Title: PSI 2016/2017 Quantum Theory - Lecture 6

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Abstract:

Quantum Mechanics

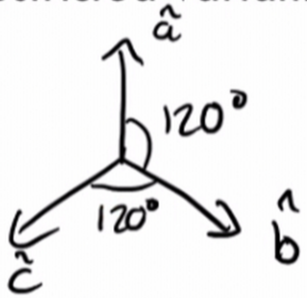
Quantum teleportation, EPR and Bell, continued…

Lecture 6

THE UNIVERSITY OF SYDNEY

Bell test

Bell proposed an experiment similar to Einstein's, but with more possibilities
We will analyse a restricted variant that conveys the main ideas



Three choices of measurement directions

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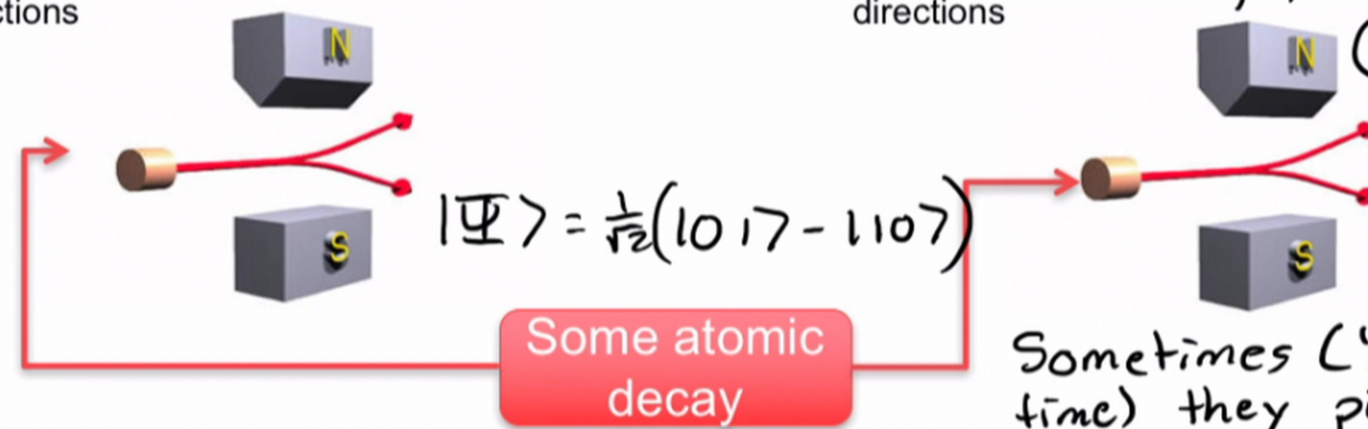
① Create entangled state
② Alice measures particle 1 in $\hat{a}, \hat{b}, \hat{c}$ at random
③ Bob measures particle 2 in $\hat{a}, \hat{b}, \hat{c}$ at random
④ Record outcomes



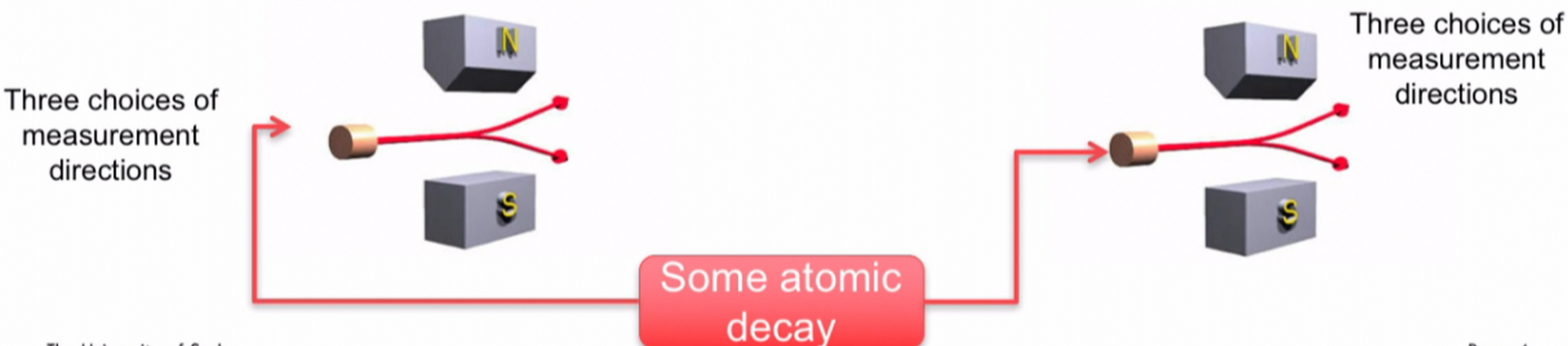
$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Sometimes (1/3 of the time) they pick the same directions.

Central idea of a Bell test

Bell tests can rule out *any* local hidden variable model

- Consider many 'runs' of the experiment, with the measurement directions at each side chosen randomly and independently
- First, show some property of the statistics that any local hidden variable model must satisfy
- Second, show that quantum mechanics predicts a violation of this property



Local hidden variable model for this experiment

Hidden variables: an 'instruction set' for each particle

– Constraint our model must satisfy: If the two measurement directions are the same (say a and a), then the outcomes must be opposite

Instruction set: list for each particle, say $(+,-,+)$

If a is measured $\to +$
b $\to -$
c $\to +$

All possible instruction sets

Particle 1	Particle 2	
$(+,+,+)$	$(-,-,-)$	$\Leftarrow$ For all possible meas. $P_{opp}=1$
$(+,+,-)$	$(-,-,+)$	$P_{opp}=5/9$
⋮	⋮	
$(-,-,+)$	$(+,+,-)$	
$(-,-,-)$	$(+,+,+)$	$\Leftarrow$ $P_{opp}=1$

Any theory that selects these instruction sets from any rule or distribution will never predict $P_{opp}<5/9$

Quantum predictions for this experiment

$|\psi$

Quantum predictions for this experiment

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

$+1 \quad -1$

Measure particle 1 in $|0\rangle, |1\rangle$

" " 2 in $|+_n\rangle = \cos\theta/2\,|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle$

$$|-_n\rangle = \sin\theta/2\,|0\rangle - e^{i\phi}\cos\theta/2\,|1\rangle$$

$$P_{+,+} = \left|\,{}_1\langle 0|\,{}_2\langle +_n|\,|\psi^-\rangle_{12}\right|^2$$

$$= \frac{1}{2}\sin^2\theta/2$$

$$P_{-,-} = \frac{1}{2}\sin^2\theta/2$$

$$P_{same} = \sin^2\theta/2$$

$$P_{opp} = \cos^2\theta/2$$

Quantum predictions for this experiment

$P_{opp} = \cos^2 \theta/2$ particle 1 along $\hat{z}$
" 2 " $\hat{n}$

If the directions are the same (e.g. $\hat{a}, \hat{a}$)
then $\theta = 0 \Rightarrow P_{opp} = 1$

If the directions are different (e.g. $\hat{a}, \hat{b}$)
then $\theta = 120° \Rightarrow P_{opp} = \cos^2 60° = 1/4$

Many experiments
Same directions 1/3 of the time
Diff " 2/3 " " "

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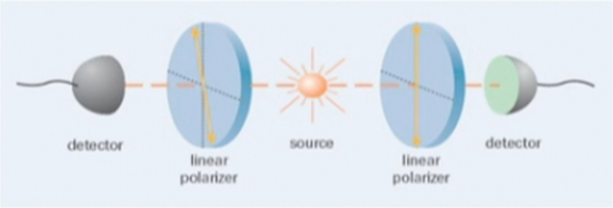
$$P_{opp} = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot\frac{1}{4} = \frac{1}{2} < 5/9$$

Quantum nonlocality

So what's missing? Was Einstein right?

- Bell proved that any realistic model that is local cannot reproduce the statistics of quantum mechanics



- Experiments now routinely prove this is true
- Any ontology of quantum mechanics must be nonlocal, but in a cleverly-designed way that does not allow us to signal faster than light.
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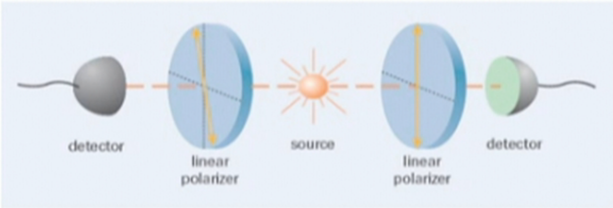

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Hilbert spaces

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inf. dim: some subtleties

Restriction to separable Hilbert spaces: countable orthonormal basis

$\{|n\rangle;\ n=1,2,3,\dots\}$ orthonormal basis for $\mathbb{H}$

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Define complex vector space $V \subset \mathbb{H}$ of all finite linear comb.

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subject to $\sum_{n=0}^{\infty} |\psi_n|^2 < \infty$ e.g.

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Dense subspaces

Subspace $\mathbb{U} \subset \mathbb{H}$ is <u>dense</u> if its completion is $\mathbb{H}$

Aside: Hilbert space $\mathcal{L}^2(\mathbb{R})$, sq. int. functions

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Smooth (differentiable) functions form a dense subspace $V \subset \mathcal{L}^2(\mathbb{R})$

Beyond the Hilbert space

Useful functions: $\psi_{x'}(x) = \delta(x - x')$

$\psi_{p'}(x) = \frac{1}{\sqrt{2\pi\hbar}}$

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These functions (actually dist^{n}s) are not in $\mathcal{L}^2(\mathbb{R})$

What are they?

Reminder: for Hilbert space $\mathbb{H}$, with vectors $|\psi\rangle$

dual space of linear functionals $\langle\phi| : \mathbb{H} \to \mathbb{C}$

$\langle\phi| \in \mathbb{H}$

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Rigged Hilbert space

However, the dual V^* of a subspace $V \subset \mathbb{H}$ is larger than $\mathbb{H}^*$

e.g. $\langle x'|$ defined as $\langle x'|\psi\rangle = \int_{\mathbb{R}} \delta(x-x')\psi(x)dx = \psi(x')$

is a well-defined functional (in V^*) on the (dense) subspace of smooth functions $V \subset \mathcal{L}^2(\mathbb{R})$.

Similarly

$\langle p'|$ defined as $\langle p'|\psi\rangle = \int_{\mathbb{R}} e^{\frac{i}{\hbar}p'x}\psi(x)dx$

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Rigged Hilbert space: $(V, \mathbb{H}, V^*)$

dense subspace of $\mathbb{H}$

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Similarly, $\exists$ dense subspace $V' \subset \mathcal{L}^2(\mathbb{R})$

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Rigged Hilbert space: $(V, \mathbb{H}, V^*)$

dense subspace of $\mathbb{H}$

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Unbounded operators: position and momentum

Rigged Hilbert space is useful for understanding eigenstates of some unbounded operators.

A on $\mathbb{H}$ is bounded if $\exists N_A$ s.t. $\| A|\psi\rangle \| < N_A \| |\psi\rangle \|$ $\forall |\psi\rangle$ in domain of A

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Operators $\hat{x}$, $\hat{p}$, $\hat{H} = \frac{p^2}{2m} + V(x)$ are unbounded.

Domain may only be a dense subspace of $\mathcal{L}^2(\mathbb{R})$

$\psi(x) \in \mathcal{L}^2(\mathbb{R})$ but $x\psi(x)$ may not be.

$p = i\hbar \frac{d}{dx}$ is not defined unless $\psi(x)$ is diff.

*

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then the map $\mathbb{H} \to \mathbb{C}$ defined by

$$|\psi\rangle \to \langle\phi|A|\psi\rangle$$

is a bounded linear functional on $\mathbb{H}$.

$\therefore \exists \langle\eta|$ s.t. $\langle\eta|\psi\rangle = \langle\phi|A|\psi\rangle$

Define the adjoint of A as the operator

$$A^{\dagger} : |\phi\rangle \to |\eta\rangle$$

i.e. $\langle\psi$

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$$A^{+} : |\phi\rangle \rightarrow |\eta\rangle$$

i.e. $\langle\psi|A^{+}|\phi\rangle = \langle\phi|A|\psi\rangle^{*}$

Then A^{+} is bounded, and $\|A^{+}\| = \|A\|$

Bounded operator is Hermitian

(equivalel

Unbounded operators

A a densely-defined linear operator on $\mathbb{H}$ with domain $D(A)$

Consider the set $D(B)$ of all $|\varphi\rangle \in \mathbb{H}$ s.t.

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is a bounded linear functional on $D(A)$.

For $|\varphi\rangle \in D(B)$ $\quad \langle\psi|B|\varphi\rangle = \langle\varphi|A|\psi\rangle^* \quad \forall |\psi\rangle \in D(A)$

The operator B is the adjoint of A, $B = A^\dagger$

A is Hermitian if $D(A) \subseteq D(A^\dagger)$ and

$$A|\varphi\rangle = A^\dagger|\varphi\rangle \quad \forall |\varphi\rangle \in D(A)$$

and self-adjoint if $D(A) = D(A^\dagger)$ so $A = A^\dagger$

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