Title: PSI 2016/2017 Quantum Theory - Lecture 8

Date: Sep 15, 2016 10:45 AM

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Abstract:

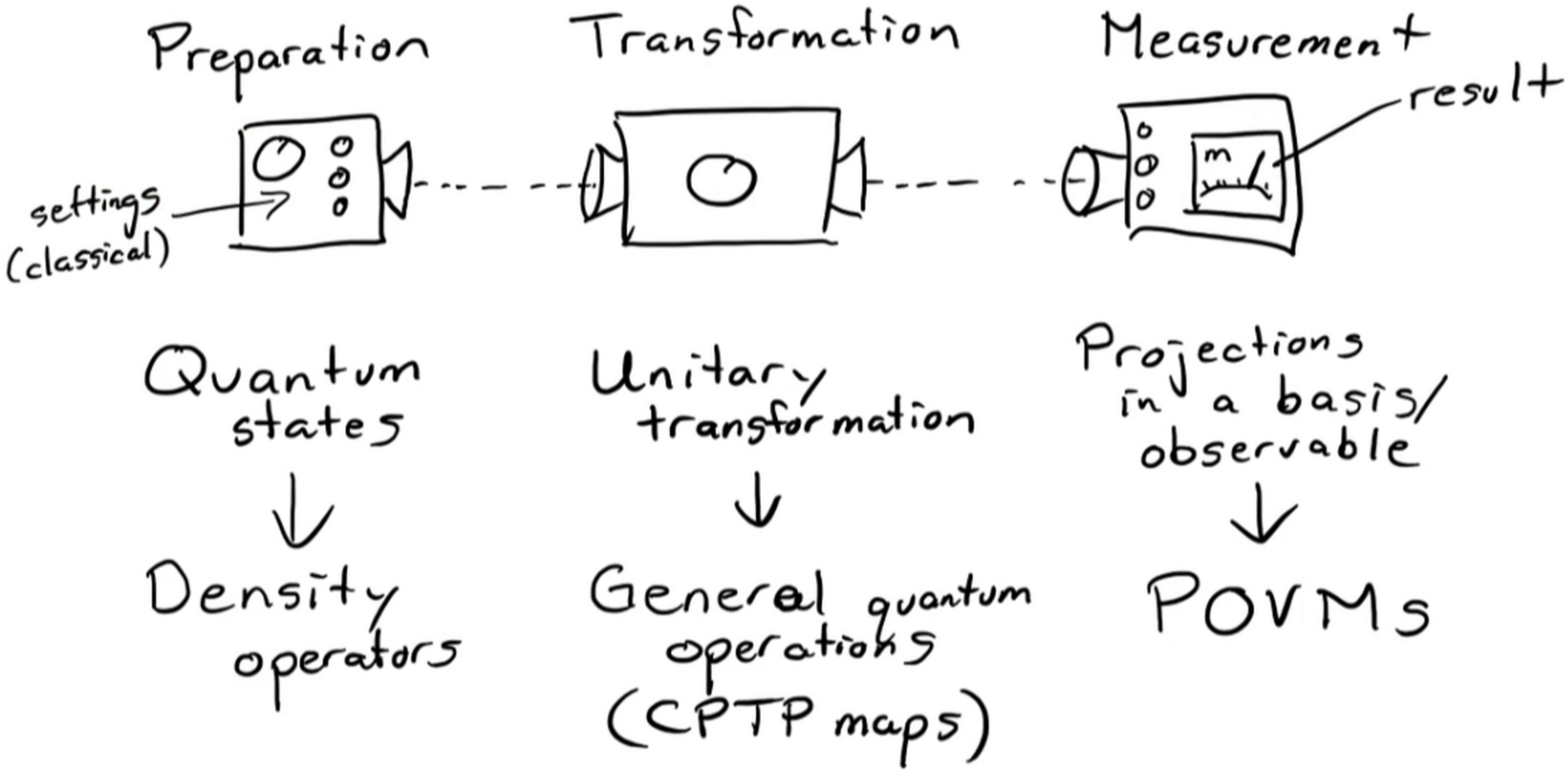
Quantum Mechanics

Quantum Operations and Quantum Noise

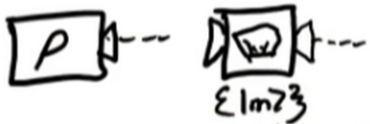
Lecture 8

THE UNIVERSITY OF SYDNEY

Operational quantum mechanics



Example: Projective measurement and more



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Many possible update rules

Luder's rule: $\rho \to \rho_m = |m\rangle\langle m|$

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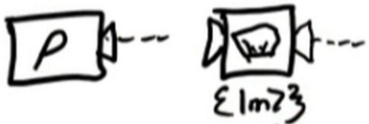

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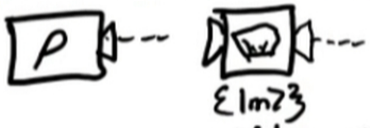

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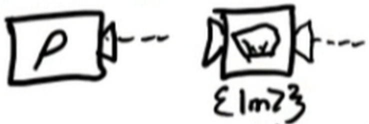

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Define measurement operators

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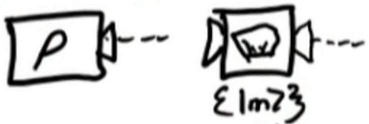

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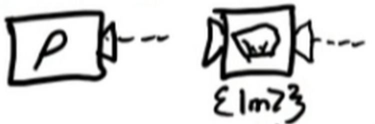

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Generalize: Given a POVM $\{E_m\}$ find a set $\{M_m\}$

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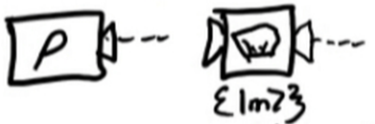

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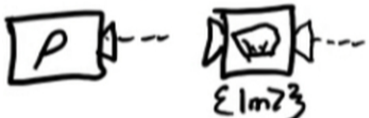

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"Forget" meas. result: $\rho \to \rho' = \sum_m p(m)\rho_m = \sum_m M_m \rho M_m^\dagger$

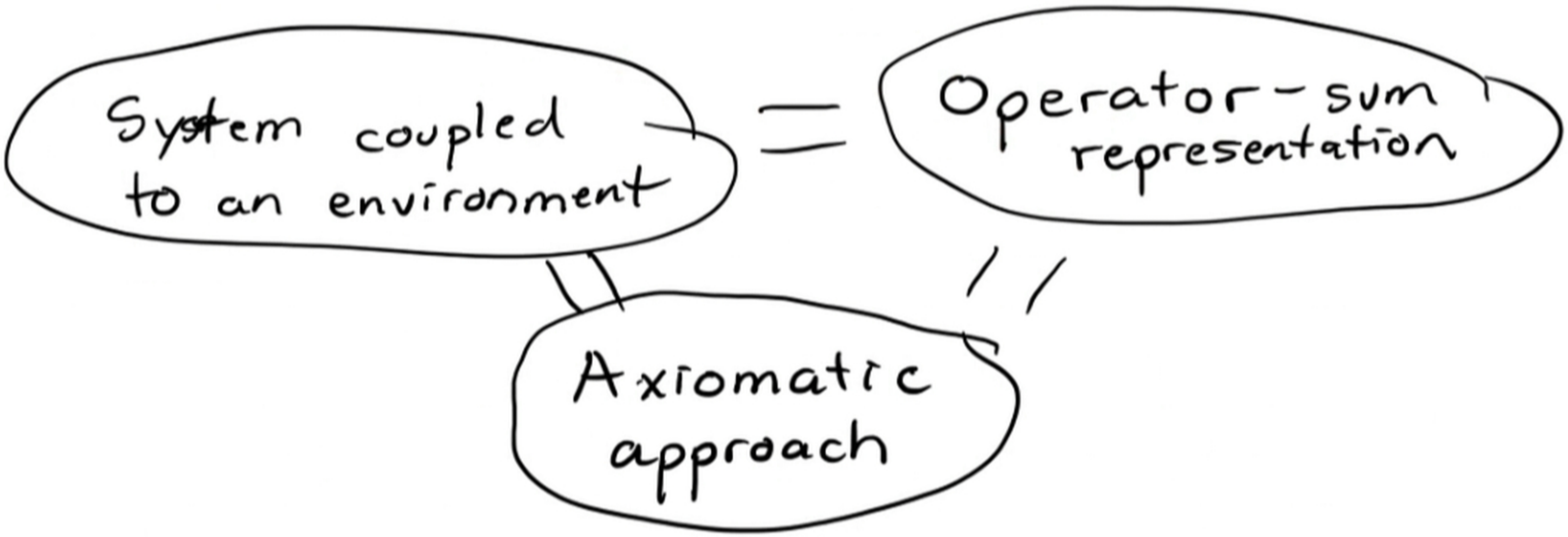
Three approaches to quantum operations

System coupled to an environment

Operator-sum representation

Axiomatic approach

Three approaches to quantum operations



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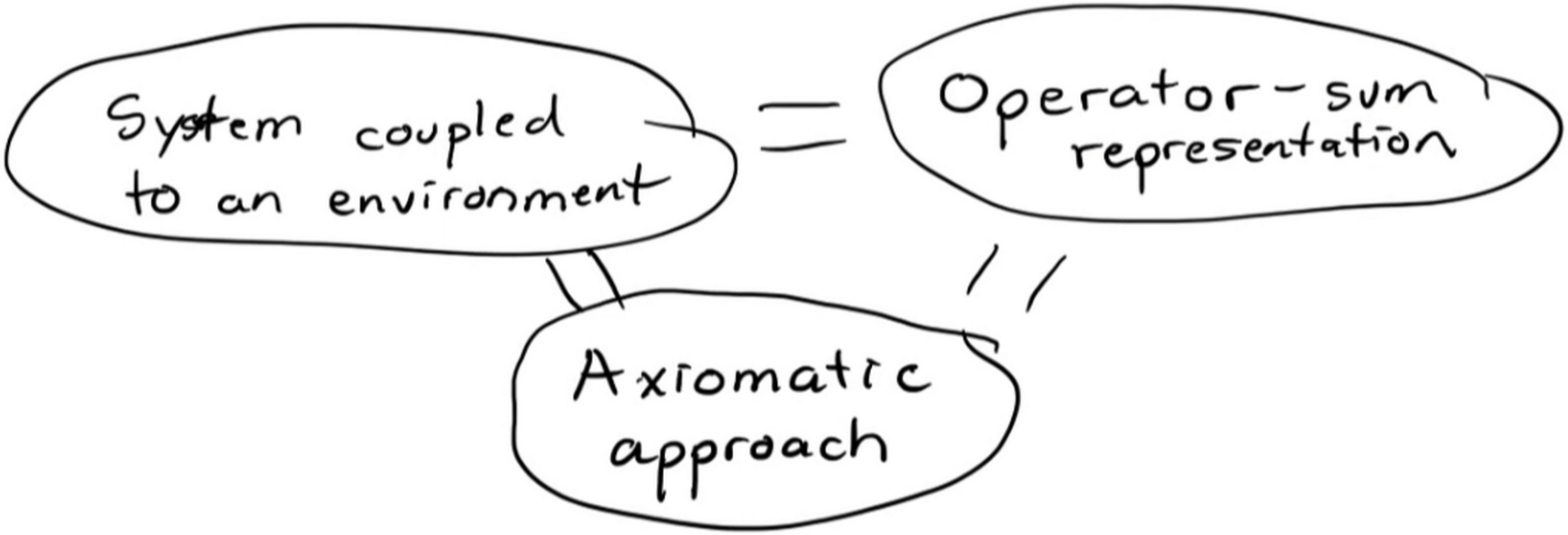

Quantum operation: $\rho' = \mathcal{E}(\rho)$

map from operators to operators satisfying some requirements

Interaction with an environment

System : initial state ρ

Environment : initial state ρ_E

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Unitary coupling $\mathcal{U}$ on sys + env

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$$\mathcal{E}(\rho) = \mathrm{Tr}_E\left[U(\rho \otimes \rho_E)U^\dagger\right]$$

Operator-sum representation

Let's derive this from the coupling with the env.

Let E be fin. dim. Choose orthonormal basis $\{|e_k\rangle_E\}$

Let $\rho_E = |e_0\rangle_E\langle e_0|$

$$\mathcal{E}(\rho) = \sum_k {}_E\langle e_k| U(\rho_s \otimes |e_0\rangle_E\langle e_0|) U^\dagger |e_k\rangle_E$$

$$= \sum_k E_k \rho E_k^\dagger$$

where $E_k = {}_E\langle e_k| U_{SE} |e_0\rangle_E$ operators on system

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What properties does $\{E_k\}$ have?

① $\mathcal{E}(\rho) \geq 0$ ✓ ② $Tr[\mathcal{E}(\rho)] = 1 = Tr(\sum_k E_k \rho E_k^+) = Tr(\sum_k E_k^+ E_k \rho)$

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$$\Rightarrow \sum_k E_k^+ E_k = I \quad \text{Trace preserving}$$

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Definition: A map $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$ is a quantum operation if $\sum_k E_k^\dagger E_k = I$

Physical interpretation of operator-sum representation

Consider coupling to an environment which is subsequently measured in the basis $|e_k\rangle$

$$\rho_k \propto \mathrm{Tr}_E\left[|e_k\rangle\langle e_k|\, U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger |e_k\rangle\langle e_k|\right] = E_k \rho E_k^\dagger$$

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Normalize: $\rho_k = \dfrac{E_k\rho E_k^\dagger}{\mathrm{Tr}[E_k\rho E_k^\dagger]}$

Prob. of outcome k occurring is

$$p(k) = \mathrm{Tr}\left[|e_k\rangle\langle e_k|\left(U(\rho\otimes|e_0\rangle\langle e_0|)U^\dagger\right)|e_k\rangle\langle e_k|\right]$$
$$= \mathrm{Tr}[E_k\rho E_k^\dagger]$$

The ensemble of outcomes is represented as a density operator

$$\mathcal{E}(\rho) = \sum_k p(k)\rho_k = \sum_k E_k\rho E_k^\dagger$$

Example: amplitude damping

Consider a 2-level system. Excited state $|1\rangle$ Ground state $|0\rangle$

If its in the excited state $|1\rangle$, there is some probability γ of transitioning to the g.s. $|0\rangle$

Qubit environment initially in $|0\rangle$ state.

$$(\alpha|0\rangle_S + \beta|1\rangle_S)|0\rangle_E \rightarrow \alpha|00\rangle_{SE} + \beta\left(\sqrt{1-\gamma}\,|10\rangle_{SE} + \sqrt{\gamma}|01\rangle_{SE}\right)$$

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$$\mathcal{E}_{AD}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

Coupling to environment from the operator-sum representation

Start with $\{E_k\}$ s.t. $\sum_k E_k^\dagger E_k = I$

Choose env. with dimension equal to the cardinality of $\{E_k\}$, pick a basis $\{|e_k\rangle\}$, inc. initial $|e_0\rangle$

Define U satisfying

$$U|\psi\rangle_S |e_0\rangle_E = \sum_k E_k |\psi\rangle_S |e_k\rangle_E \quad \forall |\psi\rangle_S$$

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Note $\forall\, |\psi\rangle_S, |\phi\rangle_S$

$$\langle\psi|\langle e_0|\mathcal{U}^\dagger \mathcal{U}|\phi\rangle|e_0\rangle = \sum_k {}_S\langle\psi|E_k^\dagger E_k|\phi\rangle = \langle\psi|\phi\rangle$$

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Then $\mathcal{U}$ can be extended to a unitary op. on S+E

Then $$\mathrm{Tr}_E\left(\mathcal{U}(\rho\otimes|e_0\rangle\langle e_0|)\mathcal{U}^\dagger\right) = \sum_k E_k\rho E_k^\dagger$$

Freedom in the operator-sum representation

Can $\{E_k\}$ and $\{F_k\}$ be such that $\mathcal{E} = \mathcal{F}$?

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In "coupling with env" picture, a final unitary on env. (or equiv. a change of basis $\{|e_k\rangle\}$) changes the op. sum rep but not the q. operation.

$$E_k = \langle e_k | U | e_0 \rangle \qquad F_k = \langle e_k | (I_S \otimes U'_E) U | e_0 \rangle$$

$$= \sum_j [I \otimes \langle e_k | U'_E | e_j \rangle] \langle e_j | U | e_0 \rangle$$

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Thm: $\mathcal{E} = \mathcal{F}$ iff $\exists$ $m \times m$ unitary U' s.t. $F_k = \sum_j U'_{kj} E_j$

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(Note: pad out smaller sets with zero operators)