Title: PSI 2016/2017 Quantum Theory - Lecture 11

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Abstract:

Quantum Mechanics

Perturbation Theory – Schrieffer-Wolff

Lecture 11

Bravyi, DiVincenzo, Loss

arXiv: 1105.0675

THE UNIVERSITY OF SYDNEY

Quantum Mechanics

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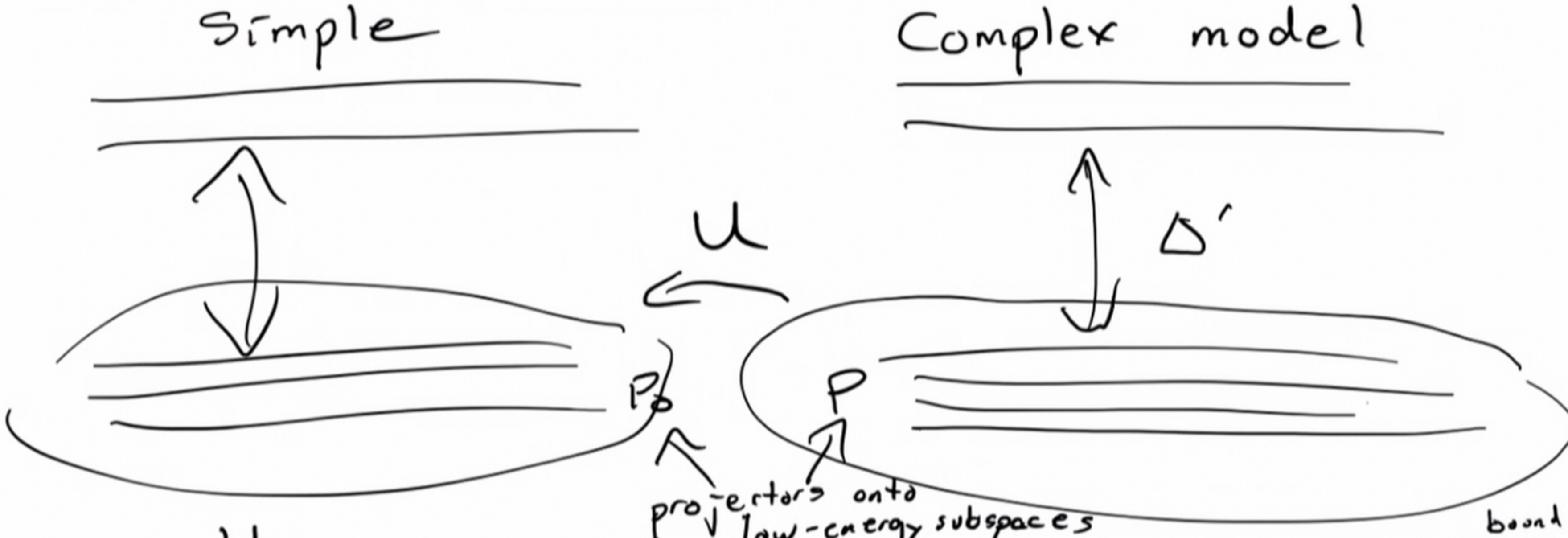
'Degenerate' Perturbation Theory



H_0
easy to solve

$\underbrace{H = H_0 + \epsilon V}_{\text{hard to solve}} \Leftarrow$ bound on $\epsilon \|V\|$

Construct H_{eff} on P_0

$$H_{eff} = U(H_0 + \epsilon V)U^{-1}\big|_{P_0}$$

as Taylor series in ϵ

Advantages of SW perturbation theory

- Works well for topological quantum many-body models
  - Standard techniques for Feynman-Dyson diagram technique require adiabatic assumption: i.e., non-degenerate ground state
- Rigorous results with proven convergence conditions
- Systematic method to construct local H_{eff} (satisfying linked cluster theorem)

Direct rotation between a pair of subspaces – one-dimensional

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Finite dimensional Hilbert space $\mathcal{H}$

For any $|\psi\rangle \in \mathcal{H}$

$$R_\psi = I - 2|\psi\rangle\langle\psi|$$

flips sign of $|\psi\rangle$
acts trivially on complement

Given a pair $|\psi\rangle, |\phi\rangle$ we want $U_{\psi\to\phi}$ mapping $|\psi\rangle \to |\phi\rangle$ (up to an overall phase)

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– identity elsewhere

$$U_{\psi\to\phi} = \sqrt{R_\phi R_\psi}$$

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Lemma: $|\phi\rangle, |\psi\rangle$ nonorthogonal $\Rightarrow$ $R_\phi R_\psi$ has no eigenvalues on real neg axis

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Proof of lemma:

Since $U_{\psi\to\phi}$ (and $R_\psi R_\phi$) acts trivially on complement of span$\{|\psi\rangle, |\phi\rangle\}$ $\Rightarrow$ restrict to qubit case $\mathbb{C}^2$

wlog $|\psi\rangle = |1\rangle$ $\quad |\phi\rangle = \sin\theta|0\rangle + \cos\theta|1\rangle$

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Write $U_{\psi\to\phi} = \exp(S)$ $\quad S$ anti-Hermitian

- $PSP = P_0 S P_0 = (1-P)S(1-P) = (1-P_0)S(1-P_0) = 0$
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$\mathcal{P}, \mathcal{P}_0 \subseteq \mathcal{H}$ linear subspaces of $\mathcal{H}$ of same dim.
Define projectors P, P_0 onto $\mathcal{P}, \mathcal{P}_0$

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$$\mathcal{U} = \sqrt{R_{\mathcal{P}_0} R_{\mathcal{P}}}$$ Direct rotation from $\mathcal{P}$ to $\mathcal{P}_0$

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Generator of the direction rotation

How do we find S?

What are some "nice" conditions defining S?

Lemma: Suppose $\|P - P_0\| < 1$.

Then $\exists$ unique an~~ti~~hermitian S s.t.

1) $e^{S} P e^{-S} = P_0$

2) S is block-offdiagonal wrt P_0 $\quad P_0 S P_0 = 0$, $(I-P_0)S(I-P_0) = 0$

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Then $U = \exp(S)$ is the direct rotation from P to P_0

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Proof sketch: We know S exists, just need to prove uniqueness.

Weak multiplicativity of the direction rotation

Bipartite system $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$

Projectors P^A, P_0^A and P^B, P_0^B with $\|P^A - P_0^A\| < 1$, $\|P^B - P_0^B\| < 1$

$$U^A P^A (U^A)^\dagger = P_0^A \qquad U^B P^B (U^B)^\dagger = P_0^B$$

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"Global" direct rotation $\mathcal{U}^{AB} : P^A \otimes P^B \rightarrow P_\partial^A \otimes P_\partial^B$

$$\mathcal{U}^{AB}(P^A \otimes P^B)(\mathcal{U}^{AB})^\dagger = P_\partial^A \otimes P_\partial^B$$

Note $\mathcal{U}^{AB} \neq \mathcal{U}^A \otimes \mathcal{U}^B$ in general

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Prop: If $\|P^A - P_\partial^A\| < 1$ and $\|P^B - P_\partial^B\| < 1$ then

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$$\|P^A \otimes P^B - P_0^A \otimes P_0^B\| < 1$$

Weak multiplicativity:

$$\mathcal{U}^{AB}(P^A \otimes P^B) = \mathcal{U}^A \otimes \mathcal{U}^B (P^A \otimes P^B)$$

Spectral gap

unperturbed Hamiltonian

Finite dim $\mathcal{H}$, H_0 Hermitian operator

$P_0 \subseteq \mathcal{H}$ subspace spanned by eigenvectors corresponding to all eigenvalues in some interval $\mathcal{I}_0$

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Restrict ϵ s.t. $|\epsilon| \leq \epsilon_c = \frac{\Delta}{2\|V\|}$

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$\mathcal{I} = \mathcal{I}_{0\,-\Delta/2}^{\;+\Delta/2}$ $\quad P \subseteq \mathcal{H}$ eigenvectors of H with eigenvalue in $\mathcal{I}$

Then for $|\epsilon| < \epsilon_c$, $\mathcal{I}$ has positive gap with rest of the spectrum

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Then for $|\epsilon| < \epsilon_c$, I has positive gap with rest of the spectrum

P and P_0 have same dimension

Define P, P_0 projectors on $\mathcal{P}, \mathcal{P}_0$

$Q = I - P \quad Q_0 = I - P_0$