Title: PSI 2016/2017 Relativity - Lecture 14

Date: Sep 23, 2016 09:00 AM

URL: http://pirsa.org/16090014

Abstract:

$$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \quad , \quad r_s = \frac{2GM}{c^2}$$

Let's drop a massive ptcle into the black hole, from r_s, see ho

to hit the singularity at $r=0$. $\left(R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda} = 12\frac{r_s^2}{r^6} \right)$

$\varphi'(x) = \varphi(x)$ For $r<r_s$, r is the timelike coordinate, t is a spac

$d\tau = dt\left(1-\left(\frac{dx}{dt}\right)^2\right)^{\frac{1}{2}}$ $\quad d\tau = dr\sqrt{\frac{1}{\left(\frac{r_s}{r}-1\right)} - \left(\frac{dt}{dr}\right)^2\left(\frac{r_s}{r}-1\right)}$: longest time is obtaine

radial $\rightarrow \dot{\theta}=\dot{\varphi}=0$ to maximise, set $\frac{dt}{dr}=0$

$t(r)$ $\quad \tau = \int_0^{r_s} dr\sqrt{\frac{r}{r_s-r}} = \frac{\pi}{2}r_s$

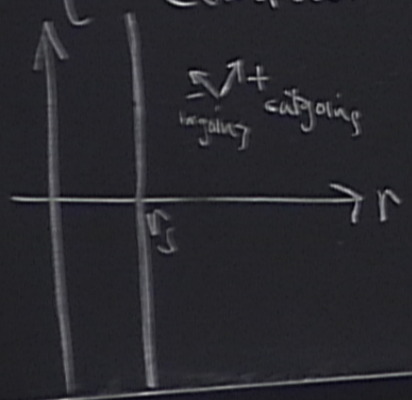
$\int_0 \frac{dx}{x^{\frac{1}{2}}} = \left[x^{\frac{1}{2}}\right]$

Kruskal coordinates for Schwarzschild

Goal: find coordinates in which metric is not singular at $r = r_s$.

Necessary that c.t. is singular at $r = r_s$!

Consider radial light rays. $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow dt = \pm \dfrac{dr}{\left(1 - \frac{r_s}{r}\right)}$

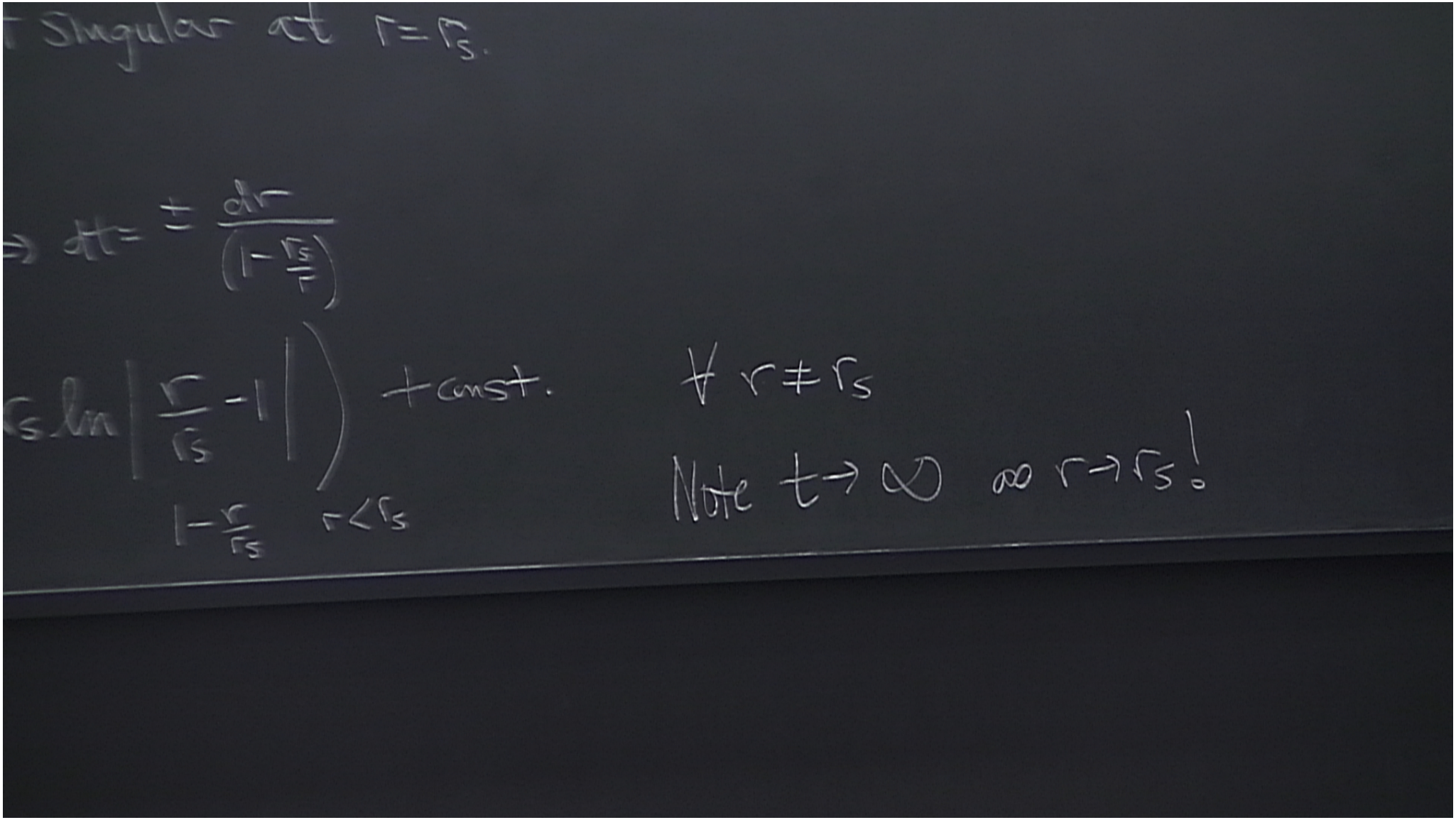


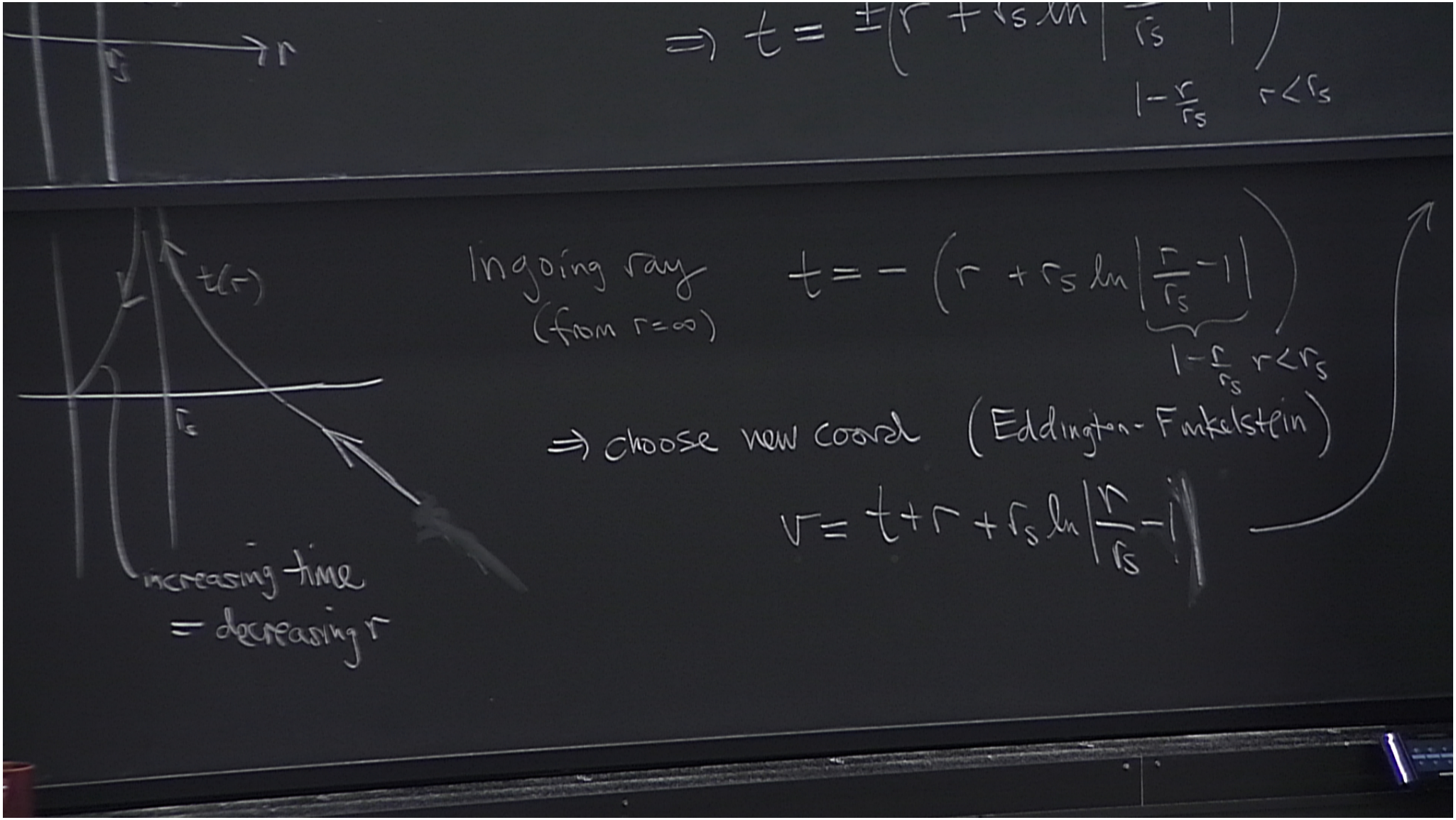
s in which metric is not singular at $r = r_s$.

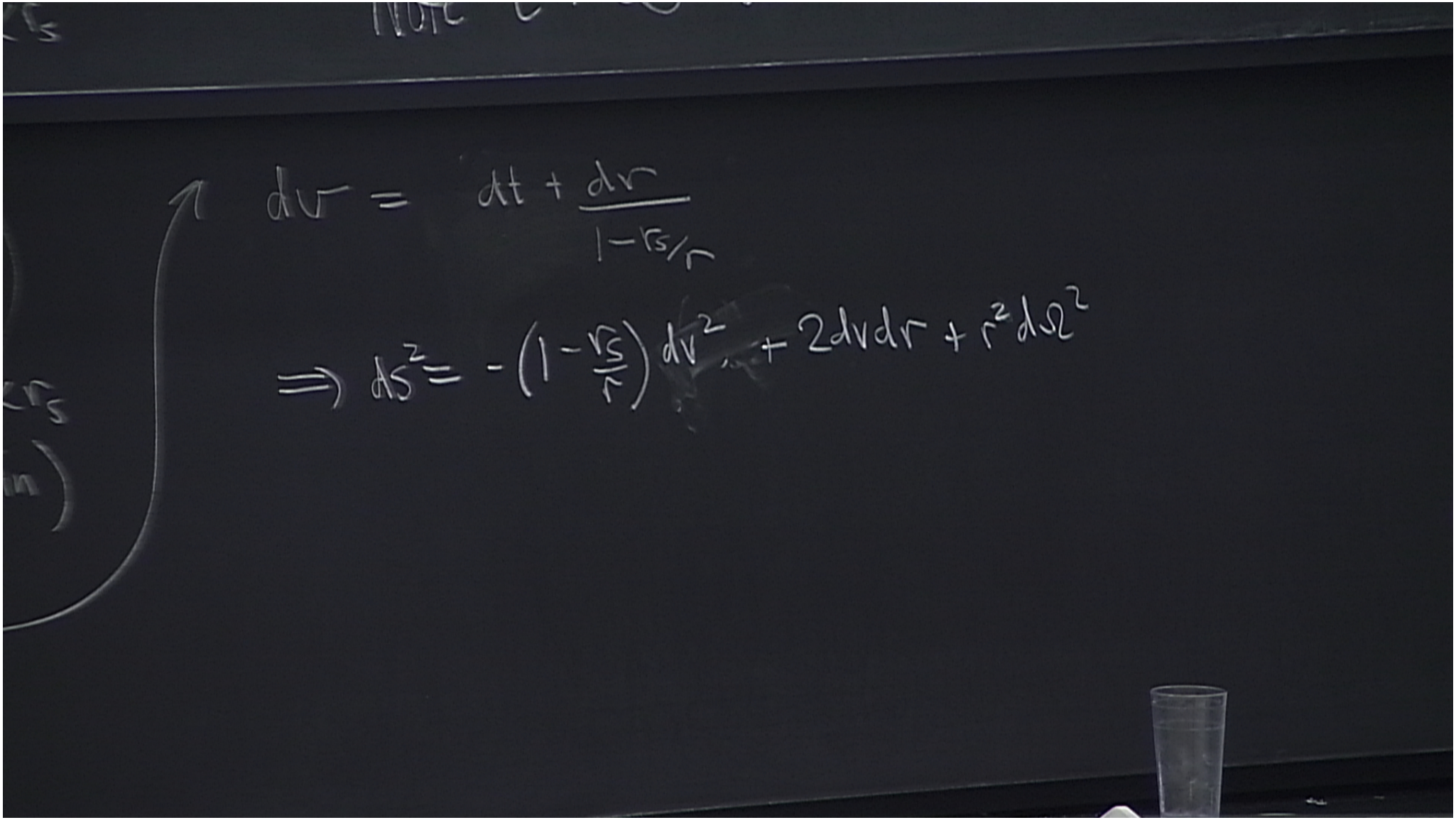
singular at $r = r_s$!

rays. $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 \Rightarrow dt = \pm \dfrac{dr}{\left(1 - \frac{r_s}{r}\right)}$

$$\Rightarrow t = \pm\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right) + \text{const.} \qquad \forall\, r \neq r_s$$


Singular at $r = r_s$.
$\Rightarrow dt = \pm \frac{dr}{\left(1 - \frac{r_s}{r}\right)}$
$r_s \ln\left|\frac{r}{r_s} - 1\right|\Big) + \text{const.}$
$\forall\, r \neq r_s$
Note $t \to \infty$ as $r \to r_s$!
$1 - \frac{r}{r_s}$ $r < r_s$


$\Rightarrow t = \pm\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right)$
$1 - \frac{r}{r_s}$ $r < r_s$
$t(r)$
r_s
Ingoing ray (from $r=\infty$)
$t = -\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right)$
$1 - \frac{r}{r_s}$ $r < r_s$
$\Rightarrow$ choose new coord (Eddington-Finkelstein)
$v = t + r + r_s \ln\left|\frac{r}{r_s} - 1\right|$
increasing time = decreasing r


$dv = dt + \frac{dr}{1 - r_s/r}$
$\Rightarrow ds^2 = -\left(1 - \frac{r_s}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2$

$= dt + \frac{dr}{1 - r_s/r}$

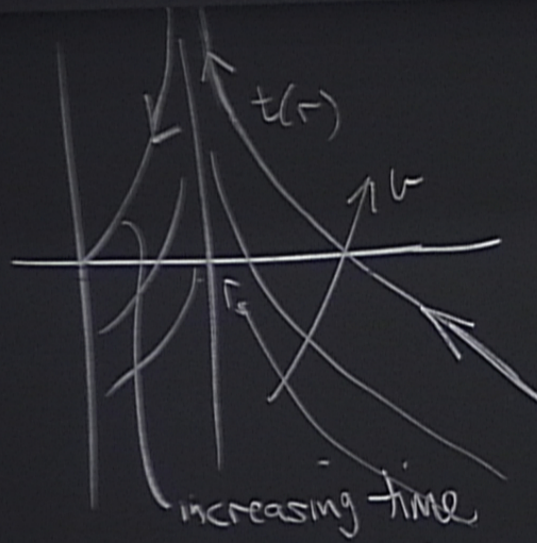
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$)

$ds^2 = -\left(1 - \frac{r_s}{r}\right) dv^2 + 2 dv dr + r^2 d\Omega^2$; <u>invertible</u> at $r = r_s$

Metric is nonsingular at $r = r_s$.

$\Rightarrow t = \pm\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right)$

$1 - \frac{r}{r_s} \quad r < r_s$



Ingoing ray (from $r=\infty$) $\quad t = -\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right)$

$1 - \frac{r}{r_s} \quad r < r_s$

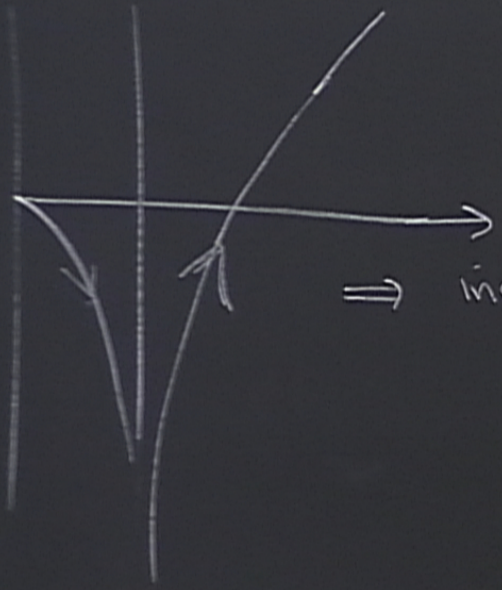
$\Rightarrow$ choose new coord (Eddington-Finkelstein)

$t \longrightarrow v = t + r + r_s \ln\left|\frac{r}{r_s} - 1\right|$

(ingoing light rays satisfy $dv = 0$)

BUT: Could have chosen outgoing coords

$$u = t - r - r_s \ln\left|\frac{r}{r_s} - 1\right|$$



$\Rightarrow$ inc time outside b.hole corresponds to increasing r inside b.hole!

$$\Rightarrow \quad ds^2 = -\left(1 - \frac{r_s}{r}\right) du^2 - 2\,du\,dr + r^2 d\Omega^2$$

sen outgoing coords

$$u = t - r - r_s \ln\left|\frac{r}{r_s} - 1\right|$$

b hole corresponds to increasing r inside b hole! (Disagrees w/ ingoing coords)

$$\Rightarrow ds^2 = -\left(1 - \frac{r_s}{r}\right)du^2 - 2du\,dr + r^2 d\Omega^2$$

again nonsingular at $r = r_s$.

$$\Rightarrow \quad ds^2 = -\left(1-\frac{r_s}{r}\right)du^2 - 2du\,dr + r^2 d\Omega^2$$ again

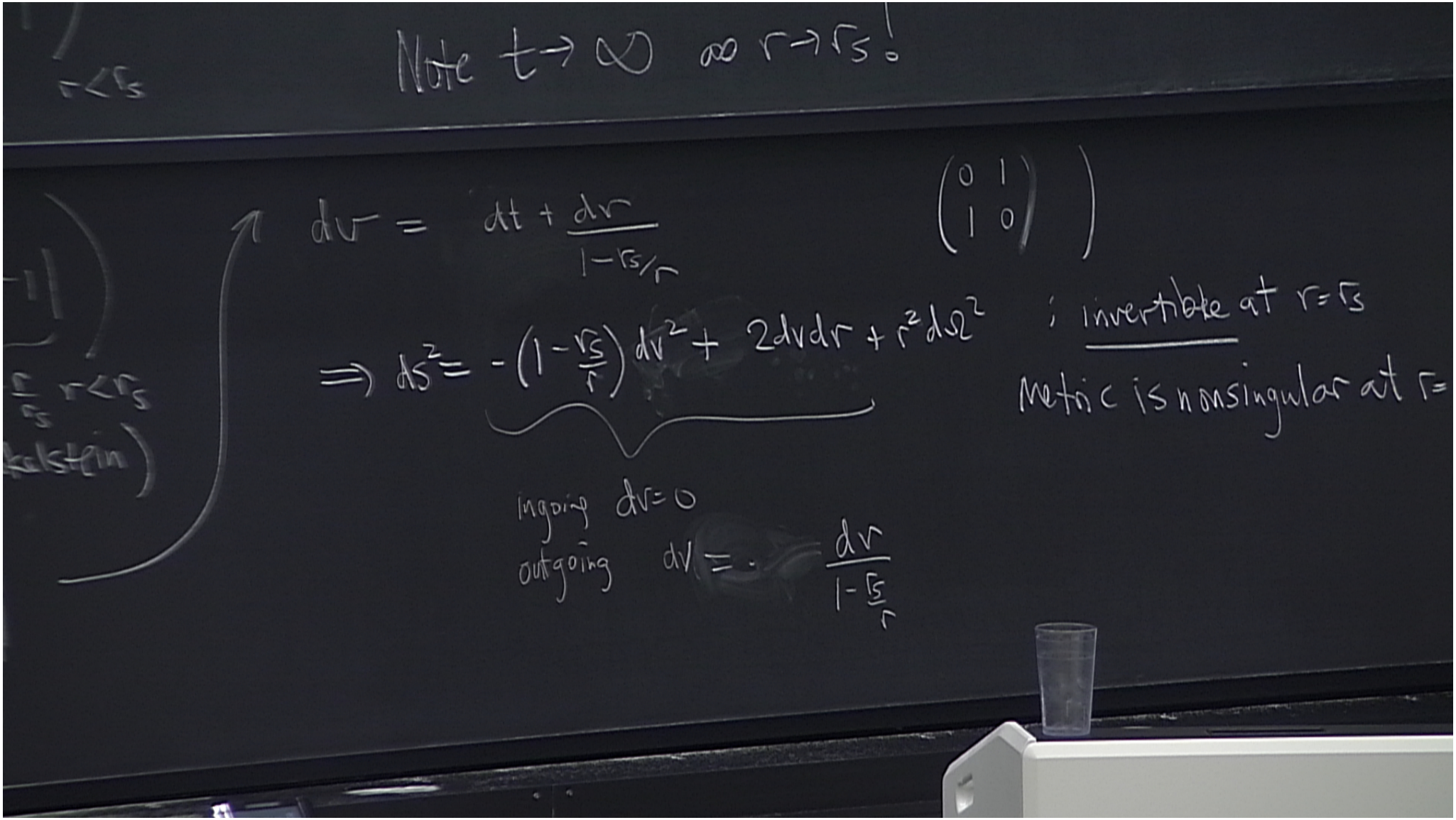
Problems: 1) 2 conflicting answers of time for $r < r_s$

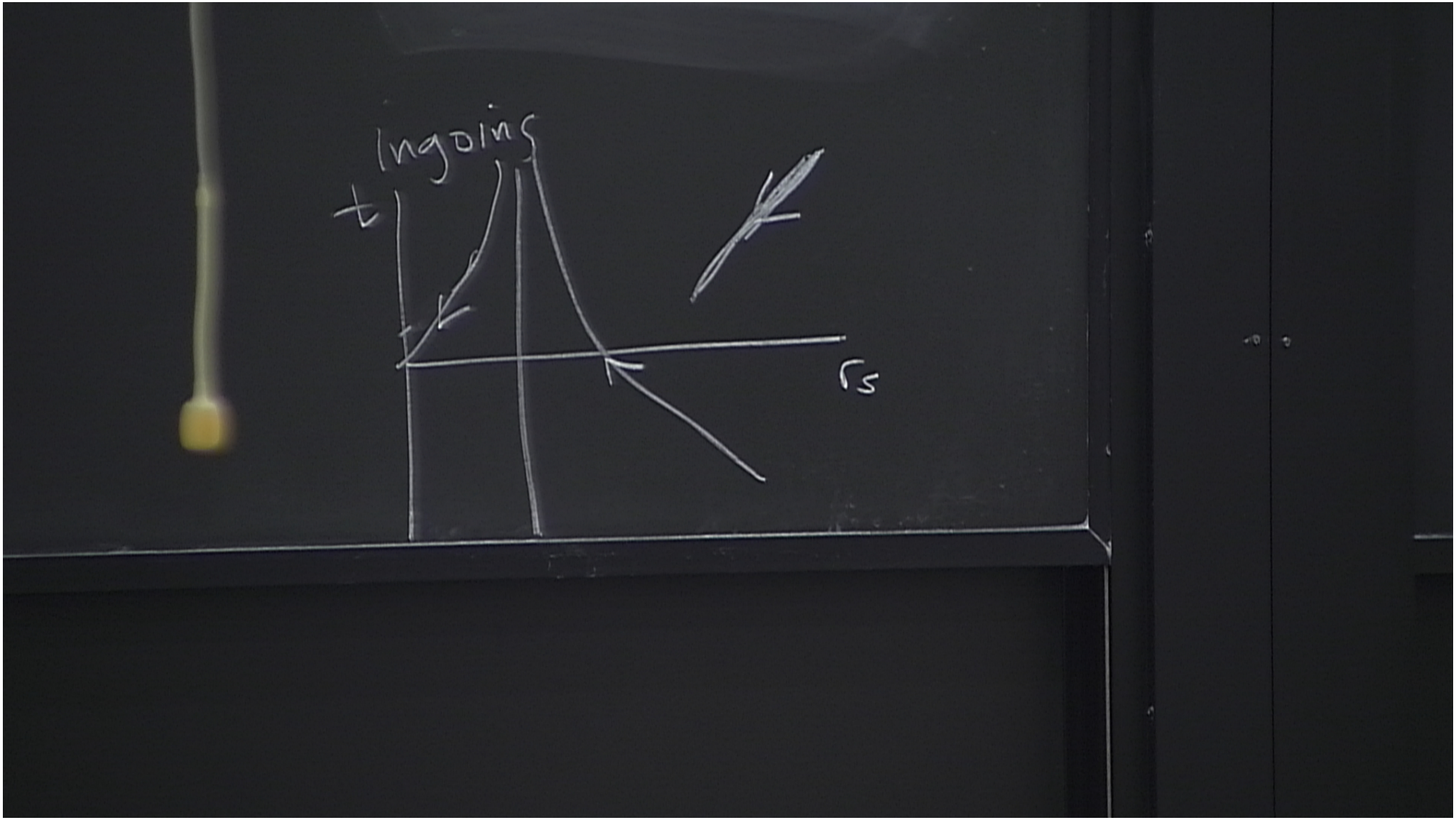
2)

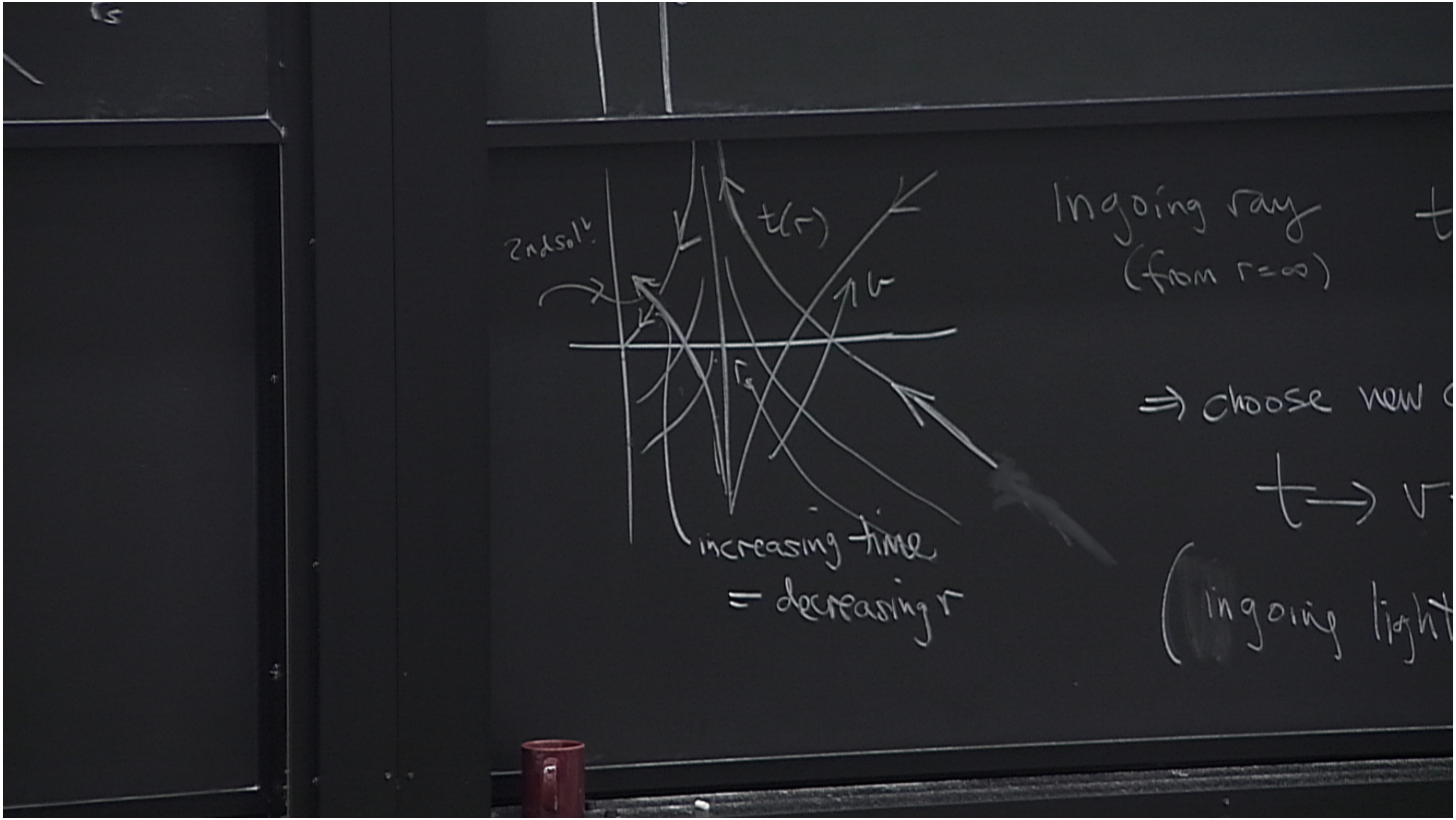
lets combine them, working outside r_s, i.e. $r > r_s$

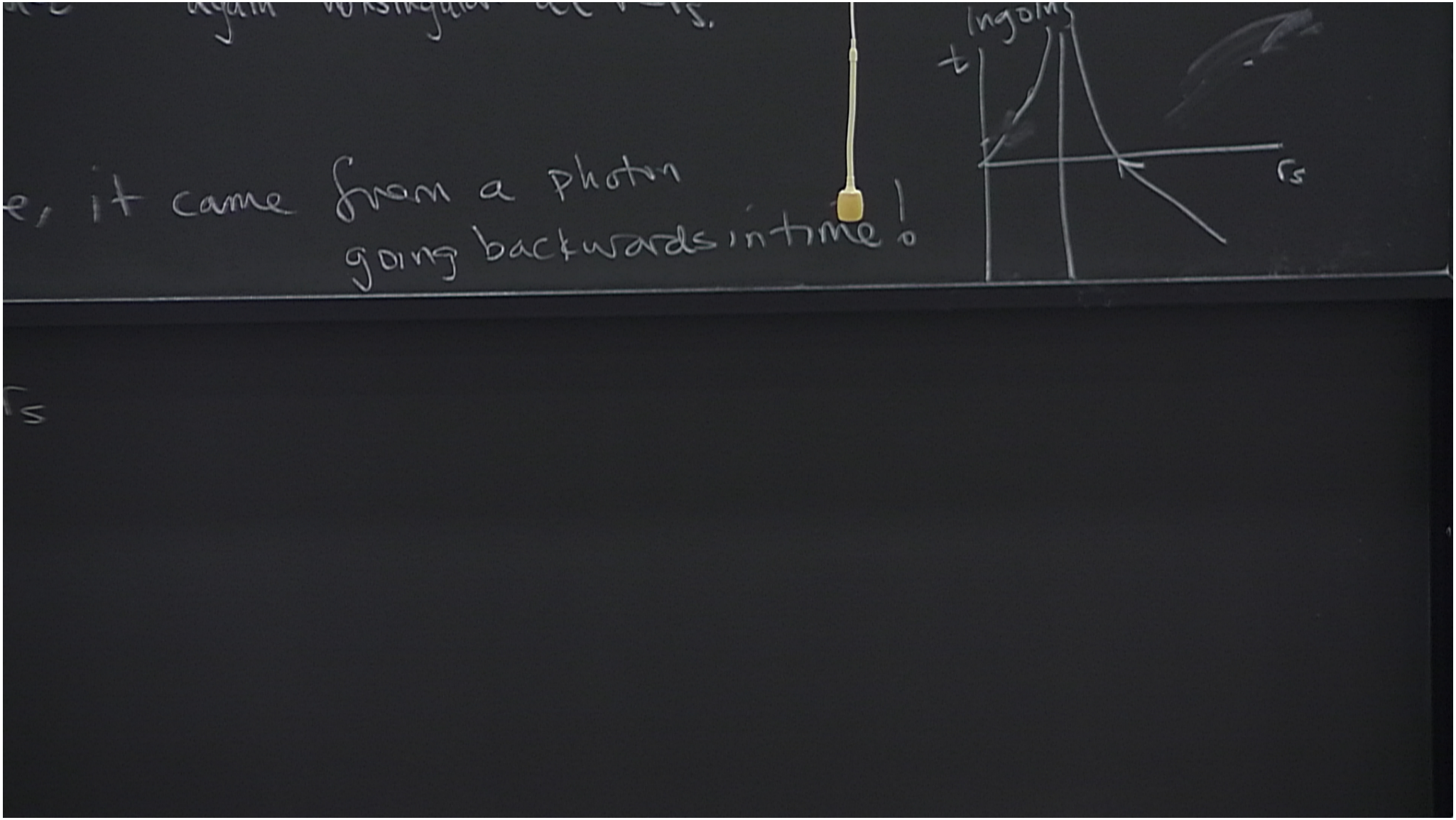
$$u = t - r - r_s \ln\left(\frac{r}{r_s} - 1\right)$$

$$v = t + r + r_s \ln\left(\frac{r}{r_s} - 1\right)$$


Note $t \to \infty$ as $r \to r_s$!
$r < r_s$
$dv = dt + \frac{dr}{1 - r_s/r}$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\Rightarrow ds^2 = -\left(1 - \frac{r_s}{r}\right) dv^2 + 2 dv dr + r^2 d\Omega^2$
; invertible at $r = r_s$
Metric is nonsingular at $r =$
ingoing $dv = 0$
outgoing $dv = \frac{dr}{1 - \frac{r_s}{r}}$


ingoing
t
r_s


2nd sol.
t(r)
r
increasing time
= decreasing r
ingoing ray
(from r=∞)
⇒ choose new
t →
(ingoing light


ingoing
t
r_s
e, it came from a photon
going backwards in time!
r_s

$$\Rightarrow \quad ds^2 = -\left(1-\frac{r_s}{r}\right)du^2 - 2du\,dr + r^2 d\Omega^2$$

Problems: 1) 2 conflicting arrows of time for $r < r_s$

2) follow 2nd soln to $ds^2 = 0$ out of b. hole, i

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lets combine them, working outside r_s, i.e. $r > r_s$

$$u = t - r - r_s \ln\left(\frac{r}{r_s} - 1\right)$$

$$v = t + r + r_s \ln\left(\frac{r}{r_s} - 1\right)$$

2) follow 2nd Soln to $ds^2 = 0$ out of b. hole

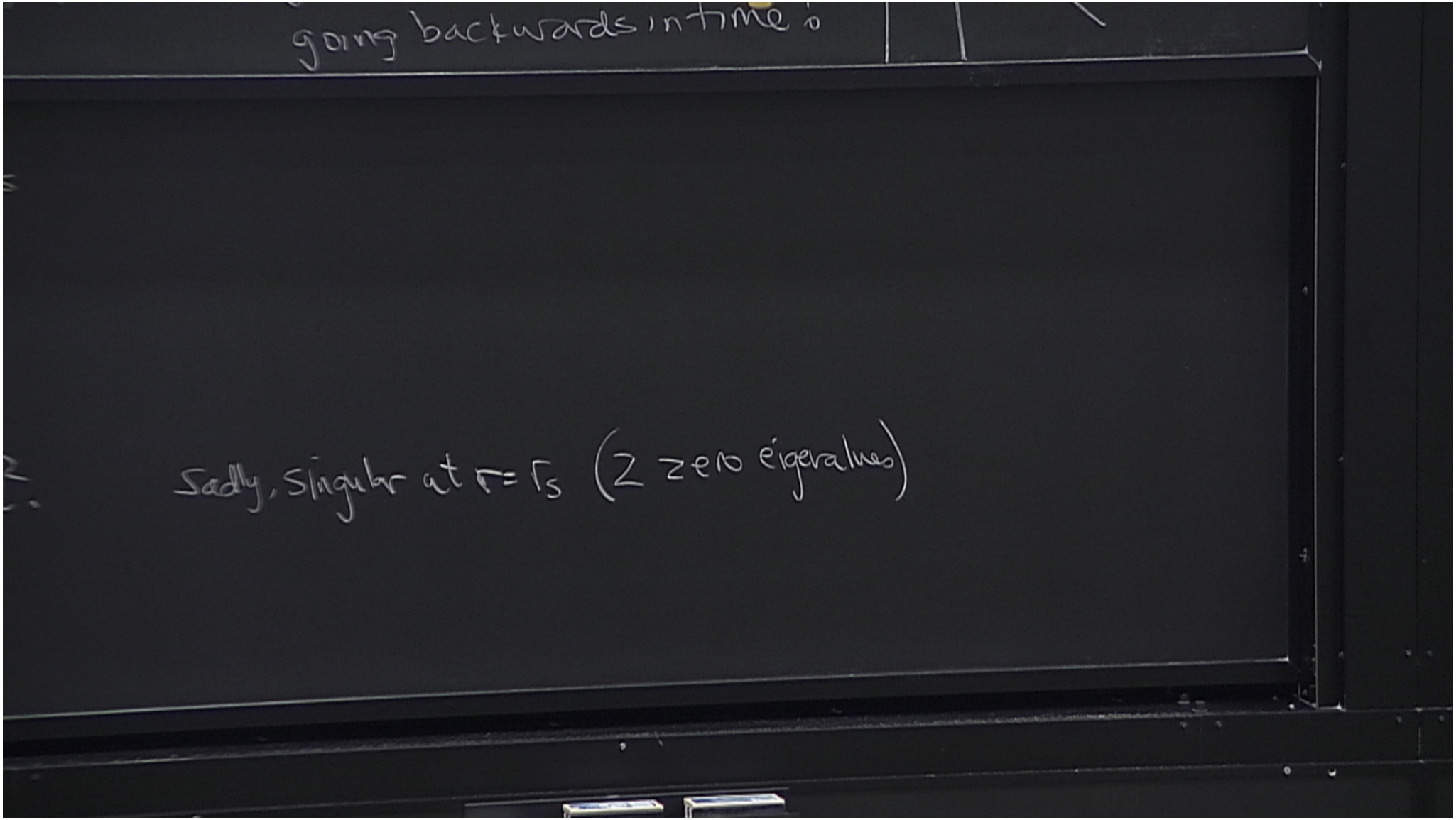
lets combine them, working outside r_s, i.e. $r > r_s$

Kruskal:

$$u = t - r - r_s \ln\left(\frac{r}{r_s} - 1\right)$$

$$v = t + r + r_s \ln\left(\frac{r}{r_s} - 1\right)$$

$$\Rightarrow \begin{cases} du = dt - \dfrac{dr}{1 - \frac{r_s}{r}} \\ dv = dt + \dfrac{dr}{1 - \frac{r_s}{r}} \end{cases} \Rightarrow ds^2 = -\left(1 - \frac{r_s}{r}\right) du\,dv + r^2 d\Omega^2.$$


going backwards in time
Sadly, singular at $r = r_s$ (2 zero eigenvalues)

r_s, ie. $r > r_s$

$$\left(\frac{r}{r_s} - 1\right) \qquad (*)$$

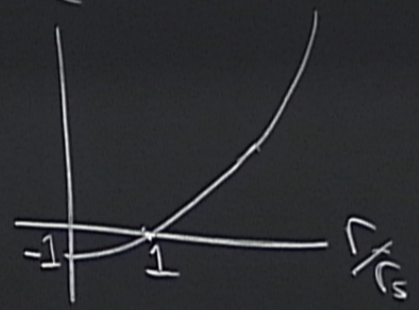
$$\left(\frac{r}{r_s} - 1\right)$$

$$-\left(1 - \frac{r_s}{r}\right) du\,dv + r^2 d\Omega^2.$$

Sadly, singular at $r = r_s$ (2 zero eigenvalues)

ss r in terms of u, v : $(*) \Rightarrow$ $r + r_s \ln\left(\frac{r}{r_s} - 1\right) = \frac{v-u}{2}$

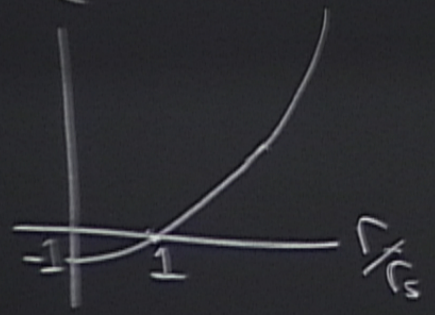
$\div r_s$, exponentiate $\Rightarrow$ $\left(\frac{r}{r_s} - 1\right) e^{\frac{r}{r_s}} = e^{\frac{v-u}{2r_s}}$



$$ds^2 = -\left(1-\frac{r_s}{r}\right)dudv + r^2 d\Omega^2.$$ Sadly, singular at $r = r_s$ (Z zero

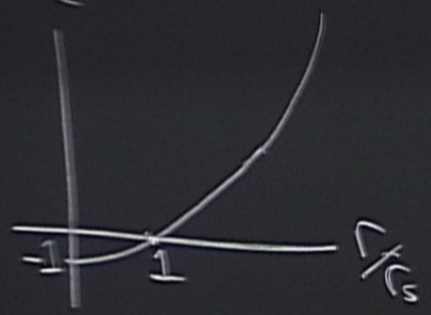
express r in terms of u, v : $(*) \Rightarrow$ $r + r_s \ln\left(\frac{r}{r_s} - 1\right) = \frac{v-u}{2}$

$$\left(\frac{r}{r_s} - 1\right) e^{r/r_s} = e^{\frac{v-u}{2r_s}}$$



$$\Rightarrow ds^2 = -\frac{r_s}{r} e^{-r/r_s} e^{\frac{v-u}{2r_s}} dudv + r^2 d\Omega^2$$

$$\left(\frac{r}{r_s}-1\right)e^{r/r_s} = e^{\frac{v-u}{2r_s}}$$



$$\Rightarrow ds^2 = -\frac{r_s}{r}e^{-r/r_s}\underbrace{e^{\frac{v-u}{2r_s}}}_{\underbrace{e^{v/2r_s}}_{dV}\,\underbrace{e^{-u/2r_s}}_{d\bar{U}}}du\,dv + r^2 d\Omega^2$$

Kruskal

$e^{\frac{v-u}{2r_s}}$

$$\Rightarrow ds^2 = -\frac{r_s}{r} e^{-r/r_s} \underbrace{e^{\frac{v-u}{2r_s}} du\,dv}_{e^{v/2r_s}dv \; e^{-u/2r_s}du} + r^2 d\Omega^2$$

$e^{v/2r_s}dv = dV$ $\quad e^{-u/2r_s}du = dU$

Kruskal : then $U = X - T$, $V = X + T$

$=$

so that $dU\,dV = -dT^2 + dX^2$

$$\Rightarrow ds^2 = -\frac{r_s}{r} e^{-r/r_s} e^{\frac{v-u}{2r_s}} du\,dv + r^2 d\Omega^2$$

Kruskal

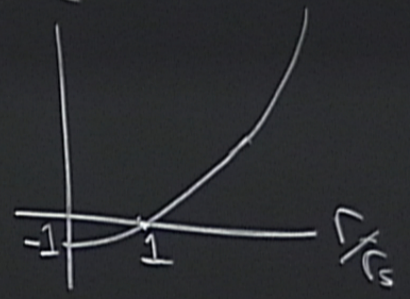
$e^{v/2r_s} dv \quad e^{-u/2r_s} du$: then $U = X - T$, $V = X + T$

$2dV \quad -2dU$

$V = r_s e^{r/2r_s} \quad U = r_s e^{-u/2r_s}$ so that $dU\,dV = -dT^2 + dX^2$

$$= 4\frac{r_s}{r} e^{-r/r_s}\left(-dT^2 + dX^2\right) + r^2 d\Omega^2$$

$\div r_s$, exponentiate $\Rightarrow \left(\frac{r}{r_s}-1\right)e^{\frac{r}{r_s}} = e^{\frac{v-u}{2r_s}}$



$\Rightarrow ds^2 = -$

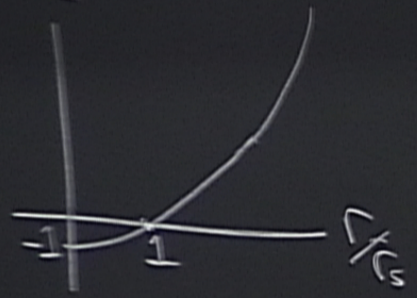
Kruskal

Express r in terms of T, X.

$$X^2 - T^2 = UV = r_s^2 e^{\frac{v-u}{2r_s}} = r_s^2 e^{\frac{r}{r_s}}\left(\frac{r}{r_s}-1\right) =$$

$$\left[t = \frac{u+v}{2} = r_s \ln\left(\frac{V}{U}\right) = r_s \ln\left(\frac{X+T}{X-T}\right) \text{ is old time coord}\right]$$

$$\left(\frac{r}{r_s}-1\right)e^{r/r_s} = e^{\frac{v-u}{2r_s}}$$



$$\Rightarrow ds^2 = -\frac{r_s}{r}e^{-r/r_s}\underbrace{e^{\frac{v-u}{2r_s}}du\,dv}_{\underbrace{e^{\frac{v}{2r_s}}dv}_{2dV}\;\underbrace{e^{\frac{-u}{2r_s}}du}_{-2dU}} + r^2d\Omega^2$$

Kruskal : then

$V = r_s e^{v/2r_s}$ $U = r_s e^{-u/2r_s}$ So th

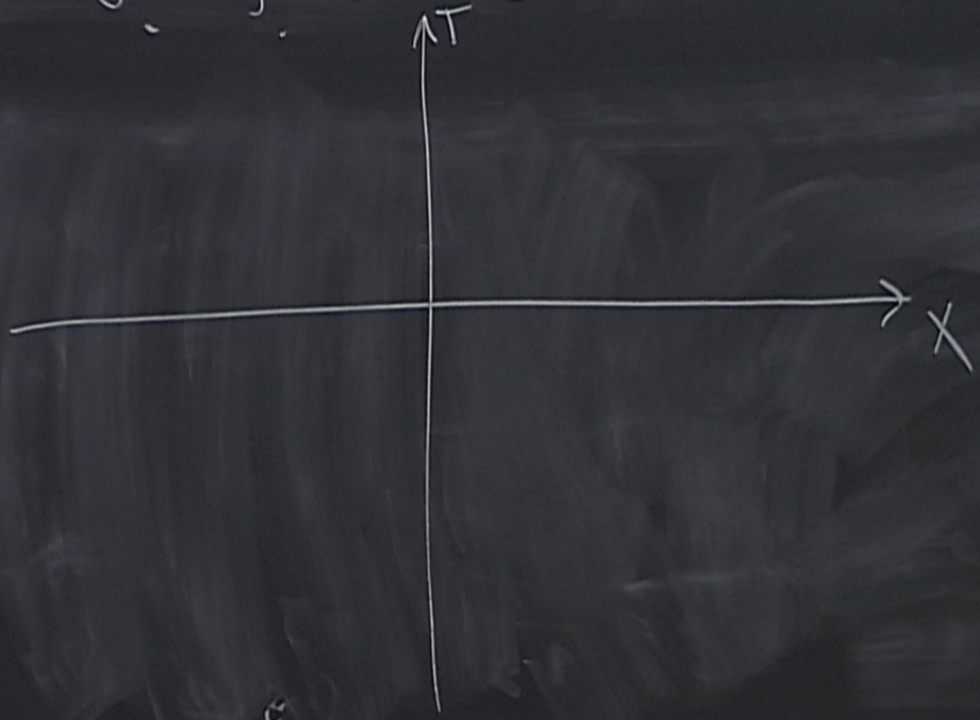
$$V = r_s^2 e^{\frac{v-u}{2r_s}} = r_s^2 e^{r/r_s}\left(\frac{r}{r_s}-1\right) = \underline{4\frac{r_s}{r}e^{-r/r_s}\left(-dT^2+dX^2\right) + r^2d\Omega^2}$$

$= r_s \ln\left(\frac{X+T}{X-T}\right)$ is old time coord]

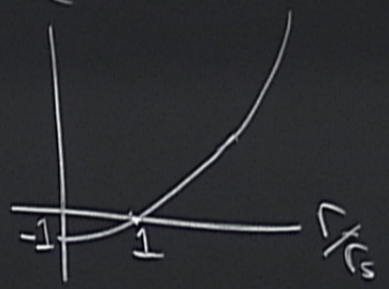
ingoing outgoing

$\rightarrow t = \pm\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right) + \text{const.}$

Clearly, ingoing + outgoing null rays are just lines at 45° in these



$\div r_s$, exponentiate $\Rightarrow$ $\left(\frac{r}{r_s}-1\right)e^{\frac{r}{r_s}} = e^{\cdots}$

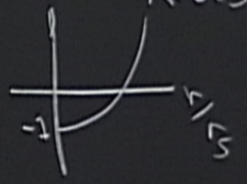


$\Rightarrow ds^2 = -\frac{r_s}{r} e^{-r/r_s} e^{\frac{v-u}{2r_s}} du$

$e^{v/2r_s} dv$

$2dV$

Kruskal



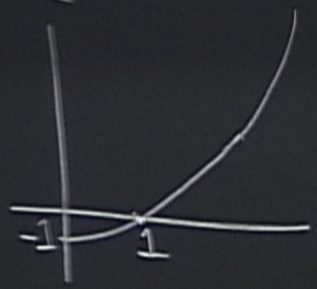
r in terms of T, X.

$r = r_s$

$$X^2 - T^2 = UV = r_s^2 e^{\frac{v-u}{2r_s}} = r_s^2 e^{r/r_s}\left(\frac{r}{r_s}-1\right) = 4\frac{r_s}{r} e^{-r/r_s}(-dT^2$$

$$\left[t = \frac{u+v}{2} = r_s \ln\left(\frac{V}{U}\right) = r_s \ln\left(\frac{X+T}{X-T}\right) \text{ is old time coord}\right]$$

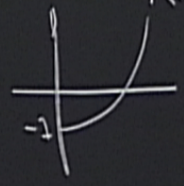
$$\Rightarrow \left(\frac{r}{r_s}-1\right)e^{r/r_s} = e^{\frac{v-u}{2r_s}}$$



$$\Rightarrow ds^2 = -\frac{r_s}{r}e^{-r/r_s}\underbrace{e^{\frac{v-u}{2r_s}}du\,dv}_{\underbrace{e^{v/2r_s}dv}_{2dV}\;\underbrace{e^{-u/2r_s}du}_{-2dU}} + r^2 d\Omega^2$$

: then

Kruskal



$$UV = r_s^2 e^{\frac{v-u}{2r_s}} = r_s^2 e^{r/r_s}\left(\frac{r}{r_s}-1\right) \quad (\#)$$

$$= \frac{4r_s}{r}e^{-r/r_s}\left(-dT^2+dX^2\right) + r^2 d\Omega^2$$

$$\frac{V}{U} = r_s \ln\left(\frac{X+T}{X-T}\right)$$ is old time coord

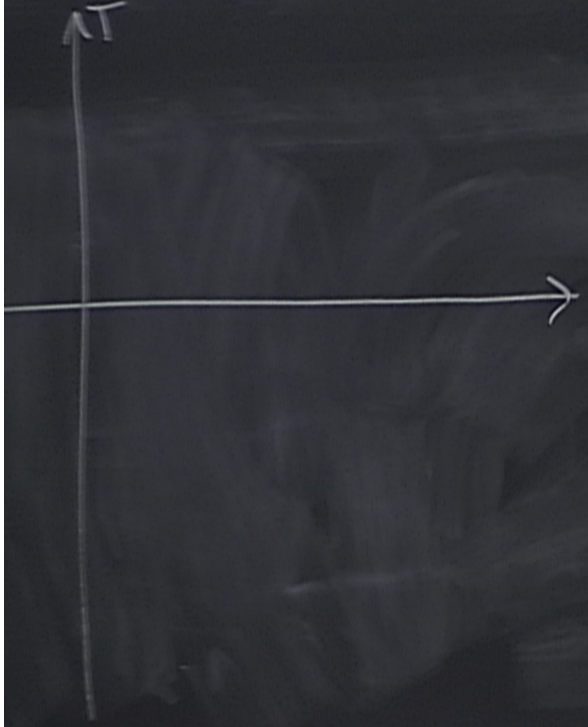
$V = r_s e^{v/2r_s}$ $U = r_s e^{-u/2r_s}$ So tha

$\rightarrow t - = \pm\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right) + \text{const.} \quad \forall\, r \neq r_s$

+ outgoing [radial null rays are just lines at 45° in these new T, X coords.

rhs of (♯) $> -r_s^2$.

$\Rightarrow T^2 < X^2 + r_s^2$



$t - = \pm\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right) + \text{const.} \qquad \forall\, r \neq r_s$

dial
null rays are just lines at 45° in these new T, X coords.

rhs of (♯) $> -r_s^2$.

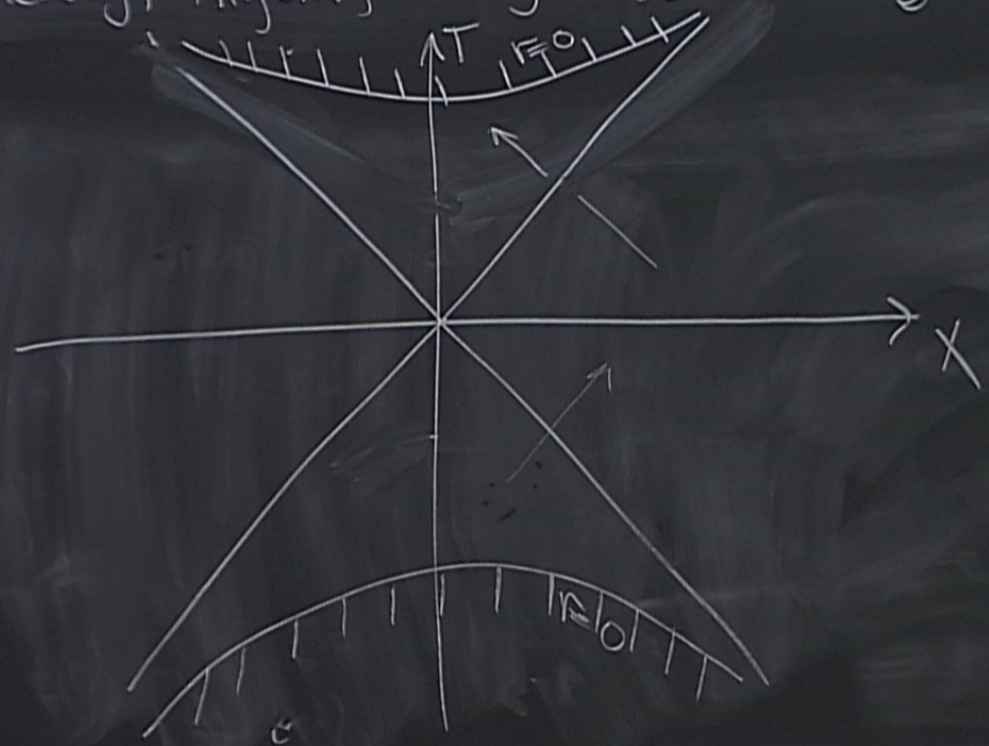
$\Rightarrow T^2 < X^2 + r_s^2 \Rightarrow -\sqrt{X^2+r_s^2} < T < \sqrt{X^2+r_s^2}$



$\pm$ ingoing, $\mp$ outgoing

$\rightarrow \pm - \pm\left(r + r_s \ln\left|\frac{r}{r_s} - 1\right|\right) + \text{const.}$ $\forall$

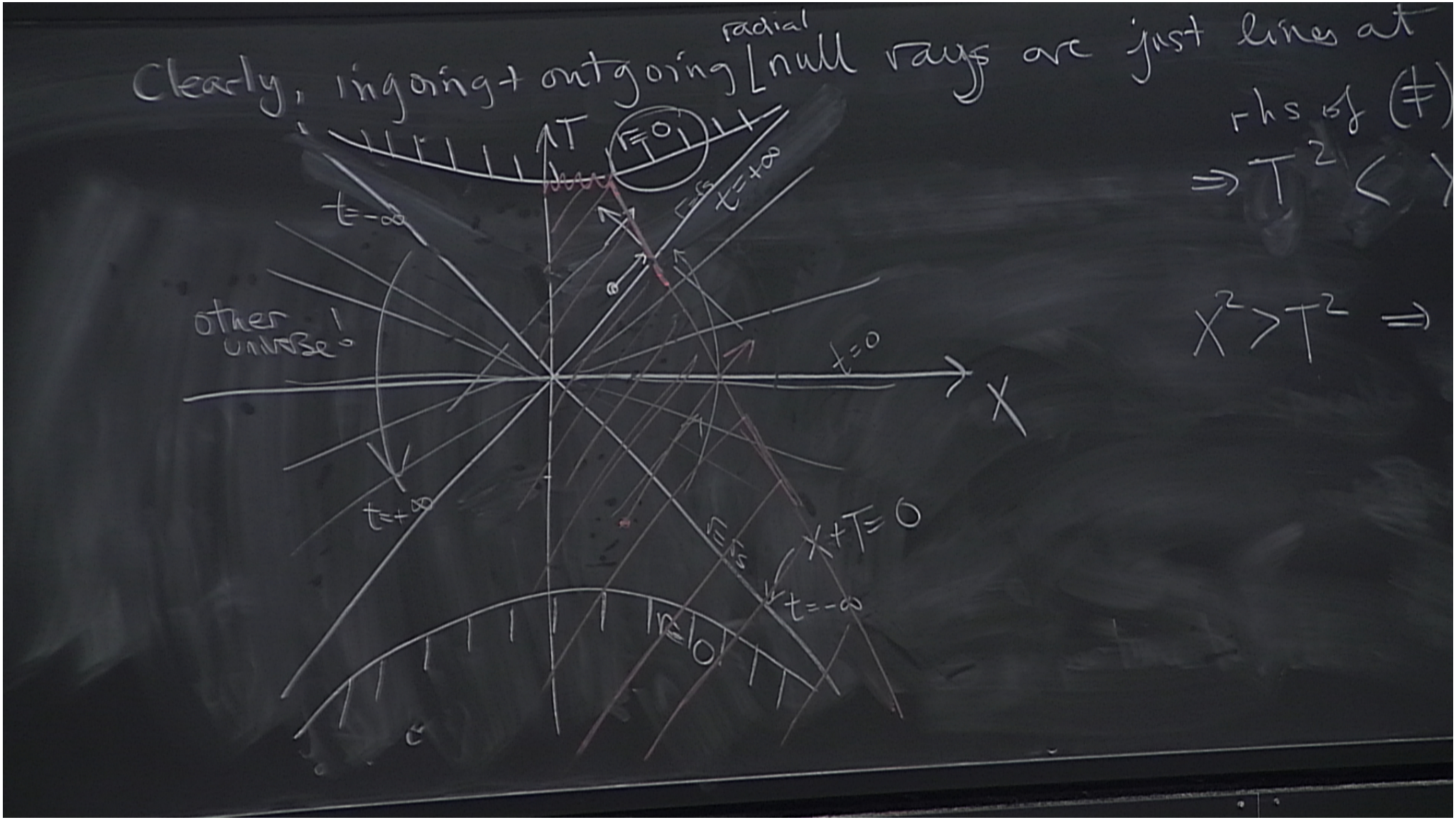
Clearly, ingoing + outgoing [radial] null rays are just lines at 45° in these n



rhs of (#) $> -r_s^2$.

$\Rightarrow T^2 < X^2 + r_s^2 \Rightarrow -\sqrt{X^2}$

$X^2 > T^2$


Clearly, ingoing + outgoing radial null rays are just lines at
rhs of (*)
$\Rightarrow T^2 <$
$X^2 > T^2 \Rightarrow$
T
r=0
$t=-\infty$
$t=+\infty$
other universe!
$t=0$
X
$t=+\infty$
$X+T=0$
$t=-\infty$
r=0