Title: Emergent network geometry and quantum statistics

Date: Sep 15, 2016 02:30 PM

URL: http://pirsa.org/16090029

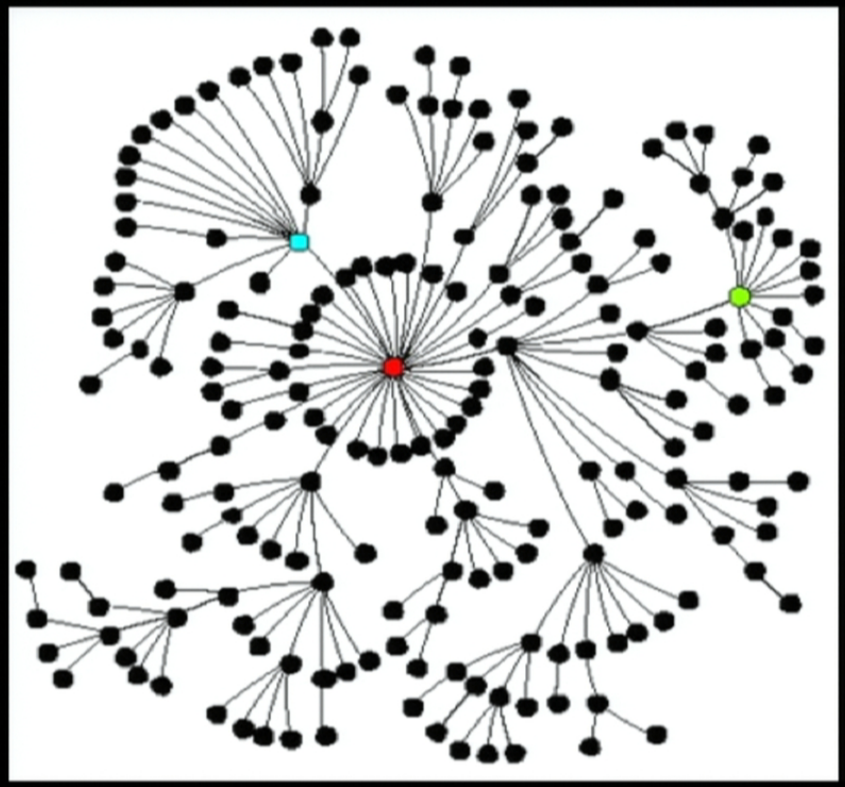
Abstract: Abstract: Complex networks describe interacting systems ranging from the brain to the Internet. While so far the geometrical nature of complex networks has been mostly neglected, the novel field of network geometry is crucial for gaining a deeper theoretical understanding of the architecture of complexity. At the same time, network geometry is at the heart of quantum gravity, since many approaches to quantum gravity assume that space-time is discrete and network-like at the quantum level. In network geometry a crucial problem is the identifications of mechanisms to describe emergent network geometry. In this talk, after an introduction to complex network theory, l will present a class of nonâ€“equilibrium models [1-4] in which geometrical properties of the networks emerge spontaneously from their dynamics. Specifically we will discuss the model called network geometry with flavor (NGF).The NGF can generate discrete geometries of different nature, ranging from chains and higher dimensional manifolds to scale-free complex networks. Interestingly the NGF with fitness of the nodes reveals relevant relations with quantum statistics. In fact the faces of the NGF have generalized degrees that follow either the Fermi-Dirac, Boltzmann or Bose-Einstein statistics depending on their flavor and on their dimensionality. Specifically, NGFs with flavor s=-1, when constructed in dimension d=3 gluing tetrahedra along their triangular faces, have the generalized degrees of the triangular faces, of the links, and of the nodes following respectively the Fermi-Dirac, the Boltzmann or the Bose-Einstein distribution.

[1] G. Bianconi, Interdisciplinary and physics challenges in network theory, EPL 111, 56001 (2015).
[2] Z. Wu, G. Menichetti C. Rahmede G. Bianconi, Emergent Complex Network Geometry, Scientific Reports 5, 10073 (2015).
[3] G. Bianconi and C. Rahmede, Complex Quantum Network Manifolds are Scale-Free in d>2, Scientific Reports 5, 13979 (2015)
[4] G. Bianconi and C. Rahmede, Network geometry with flavor: from complexity to quantum geometry, arxiv:1511.04539 (2015)

Networks
are
everywhere

Networks

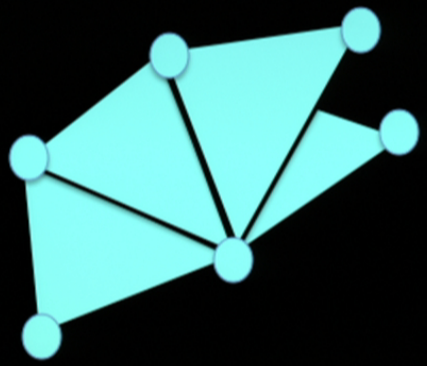
Pairwise interactions define networks

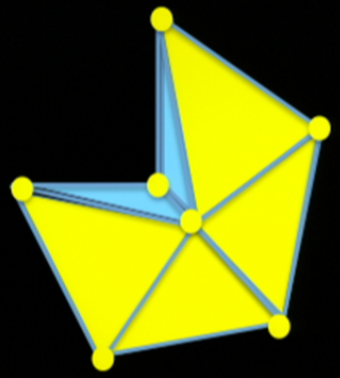


Simplicial complexes

Interactions between two or more nodes define simplicial complexes

Simplicial complexes are not only formed by nodes and links but also by triangles, tetrahedra etc.





Quantum Spacetime as a network

My own view is that ultimately physical laws
should find their most natural expression
in terms of essentially combinatorial principles…
Thus, in accordance with such a view,
should emerge some form
of
discrete or combinatorial spacetime.

Roger Penrose
in
On the Nature of Quantum Geometry

Quantum gravity approaches that use network-like structure for space-time

- Loop Quantum Gravity
- Causal-Dynamical-Triangulations
- Causal sets
- Spin-Foams
- Group field theory
- ...

The Universe and the Complex: a case of cross fertilization?

Most surprising and far-reaching analogies revealed themselves between apparently quite disparate natural processes.
(Maxwell, Scientific papers, Vol I)

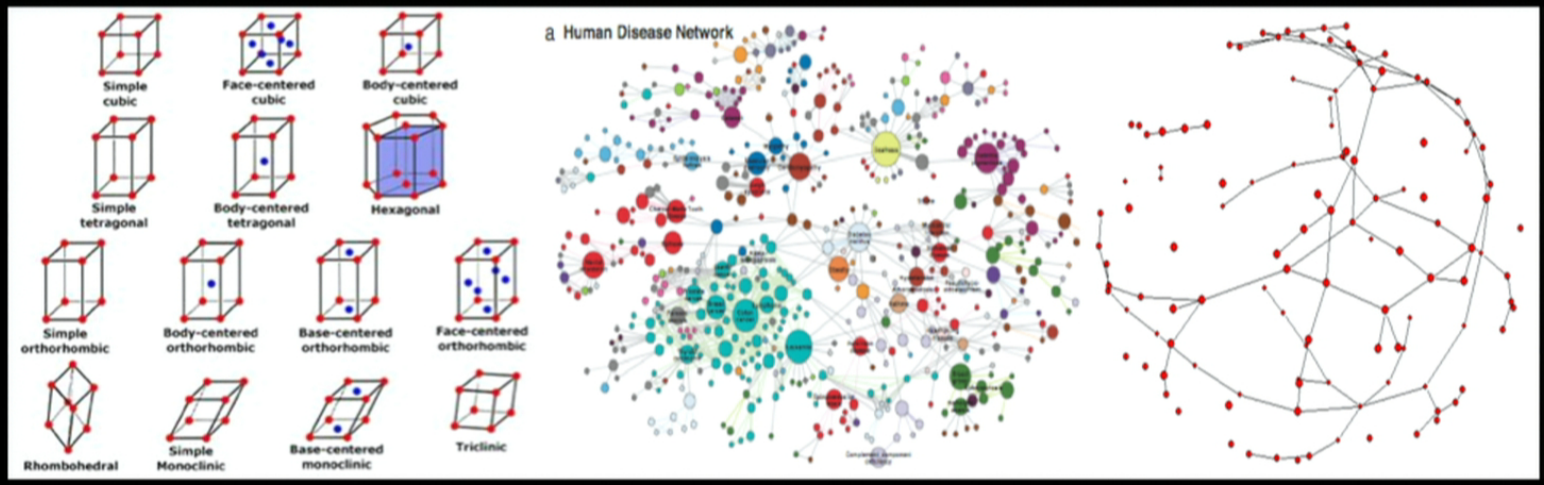
I think the next century will be the century of complexity
(Stephen Hawking Jan 2000)

A theory of quantum cosmology cannot be logically consistent if it does not describe a complex universe. […]
A theory of cosmology
must, if it is to be consistent, be a theory of self-organization.
(*The Life Of the Cosmos*, Lee Smolin)

Complex networks

Between randomness and order

LATTICES | COMPLEX NETWORKS | RANDOM GRAPHS

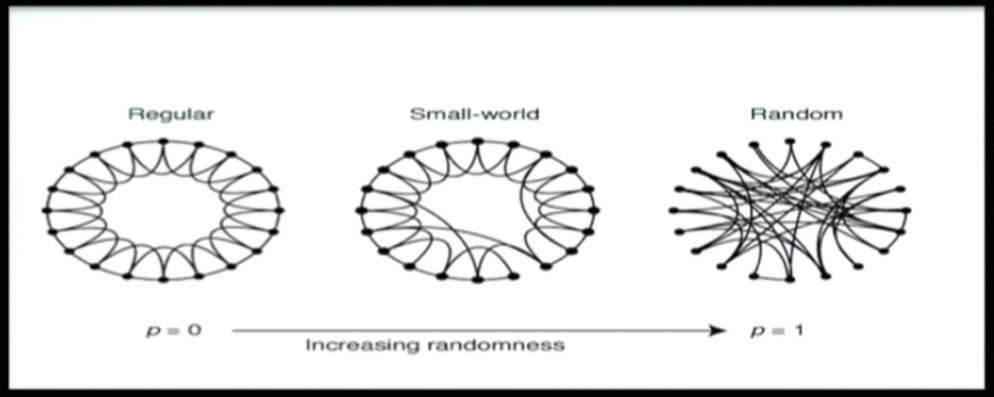


Regular networks
Symmetric

Scale free networks
Small world
With communities
ENCODING INFORMATION IN THEIR STRUCTURE

Totally random
Poisson degree distribution

Small world networks



For a very wide range of p values the Hausdorff dimension is infinite while the network preserves a notion of locality (high clustering coefficient)

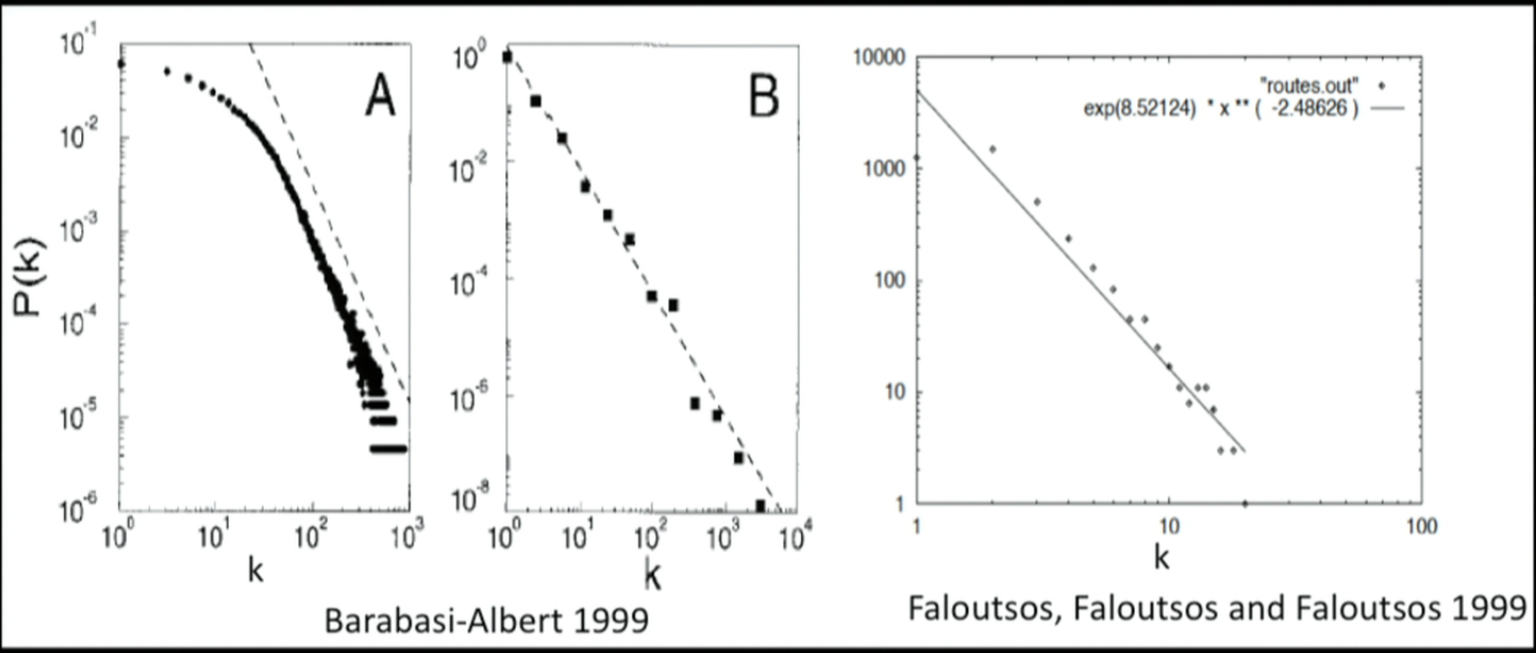
$$\ell \approx \ln N$$

$$N \approx e^{\ell}$$

$$N \approx \ell^{d_H} \Rightarrow d_H = \infty$$

Scale-free networks

Actor networks WWW Internet



Barabasi-Albert 1999

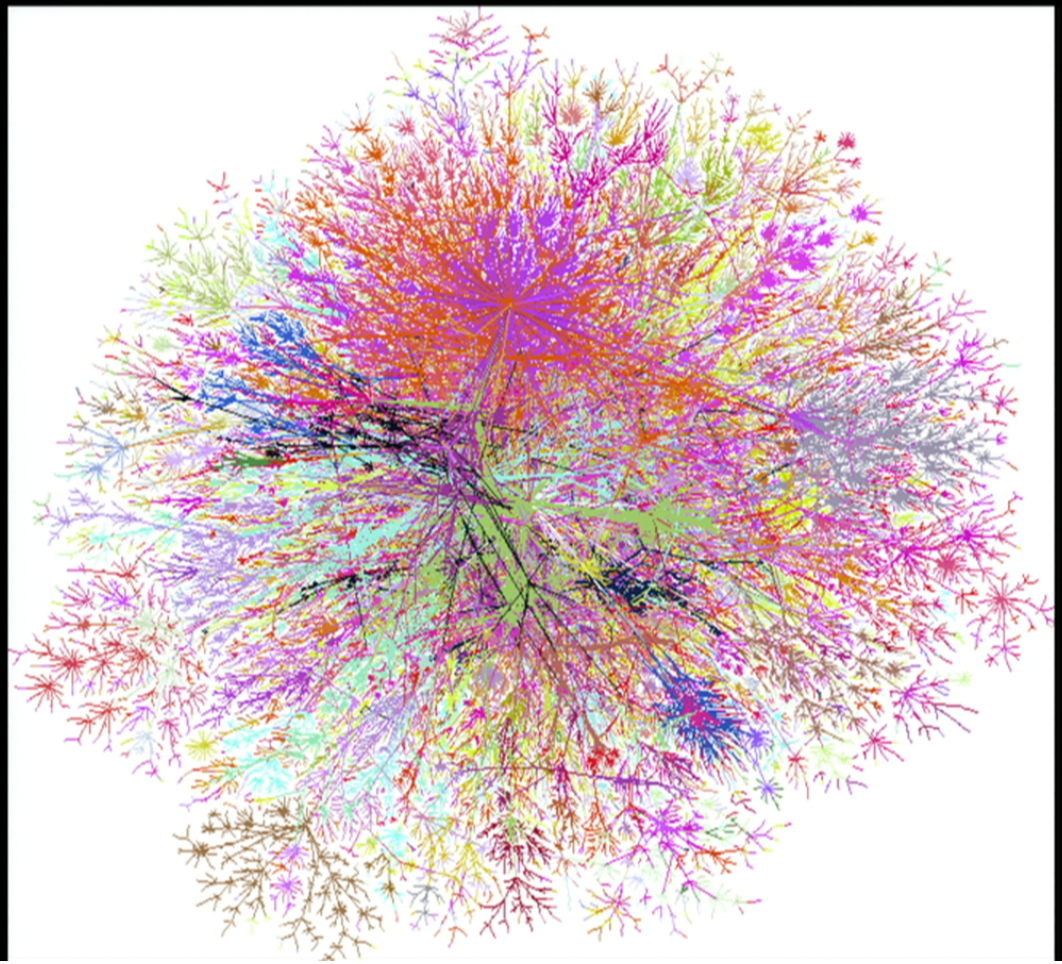
Faloutsos, Faloutsos and Faloutsos 1999

$$P(k) \propto k^{-\gamma} \qquad \gamma \in (2,3]$$

$$\langle k \rangle \quad finite$$

$$\langle k^2 \rangle \to \infty$$

Scale-free networks



Technological networks
Internet
World-Wide Web

Biological networks
Metabolic networks,
Protein-interaction networks,
Transcription networks

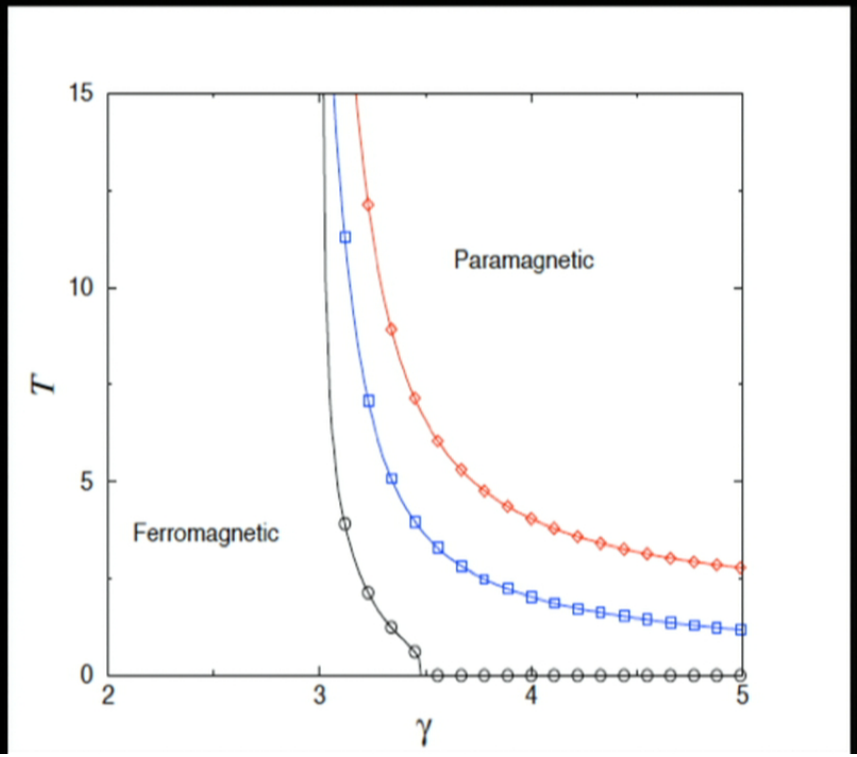
Brain functional networks

Social networks
Collaboration networks
Citation networks
Facebook

Economical networks
Networks of shareholders
The World Trade Web

The Ising model on scale-free networks

$$P(k) \propto k^{-\gamma}$$



Critical Temperature

$$\frac{T_c}{J} \propto \frac{\langle k(k-1) \rangle}{\langle k \rangle}$$

Bianconi 2002,Leone et al. 2002
Dorogovtsev et al. 2002

For $\gamma \leq 3$ the critical temperature diverges. The system is in the Ferromagnetic phase at every temperature

Barabasi-Albert scale-free model

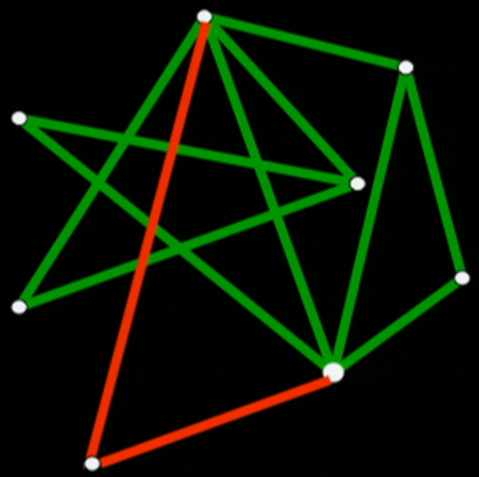
(1) GROWTH :

At every timestep we add a new node with *m* edges (connected to the nodes already present in the system).

(2) PREFERENTIAL ATTACHMENT :

The probability $\Pi(k_i)$ that a new node will be connected to node *i* depends on the connectivity k_i of

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}$$



Barabasi Albert Science 1999

Growth with uniform attachment

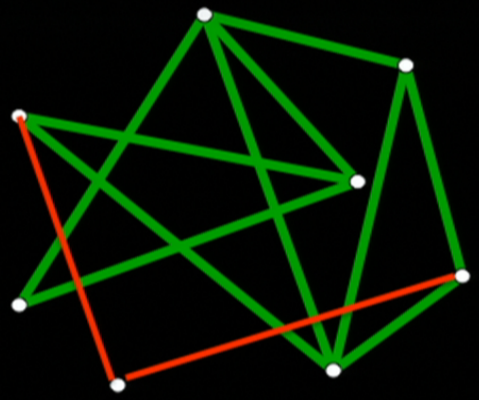
(1) GROWTH :

At every timestep we add a new node with *m* links connected to the nodes already present in the system

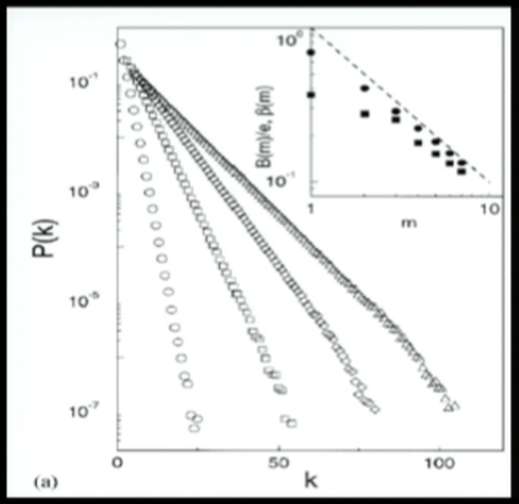
(2) UNIFORM ATTACHMENT :

The probability Π_i that a new node will be connected to node *i is uniform*

$$\Pi_i = \frac{1}{N}$$



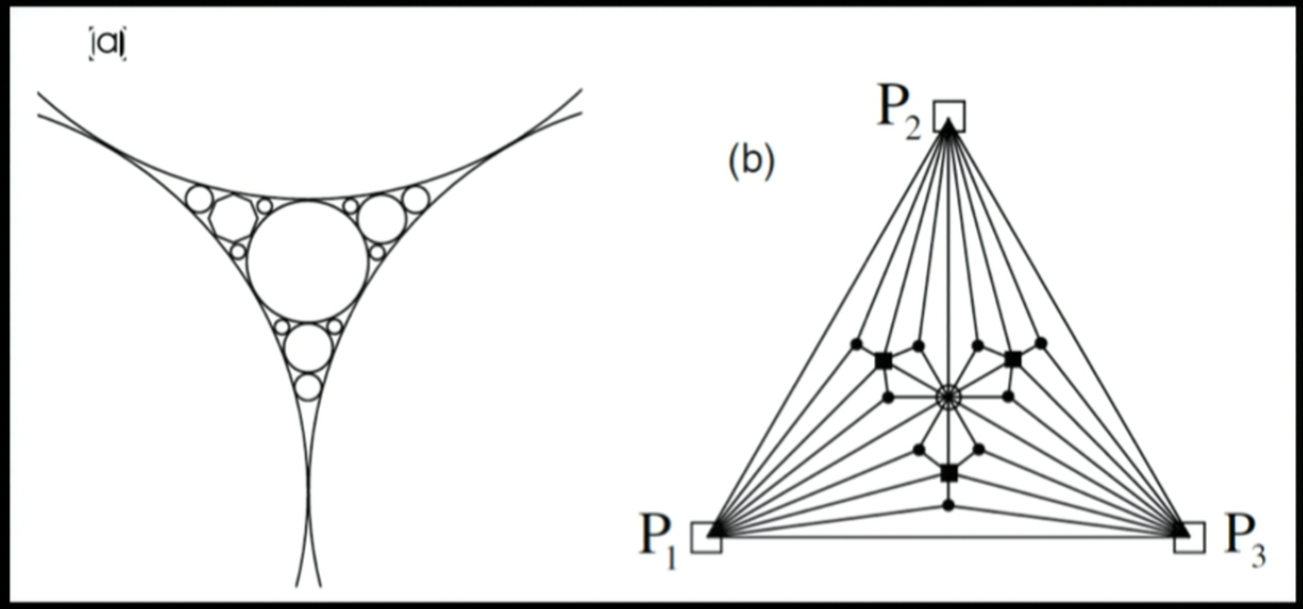
Exponential



Barabási & Albert, Physica A (1999)

Apollonian networks

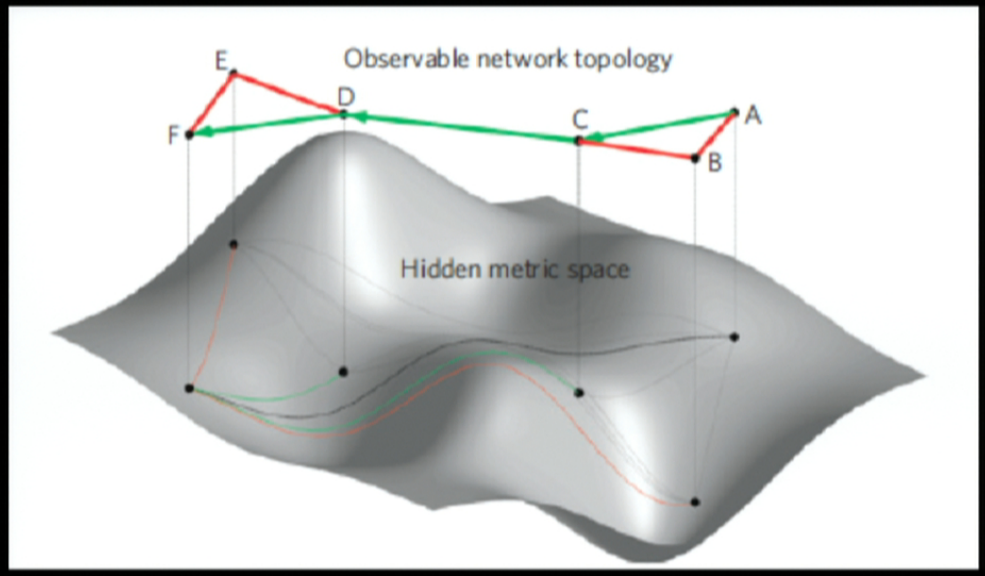
Apollonian networks are formed by linking the centers of an Apollonian sphere packing
They are scale-free and are described by the Lorentz group



Andrade et al. PRL 2005
Soderberg PRA 1992

Network Geometry

aims at unveiling
the hidden metric space of networks



Boguna, Krioukov, Claffy Nature Physics (2008)

Hyperbolic geometry

Sum of the angles of a triangle Σ

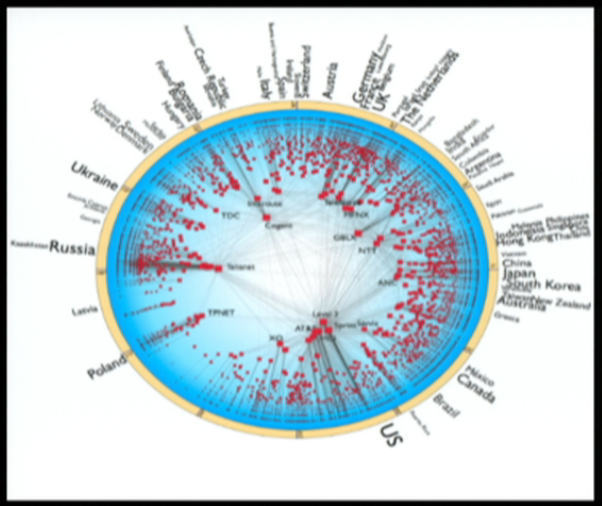
$\Sigma < \pi$

Number of nodes

$N \simeq e^{D}$

Hyperbolic networks

A large variety of networks display an hyperbolic network geometry strongly affecting their navigability



Boguna et al. Nature Communication (2010)

Hyperbolic geometry of complex networks could contribute to improve routing algorithms

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE INFOCOM 2007 proceedings.

Geographic Routing Using Hyperbolic Space

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*Supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship.
Portions of this work were completed while the author was a postdoctoral fellow at UC Berkeley.

nature COMMUNICATIONS

ARTICLE

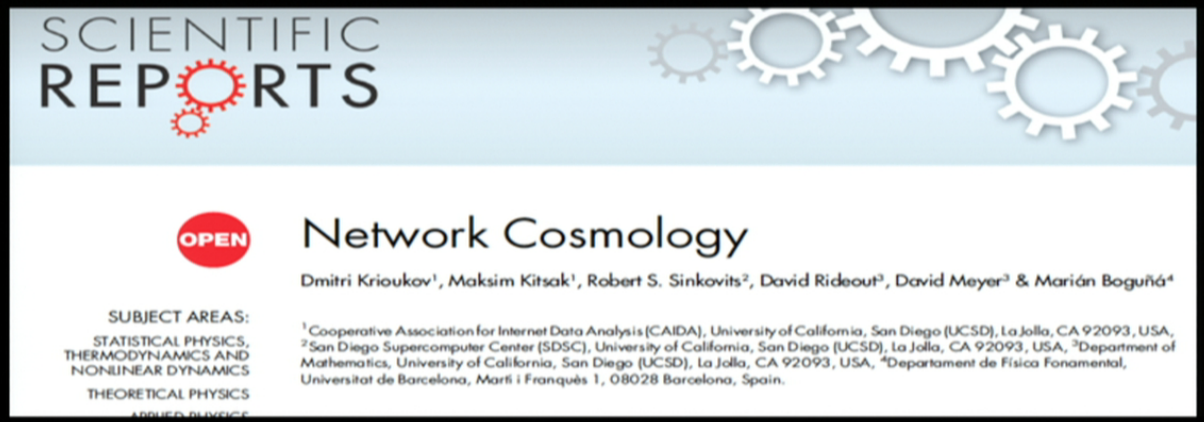
Received 6 Apr 2010 | Accepted 6 Aug 2010 | Published 7 Sep 2010

DOI: 10.1038/ncomms1063

Sustaining the Internet with hyperbolic mapping

Marián Boguñá[1], Fragkiskos Papadopoulos[2] & Dmitri Krioukov[3]

Complex networks and causal sets



SCIENTIFIC REPORTS

OPEN

Network Cosmology

Dmitri Krioukov[1], Maksim Kitsak[1], Robert S. Sinkovits[2], David Rideout[3], David Meyer[3] & Marián Boguñá[4]

[1]Cooperative Association for Internet Data Analysis (CAIDA), University of California, San Diego (UCSD), La Jolla, CA 92093, USA, [2]San Diego Supercomputer Center (SDSC), University of California, San Diego (UCSD), La Jolla, CA 92093, USA, [3]Department of Mathematics, University of California, San Diego (UCSD), La Jolla, CA 92093, USA, [4]Departament de Física Fonamental, Universitat de Barcelona, Martí i Franquès 1, 08028 Barcelona, Spain.

SUBJECT AREAS:
STATISTICAL PHYSICS, THERMODYNAMICS AND NONLINEAR DYNAMICS
THEORETICAL PHYSICS

What is the dimension of citation space?

James R. Clough[1], Tim S. Evans[2]

Centre for Complexity Science, Imperial College London,
South Kensington campus, London, SW7 2AZ, U.K.
30th July 2014

Emergent geometry

It is possible that
the hyperbolic geometry is an
emergent property
of the network evolution
which follows dynamical rules
that make no use of the hidden geometry?

Growing networks describe the emergence of complexity

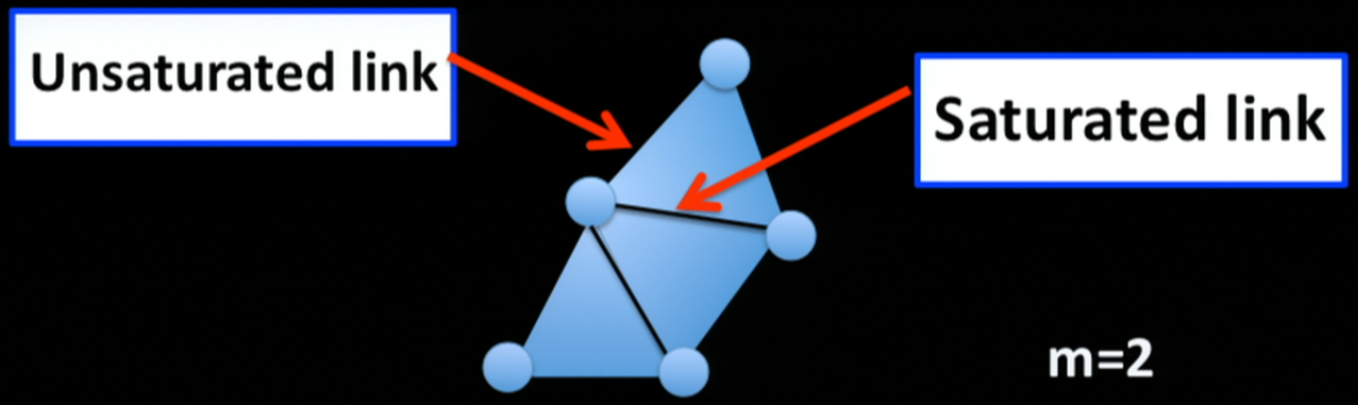
Would growing simplicial complexes
describe
the emergence of geometry?

Emergent Network Geometry

The model describes the underlying structure of a simplicial complex constructed by gluing together triangles by a non-equilibrium dynamics.

Every link is incident to at most m triangles with m>1.

Saturated and Unsaturated links

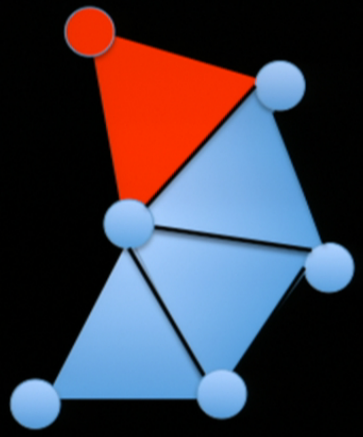


- if the link is unsaturated, i.e. less than m triangles are incident on it
- if the link is saturated, i.e. the number of incident triangles is given by m

Process (a)

We choose a unsaturated link and
we glue a new triangle the link

Growing
Simplicial
Complex

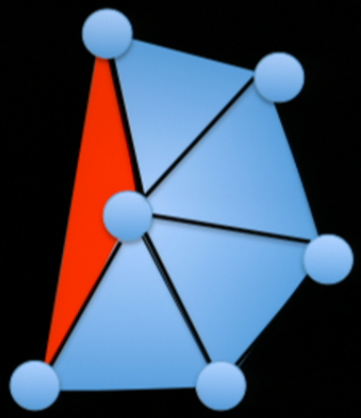


Growing
Geometrical
Network

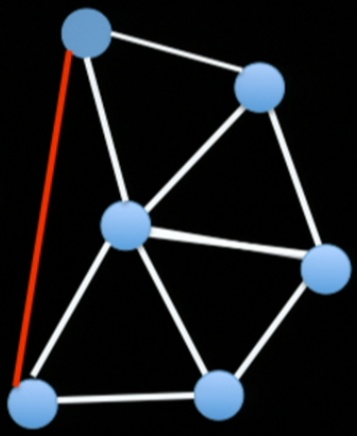
Process (b)

We choose two adjacent unsaturated links and we add the link between the nodes at distance 2 and all triangles that this link closes as long that this is allowed.

Growing Simplicial Complex



Growing Geometrical Network



The model

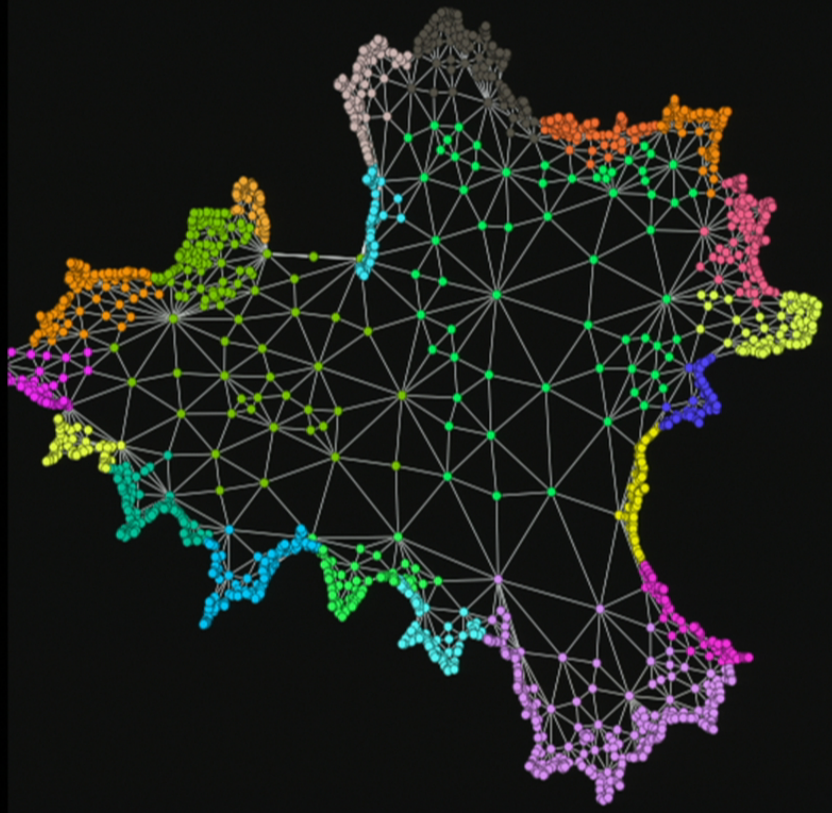
Starting from an initial triangle,

At each time

- process (a) takes place and
- process (b) takes place with probability $p<1$

Z. Wu, G. Menichetti, C. Rahmede, G. Bianconi, Scientific Reports 5, 10073 (2015)

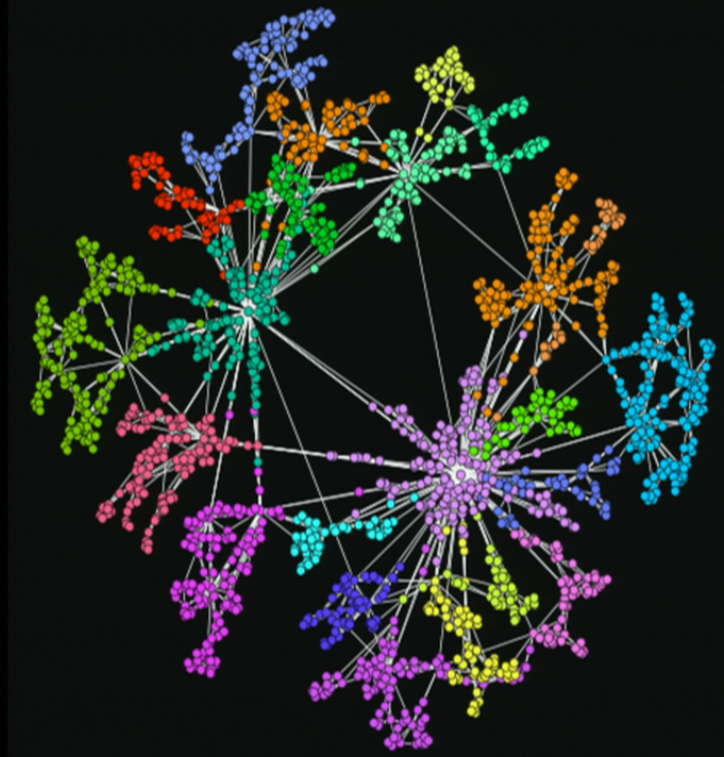
Discrete Manifolds



A discrete manifold of dimension d=2 is a simplicial complex formed by triangles such that every link is incident to at most two triangles.

Exponential degree distribution

Scale-free networks



For $m = \infty$ a **scale-free**, small-world network with high clustering coefficient and significant community structure is generated.

Degree distribution

- For *m=2* and *p=0* we can calculate the degree distribution given by

$$P(k) = \frac{1}{2}\left(\frac{2}{3}\right)^{k-1}$$

- For $m = \infty$ and *p=0* we can calculate the degree distribution given by

$$P(k) = \frac{12}{(k+2)(k+1)k}$$

Combinatorial Curvature

The combinatorial curvature for a node i of a planar triangulation is

$$R_i = 1 - \frac{k_i}{2} + \frac{T_i}{3}$$

- k_i is the degree of the node i, T_i is the number of triangles to which node i belongs

For a node in the bulk

$$R_i = \frac{6 - k_i}{6}$$

For a node at the surface

$$R_i = \frac{4 - k_i}{6}$$

Distribution of local curvature

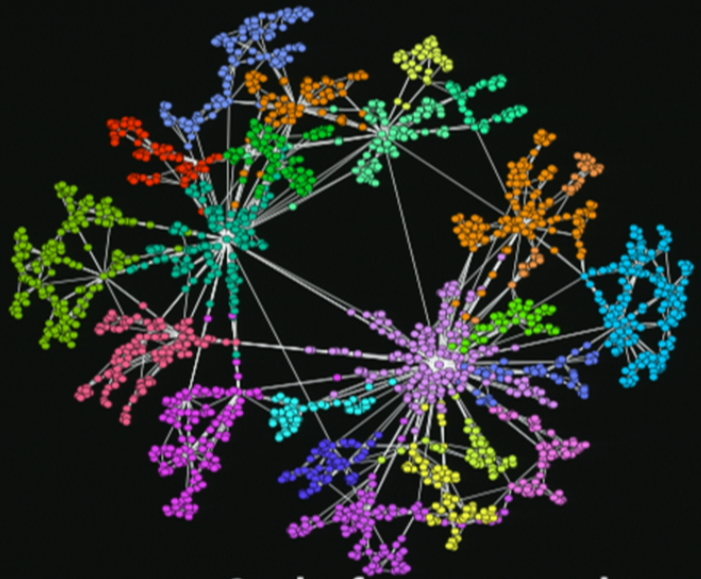
m=2 p=0.9

Exponential network

$$\langle R \rangle = \frac{1}{N}$$

$$\langle R^2 \rangle < \infty$$

m=infinity p=0

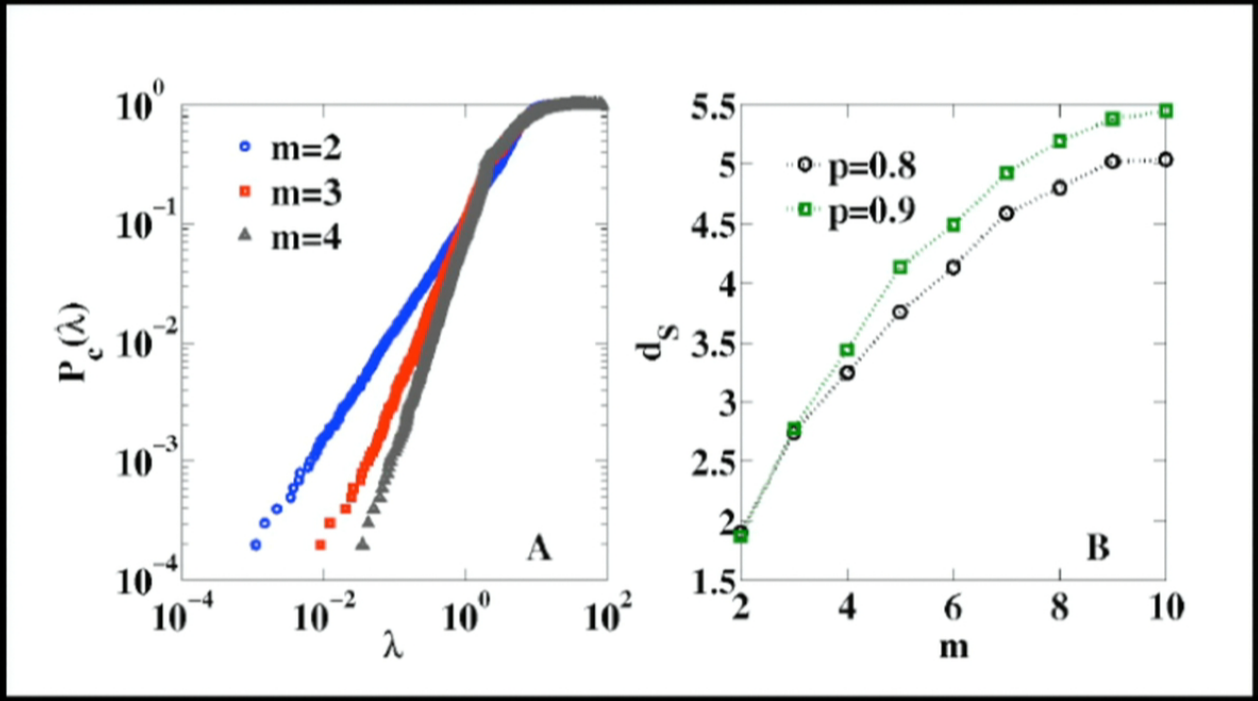


Scale-free network

$$\langle R \rangle = \frac{1}{N}$$

$$\langle R^2 \rangle = \infty$$

Finite spectral dimension



$$L_{ij} = k_i\,\delta_{ij} - a_{ij}$$

$$\rho(\lambda) \approx \lambda^{(d/2-1)}$$

$$P_c(\lambda) \approx \lambda^{d/2}$$

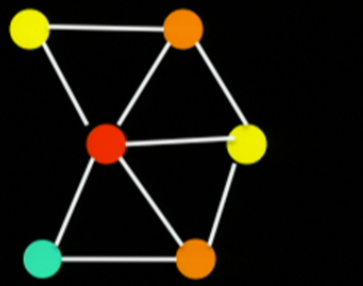
The d-dimensional simplicial complexes

In dimension d the growing simplicial complex

built by gluing simplices of dimension d along their (d-1)-face

In d=1 the simplices are links

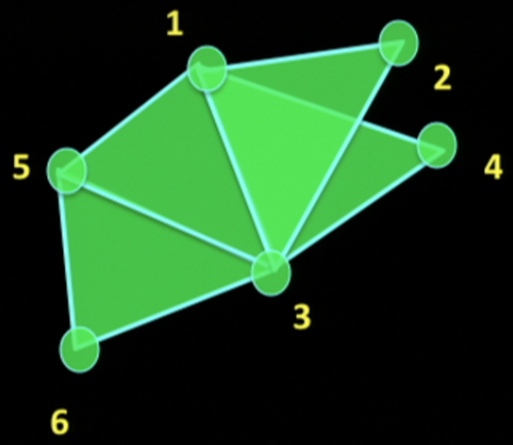
In d=2 the simplices are triangles



In d=3 they are tetrahedra.

Generalized degrees

The generalized degree $k_{d,\delta}(\alpha)$ of a δ-face α in a d-dimensional simplicial complex is given by the number of d-dimensional simplices incident to the δ-face α.



Number of triangles incident

- **to the nodes $k_{2,0}$**
- **to the links $k_{2,1}$**

i	$k_{2,0}(i)$
1	3
2	1
3	4
4	1
5	2
6	1

(i,j)	$k_{2,1}(i,j)$
(1,2)	1
(1,3)	3
(1,4)	1
(1,5)	1
(2,3)	1
(3,4)	1
(3,5)	2
(3,6)	1
(5,6)	1

The incidence number

To each (d-1)-face α we associate the incidence number

$$n_\alpha = k_{d,d-1}(\alpha) - 1$$

given by

the number of d-dimensional simplices incident to the face minus one

If n_α takes only values n_α=0,1 the *simplicial complex is a discrete manifold.*

Network Geometry with Flavor

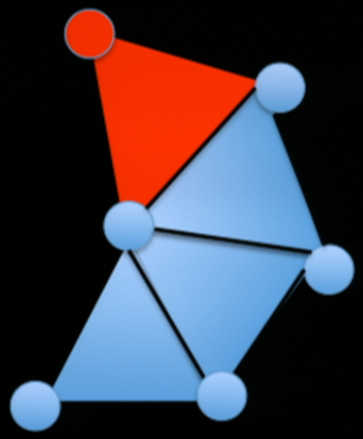
We start with a d-dimensional simplex,
At each time we choose a (d-1)-face
with probability

$$\Pi_{\alpha}^{[s]} = 1 + s\, n_{\alpha}$$

and we glue to it a new d-dimensional simplex

The flavor s=-1,0,1

Growing Simplicial



Network Geometry with Flavor

Flavor s=-1,0,1 and attachment probability

$$\Pi_{\alpha}^{[s]} = \frac{(1+s\,n_{\alpha})}{\sum_{\alpha' \in Q_{d,d-1}} (1+s n_{\alpha'})} = \begin{cases} \dfrac{(1-n_{\alpha})}{Z^{[-1]}} & s=-1 \\ \dfrac{1}{Z^{[0]}} & s=0 \\ \dfrac{k_{d,d-1}(\alpha)}{Z^{[1]}} & s=1 \end{cases}$$

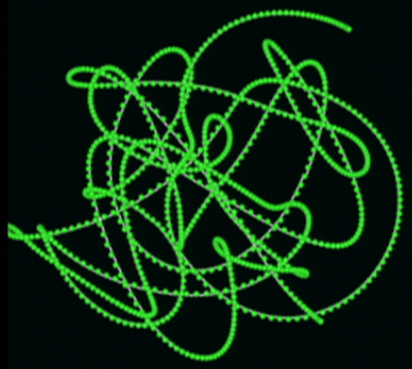
s=-1	Manifold	n_{α}=0,1
s=0	Uniform attachment	n_{α}=0,1,2,3,4...
s=1	Preferential attachment	n_{α}=0,1,2,3,4...

Dimension d=1

Manifold

Chain

s=-1



Chain

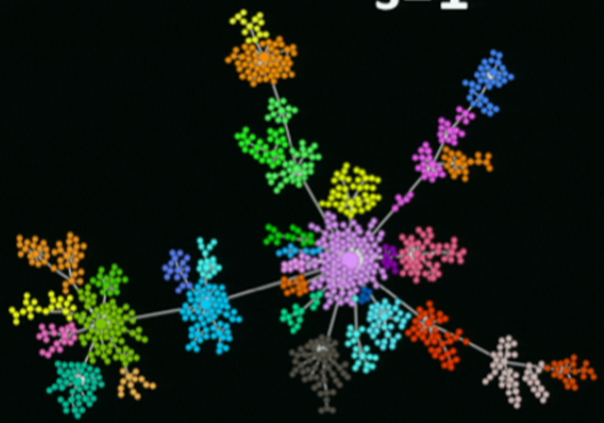
Uniform attachment

s=0

Exponential

Preferential attachment

s=1



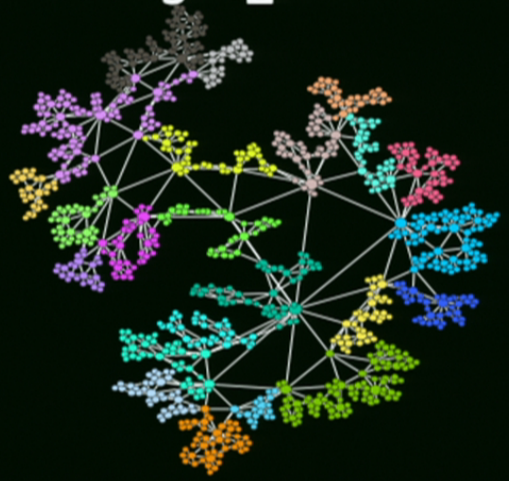
BA model

Scale-free

Dimension d=2

Manifold

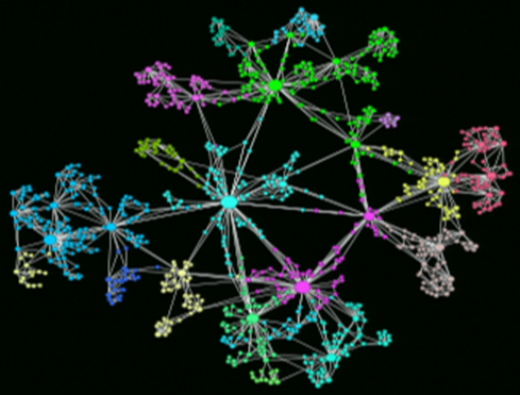
s=-1



Exponential

Uniform attachment

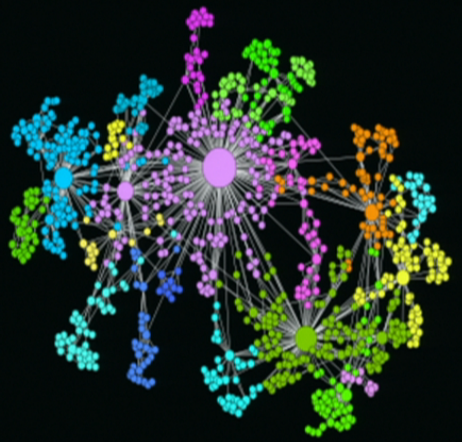
s=0



Scale-free

Preferential attachment

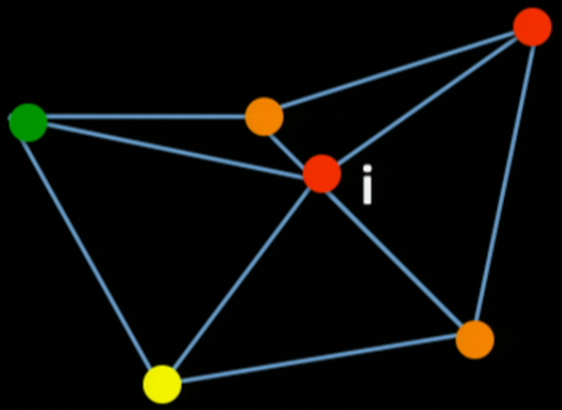
s=1



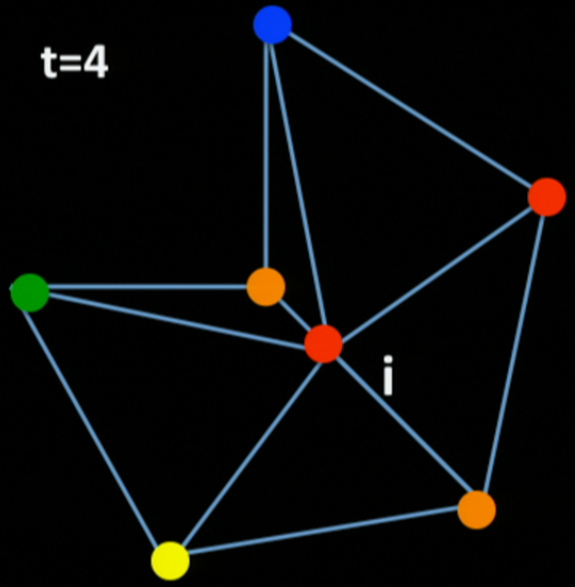
Scale-free

Effective preferential attachment emerging in d=3 dimensions

t=3



t=4



Node i has generalized degree 3

Node i is incident to 5 unsaturated faces

Node i has generalized degree 4

Node i is incident to 6 unsaturated faces

Degree distribution of NGF

- The NGF with s+d=0, i.e. with (s,d)=(-1,1) are chains
- For s+d=1, i.e. (s,d)=(-1,2) and (0,1) we have

$$P_d^{[s]} = \left(\frac{d}{d+1}\right)^{k-d} \frac{1}{d+1}$$

- For s+d>1 we have

$$P_d^{[s]} = \frac{d+s}{2s+s} \frac{\Gamma[1+(2d+s)/(d+s-1)]}{\Gamma[d/d+s-1]} \frac{\Gamma[k-d+d/(d+s-1)]}{\Gamma[k-d+1+(2d+s)/(d+s-1)]}$$

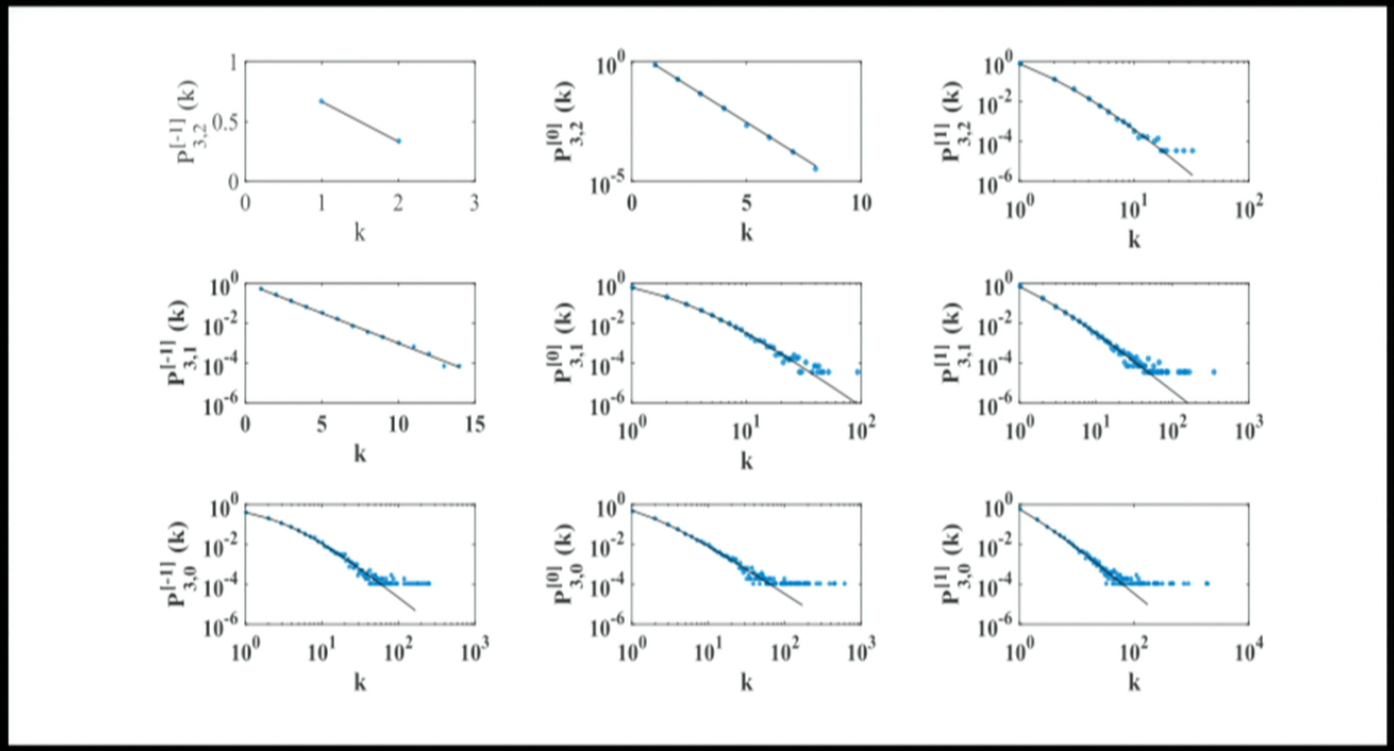
Generalized degree distribution for Network Geometry with Flavor (NGF)

flavor	$s=-1$	$s=0$	$s=1$
$\delta = d-1$	Binomial	Exponential	Power-law
$\delta = d-2$	Exponential	Power-law	Power-law
$\delta \leq d-3$	Power-law	Power-law	Power-law

- For s=1 the NGF are always scale-free
- For s=0 the NGF are scale-free for d>1
- For s=-1 the NGF are scale-free for d>2

G. Bianconi, C. Rahmede, Scientific Reports (2015);PRE (2016).

Generalized degree distributions of NGF



Theory versus simulation in NGF s=-1,0,1 and d=3

Hausdorff dimension and Area

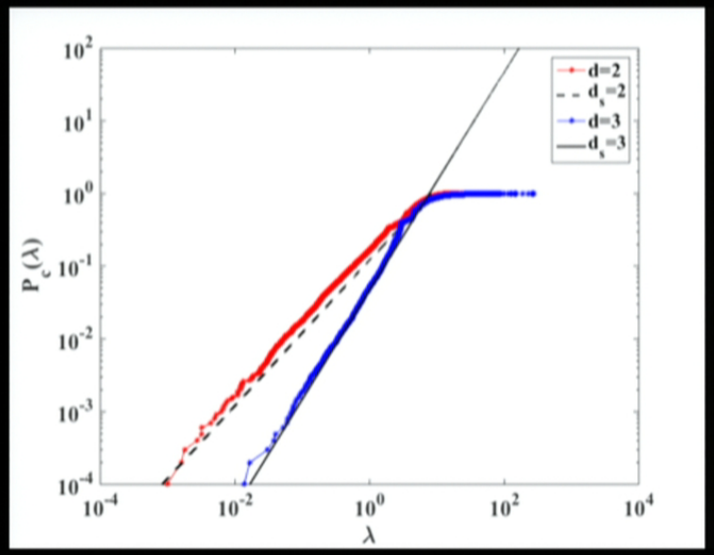
The NGF have Hausdorff dimension

$$d_H = \infty$$

The Area A of the NGFs is
the number of (d-1)-faces with n_α=0
we have

$$A \propto N$$

The Spectral dimensions of NGF with s=-1 is equal to the dimension of the simplex



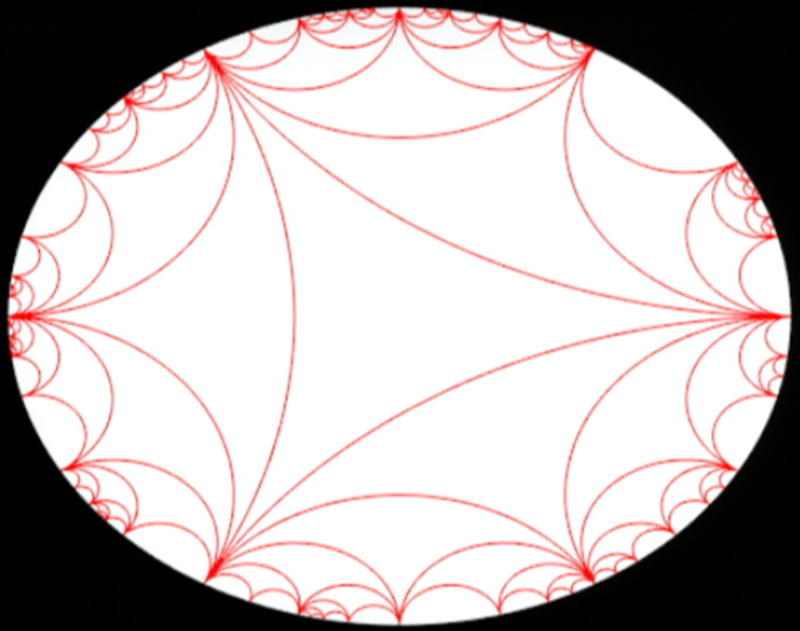
$$L_{ij} = k_i \, \delta_{ij} - a_{ij}$$

$$\rho(\lambda) \approx \lambda^{d_s/2-1}$$

$$P_c(\lambda) \approx \lambda^{d_s/2}$$

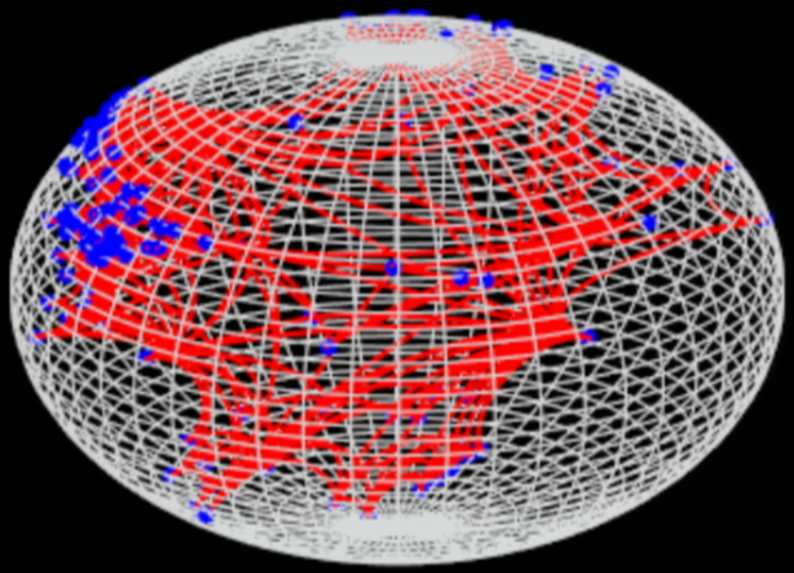
Emergent hyperbolic geometry

The emergent hidden geometry is the hyperbolic H^d space
Here all the links have equal length



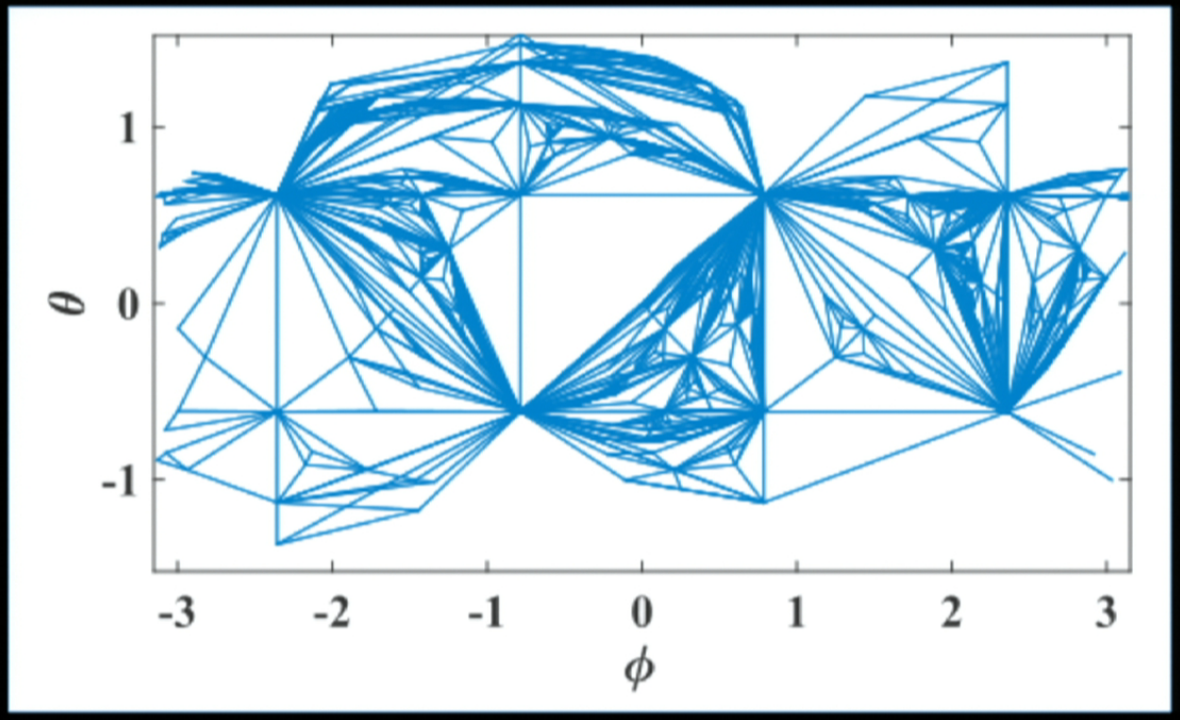
d=2

Emergent hyperbolic geometry



d=3

The pseudo-fractal geometry of the surface of the 3d manifold (random Apollonian network)



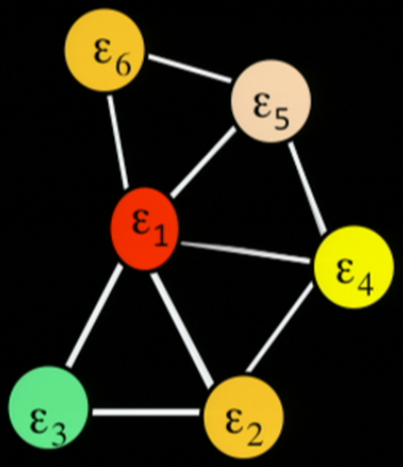
Energies of the nodes

Not all the nodes are the same!

Let assign to each node i

an energy ε *from a*

$g(\varepsilon)$ *distribution*



Energy of the δ-faces

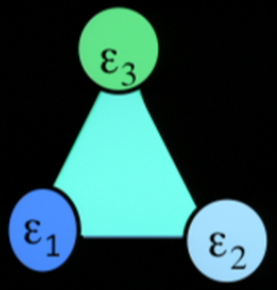
Every δ-face α is associated to an energy which is the sum of the energy of the nodes belonging to α

$$\varepsilon_\alpha = \sum_{i \subset \alpha} \varepsilon_i$$

For example, in d=3

the energy of a link

is $\varepsilon_1+\varepsilon_2$

the energy of a face 

is $\varepsilon_1+\varepsilon_2+\varepsilon_3$

Fitness of the δ-faces

The fitness of a δ−face α is given by

$$\eta_\alpha = e^{-\beta \varepsilon_\alpha}$$

where β=1/T is the inverse temperature

If β=0 all the nodes have same fitness

If β>>1 small differences in energy have large impact on the fitness of the faces

Network Geometry with Flavor

Starting from a d-dimensional simplex
We choose a (d-1)-dimensional face α,
with probability

$$\Pi_\alpha = \frac{e^{-\beta\varepsilon_\alpha}(1+sn_\alpha)}{\sum_{\alpha'\in Q_{d,d-1}} e^{-\beta\varepsilon_{\alpha'}}(1+sn_{\alpha'})}$$

and glue a new d-dimensional simplex to it.

The parameter $\beta=1/T$ is the inverse temperature.
If $\beta=0$ we are recast into the previous model
The flavor s=-1,0,1

Statistical mechanics of the manifold evolution

Each manifold evolution is characterized by the sequence

$$\left\{\alpha(t')\right\}_{t'\le t}$$

of (d-1)-faces chosen for growing the manifold

Each network history up to time t=N has probability

$$P\left(\left\{\alpha(t')\right\}_{t'\le t}\middle|\left\{\varepsilon(t')\right\}_{t'\le t}\right)=\frac{e^{-\beta E}}{\tilde{Z}(t)}=\frac{e^{-\beta(E-\mu N)}}{N!}$$

where the total energy E associated to the manifold is given by

$$E=\sum_{\alpha\in Q_{d,d-1}}\varepsilon_\alpha n_\alpha$$

Generalized Area law

Each given network G is obtained at time t with probability

$$P\left(G\middle|\left\{\varepsilon_i\right\}_{i\leq N}\right) = {}^{-\beta(E-\nu N)}$$

Where this expression is obtained from the probability of each single history summing over all the histories resulting in the same network G

The change in entropy of the network satisfies

$$\Delta S = \beta(\langle \Delta E \rangle - \nu \Delta N) = [\beta(\langle e \rangle - \nu)]\Delta N \propto [\beta(\langle e \rangle - \nu]\Delta A$$

because N=V is proportional to the area A of the manifold.

Generalized Area law

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Energy of manifolds and Regge curvature

Assuming that the simplices are flat the Regge curvature R_α of a (d-2)-face α is given by

$$R_\alpha = \pi - \theta_d k_{d,d-2}(\alpha)$$

The energy E can be expressed in term of the curvature as

$$E = \sum_{\alpha \in S_{d,d-1}} \varepsilon_\alpha n_\alpha = -\frac{B_d}{\theta_d}\left(\sum_{\alpha \in S_{d,d-2}} \varepsilon_\alpha R_\alpha - \Lambda \right)$$

$$\Lambda = (\pi - \frac{\theta_d}{2}) \sum_{\alpha \in S_{d,d-2}} \varepsilon_\alpha$$

Bianconi-Barabasi model or the fitness model
Is the NGF d=1 s=1

$$\Pi_i^{[-1]} = \frac{e^{-\beta\varepsilon_i} k_i}{\sum_{j=1,N} e^{-\beta\varepsilon_j} k_j}$$

The average degree of the nodes with energy ε follows the Bose-Einstein statistics

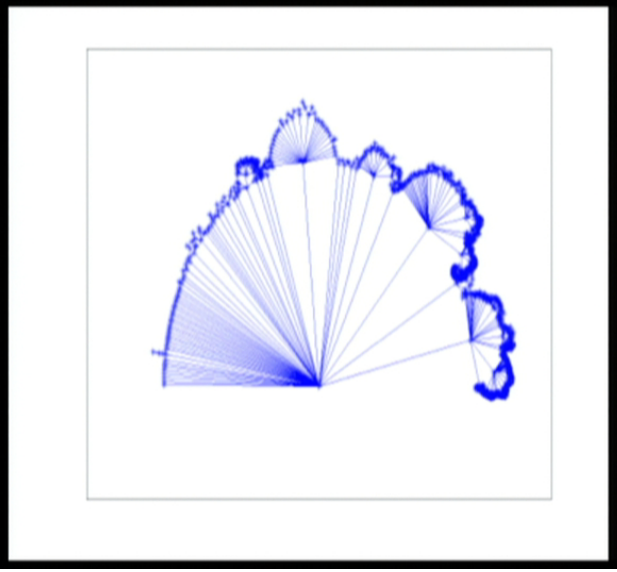
$$\left\langle [k_i - 1] \middle| \varepsilon \right\rangle = \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$$

G. Bianconi A.L. Barabasi PRL (2001)

Bose-Einstein condensation in complex networks

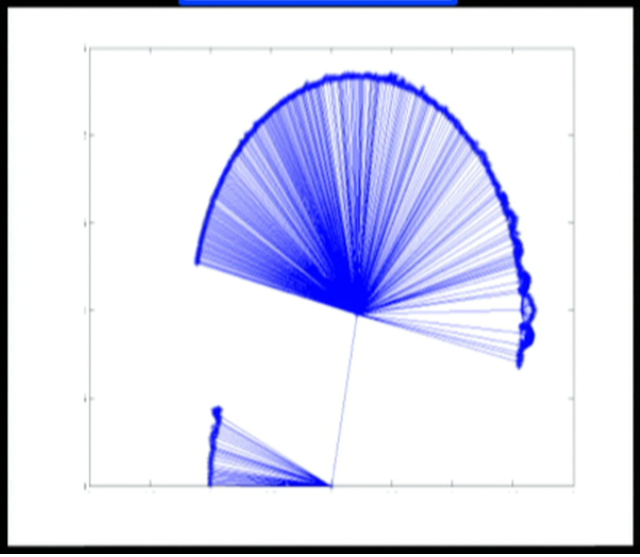
Scale-Free phase

$\beta < \beta_C$



Bose-Einstein condensate phase

$\beta > \beta_C$



Quantum statistics
and
Network Geometry with Flavor

What is the combined effect of flavor and dimensionality on the emergence of quantum statistics?

The average of the generalized degree of the NGF over δ-faces of energy ε

$$\left\langle \left[k_{d,\delta} - 1 \right] \middle| \varepsilon \right\rangle$$

follows

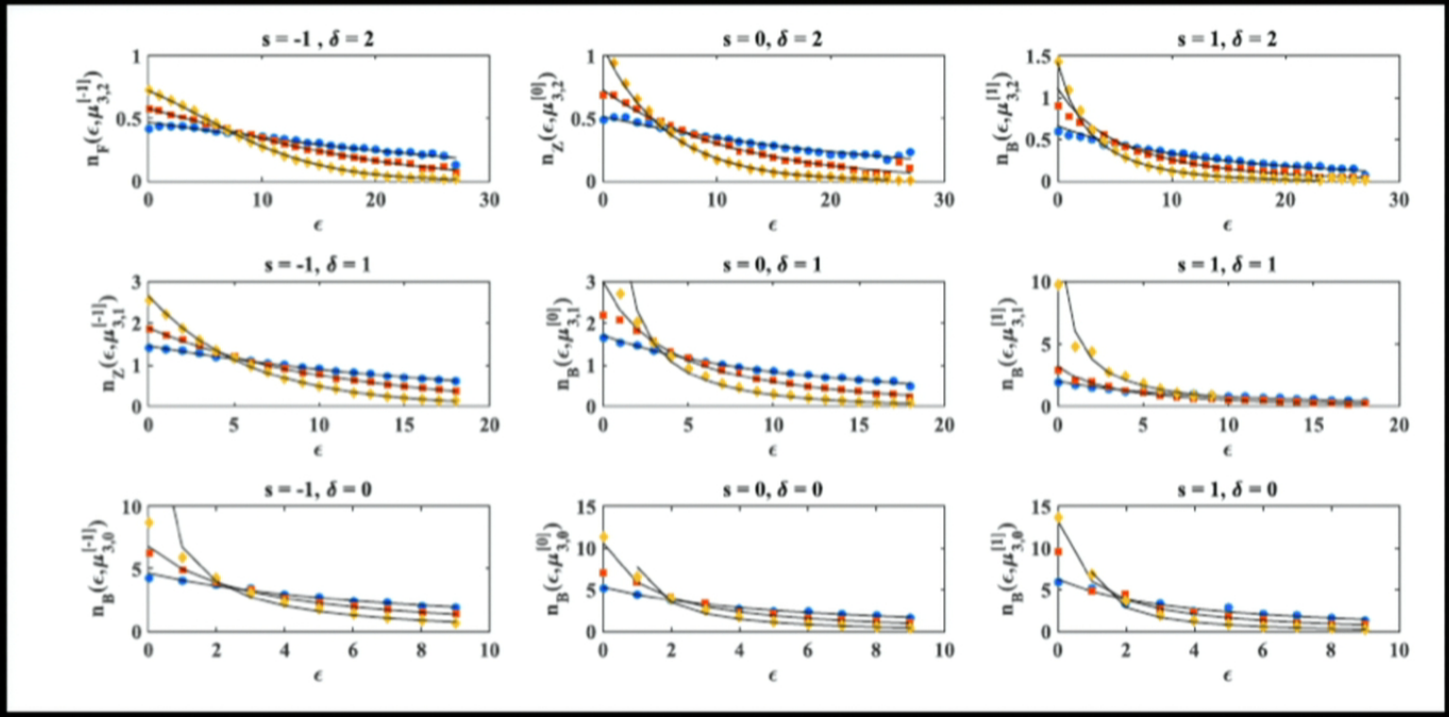
flavor	$s=-1$	$s=0$	$s=1$
$\delta = d-1$	Fermi-Dirac	Boltzmann	Bose-Einstein
$\delta = d-2$	Boltzmann	Bose-Einstein	Bose-Einstein
$\delta \leq d-3$	Bose-Einstein	Bose-Einstein	Bose-Einstein

G. Bianconi, C. Rahmede, Scientific Reports (2015);PRE (2016).

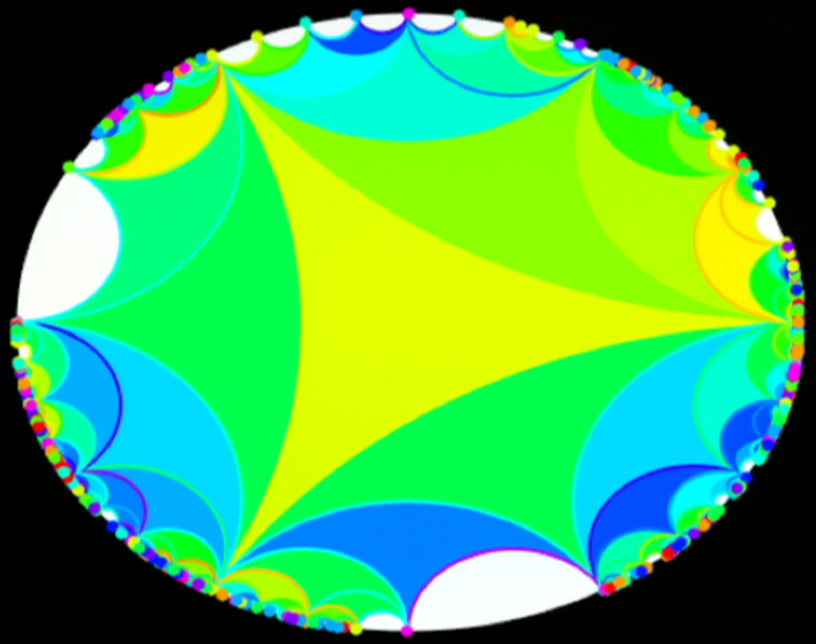
Manifolds in d=3

In NGF with s=-1 and d=3
also called
Complex Quantum Network Manifolds
the average of the generalized degree follow
the ***Fermi-Dirac, Boltzmann and Bose-Einstein***
distribution
respectively for
triangular faces, links and nodes

Theory and simulations for NGF in d=3



Emergent geometry at high temperature



d=2

β=0.01

Conclusions

Growing simplicial complexes called Network Geometry with Flavor (NGF) display:

- Non-trivial distribution of the curvature
- *strong dependence on the dimensionality*
  - *NGF with s=-1 are manifolds and are scale-free for d>2*
  - *NGF with s=0 (uniform attachment) are scale-free for d>1*
- *emergent hyperbolic geometry*

NGF at finite temperature (with fitness and energy)

- *have generalized degrees following the Fermi-Dirac, Boltzmann or Bose-Einstein distribution depending on the dimensionality of the δ-face and the flavor s.*
- *NGF can undergo phase transition strongly affecting their hidden hyperbolic geometry.*

Collaboration

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Christoph Rahmede

Zhihao Wu

Giulia Menichetti

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