Title: PSI 2016/2017 Quantum Theory - Lecture 7

Date: Sep 14, 2016 10:45 AM

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Abstract:

Harmonic Oscillator as a model for everything

Everything (just about) in the universe can be modeled by harmonic oscillators

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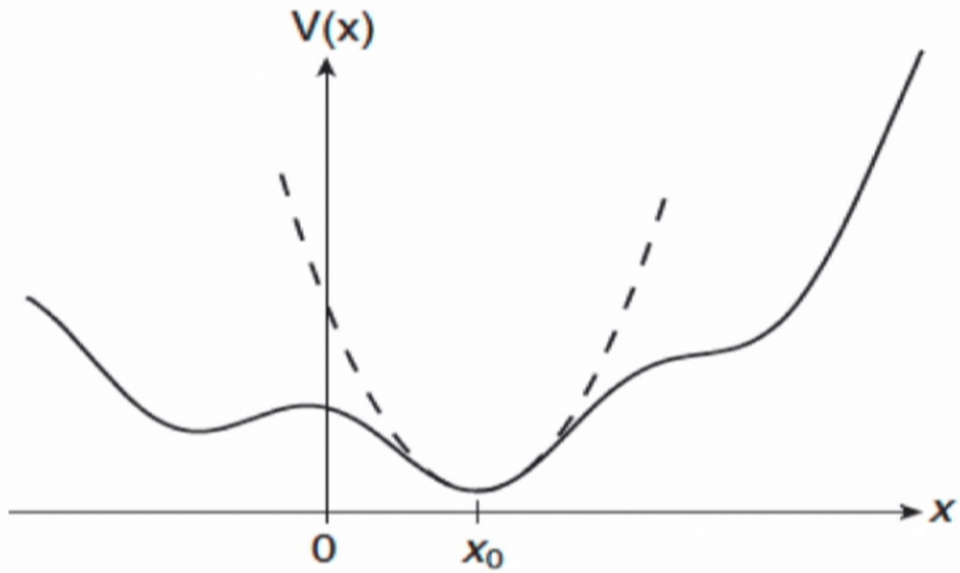

FIGURE 9.1 A general potential energy function (solid) is approximated by a quadratic harmonic potential (dashed) in the vicinity of the potential minimum.

Harmonic Oscillator - Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \qquad \omega^2 = k/m$$

Convenient operators

Define

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right)$$

a not Hermitian

$a^\dagger \neq a$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}\right.$$

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Invert:

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

$$p = -i\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right)$$

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$$[a, a^\dagger] =$$

The Hamiltonian reexpressed

$$a^{\dagger}a = \frac{m\omega}{2\hbar}\left(x - i\frac{p}{m\omega}\right)\left(x + i\frac{p}{m\omega}\right)$$

$$= \frac{m\omega}{2\hbar}\left(x^2 + \frac{p^2}{m^2\omega^2} + \frac{i}{m\omega}[x,p]\right)$$

$$= \frac{m\omega}{2\hbar}\left(x^2 + \frac{p^2}{m^2\omega^2} + \frac{i}{m\omega}(i\hbar)\right)$$

$$= \frac{1}{\hbar\omega}\underbrace{\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2\right)}_{\text{H of QHO}} - \frac{1}{2}$$

$$H = \hbar\omega(a^{\dagger}a + 1/2)$$

$$= \hbar\omega(N + 1/2)$$

Define

$N = a^{\dagger}a$

Numbr

The Hamiltonian reexpressed

Usef

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Eigenstates of N

$N|\nu_n^i\rangle = \nu_n|\nu_n^i\rangle$

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$$[N,a] = [a^\dagger,a]a$$
$$= -a$$
$$[N,a^\dagger] = a^\dagger$$

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$$[a, a^\dagger] = 1$$

Spectrum of the number operator

Lemma 1: Eigenvalues ν_n of N are nonnegative

$$\| a|\nu_n^i\rangle \|^2 = \langle \nu_n^i | a^\dagger a | \nu_n^i \rangle = \nu_n \langle \nu_n^i | \nu_n^i \rangle = \nu_n \geq 0$$

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Lemma 2:

a) If $\nu_n = 0$ then $a|\nu_n^i\rangle = 0$

Proof: from Lemma 1 $\| a|\nu_n^i\rangle \|^2 = \nu_n = 0 \Rightarrow a|\nu_n^i\rangle = 0$ for $\nu_n = 0$

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Proof:
$$\begin{aligned} N(a|\nu_n^i\rangle) &= (aN + [N,a])|\nu_n^i\rangle \\ &= a(N|\nu_n^i\rangle) - a|\nu_n^i\rangle \\ &= \nu_n a|\nu_n^i\rangle - a|\nu_n^i\rangle \\ &= (\nu_n - 1) a|\nu_n^i\rangle \end{aligned}$$

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Spectrum of the number operator

Lemma 3 (properties of $a^+|\nu_n^i\rangle$)

a) The ket $a^+|\nu_n^i\rangle$ is always nonzero

b) $a^+|\nu_n^i\rangle$ is an eigenvector of N with eigenvalue ν_n+1

Proof of b)

$$N\left(a^+|\nu_n^i\rangle\right) = \left(a^+N + [N,a^+]\right)|\nu_n^i\rangle$$
$$= \nu_n a^+|\nu_n^i\rangle + a^+|\nu_n^i\rangle$$
$$= (\nu_n+1)\, a^+|\nu_n^i\rangle$$

Spectrum of the number operator

Nonnegative ν_n implies $\exists$ ground state $|0\rangle = |\nu_n = 0\rangle$ satisfying $a|0\rangle = 0$

Proof sketch: If $a|0\rangle \neq 0$ then by Lemma 2 it would be an nonzero eigenvector with $\nu_{n-1} = \nu_n - 1 < 0 \Rightarrow$ contradiction

Ground state $|0\rangle$ satisfying $a|0\rangle = 0$ is unique.

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Generate set of unique eigenstates $\nu_n = n$ using Lemma 3 starting from g.s. $|0\rangle$

$$a^\dagger |n\rangle = C_n |n+1\rangle \qquad C_n \in \mathbb{C}$$

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Spectrum of the number operator

$$\langle n+1 | a^\dagger | n \rangle = C_n$$

$$\langle n | a | n+1 \rangle = C_n^* \equiv C_n$$ choose C_n real, nonneg. wlog.

$$\therefore \; a|n+1\rangle = C_n |n\rangle$$

Use $[a, a^\dagger] = 1 \Rightarrow |n\rangle = [a, a^\dagger]|n\rangle = (aa^\dagger - a^\dagger a)|n\rangle$

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$$C_n^2 - C_{n-1}^2 = 1$$

Using $a|0\rangle = 0$ $\quad C_0^2 = 1,\ C_1^2 = 2,\ \ldots\ C_n^2 = n$

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$N|n\rangle = n|n\rangle$$

$$H|n\rangle = \hbar\omega(n + 1/2)|n\rangle$$

Expectations values

$$\langle 0|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 0|(a+a^\dagger)|0\rangle$$
$$= 0$$

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$$\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}\langle 0|(a^2 + \underline{\underline{aa^\dagger}} + a^\dagger a + (a^\dagger)^2)|0\rangle$$

$$= \frac{\hbar}{2m\omega}$$

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\Delta p = \sqrt{\frac{\hbar m\omega}{2}}$$

$$\Rightarrow (\Delta x \cdot \Delta p)_{g.s.} = \hbar/2$$

Coherent states

Coherent states : $|\alpha\rangle_{cs} = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$

3 equiv. def.

(1) Eigenstates of a : $a|\alpha\rangle_{cs} = \alpha|\alpha\rangle_{cs}$ $\alpha \in \mathbb{C}$

(2) Group of phase space disp. $D(x,p) = e^{-\frac{i}{\hbar} x \hat{p}} e^{\frac{i}{\hbar} p \hat{x}}$

$$U(\alpha, \alpha^*) = e^{\alpha a^\dagger - \alpha^* a} = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a}$$

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satisfy $\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$

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