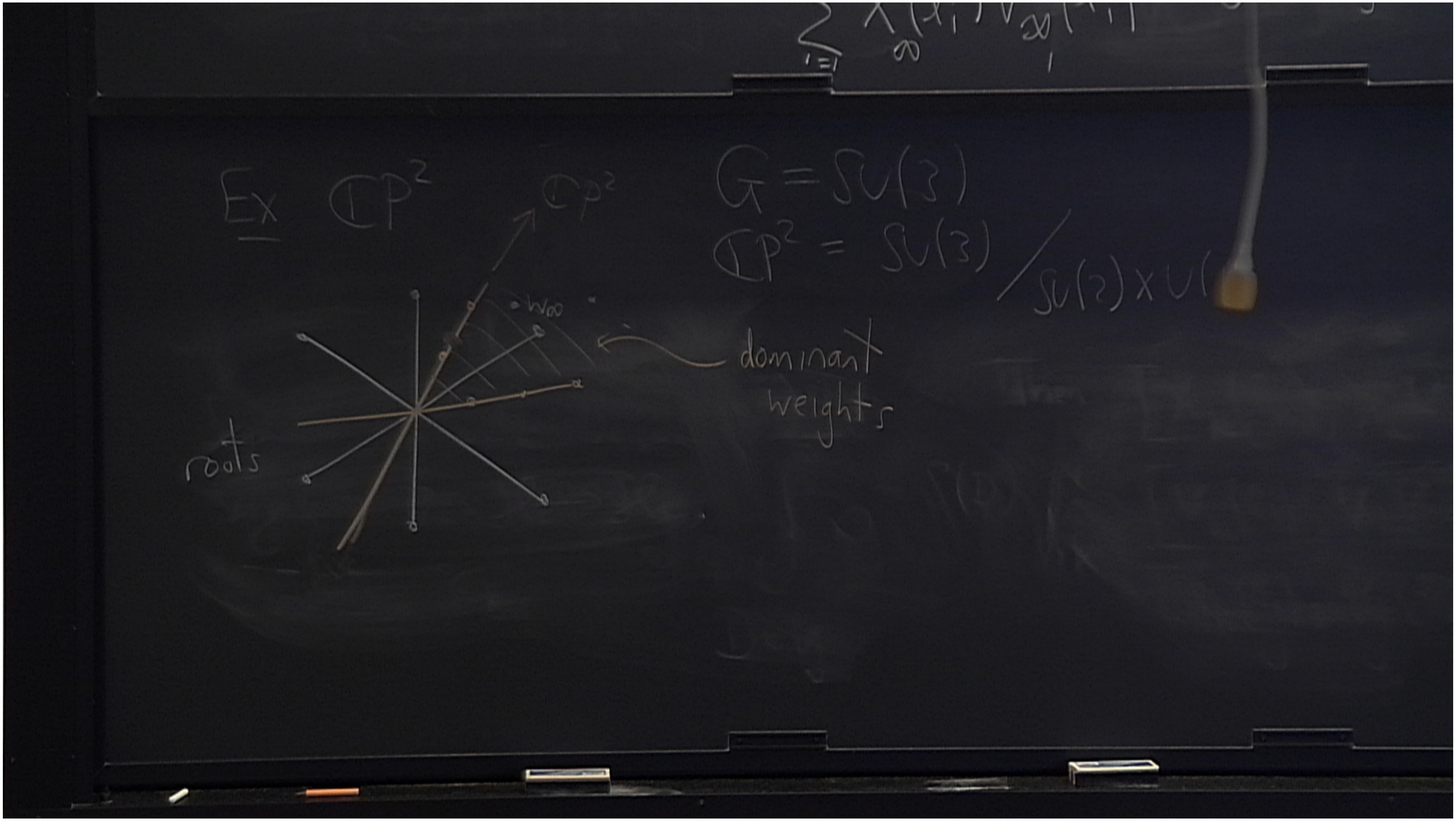
$w_\infty \to$ Irrep of $G \ltimes \tilde{L}$, induced from $\chi(t) = e^{i w_\infty(t)}$ & trivial rep of $\mathrm{Stab}(\chi)$

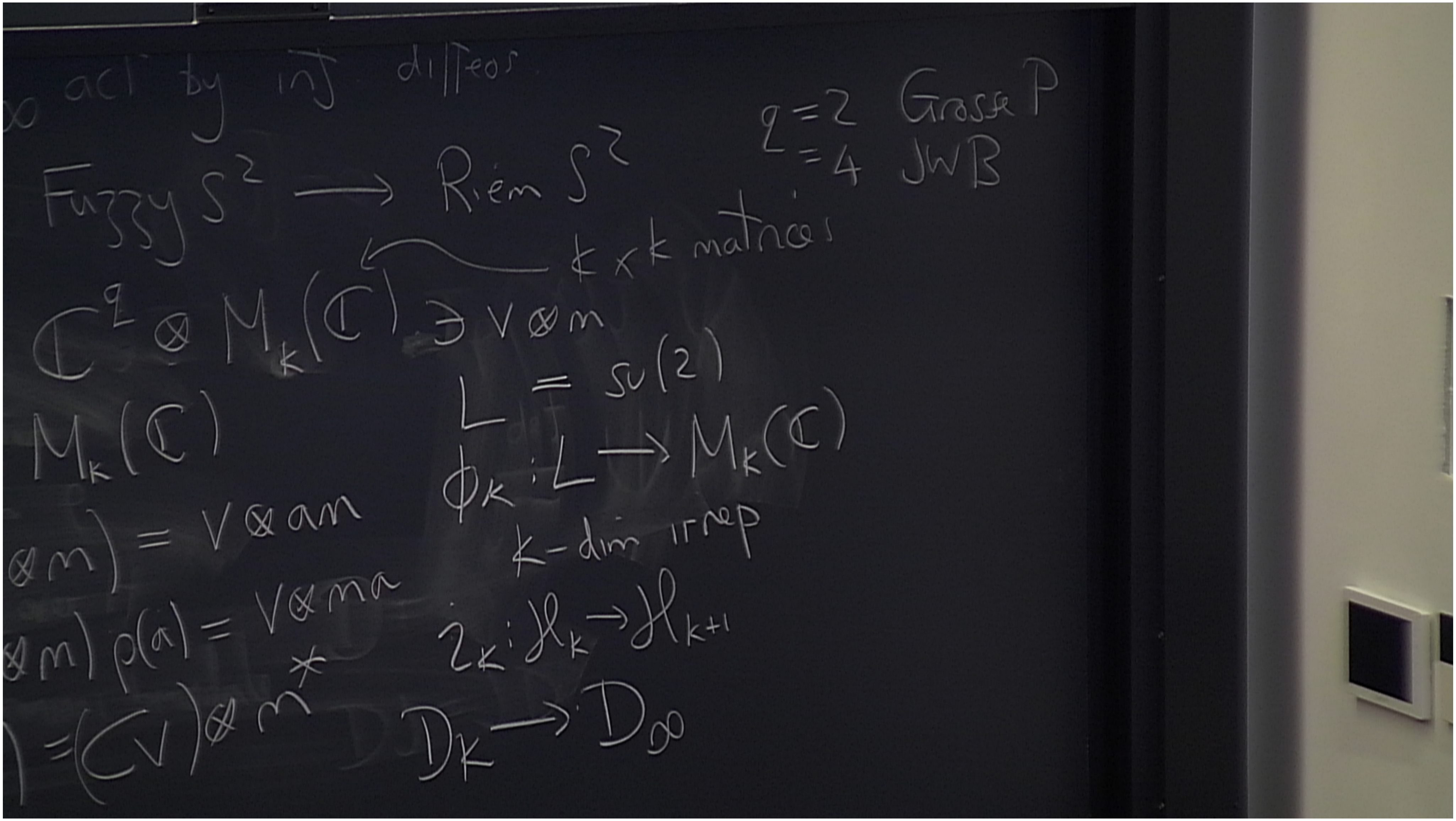
L as Abl. Gp $\nearrow$ $\quad$ character on $\tilde{L}$

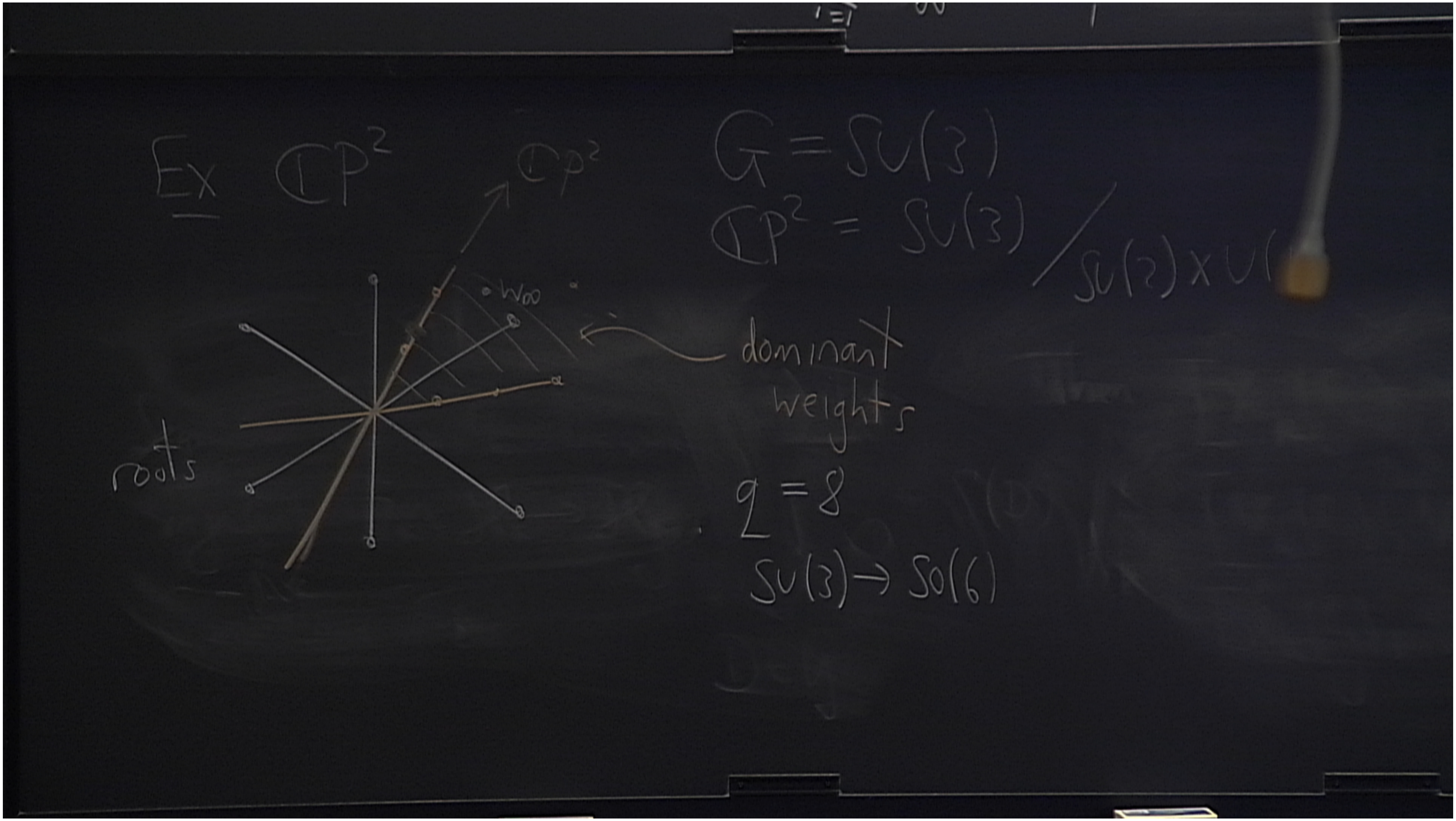
$\mathcal{D}_k \to \mathcal{D}_\infty$

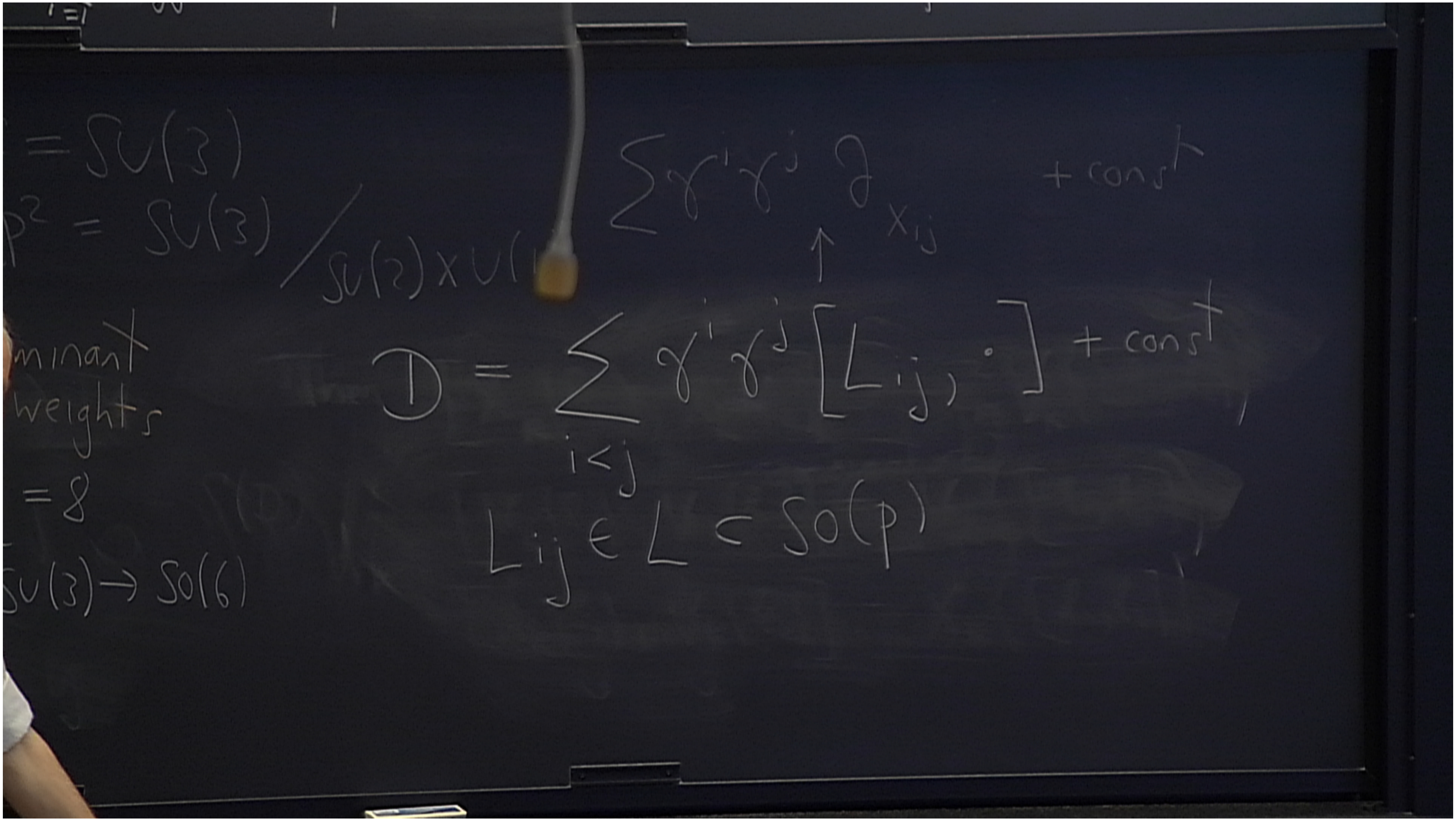

Ex $\mathbb{C}P^2$
H_c^*

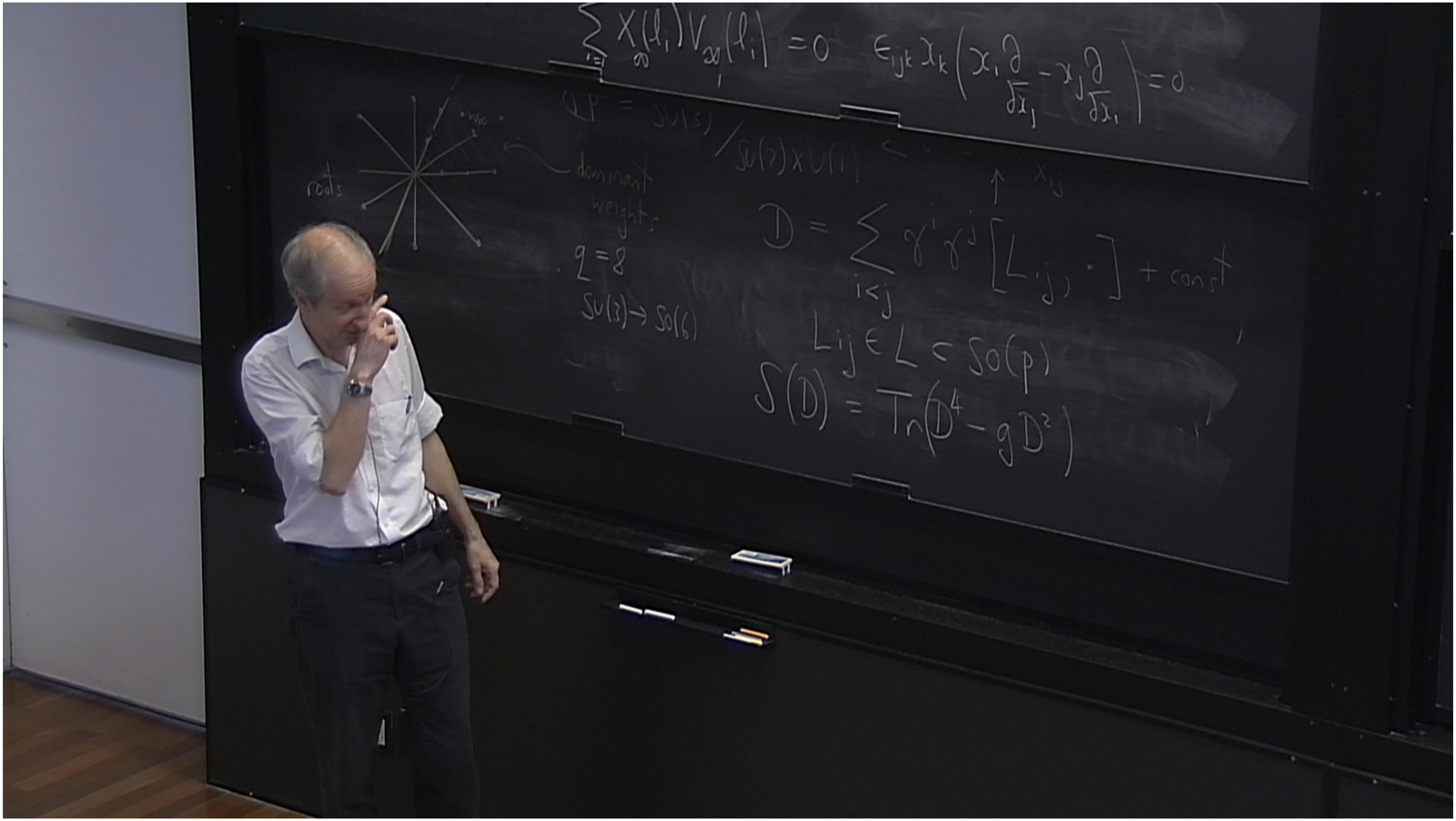

Ex $\mathbb{CP}^2$
H_c^*
w_∞
dominant weights
roots


Ex $\mathbb{CP}^2$
$\mathbb{CP}^2$
$G = SU(3)$
$\mathbb{CP}^2 = SU(3) / SU(2) \times U($
w_∞
dominant weights
roots


act by inj diffeos
q=2 Grosse P
=4 JWB
Fuzzy S^2 → Riem S^2
k×k matrices
C^q ⊗ M_k(C) ∋ v⊗m
L = su(2)
M_k(C)
ϕ_k : L → M_k(C)
k-dim irrep
⊗m) = v⊗am
⊗m) ρ(a) = v⊗ma
ι_k : H_k → H_{k+1}
= (Cv)⊗m*
D_k → D_∞


Ex CP²
CP²
w∞
dominant weights
roots
G = SU(3)
CP² = SU(3) / SU(2) x U(
q = 8
SU(3) → SO(6)


= SU(3)
P² = SU(3) / SU(2) x U(1
minant
weights
= 8
SU(3) → SO(6)
∑ γ^i γ^j ∂_{x_ij} + const
D = ∑_{i<j} γ^i γ^j [L_ij, ·] + const
L_ij ∈ L ⊂ SO(p)


roots
dominant weights
g = 8
SU(3) → SO(6)
D = ∑ γ^i γ^j [L_ij, ·] + const
i<j
L_ij ∈ L ⊂ SO(p)
S(D) = Tr(D^4 − g D^2)