

Title: PSI 2016/2017 Relativity - Lecture 13

Date: Sep 22, 2016 09:00 AM

URL: <http://pirsa.org/16090013>

Abstract:

$$dr^2 + r^2 d\Omega^2$$

interp: $r_s = \frac{2GM}{c^2}$

Schwarzschild

Solves $R_{\mu\nu} = 0$
(except at $r=0$)
and should be sceptical
about $r=r_s$.

$$- \left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

orbits of a massive particle (Mercury) around

→ solve geodesic eqⁿ, determine shape of orbit
For spher., symm, static metric, enough symm to get orbit

$$- \left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

interp: $r_s = \frac{2GM}{c^2}$

orbits of a massive particle (Mercury) around the Sun.

→ solve geodesic eqⁿ, determine shape of orbit.

$$u^\mu = \frac{dx^\mu}{dt}$$

For spher. symm, static metric, enough symm to get orbit just from $g_{\mu\nu} u^\mu u^\nu = -1$.

interp: $\Gamma_S = \frac{2GM}{c^2}$

Sum.

$$\frac{d}{d\tau} (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu) = 0$$

using geod. eqⁿ.

$$u^\mu = \frac{dx^\mu}{d\tau}$$

$$\dot{} = \frac{d}{d\tau}$$

from $g_{\mu\nu} u^\mu u^\nu = -1$,

$$- \left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

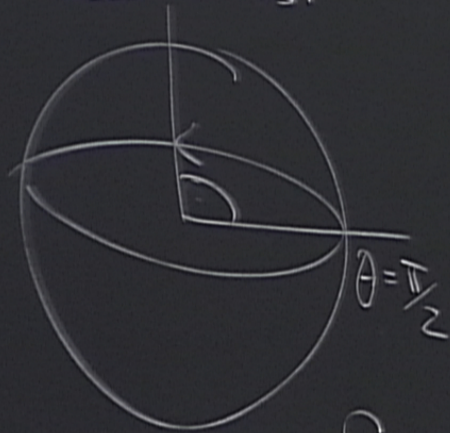
orbits of a massive particle (Mercury) around the Sun.

→ solve geodesic eqⁿ, determine shape of orbit.

For spher. symm, static metric, enough symm to get orbit just from $g_{\mu\nu}$

$$- \left(1 - \frac{r_s}{r}\right) \dot{t}^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = -1$$

Solves $R_{\mu\nu} = 0$
 (except at $r=0$)
 and should be sceptical
 about $r=r_s$.



orbits of a massive particle (Mercury)

→ solve geodesic eqⁿ, determine shape of

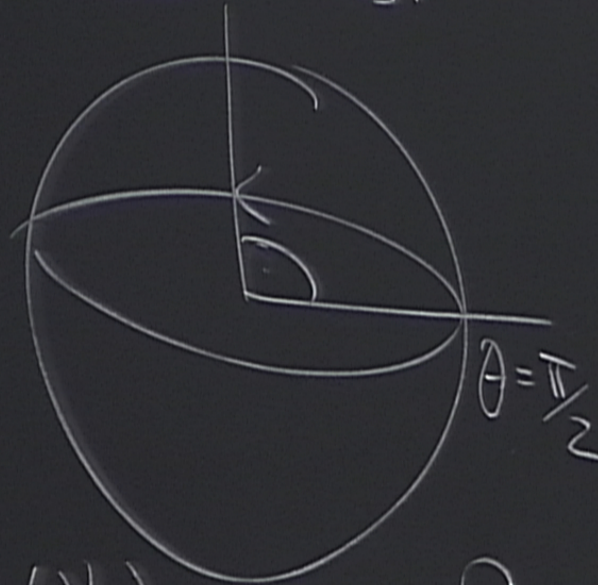
For spher., symm, static metric, enough symm to get

$$-\left(1 - \frac{r_s}{r}\right) \dot{t}^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 + r^2 \left(\dot{\theta}^2 + \underbrace{\sin^2 \theta}_{1} \dot{\phi}^2 \right) = -1$$

$E =$

$$-m \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

about $r=r_s$.



For spher., symm, static met

$$- \left(1 - \frac{r_s}{r}\right) \dot{t}^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 +$$

$$E = \frac{\partial L}{\partial \dot{x}^0}$$

$$\frac{\partial L}{\partial x^0} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^0} \right)$$

$$= m \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

For spher. symm, static metric, $\exists$ enough symm to get orbit just

$$\left(1 - \frac{r_s}{r}\right) \dot{t}^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 + r^2 \left(\cancel{\dot{\theta}^2} + \underbrace{\sin^2 \theta}_{1} \dot{\phi}^2 \right) = -1$$

$$E = \frac{\partial L}{\partial \left(\frac{dx^0}{d\lambda} \right)} = m g_{00} \frac{dx^0}{d\tau}, \quad L = m g$$

$$\frac{\partial L}{\partial x^0} = \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \frac{dx^0}{d\lambda}} \right)$$

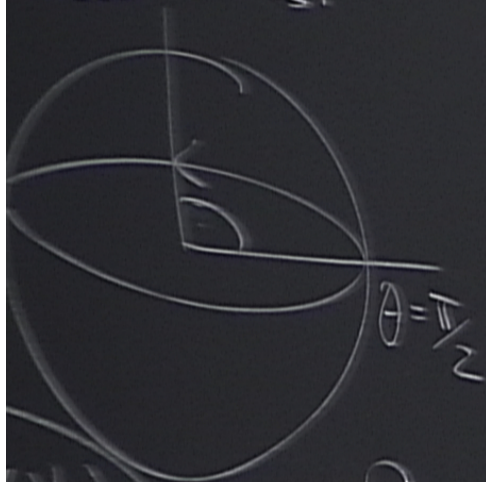
$$-m \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

$$E = \frac{\partial L}{\partial \left(\frac{dx^0}{d\lambda} \right)} = m g_{00} \frac{dx^0}{d\tau}$$

$$d\tau = d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

$$\frac{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}{\sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}} = -g_{\mu\nu} \frac{dx^\mu}{d\tau}$$

...spherical
about $r=r_s$.



For spher., symm, static metric, enough symm to get arb

$$-\left(1-\frac{r_s}{r}\right)\dot{t}^2 + \left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2 + r^2(\underbrace{\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2}_1) = -1$$

Noether

$$\rightarrow E = \frac{\partial L}{\partial \left(\frac{dx^0}{d\lambda}\right)} = m g_{0\mu} \frac{dx^\mu}{d\tau}, \quad J = m g_{\phi\mu} \frac{dx^\mu}{d\tau}$$

$$= -\left(1-\frac{r_s}{r}\right) \frac{dx^0}{d\tau}$$

$$-m \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

$$\frac{-g_{0\nu} \frac{dx^\nu}{d\lambda}}{\sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}} = -g_{0\nu} \frac{dx^\nu}{d\tau}$$

$$\rightarrow E = \frac{\partial L}{\partial \left(\frac{dx^0}{d\lambda} \right)} = m g_{0\mu} \frac{dx^\mu}{d\tau}, \quad J = m g_{\phi\mu} \frac{dx^\mu}{d\tau} = + m r^2 \sin^2 \theta \frac{d\phi}{d\tau}$$

$$-m \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda = -m \left(1 - \frac{r_s}{r} \right) \frac{dx^0}{d\tau} = + m r^2 \sin^2 \theta \frac{d\phi}{d\tau}$$

drop hats

$$\Rightarrow -\frac{E^2}{\left(1 - \frac{r_s}{r}\right)} + \frac{\dot{r}^2}{\left(1 - \frac{r_s}{r}\right)} + \frac{J^2}{r^2} = -1$$

$$\times \left(1 - \frac{r_s}{r}\right) \quad \left[\underbrace{\dot{r}^2}_{\text{K.E.}} + \underbrace{J^2 \left(\frac{1}{r^2} - \frac{r_s}{r^3} \right)}_{\text{P.E. } V(r)} - \frac{r_s}{r} \right] = E^2 - 1$$

just Newton

Einstein's correction, down by $\frac{r_s}{r}$.

$$\frac{-g_{0\nu} \frac{dx^\nu}{d\lambda}}{\sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}} = -g_{0\nu} \frac{dx^\nu}{d\tau}$$

drop hats

$$\Rightarrow -\frac{E^2}{\left(1-\frac{r_s}{r}\right)} + \frac{\dot{r}^2}{\left(1-\frac{r_s}{r}\right)} + \frac{J^2}{r^2} = -1$$

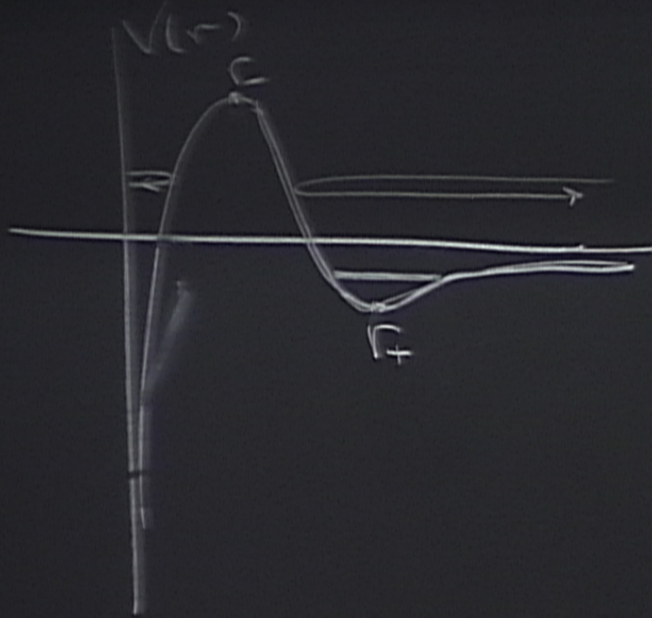
$\times \left(1-\frac{r_s}{r}\right)$

$$\underbrace{\dot{r}^2}_{\text{K.E.}} + \underbrace{J^2 \left(\frac{1}{r^2} - \frac{r_s}{r^3} \right)}_{\text{P.E. } V(r)} - \frac{r_s}{r} = E^2 - 1.$$

just Newton

Einstein's correction, down by $\frac{r_s}{r}$.
 $(r_s(M_0) \sim 4 \text{ km})$

$$\frac{-g_{0\nu} \frac{dx^\nu}{d\lambda}}{\sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}} = -g_{0\nu} \frac{dx^\nu}{dt}$$

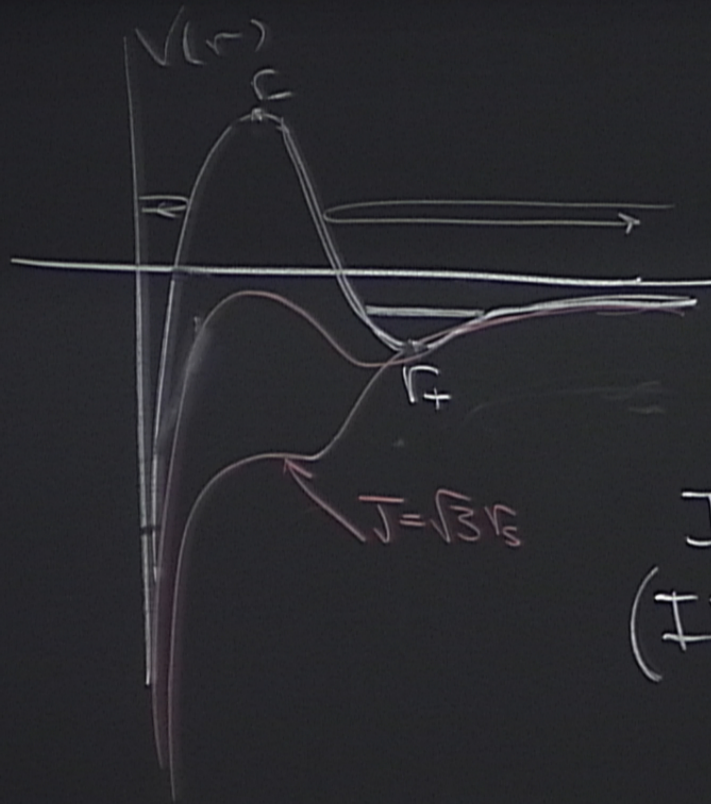


circular orbits are at $\frac{\partial V}{\partial r} = 0$
 $= \frac{r_s}{r^2} - \frac{2J^2}{r^3}$

are at $\frac{\partial V}{\partial r} = 0$

$$= \frac{r_s}{r^2} - \frac{2J^2}{r^3} + \frac{3r_s J^2}{r^4} \Rightarrow r_{\pm} = \frac{J}{r_s} \left(1 \pm \sqrt{1 - \frac{3r_s^2}{J^2}} \right)$$

to have real solⁿs, must have $J > \sqrt{3} r_s$



circular orbits are at

Innermost stable circular orbit.
(ISCO)

is at $\frac{\partial V}{\partial r} = 0$

$$= \frac{r_s}{r^2} - \frac{2J^2}{r^3} + \frac{3r_s J^2}{r^4} \Rightarrow r_{\pm} = \frac{J^2}{r_s} \left(1 \pm \sqrt{1 - \frac{3r_s^2}{J^2}} \right)$$

to have real solⁿs, must have $J > \sqrt{3} r_s$

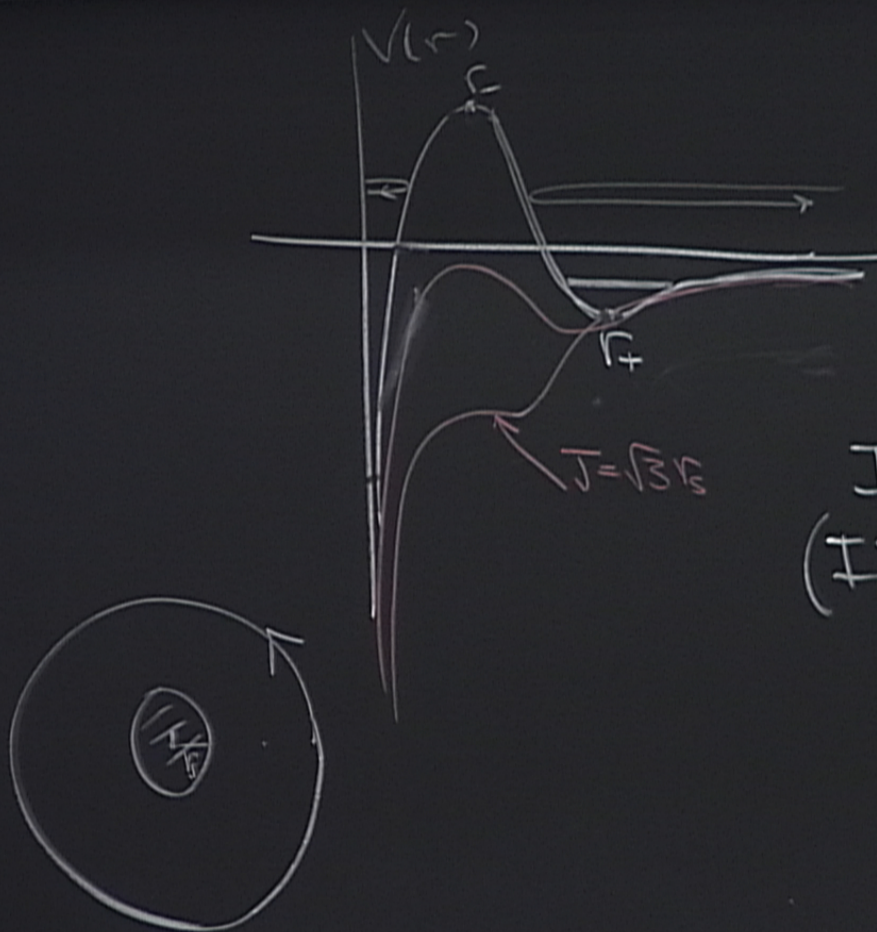
horiz. orbit.

circular orbits are at $\frac{\partial V}{\partial r} = 0$

$$= \frac{r_s}{r^2} - \frac{2J^2}{r^3} + \frac{3r_s J^2}{r^4} \Rightarrow r_{\pm} = \frac{J^2}{r_s}$$

to have real solⁿs, must have $J > \sqrt{3} r_s$

Innermost stable circular orbit.
(ISCO) is at $r_{\text{ISCO}} = 3r_s$



circular orbits are at

Innermost stable circular orbit.
(ISCO) is at $r_{\text{ISCO}} = 3r_s$

To find trajectory, in regime where GR corr's small, i.e. $r \gg R$,
follow Newton: define $r(\varphi)$: $r' \equiv \frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = \frac{\dot{r} r^2}{J}$

$\Rightarrow$

GR corr's small, i.e. $r \gg r_s$,

$$r' \equiv \frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} = \frac{\dot{r} r^2}{J}$$

$$-\frac{r_s}{r} + \frac{J^2}{r^2} \left(1 - \frac{r_s}{r}\right) = \text{const.}$$

$$u^{1/2} - \frac{r_s}{J^2} u + u^2 - r_s u^3 = \text{const}$$

SHO

$$u^{1/2} + \left(u - \frac{r_s}{2J^2}\right)^2 - r_s u^3 = \text{const}$$

SHO

treat last term as a perturbation

$$u = \underbrace{u^{(0)}}_{\text{solves SHO}} + \underbrace{u^{(1)}}_{\text{order } \frac{r_s}{J^2}}$$

$e \cos \varphi) \Rightarrow$ ellipse

$$r = \frac{2J^2}{r_s(1 + e \cos \varphi)}$$

pert.

$$u'' = -\left(u - \frac{r_s}{2J^2}\right) + \frac{3r_s}{2}u^2$$

$u^{(0)}$ solves.

resonant driving force.

$$\underline{u^{(1)}} = -u^{(1)} + \frac{3}{2}r_s u^{(0)2} = -\underline{u^{(1)}} + \frac{3}{8} \frac{r_s^3}{J^4} e \cos \varphi$$

$\Rightarrow$ secular growth

$\rightarrow$ breakdown of pert. th.

$$\underline{u^{(1)}} = -u^{(1)} + \frac{3}{2} \Gamma_5 u^{(1)} = -u^{(1)} + \frac{3}{8} \frac{\Gamma_5}{J^4} e \cos \varphi$$

Solⁿ is $u^{(1)} = \frac{3\Gamma_5^3}{8J^4} e \varphi \sin \varphi$

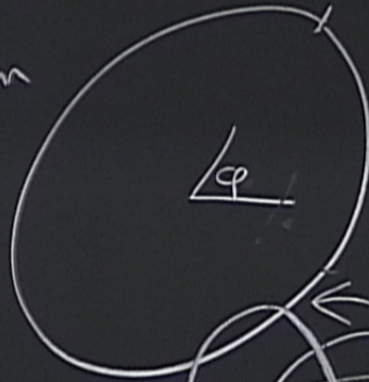
↑
secular growth

$$u = \frac{r_s}{2J^2} (1 + e \cos \varphi) + \frac{3r_s^3}{8J^4} e \varphi \sin \varphi + \dots$$

$$\approx \frac{r_s}{2J^2} (1 + e \cos((1-\epsilon)\varphi))$$

, where $\epsilon = \frac{3r_s^2}{4J^2}$
 (resummation of pert.ⁿ theory
 to remove secular term)

Newton



Einstein



precession of
perihelion

$\Rightarrow$ full sol'n is $u = \frac{\sqrt{s}}{2J^2}$

$$\frac{1}{r} = \frac{\sqrt{s}}{2J^2}$$

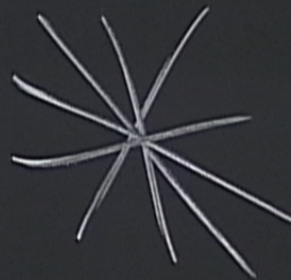
soln is $u = \frac{r_s}{2J^2} (1 + e \cos \varphi) + \frac{3r_s^3}{8J^4} e \varphi \sin \varphi + \dots$

$$\approx \frac{r_s}{2J^2} (1 + e \cos((1-e)\varphi)) \quad , \text{ where } e = \frac{3r_s^2}{4J^2}$$

(agrees to 1st order in e ,)

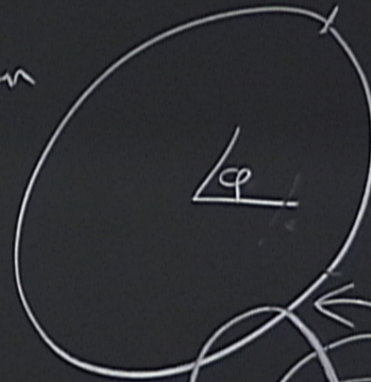
2nd expⁿ does not have secular growth.

(resummation of pertⁿ
to remove secular)

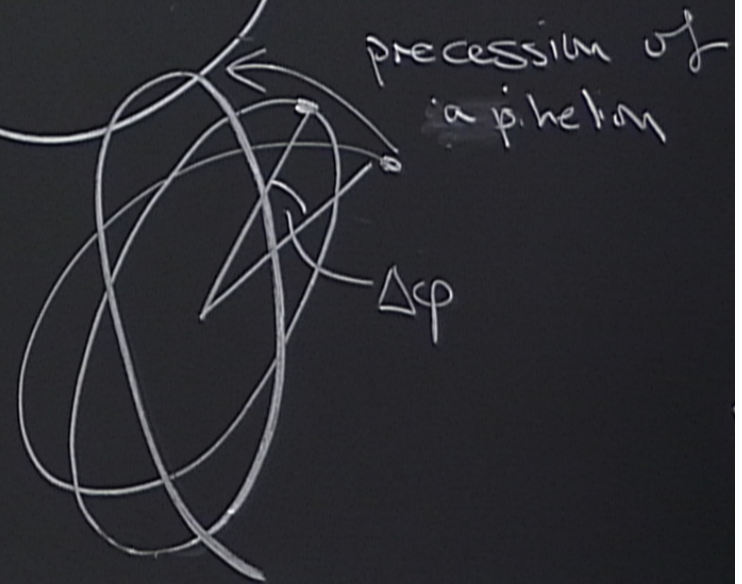


Newton

$\Rightarrow$ full



Einstein



$$u = \frac{r_s}{2J^2} (1 + e \cos \varphi) + \frac{3r_s^3}{8J^4} e \varphi \sin \varphi + \dots$$

$$\approx \frac{r_s}{2J^2} \left(1 + e \cos \left((1 - \epsilon) \varphi \right) \right), \text{ where } \epsilon = \frac{3r_s^2}{4J^2}$$

(agrees to 1st order in ϵ ,)
 2nd expⁿ does not have secular growth.

(resummation of pertⁿ theory to remove secular term)

$$\Delta \varphi_{\text{per orbit}} = 2\pi\epsilon = \frac{3\pi r_s^2}{2J^2} \Rightarrow \frac{43''}{45\Delta\varphi} \text{ per century for Mercury, } \underbrace{}_{415 \text{ orbits.}}$$

$\Rightarrow$ full soln is $u = \frac{r_s}{2J^2} (1 + e \cos \varphi) + \frac{3r_s^3}{8J^4} e \varphi \sin$

$\approx \frac{r_s}{2J^2} (1 + e \cos((1-\epsilon)\varphi))$
 $2\pi(1-\epsilon)$

(agrees to 1st order in ϵ .)

2nd expⁿ does not have secular growth.

$\Delta\varphi = \frac{2\pi\epsilon}{\text{per orbit}} = \frac{3\pi r_s^2}{2J^2} \Rightarrow 43'' \text{ per century}$
 $415\Delta\varphi$ 415

precession of
a planet

Newton
 $\frac{GM}{J^2}$
 $J = r^2 \frac{d\varphi}{dt}$
 $\approx \frac{d\varphi}{dt}$

$\Delta\varphi$

Bending of light

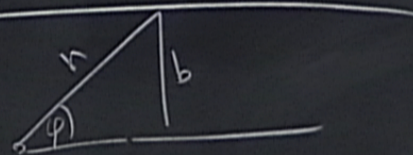


$$g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu = 0 \Rightarrow u'^2 + u^2 - r_s u^3 = \text{const}$$

Solve this first

$$u^{(0)} = \frac{1}{b} \sin \varphi = \frac{1}{r^{(0)}} \Rightarrow r(\varphi) = \frac{b}{\sin \varphi}$$

ght



$$u^2 + u^2 - r^2 u^3 = \text{const}$$

Solve this first

$$u(\varphi) = \frac{1}{b} \sin \varphi = \frac{1}{r(\varphi)} \Rightarrow r(\varphi) = \frac{b}{\sin \varphi} \quad \text{straight line}$$

Bending of light

star = Newton

(Sun)



$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 \Rightarrow u'^2 + u^2 - r_s u^3 = \text{const}$$

Solve this first

$$u^{(0)} = \frac{1}{b} \sin \varphi = \frac{1}{r^{(0)}} \Rightarrow r(\varphi) = \frac{b}{\sin \varphi}$$

straight line

$$\frac{1}{u'} \frac{d}{d\varphi}$$

$$2u'(u'' + u) - 3r_s u^2 u' = 0$$

$$u'' = -u + \frac{3r_s}{2} u^2$$

deflection of light

Sun

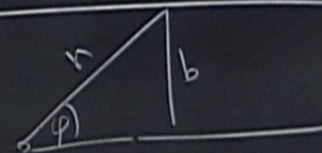
$$\Rightarrow u'^2 + u^2 - r_s u^3 = \text{const}$$

Solve this first

$$u^{(0)} = \frac{1}{b} \sin \varphi = \frac{1}{r(\varphi)} \Rightarrow r(\varphi) = \frac{b}{\sin \varphi} \quad \text{straight line}$$

$$u'' = -u + \frac{3}{2} r_s u^2$$

$$u = \frac{1}{b} \sin \varphi + \frac{3}{4} \frac{r_s}{b^2} (1 + \frac{1}{3} \cos 2\varphi) + \dots$$



$$-3\frac{r_s^2}{2}u$$

$$u'' = -u + \frac{3r_s}{2}u^2$$

$$u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) + \dots$$

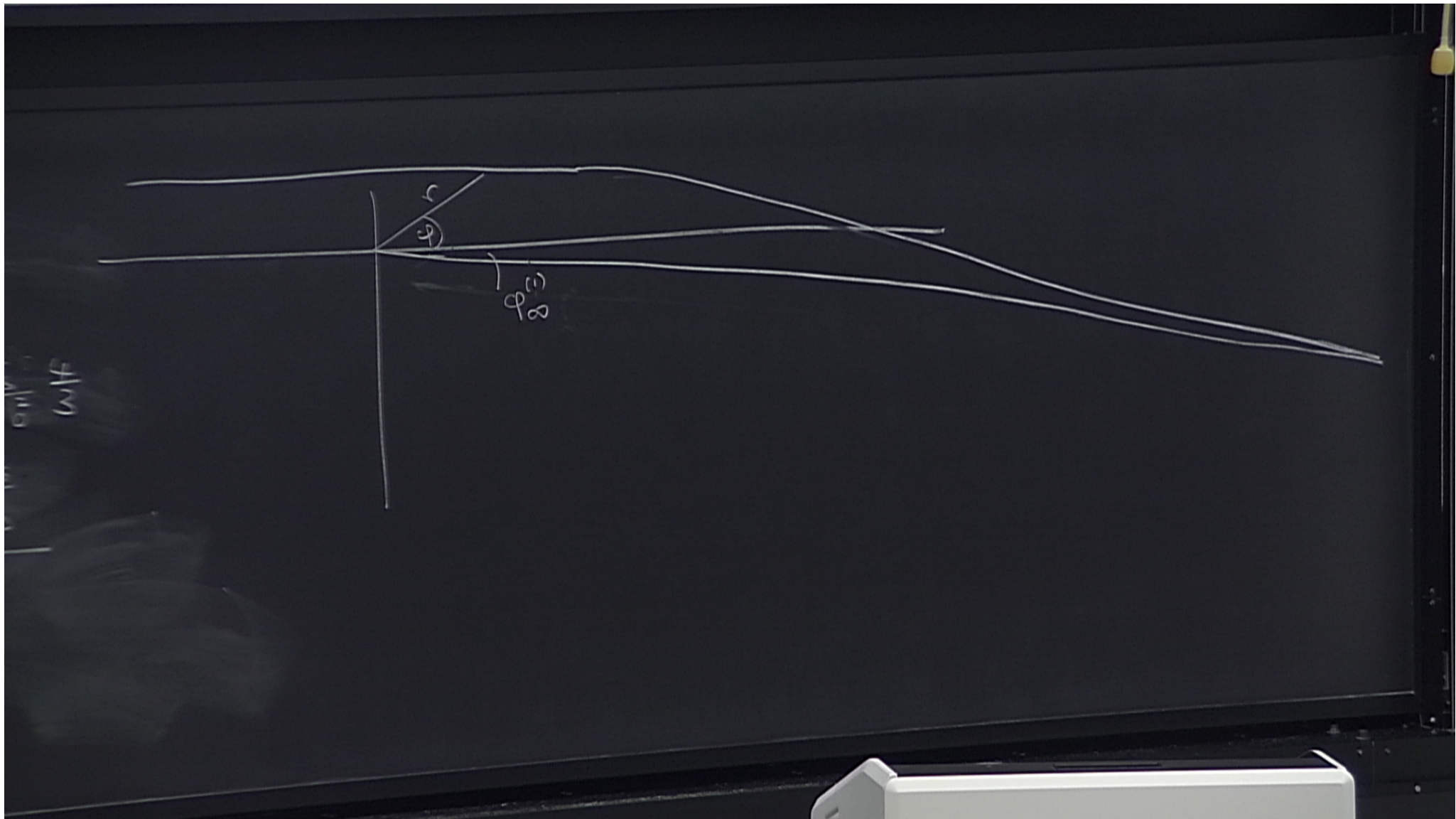
(≠)

shape of line: $r \rightarrow \infty$ $u \rightarrow 0$ $\varphi = 0$ and $\varphi = \pi$

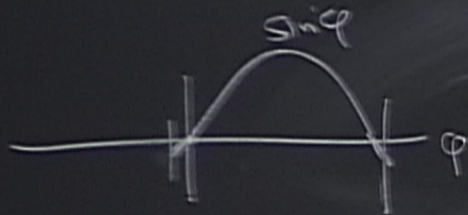
$$\varphi_\infty = \varphi_\infty^{(0)} + \underbrace{\varphi_\infty^{(1)}}_{\text{small}}$$

$$\varphi = 0 + \varphi_\infty^{(1)}, \quad (≠) \quad u=0 = \frac{1}{b}\varphi_\infty^{(1)} + \frac{3}{4}\frac{r_s}{b^2}\frac{4}{3}$$

$$\Rightarrow \varphi_\infty^{(1)} = -\frac{r_s}{b}$$



Shape of line: $r \rightarrow \infty$ $u \rightarrow 0$ $\varphi = 0$ and $\varphi = \pi$ $u^{(0)}$



$$\varphi = \varphi_{\infty}^{(0)} + \underbrace{\varphi_{\infty}^{(1)}}_{\text{small}}$$

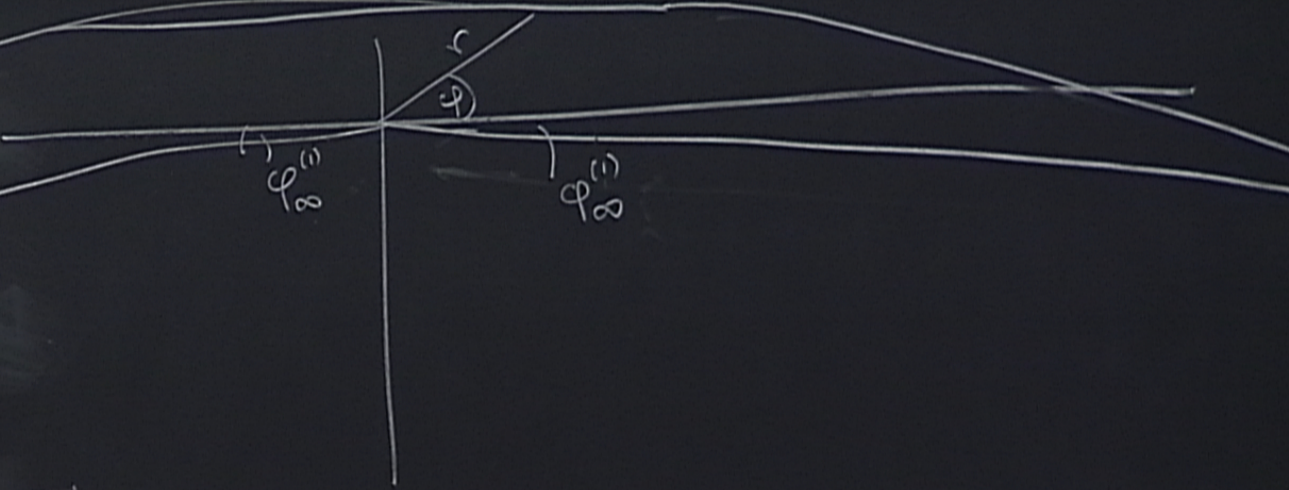
$$\varphi = 0 + \varphi_{\infty}^{(1)}, \quad (\neq)$$

$$u=0 = \frac{1}{b} \varphi_{\infty}^{(1)} + \frac{3}{4} \frac{r_s}{b} \frac{4}{3}$$

$$\Rightarrow \varphi_{\infty}^{(1)} = - \frac{r_s}{b}$$

$$\varphi = \pi + \varphi_{\infty}^{(1)} \rightarrow \varphi_{\infty}^{(1)} = + \frac{r_s}{b}$$

$u^{(2)}$
 $\varphi = 0$ and $\varphi = \pi$

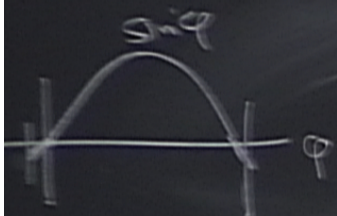


$$u=0 = \frac{1}{b} \varphi_{\infty}^{(1)} + \frac{3}{4} \frac{r_s}{b} = \frac{4}{3}$$

$$\Rightarrow \varphi_{\infty}^{(1)} = - \frac{r_s}{b}$$

$$\varphi_{\infty}^{(2)} = + \frac{r_s}{b}$$

shape of line: $r \rightarrow \infty$ $u \rightarrow 0$ $\varphi = 0$ and $\varphi = \pi$



$$\varphi_{\infty} = \varphi_{\infty}^{(0)} + \underbrace{\varphi_{\infty}^{(1)}}_{\text{small}}$$

$$\varphi = 0 + \varphi_{\infty}^{(1)}, \quad u=0 = \frac{1}{b} \varphi_{\infty}^{(1)} + \frac{3}{4} \frac{r_s}{b} \frac{4}{3}$$

$$\Rightarrow \varphi_{\infty}^{(1)} = - \frac{r_s}{b}$$

$$\varphi = \pi + \varphi_{\infty}^{(1)} \rightarrow \varphi_{\infty}^{(1)} = + \frac{r_s}{b}$$

$$\text{total deflection angle} = \frac{2r_s}{b} = \frac{4GM}{bc^2}$$

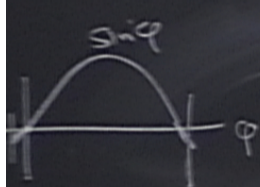
in radians

$b = \text{impact parameter.}$

$$u'' = -u + \frac{3r_s}{2}u^2 \quad ; \quad u = \frac{1}{b}\sin\varphi + \frac{3}{4}\frac{r_s}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right) + \dots$$

($\neq$)

shape of line: $r \rightarrow \infty \quad u \rightarrow 0 \quad \varphi = 0 \text{ and } \varphi = \pi$



$$\varphi_{\infty} = \varphi_{\infty}^{(0)} + \underbrace{\varphi_{\infty}^{(1)}}_{\text{small}}$$

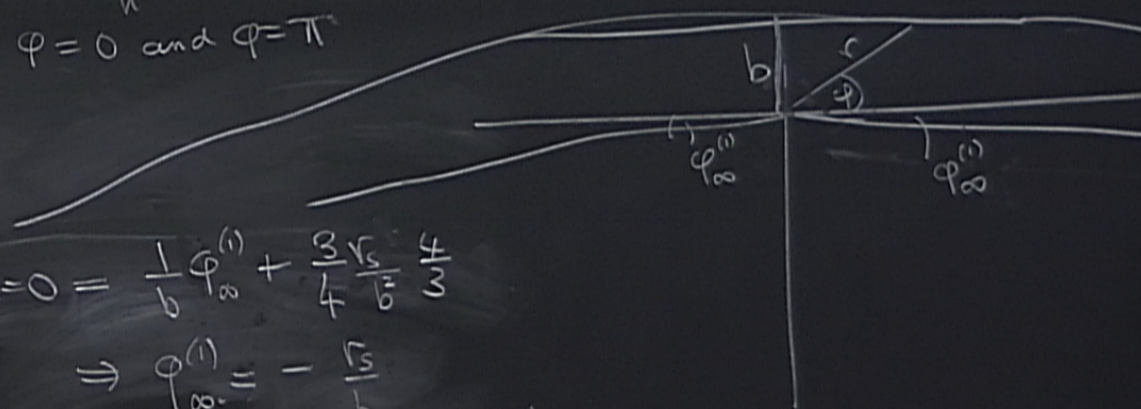
$$\varphi = 0 + \varphi_{\infty}^{(1)} \quad , \quad u=0 = \frac{1}{b}\varphi_{\infty}^{(1)} + \frac{3}{4}\frac{r_s}{b^2}\frac{4}{3}$$

$$\Rightarrow \varphi_{\infty}^{(1)} = -\frac{r_s}{b}$$

$$\varphi = \pi + \varphi_{\infty}^{(1)} \rightarrow \varphi_{\infty}^{(1)} = +\frac{r_s}{b}$$

total deflection angle = $\frac{2r_s}{b} = \frac{4GM}{bc^2}$, $b = \text{impact parameter.}$

in radians



$$\phi = \pi/2$$

$$\frac{1}{r} = \frac{1}{b} + \frac{1}{2} \frac{r_s}{b^2} \approx \frac{1}{b} \left(1 + \frac{1}{2} \frac{r_s}{b} \right)$$

$$\Rightarrow b \approx \left(1 + \frac{1}{2} \frac{r_s}{b} \right) r_{\text{closest}}$$

$$\approx \left(1 + \frac{1}{2} \frac{r_s}{r_{\text{closest}}} \right) r_{\text{closest}}$$

$$= \underline{r_{\text{closest}} + \frac{1}{2} r_s}$$

$$=0 = \frac{1}{b} \varphi_{\infty} + \frac{3}{4} \frac{r_s}{b^2} \frac{4}{3}$$

$$\Rightarrow \varphi_{\infty}^{(1)} = - \frac{r_s}{b}$$

$$+ \frac{r_s}{b}$$

$$+ \frac{GM}{bc^2}$$

$b \approx$ radius of closest approach.
for Sun, get $\Delta \varphi_{\text{total}} = \frac{4GM_0}{r_0 c^2} = 1.75''$

$$\varphi = \pi/2$$

$$\frac{1}{r} = \frac{1}{b} + \frac{1}{2} \frac{r_s}{b^2} \approx \frac{1}{b}$$

we at $\frac{\partial V}{\partial r} = 0$

$$= r_s$$

$$2J^2$$

$$3r_s J^2$$

$$\Rightarrow$$

$$r_{\pm} =$$

$$\frac{J^2}{r_s} \left(1 \pm \sqrt{1 - \frac{3r_s^2}{J^2}} \right)$$

$$\phi_{\infty}^{(1)} = +\frac{r_s}{b}$$

$$\Delta\phi = \frac{4GM}{bc^2}$$

$b \approx$ radius of closest approach.

For Sun, get $\Delta\phi_{\text{total}} = \frac{4GM_{\odot}}{rc^2} = 1.75''$

