Title: PSI 2016/2017 Quantum Theory - Lecture 4

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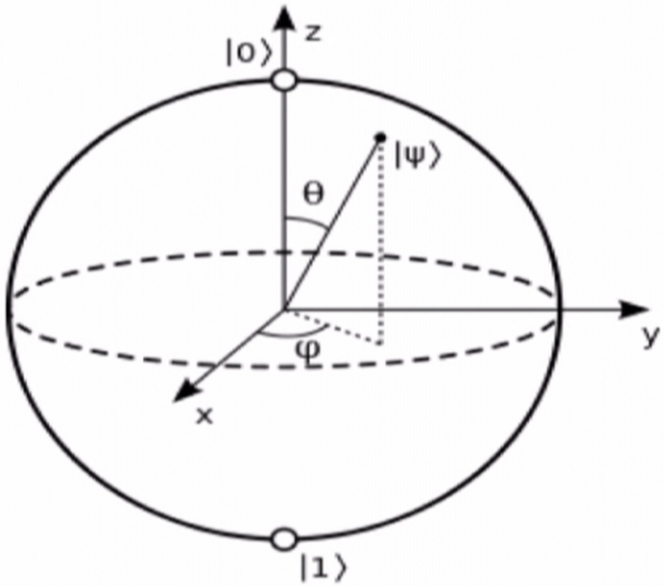
Abstract:

The Bloch sphere for mixed states of a qubit



Note $\{\overset{\sigma_0}{I}, \overset{\sigma_1}{X}, \overset{\sigma_2}{Y}, \overset{\sigma_3}{Z}\}$ form an orthonormal basis of operators

$$\mathrm{Tr}[\sigma_i \sigma_j] = 2\delta_{ij}$$

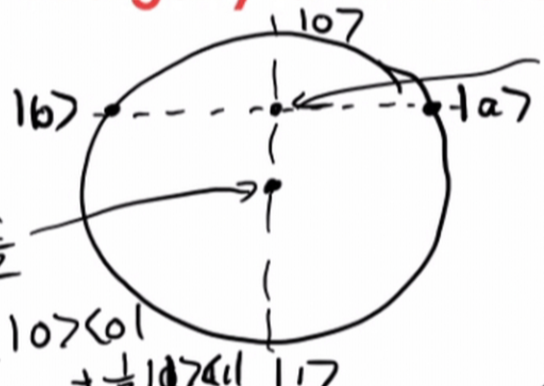
$\vec{\sigma} = (X, Y, Z)$

$$\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) \qquad |\vec{r}| \leq 1$$

So $\mathrm{Tr}[\rho] = 1$

$$\mathrm{Tr}[\rho \vec{\sigma}] = \vec{r}$$

Ambiguity of mixtures



$$\rho = \frac{I}{2} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$$

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$

Define $|a\rangle = \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle$

$$|b\rangle = \sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle$$

$$\rho' = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b| = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| = \rho$$

* Ensembles with the same ρ are indistinguishable by any measurement

Many such equivalent decompositions

No priviledged ensemble

The reduced density operator

Two quantum systems $A + B$, joint system AB

Density operator ρ_{AB} of joint system. How do I describe just A?

$$\rho_A = \mathrm{Tr}_B[\rho_{AB}] \qquad \mathrm{Tr}_B \text{ partial trace}$$

Defined by

$$\mathrm{Tr}_B[\,|a_1\rangle_A\langle a_2| \otimes |b_1\rangle_B\langle b_2|\,] = |a_1\rangle_A\langle a_2|\,\mathrm{Tr}(|b_1\rangle_B\langle b_2|) = |a_1\rangle_A\langle a_2|\,(\langle b_2|b_1\rangle)$$

and extend by linearity to full operator space

$$\text{So } \mathrm{Tr}_B[\rho_A \otimes \sigma_B] = \rho_A(\mathrm{Tr}(\sigma_B)) = \rho_A$$

Why the partial trace?

For any POVM $\{E_m^A\}$ on just system A

$$p_m = \mathrm{Tr}[(E_m^A \otimes I_B)\rho_{AB}]$$

$$= \mathrm{Tr}(E_m^A \rho_A) \quad \text{with } \rho_A = \mathrm{Tr}_B(\rho_{AB})$$

Reduced density operators of Bell states

$|\Phi_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

What about $\rho_{AB} = |\Phi_{00}\rangle\langle\Phi_{00}|$

$$\rho_A = \mathrm{Tr}_B\left(\tfrac{1}{2}|00\rangle\langle 00| + \tfrac{1}{2}|11\rangle\langle 00| + \tfrac{1}{2}|00\rangle\langle 11| + \tfrac{1}{2}|11\rangle\langle 11|\right)$$

$$= \tfrac{1}{2}|0\rangle_A\langle 0|\,\langle 0|0\rangle + \tfrac{1}{2}|1\rangle_A\langle 0|\,\cancel{\langle 0|1\rangle} + \tfrac{1}{2}|0\rangle_A\langle 1|\,\cancel{\langle 1|0\rangle} + \tfrac{1}{2}|1\rangle\langle 1|\,\langle 1|1\rangle \Big)$$

$$= \tfrac{1}{2}\left(|0\rangle_A\langle 0| + |1\rangle_A\langle 1|\right) = \tfrac{1}{2}I_A \quad \text{completely mixed state}$$

Bob measure his system B in Z basis, gets outcome $|0\rangle$ (+1)

If Alice measures her system A in Z basis, will get $|0\rangle$ (+1) with certainty

Which description should Alice use?

$$\rho_A = \tfrac{1}{2}I_A \quad \text{or} \quad \rho_A = |0\rangle_A\langle 0|$$

Depends on her knowledge of Bob's meas.

If Bob measured in X basis, gets $|+\rangle_B$ (+1)

$$\rho_A = \tfrac{1}{2}I_A \quad \text{or} \quad \rho_A = |-\rangle_A\langle -|$$

The Schmidt decomposition

Suppose $|\psi\rangle_{AB}$ is a pure state of a composite system AB

Then there exist orthonormal states $|i\rangle_A$ for A, $|i\rangle_B$ for B, s.t.

$$|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\, |i\rangle_A |i\rangle_B$$

Schmidt coefficients

where $\lambda_i \geq 0$ real noneg. numbers satisfying $\sum_i \lambda_i = 1$.

Proof: Pick any bases $|j\rangle_A$; $|k\rangle_B$

$$|\psi\rangle_{AB} = \sum_{j,k} a_{j,k} |j\rangle_A |k\rangle_B \qquad \text{SVD on } A = UDV$$

diagonal, nonneg.

unitaries

Define $\underbrace{|i\rangle_A = \sum_j u_{ji} |j\rangle_A \qquad |i\rangle_B = \sum_k v_{ik} |k\rangle_B}_{\text{orthonormal due to unitarity of } U, V}$ and $\sqrt{\lambda_i} = d_i$

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_i \lambda_i |i\rangle_A\langle i| \qquad \rho_B = \mathrm{Tr}_A(\rho_{AB}) = \sum_i \lambda_i |i\rangle_B\langle i|$$

Purifications

Given ρ_A, find orthogonal decomp. $\rho_A = \sum_i p_i |i\rangle_A\langle i|$

Introduce new system R, choose any $r \geq \text{rank}(\rho_A)$ orthogonal states $|i\rangle_R$

$$|\Psi\rangle_{AR} = \sum_i \sqrt{p_i}\,|i\rangle_A |i\rangle_R \quad \text{satisfies}$$

$$\text{Tr}_R[|\Psi\rangle_{AR}\langle\Psi|] = \rho_A$$

Different choices of basis on R $\Leftrightarrow$ unitary U_R

$$|k\rangle_R = \sum_i u_{ki} |i\rangle_R$$

$|\Psi\rangle_{AR}$ and $|\Psi'\rangle_{AR} = (I_A \otimes U_R)|\Psi\rangle_{AR}$

different purifications both satisfying

$$\rho_A = \text{Tr}_R[|\Psi\rangle_{AR}\langle\Psi|] = \text{Tr}_R[|\Psi'\rangle_{AR}$$