Title: PSI 2016/2017 Quantum Theory - Lecture 6

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Abstract:

Quantum Mechanics

Quantum teleportation,

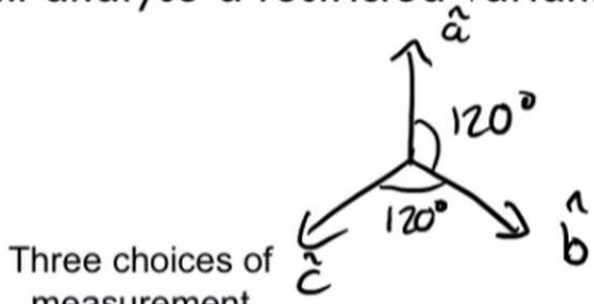
EPR and Bell, continued…

Lecture 6

THE UNIVERSITY OF SYDNEY

Bell test

Bell proposed an experiment similar to Einstein's, but with more possibilities
We will analyse a restricted variant that conveys the main ideas



Three choices of measurement directions

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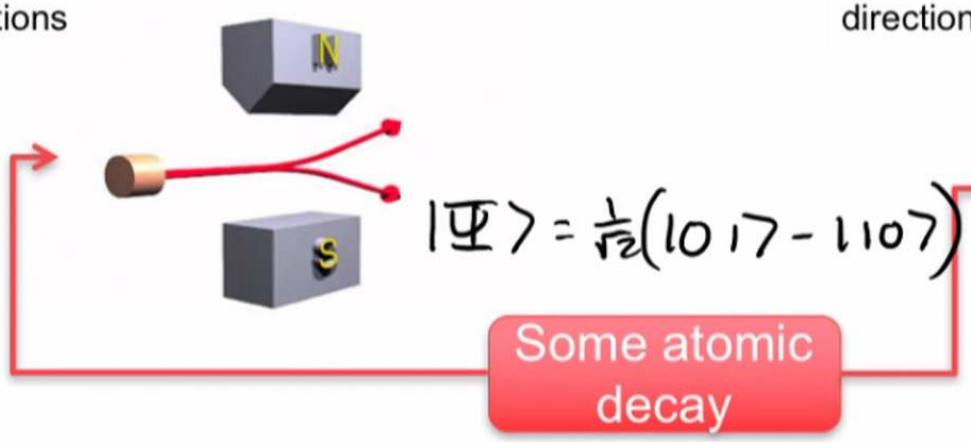


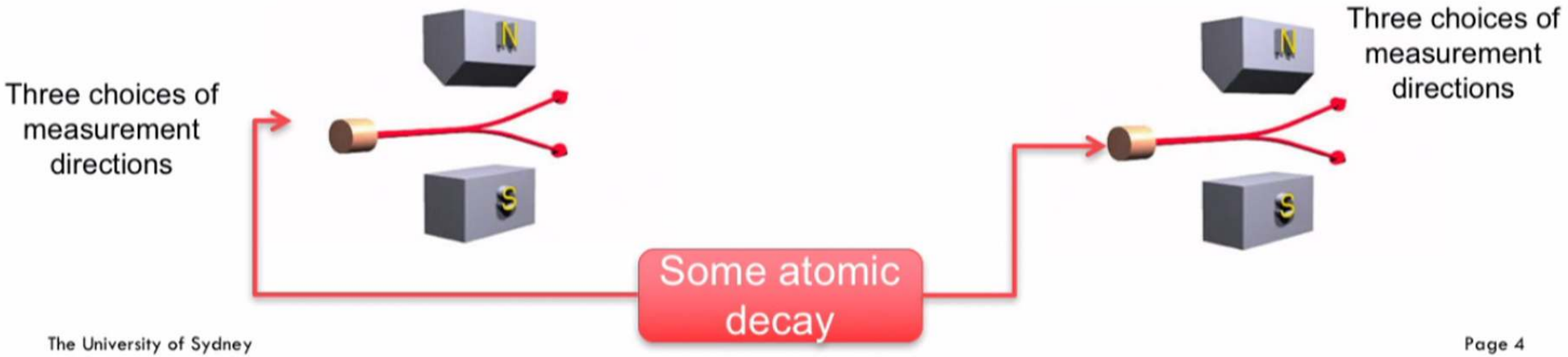
① Create entangled state
② Alice measures particle 1 in â, b̂, ĉ at random
③ Bob measures particle 2 in â, b̂, ĉ at random
④ Record outcomes

Sometimes (1/3 of the time) they pick the same directions.

Central idea of a Bell test

Bell tests can rule out *any* local hidden variable model

- Consider many 'runs' of the experiment, with the measurement directions at each side chosen randomly and independently
- First, show some property of the statistics that any local hidden variable model must satisfy
- Second, show that quantum mechanics predicts a violation of this property



Local hidden variable model for this experiment

Hidden variables: an 'instruction set' for each particle

– Constraint our model must satisfy: If the two measurement directions are the same (say a and a), then the outcomes must be opposite

Instruction set: list for each particle, say (+, −, +)
if a is measured → +
b → −
c → +

All possible instruction sets

Particle 1	Particle 2	
(+, +, +)	(−, −, −)	← For all possible meas. $P_{opp} = 1$
(+, +, −)	(−, −, +)	
⋮	⋮	$P_{opp} = 5/9$
(−, −, +)	(+, +, −)	
(−, −, −)	(+, +, +)	← $P_{opp} = 1$

Any theory that selects these instruction sets from any rule or distribution will never predict $P_{opp} < 5/9$

Quantum predictions for this experiment

$|\Psi\rangle$

Quantum predictions for this experiment

$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ $\quad +1 \quad -1$

Measure particle 1 in $|0\rangle, |1\rangle$

" " 2 in $|+_n\rangle = \cos\theta/2\,|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle$

$|-_n\rangle = \sin\theta/2\,|0\rangle - e^{i\phi}\cos\theta/2\,|1\rangle$

$P_{+,+} = |\langle 01|\,\langle +_n|\,|\psi^-\rangle_{12}|^2$

$= \frac{1}{2}\sin^2\theta/2$

$P_{-,-} = \frac{1}{2}\sin^2\theta/2$

$P_{same} = \sin^2\theta/2$

$P_{opp} = \cos^2\theta/2$

Quantum predictions for this experiment

$P_{opp} = \cos^2 \theta/2$ particle 1 along $\hat{z}$
" 2 " $\hat{n}$

If the directions are the same (e.g. $\hat{a}, \hat{a}$)
then $\theta = 0 \Rightarrow P_{opp} = 1$

If the directions are different (e.g. $\hat{a}, \hat{b}$)
then $\theta = 120° \Rightarrow P_{opp} = \cos^2 60° = 1/4$

Many experiments
Same directions 1/3 of the time
Diff " 2/3 " " "

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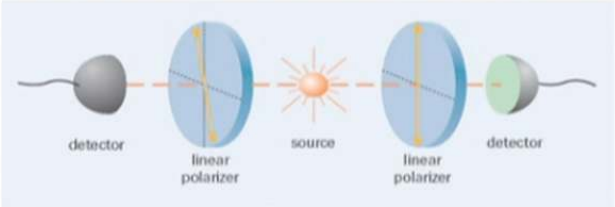
$$P_{opp} = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{2} < 5/9$$

Quantum nonlocality

So what's missing? Was Einstein right?

- Bell proved that any realistic model that is local cannot reproduce the statistics of quantum mechanics



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- Any ontology of quantum mechanics must be nonlocal, but in a cleverly-designed way that does not allow us to signal faster than light.
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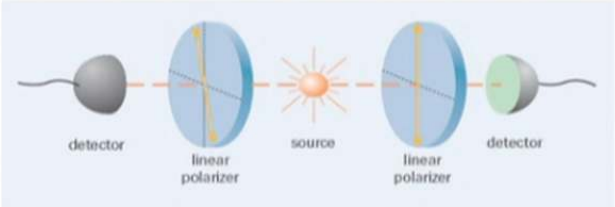

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Finite dim: Hilbert space = complex vector space with

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Inf. dim: some subtleties

Restriction to separable Hilbert spaces: countable orthonormal basis

$\{|n\rangle;\ n=1,2,3,\ldots\}$ orthonormal basis for $\mathcal{H}$

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Dense subspaces

Subspace $U \subset H$ is <u>dense</u> if its completion is H

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Consider $f(x) = \begin{cases} f_k & \text{if } x = x_k \quad k = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$ Nonzero values only on finite set of points

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Smooth (differentiable) functions form a dense subspace $V \subset L^2(\mathbb{R})$

Beyond the Hilbert space

Useful functions:

$$\psi_{x'}(x) = \delta(x - x')$$

$$\psi_{p'}(x) = \frac{1}{\sqrt{2\pi\hbar}}$$

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These functions (actually distⁿs) are not in $L^2(\mathbb{R})$

What are they?

Reminder: for Hilbert space $\mathcal{H}$, with vectors $|\phi\rangle$

dual space of linear functionals $\langle\phi| : \mathcal{H} \to \mathbb{C}$

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Rigged Hilbert space

However, the dual V^* of a subspace $V \subset \mathcal{H}$ is larger than $\mathcal{H}^*$

e.g. $\langle x'|$ defined as $\langle x'|\psi\rangle = \int_{\mathbb{R}} \delta(x-x')\,\psi(x)\,dx = \psi(x')$ is a well-defined functional (in V^*) on the (dense) subspace of smooth functions $V \subset \mathcal{L}^2(\mathbb{R})$.

Similarly

$\langle p'|$ defined as $\langle p'|\psi\rangle = \int_{\mathbb{R}} e^{\frac{i}{\hbar}p'x}\,\psi(x)\,dx$ is in V^*.

Rigged Hilbert space: $(V, \mathcal{H}, V^*)$

V: dense subspace of $\mathcal{H}$

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Unbounded operators: position and momentum

Rigged Hilbert space is useful for understanding eigenstates of some unbounded operators.

A on $\mathcal{H}$ is bounded if $\exists N_A$ s.t. $\|A|\psi\rangle\| < N_A \||\psi\rangle\|$ $\forall |\psi\rangle$ in domain of A

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Operators $\hat{x}$, $\hat{p}$, $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$ are unbounded.

Domain may only be a dense subspace of $\mathcal{L}^2(\mathbb{R})$

$\psi(x) \in \mathcal{L}^2(\mathbb{R})$ but $x\psi(x)$ may not be.

$p = i\hbar \frac{d}{dx}$ is not defined unless $\psi(x)$ is diff.

*

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If A is a bounded operator on $\mathcal{H}$, and $|\psi\rangle \in \mathcal{H}$ then the

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$$|\psi\rangle \to \langle\phi|A|\psi\rangle$$

is a bounded linear functional on $\mathcal{H}$.

$\therefore \exists \langle\eta|$ s.t. $\langle\eta|\psi\rangle = \langle\phi|A|\psi\rangle$

Define the adjoint of A as the operator

$$A^\dagger : |\phi\rangle \to |\eta\rangle$$

i.e. $\langle\psi$

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$$A^+ : |\phi\rangle \to |\eta\rangle$$

i.e. $\langle\chi|A^+|\phi\rangle = \langle\phi|A|\chi\rangle^*$

Then A^+ is bounded, and $\|A^+\| = \|A\|$

Bounded operator is Hermitian (equivalent)

Unbounded operators

A a densely-defined linear operator on $\mathcal{H}$ with domain $D(A)$

Consider the set $D(B)$ of all $|\varphi\rangle \in \mathcal{H}$ s.t.

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is a bounded linear functional on $D(A)$.

For $|\varphi\rangle \in D(B)$ $\langle\chi|B|\varphi\rangle = \langle\varphi|A|\chi\rangle^*$ $\forall |\chi\rangle \in D(A)$

The operator B is the adjoint of A, $B = A^\dagger$

A is Hermitian if $D(A) \subseteq D(A^\dagger)$ and

$$A|\varphi\rangle = A^\dagger|\varphi\rangle \quad \forall |\varphi\rangle \in D(A)$$

and self-adjoint if $D(A) = D(A^\dagger)$ so $A = A^\dagger$

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