Title: PSI 2016/2017 Relativity - Lecture 8

Date: Sep 15, 2016 09:00 AM

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Abstract:

reminder

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\alpha}\left(g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\nu\lambda,\alpha}\right)$$

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\sigma_{\rho\mu} - \Gamma^\alpha_{\rho\mu}\Gamma^\sigma_{\alpha\nu} + \Gamma^\alpha_{\rho\nu}\Gamma^\sigma_{\alpha\nu}$$

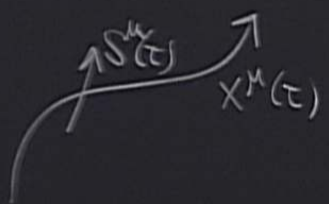
Guttfreund + Renn , "Road to Relativity"

$\Gamma^{\sigma}_{\alpha\nu}$

reminder $\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\alpha}\left(g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\nu\lambda,\alpha}\right)$

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\sigma_{\rho\mu} - \Gamma^\alpha_{\rho\mu}\Gamma^\sigma_{\alpha\nu} + \Gamma^\alpha_{\rho\nu}\Gamma^\sigma_{\alpha\nu}$$

Parallel transport along a curve



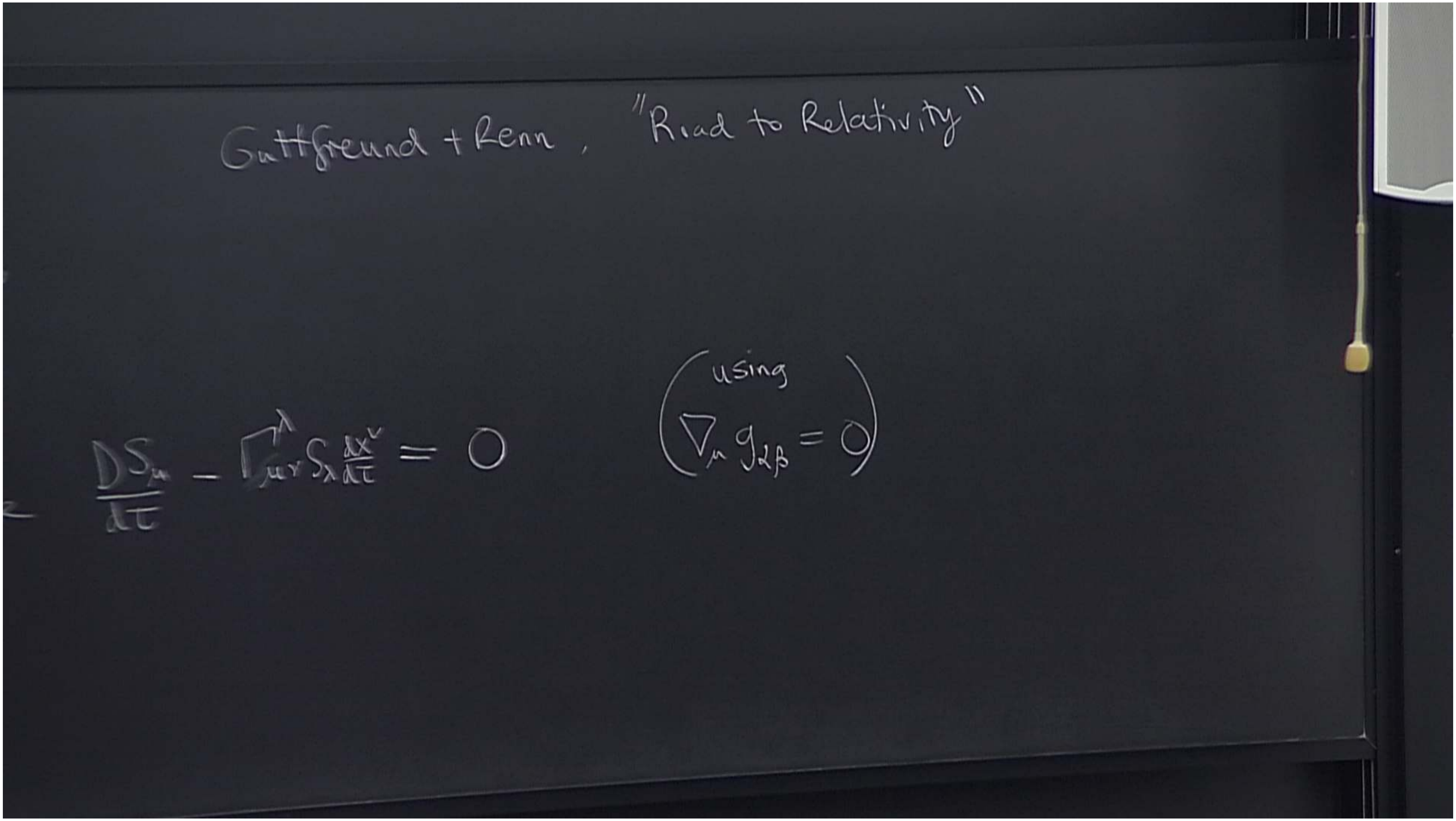
parallel transport: $\frac{DS^\mu}{D\tau} \equiv \frac{dS^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} S^\nu \frac{dx^\lambda}{d\tau} = 0$

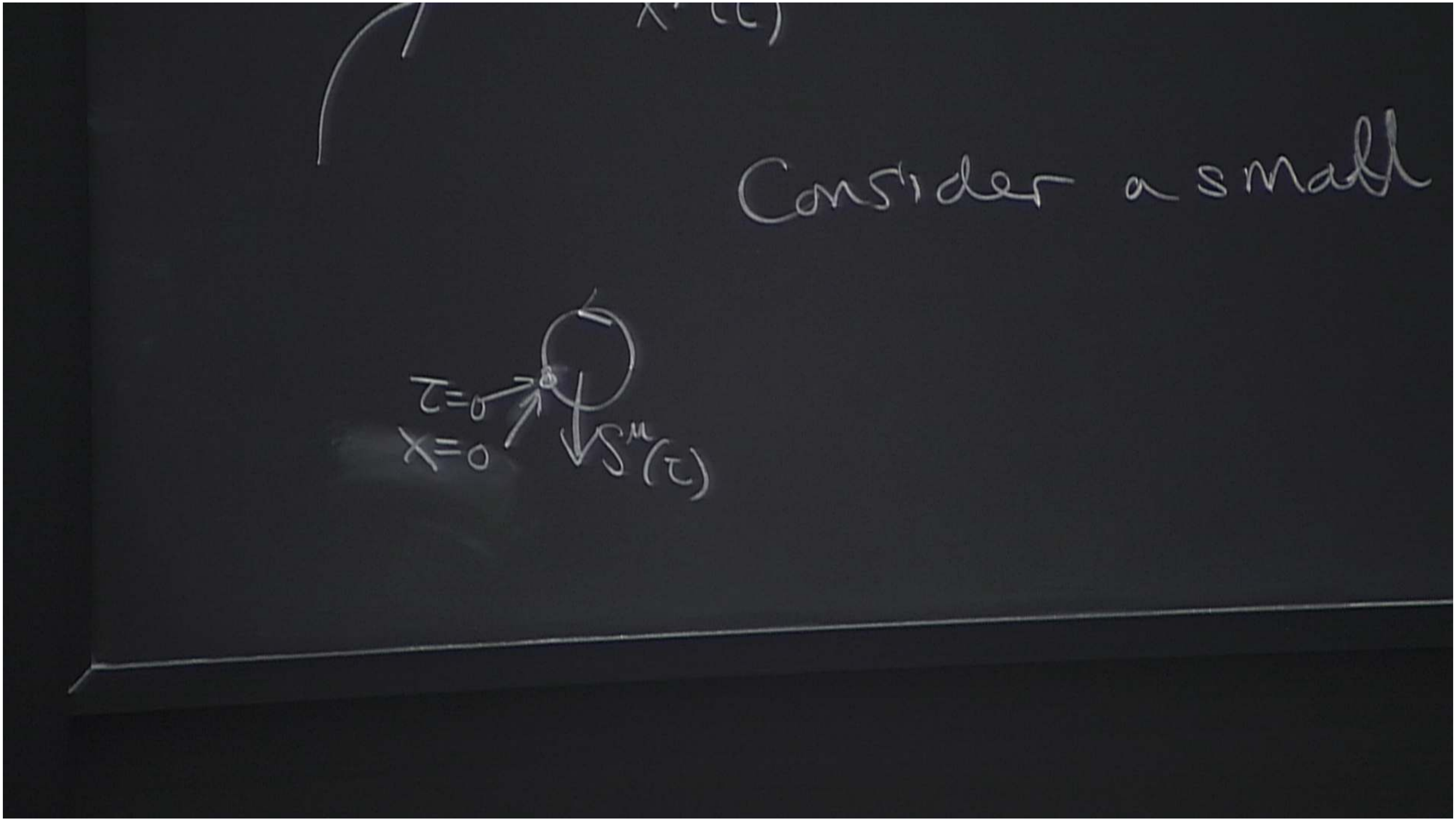
Guttfreund + Renn ,

$$= \frac{1}{2} g^{\mu\alpha}\left(g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\nu\lambda,\alpha}\right)$$

$$= \partial_\mu \Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\sigma_{\rho\mu} - \Gamma^\alpha_{\rho\mu}\Gamma^\sigma_{\alpha\nu} + \Gamma^\alpha_{\rho\nu}\Gamma^\sigma_{\alpha\nu}$$

$$\frac{DS^\mu}{D\tau} \equiv \frac{dS^\mu}{dt} + \Gamma^\mu_{\nu\lambda} S^\nu \frac{dx^\lambda}{d\tau} = 0 \text{ ; likewise } \frac{DS_\mu}{d\tau} - \Gamma^\lambda_{\mu\nu} S_\lambda \frac{dx^\nu}{d\tau} = 0$$

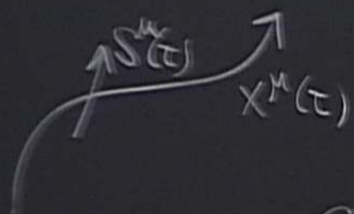

Guttfreund + Renn , "Road to Relativity"
$\frac{DS_\mu}{d\tau} - \Gamma^{\lambda}_{\mu\nu} S_\lambda \frac{dx^\nu}{d\tau} = 0$
(using $\nabla_\mu g_{\alpha\beta} = 0$)


Consider a small
τ=0
x=0
$\delta^{\mu}(\tau)$

reminder $\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\alpha} (g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\nu\lambda,\alpha})$

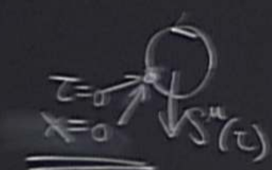
$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\sigma_{\rho\mu} - \Gamma^\alpha_{\rho\mu}\Gamma^\sigma_{\alpha\nu} + \Gamma^\alpha_{\rho\nu}\Gamma$$

Parallel transport along a curve



parallel transport: $\frac{DS^\mu}{D\tau} \equiv \frac{dS^\mu}{dt} + \Gamma^\mu_{\nu\lambda} S^\nu \frac{dx^\lambda}{d\tau} = 0$; likew

Consider a small closed loop (x^μ small)



$$\Gamma^\gamma_{\mu\alpha} = \Gamma^\gamma_{\mu\alpha}(0) + x^\rho(\partial_\rho \Gamma^\gamma_{\mu\alpha})(0) + \dots$$

$\ldots g_{\lambda\alpha,\nu} - g_{\nu\lambda,\alpha})$

$-\partial_\nu \Gamma^\sigma_{\rho\mu} - \Gamma^\alpha_{\rho\mu}\Gamma^\sigma_{\alpha\nu} + \Gamma^\alpha_{\rho\nu}\Gamma^\sigma_{\alpha\nu}$

$$\frac{dS^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} S^\nu \frac{dx^\lambda}{d\tau} = 0 \quad ; \text{ like } \frac{DS_\mu}{d\tau} = \frac{dS_\mu}{d\tau} - \Gamma^\lambda_{\mu\nu} S_\lambda \frac{dx^\nu}{d\tau} = 0$$

$\left(\text{using } \nabla_\mu g_{\alpha\beta} = 0\right)$

$(x^\mu$ small$)$

$$\Rightarrow \frac{dS_\mu}{d\tau} = \Gamma^\lambda_{\mu\nu} S_\lambda \frac{dx^\nu}{d\tau}$$

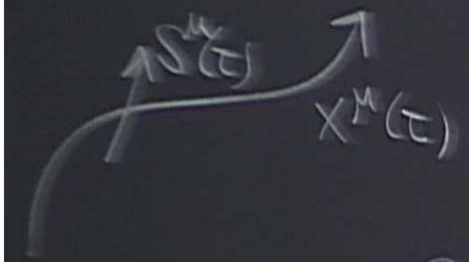
$\ldots)(0) + \ldots$

$$\frac{DS_\mu}{d\tau} = \frac{dS_\mu}{d\tau} - \Gamma^{\lambda}_{\mu\nu} S_\lambda \frac{dx^\nu}{d\tau} = 0$$

(using $\nabla_\mu g_{\alpha\beta} = 0$)

$$\Rightarrow \frac{dS_\mu}{d\tau} = \Gamma^{\lambda}_{\mu\nu} S_\lambda \frac{dx^\nu}{d\tau} \Rightarrow \frac{\partial S_\mu}{\partial x^\nu} = \Gamma^{\lambda}_{\mu\nu} S_\lambda$$

(Pa)rallel transport along a curve



parallel transport: $\frac{DS^\mu}{D\tau} \equiv \frac{dS^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} S^\nu \frac{dx^\lambda}{d\tau} = 0$

Consider a small closed loop (x^μ small)

$$\Gamma^\gamma_{\mu\alpha} = \Gamma^\gamma_{\mu\alpha}(0) + x^\rho \left(\partial_\rho \Gamma^\gamma_{\mu\alpha}\right)(0) + \ldots$$

$$S_\mu = S_\mu(0) + \Gamma^\gamma_{\mu\alpha} S_\gamma x^\alpha + \ldots$$

$$= S_\mu(0) + \Gamma^{\gamma}_{\mu\alpha} S_\gamma x^\alpha + \dots$$

$$\ldots) + \int_0^\tau d\tau \left(\Gamma^{\gamma}_{\mu\nu}(0) + x^\rho(\tau)\, \partial_\rho \Gamma^{\gamma}_{\mu\nu}(0) \right) \left(S_\gamma(0) + \Gamma^{\beta}_{\gamma\alpha}(0)\, x^\alpha S_\beta(0) \right) \frac{dx^\nu}{d\tau}$$

, we get $\int \frac{dx^\mu}{d\tau}\, d\tau$

$$= \partial_\mu \Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\sigma_{\rho\mu} - \Gamma^\alpha_{\rho\mu}\Gamma^\sigma_{\alpha\nu} + \Gamma^\alpha_{\rho\nu}\Gamma^\sigma_{\alpha\nu}$$

$$\frac{DS^\mu}{D\tau} \equiv \frac{dS^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} S^\nu \frac{dx^\lambda}{d\tau} = 0$$

; like $\frac{DS_\mu}{d\tau} = \frac{dS_\mu}{d\tau} - \Gamma^\lambda_{\mu\nu} S_\lambda \frac{dx^\nu}{d\tau} = 0$ (using $\nabla_\mu g_{\alpha\beta} = 0$)

closed loop (x^μ small)

$$\Rightarrow \frac{dS_\mu}{d\tau} = \Gamma^\lambda_{\mu\nu} S_\lambda \frac{dx^\nu}{d\tau} \quad (*) \quad \Rightarrow \frac{\partial S_\mu}{\partial x^\nu} = \Gamma^\lambda_{\mu\nu} S_\lambda$$

$$(0) + x^\rho (\partial_\rho \Gamma^\gamma_{\mu\alpha})(0) + \dots$$

$$S_\mu(0) + \Gamma^\gamma_{\mu\alpha} S_\gamma x^\alpha + \dots$$

$$\int_0^\tau d\tau \left(\Gamma^\gamma_{\mu\nu}(0) + x^\rho(\tau)\,\partial_\rho \Gamma^\gamma_{\mu\nu}(0)\right)\left(S_\gamma(0) + \Gamma^\beta_{\gamma\alpha}(0)\, x^\alpha S_\beta(0)\right)\frac{dx^\nu}{d\tau}$$

$$\int \frac{dx^\mu}{d\tau}$$

$$(0) + \int_0^\tau d\tau \left(\Gamma^\gamma_{\mu\nu}(0) + x^\rho(\tau)\, \partial_\rho \Gamma^\gamma_{\mu\nu}(0) \right) \left(S_\gamma(0) + \Gamma^\beta_{\gamma\alpha}(0)\, x^\alpha S_\beta(0) \right) \frac{dx^\nu}{d\tau}$$

x, we get $\int \frac{dx^\mu}{d\tau} d\tau$, but $\oint dx^\mu = 0$.

$$= \left[x^\mu(\tau) \right]_0^0$$

...ishing contribution :
...nd closed loop

$$S_\mu(0) + \left(\partial_\rho \Gamma^\gamma_{\mu\nu} + \Gamma^\beta_{\mu\nu} \Gamma^\gamma_{\beta\rho} \right)(0)\, S_\gamma(0) \oint x^\rho \frac{dx^\nu}{d\tau}$$

$$+ \int_0^\tau d\tau \left(\Gamma^\gamma_{\mu\nu}(0) + x^\rho(\tau)\,\partial_\rho\Gamma^\gamma_{\mu\nu}(0)\right)\left(S_\gamma(0) + \Gamma^\beta_{\gamma\alpha}(0)\, x^\alpha S_\beta(0)\right)\frac{dx^\nu}{d\tau}$$

we get $\int \frac{dx^\mu}{d\tau}\, d\tau$, but $\oint dx^\mu = 0$.

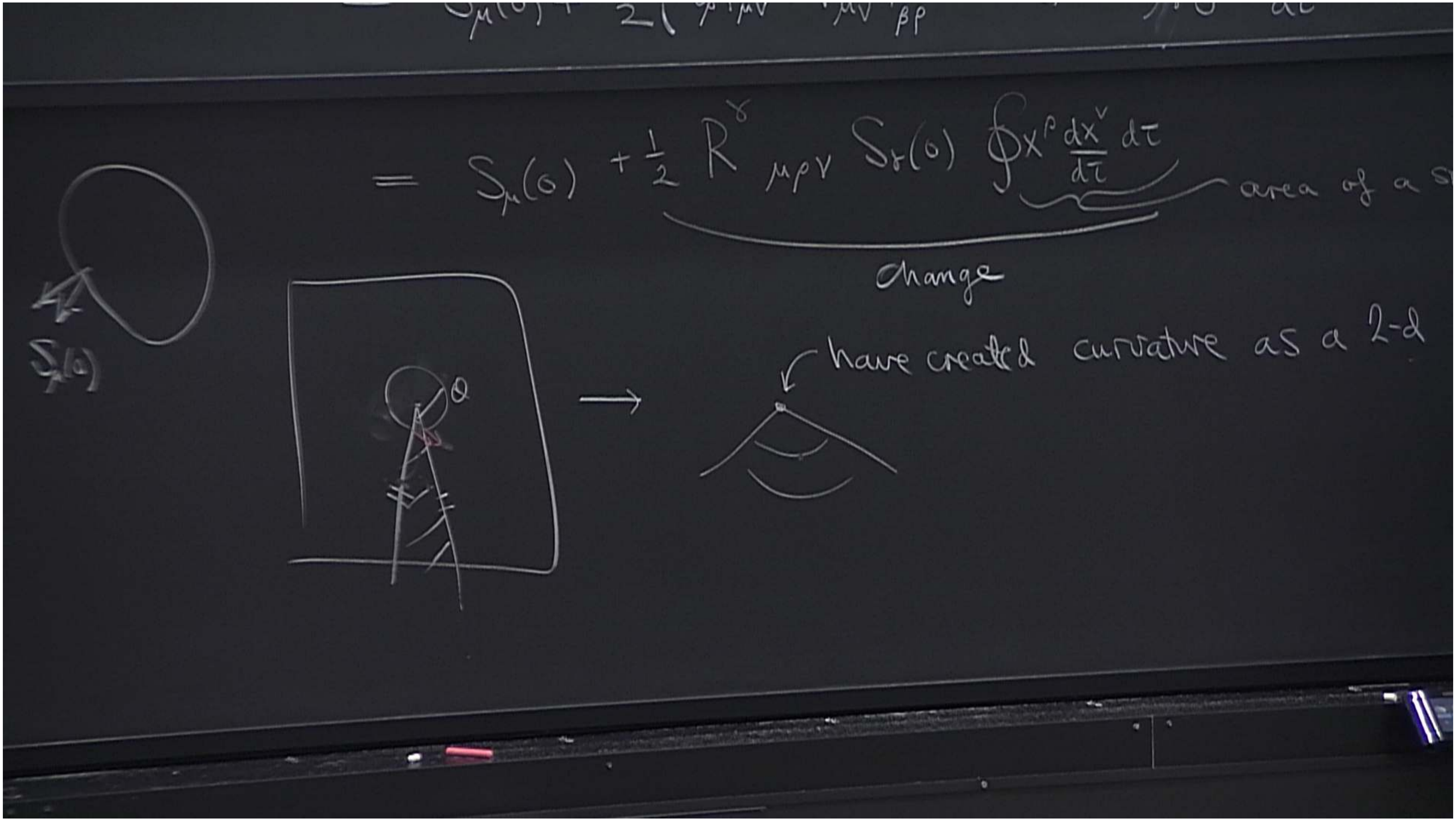
$$= \left[x^\mu(\tau)\right]_0^0$$

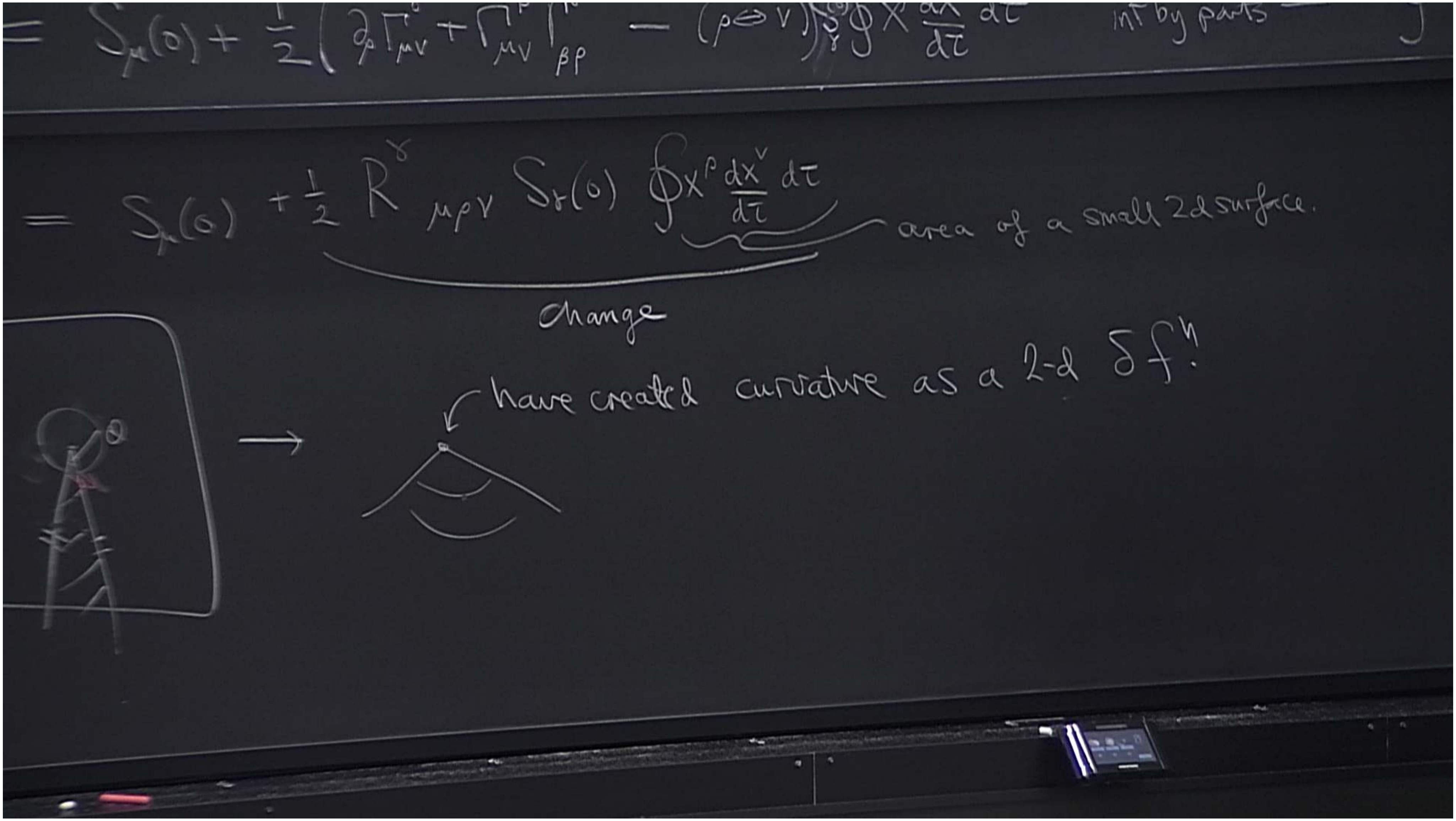
[…] contribution :
[cl]osed loop

$$S_\mu(0) + \left(\partial_\rho\Gamma^\gamma_{\mu\nu} + \Gamma^\beta_{\mu\nu}\Gamma^\gamma_{\beta\rho}\right)(0)\, S_\gamma(0) \oint x^\rho \frac{dx^\nu}{d\tau}\, d\tau$$

$$= \frac{1}{2}\left(\partial_\rho\Gamma^\gamma_{\mu\nu} + \Gamma^\beta_{\mu\nu}\Gamma^\gamma_{\beta\rho} - (\rho\leftrightarrow\nu)\right) S_\gamma(0) \oint x^\rho \frac{dx^\nu}{d\tau}\, d\tau$$

int by parts $= -\oint x^\nu \frac{dx^\rho}{d\tau}\, d\tau$


$= S_\mu(0) + \frac{1}{2} R^{\gamma}{}_{\mu\rho\nu} S_\gamma(0) \oint x^\rho \frac{dx^\nu}{d\tau} d\tau$
area of a s
change
have created curvature as a 2-d
$S_\mu(0)$
θ

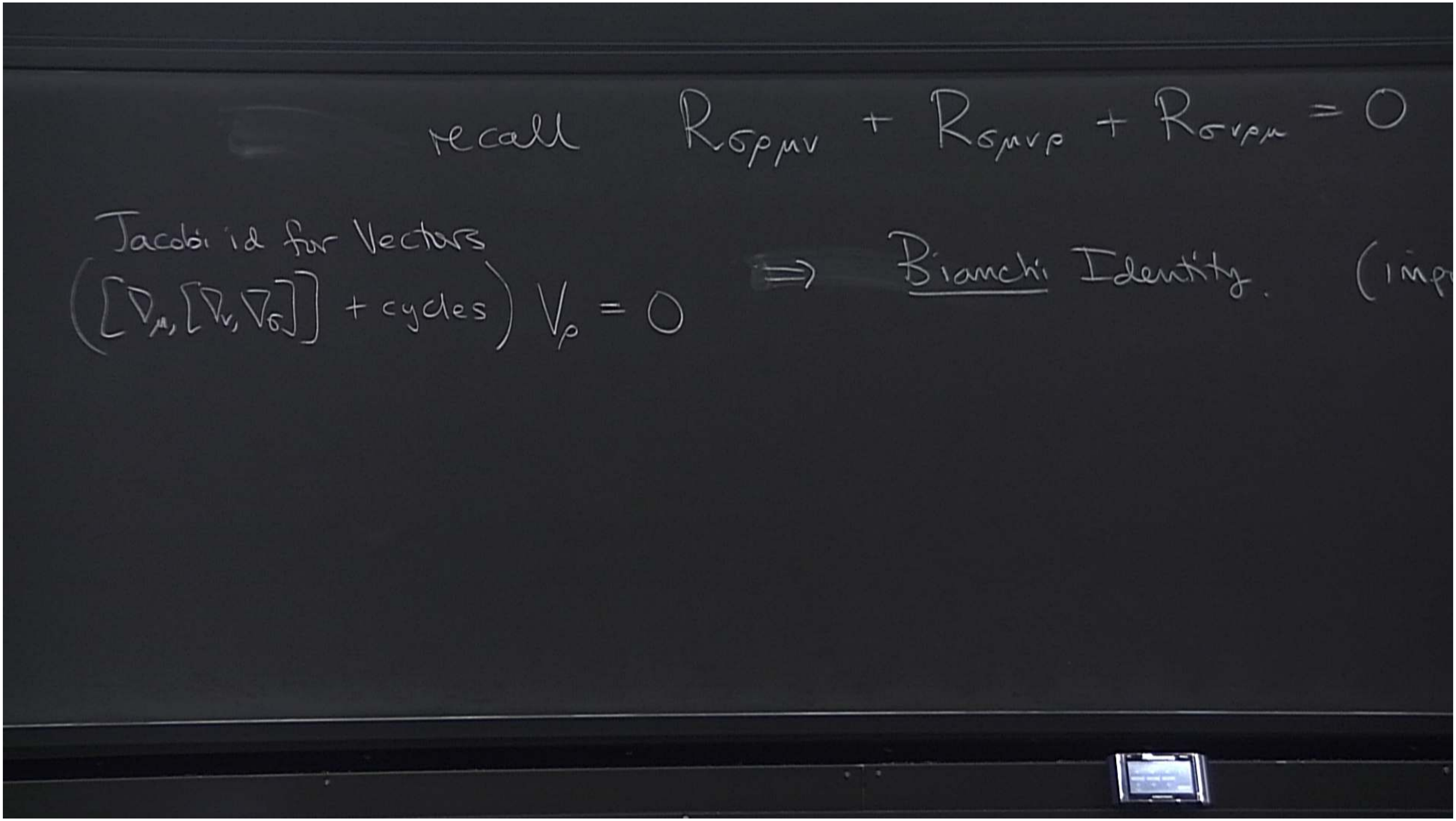


$= S_\mu(0) + \frac{1}{2}\left(\partial_\rho \Gamma^{\gamma}_{\mu\nu} + \Gamma^{\rho}_{\mu\nu}\Gamma^{\gamma}_{\beta\rho} - (\rho \leftrightarrow \nu)\right) S_\gamma \oint x^\rho \frac{dx^\nu}{d\tau} d\tau$
int by parts
$= S_\mu(0) + \frac{1}{2} R^{\gamma}{}_{\mu\rho\nu} S_\gamma(0) \oint x^\rho \frac{dx^\nu}{d\tau} d\tau$
area of a small 2d surface.
change
have created curvature as a 2-d δ f^n!

Prove that $R^{\mu}{}_{\nu\rho\lambda} = 0 \Leftrightarrow \exists$ coords in which $g_{\mu\nu} = \eta_{\mu\nu}$

using this result (*):

S. Weinberg "Gravitation + Cosmology" p. 138


recall $R_{\sigma\rho\mu\nu} + R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} = 0$
Jacobi id for Vectors
$\left([\nabla_\mu, [\nabla_\nu, \nabla_\sigma]] + \text{cycles} \right) V_\rho = 0$
$\Rightarrow$ Bianchi Identity.
(imp

$$- R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} = 0$$

<u>Bianchi</u> Identity. (important for construction of Einstein equations)

$$\ldots + R_{\sigma\nu\rho\mu} = 0$$

Identity. (important for construction of Einstein equations)

$$= \frac{1}{2}\left(g_{\nu\sigma,\rho\mu} - g_{\rho\nu,\sigma\mu} - g_{\mu\sigma,\rho\nu} + g_{\rho\mu,\sigma\nu}\right) + \underbrace{g_{\alpha\beta}\left(\Gamma^{\alpha}_{\nu\sigma}\Gamma^{\beta}_{\rho\mu} - \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\beta}_{\rho\nu}\right)}_{\text{quadratic in } \Gamma}$$

[Exact, Weinberg p.143]

$$R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} = 0$$

Bianchi Identity. (important for construction of Einstein eq

$$R_{\sigma\rho\mu\nu} = \frac{1}{2}\left(g_{\nu\sigma,\rho\mu} - g_{\rho\nu,\sigma\mu} - g_{\mu\sigma,\rho\nu} + g_{\rho\mu,\sigma\nu}\right) + g_{\alpha\beta}\left(\Gamma^{\alpha}_{\nu\sigma}\Gamma^{\beta}_{\rho\mu} - \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\beta}_{\rho\nu}\right)$$

quadratic in Γ

$$R_{\sigma\rho\mu\nu;\lambda} = \frac{1}{2}\partial_\lambda\left(\quad \right)$$

add cycles, find that

$$R_{\sigma\rho\mu\nu;\lambda} + R_{\sigma\rho\nu\lambda;\mu} + R_{\sigma\rho\lambda\mu;\nu} = 0$$

Bianchi Identity

$$\nabla_\mu (f \quad g)$$

$$= (\nabla_\mu f)\, g + f\, \nabla_\mu g .$$

find that $\ldots_{\rho\nu\lambda;\mu} + R_{\sigma\rho\lambda\mu;\nu}$ [illegible]

$$\times g^{\sigma\mu} : \quad R_{\rho\nu;\lambda} + R^{\sigma}{}_{\rho\nu\lambda;\sigma} - R_{\rho\lambda;\nu} = 0$$

$$\times g^{\rho\nu} \quad R_{;\lambda} - R^{\sigma}{}_{\lambda;\sigma} - R^{\sigma}{}_{\lambda;\sigma} = 0$$

$$\left(R^{\sigma}{}_{\lambda} - \tfrac{1}{2} R \delta^{\sigma}{}_{\lambda}\right)_{;\sigma} = 0$$

Einstein tensor $G^{\sigma}{}_{\lambda}$

$$G^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R$$

$: \; G_{\sigma\lambda} = R_{\sigma\lambda} - \tfrac{1}{2} g_{\sigma\lambda} R$, obeys $\nabla_{\mu}\left(G^{\mu\nu}\right) = 0$ "divergence-free"

...uations

... gravity in the limit of small velocities and weak gravitational fields:

$|\vec{v}|/c \ll 1 \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$

$$(1+A)^{-1} = 1 - A + A^2 - A^3 \dots$$

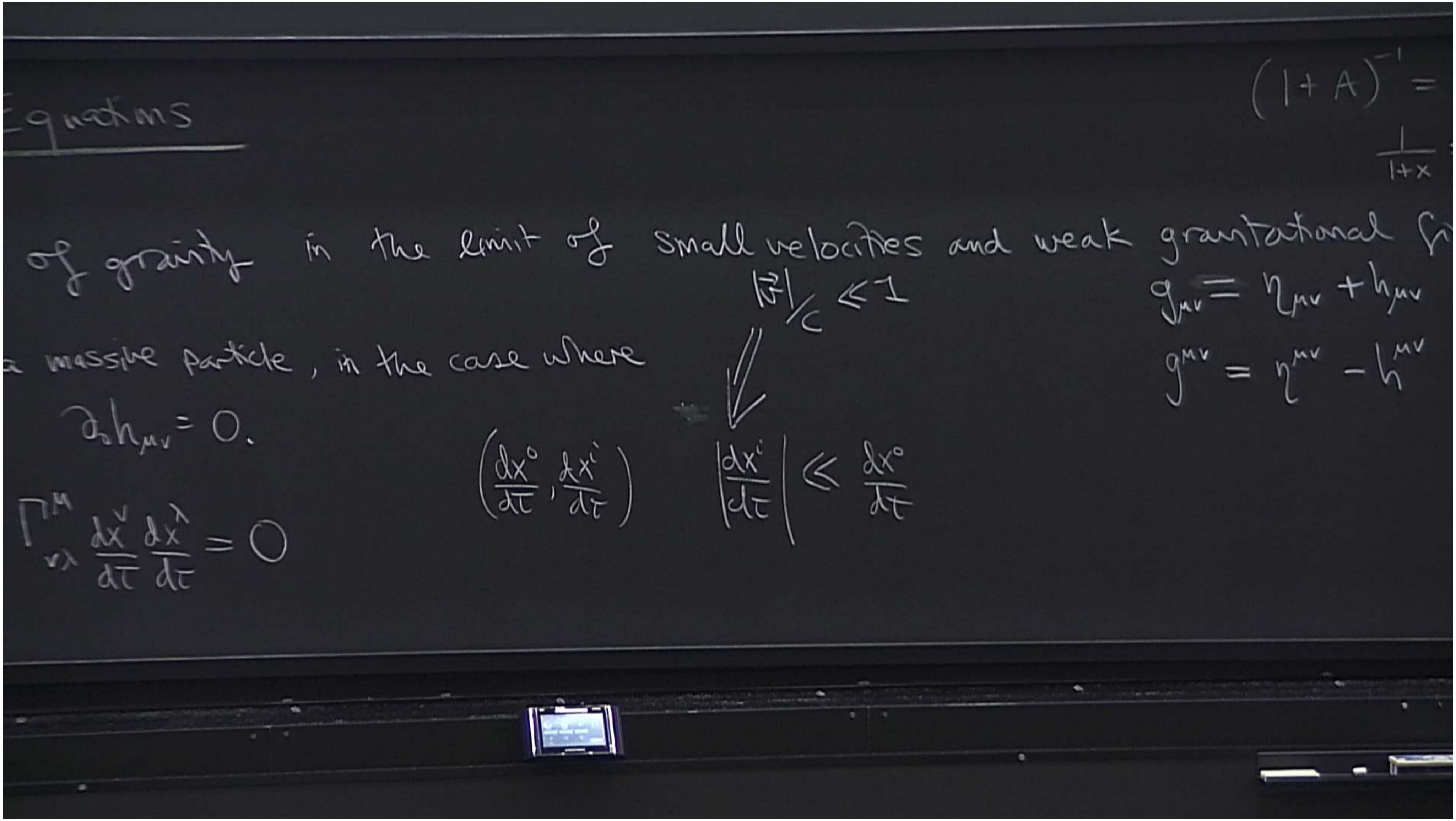
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 \dots$$

Small velocities and weak gravitational fields:

$$|\vec{v}|/c \ll 1$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2)$$


Equations
$(1+A)^{-1} =$
$\frac{1}{1+x}$
of gravity in the limit of small velocities and weak gravitational fi
$|\vec{v}|/c \ll 1$
$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$
$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$
a massive particle, in the case where
$\partial_0 h_{\mu\nu} = 0.$
$\left(\frac{dx^0}{d\tau}, \frac{dx^i}{d\tau}\right)$
$\left|\frac{dx^i}{d\tau}\right| \ll \frac{dx^0}{d\tau}$
$\Gamma^{\mu}_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$

Einstein Field Equations

Want to reproduce Newton's theory of gravity in the limit of s

Consider the motion of a massive particle, in the case where

$h_{\mu\nu}$ is indept of t $\quad \partial_0 h_{\mu\nu} = 0.$

$\left(\frac{dx^0}{d\tau}, \frac{dx^i}{d\tau}\right)$

Geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

$\times g^{\sigma\mu}$: R_ρ

$$(1+A)^{-1} = 1 - A + A^2 - A^3 \dots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 \dots$$

limit of small velocities and weak gravitational fields:

$|\vec{v}|/c \ll 1$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2)$$

case where

$$\left(\frac{dx^0}{d\tau}, \frac{dx^i}{d\tau}\right) \qquad \left|\frac{dx^i}{d\tau}\right| \ll \frac{dx^0}{d\tau}$$

tensor G_λ . $G_{\sigma\lambda} = R_{\sigma\lambda} - \frac{1}{2} g_{\sigma\lambda} R$,

$$\frac{d^2 x^i}{d\tau^2} = - \Gamma^i_{00}\, c^2 \left(\frac{dt}{d\tau}\right)^2$$

$$\frac{d^2 ct}{d\tau^2} = - \Gamma^0_{00}\, c^2 \left(\frac{dt}{d\tau}\right)^2$$

$$\Gamma^0_{00} = \frac{1}{2} g^{0\alpha} \left(g_{0\alpha,0} + g_{0\alpha,0} - g_{00,\alpha} \right)$$

$g^{0\alpha} = \eta^{0\alpha} - h^{0\alpha} + o(h^2)$

$g_{00,\alpha} \rightarrow h_{00,\alpha}$

$\eta^{0\alpha}$ zero unless $\alpha = 0$ but $\partial_0 h_{00} = 0$ by assumption

$\Rightarrow \Gamma^0_{00} = o(h^2)$, negligible to 1st order.

G.E. becomes
$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00}\, c^2 \left(\frac{dt}{d\tau}\right)^2 \qquad (1)$$

$$\frac{d^2ct}{d\tau^2} = -\Gamma^0_{00}\, c^2 \left(\frac{dt}{d\tau}\right)^2 \qquad (2)$$

$(2) \Rightarrow t \propto \tau$

$$(1) \Longrightarrow \frac{d^2x^i}{dt^2} = -c^2\Gamma^i_{00}$$

$$g^{0\alpha} = \eta^{0\alpha} - h^{0\alpha} + O(h^2)$$

$$\Gamma^0_{00} = \frac{1}{2} g^{0\alpha}\left(\cancel{g_{0\alpha,0}} + \cancel{g_{0\alpha,0}} - g_{00,\alpha} \right)$$

$$g_{00,\alpha} \to h_{00,\alpha}$$

$\eta^{0\alpha}$ zero unless $\alpha = 0$ but $\partial_0 h_{00} = 0$ by assumption

$$\Rightarrow \Gamma^0_{00} = O(h^2)$$, negligible to 1st order.

$$\Gamma^i_{00} = \frac{1}{2}\delta^{ij}\left(\cancel{g_{0j,0}} + \cancel{g_{0j,0}} - g_{00,j} \right) = -\frac{1}{2} h_{00,i}$$

$$\frac{d^2 x^i}{d\tau^2} = -\Gamma^i_{00}\, c^2 \left(\frac{dt}{d\tau}\right)^2 \quad (1)$$

$$\Gamma^0_{00} = \frac{1}{2} g^{0\alpha}\left(g\right.$$

$$\frac{d^2 ct}{d\tau^2} = -\Gamma^0_{00}\, c^2 \left(\frac{dt}{d\tau}\right)^2 \quad (2)$$

$\eta^{0\alpha}$ zero unless $\alpha =$

$(2) \Rightarrow t \propto \tau$

$$\Rightarrow \Gamma^0_{00} = 0$$

$$(1) \Rightarrow \frac{d^2 x^i}{dt^2} = -c^2 \Gamma^i_{00}$$

$$\Gamma^i_{00} = \frac{1}{2}\delta^{ij}\left(\right.$$

$$= \frac{1}{2} h_{00,i}\, c^2$$

G.E. becomes $$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00}\, c^2 \left(\frac{dt}{d\tau}\right)^2 \qquad (1)$$

$\eta^{ij} = \delta^{ij}$ $$\frac{d^2t}{d\tau^2} = -\Gamma^0_{00}\, c^2 \left(\frac{dt}{d\tau}\right)^2 \qquad (2)$$

$(2) \Rightarrow t \propto \tau$

$$(1) \Longrightarrow \frac{d^2x^i}{dt^2} = -c^2\,\Gamma^i_{00}$$

$$= \frac{1}{2} h_{00,i}\, c^2$$

cf Newton $\quad m \frac{d^2 \vec{x}}{dt^2} = \vec{F}_g = -m \underline{\nabla} \phi \quad ; \; \phi$ determined

$\Longrightarrow$ for weak, static fields, $\quad h_{00} = -\frac{2\phi}{c^2}$

e.g. around a mass M, $\quad \phi = -\frac{GM}{r}$

x, we get $\int \frac{dx^\mu}{d\tau} d\tau$, but $\oint dx^\mu = 0$.

$$m \frac{d^2 \vec{x}}{dt^2} = \vec{F}_g = -m \nabla \phi$$; ϕ determined by Poisson eq^n: $\vec{\nabla}$

$\Longrightarrow$ for weak, static fields, $h_{00} = -\frac{2\phi}{c^2}$

e.g. around a spherical mass M, $\phi = -\frac{GM}{r}$ $\Rightarrow$ $h_{00} = +\frac{2GM}{rc^2}$

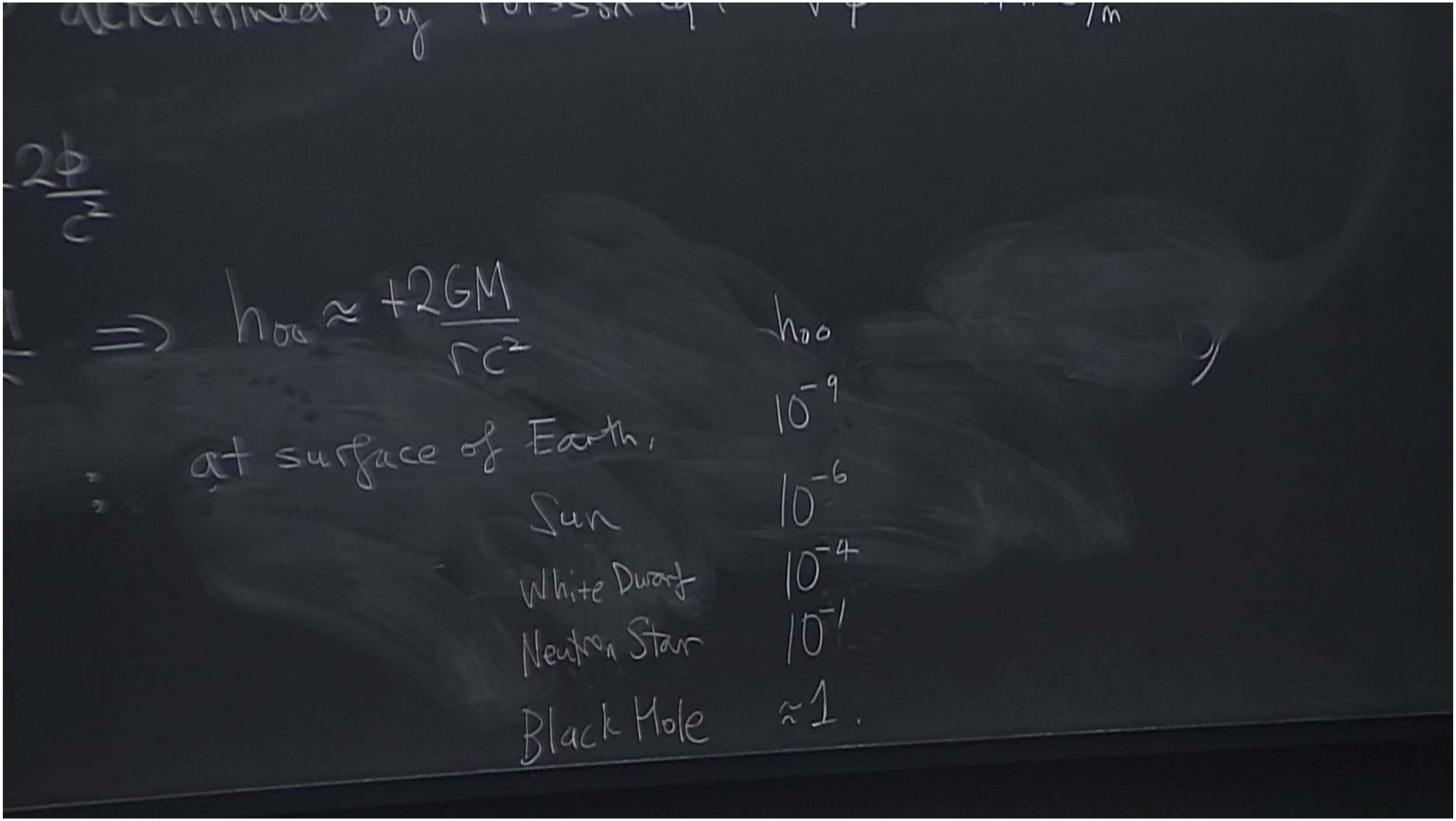
$$m\frac{d^2\vec{x}}{dt^2} = \vec{F}_g = -m\underline{\nabla}\phi$$

; ϕ determined by Poisson eq^n: $\vec{\nabla}$

$\Longrightarrow$ for weak, static fields, $h_{00} = -\frac{2\phi}{c^2}$

e.g. around a spherical mass M, $\phi = -\frac{GM}{r} \Rightarrow h_{00} = +\frac{2GM}{rc^2}$

$$g_{00} = -1 + h_{00} = -\left(1 - \frac{2GM}{rc^2}\right)$$



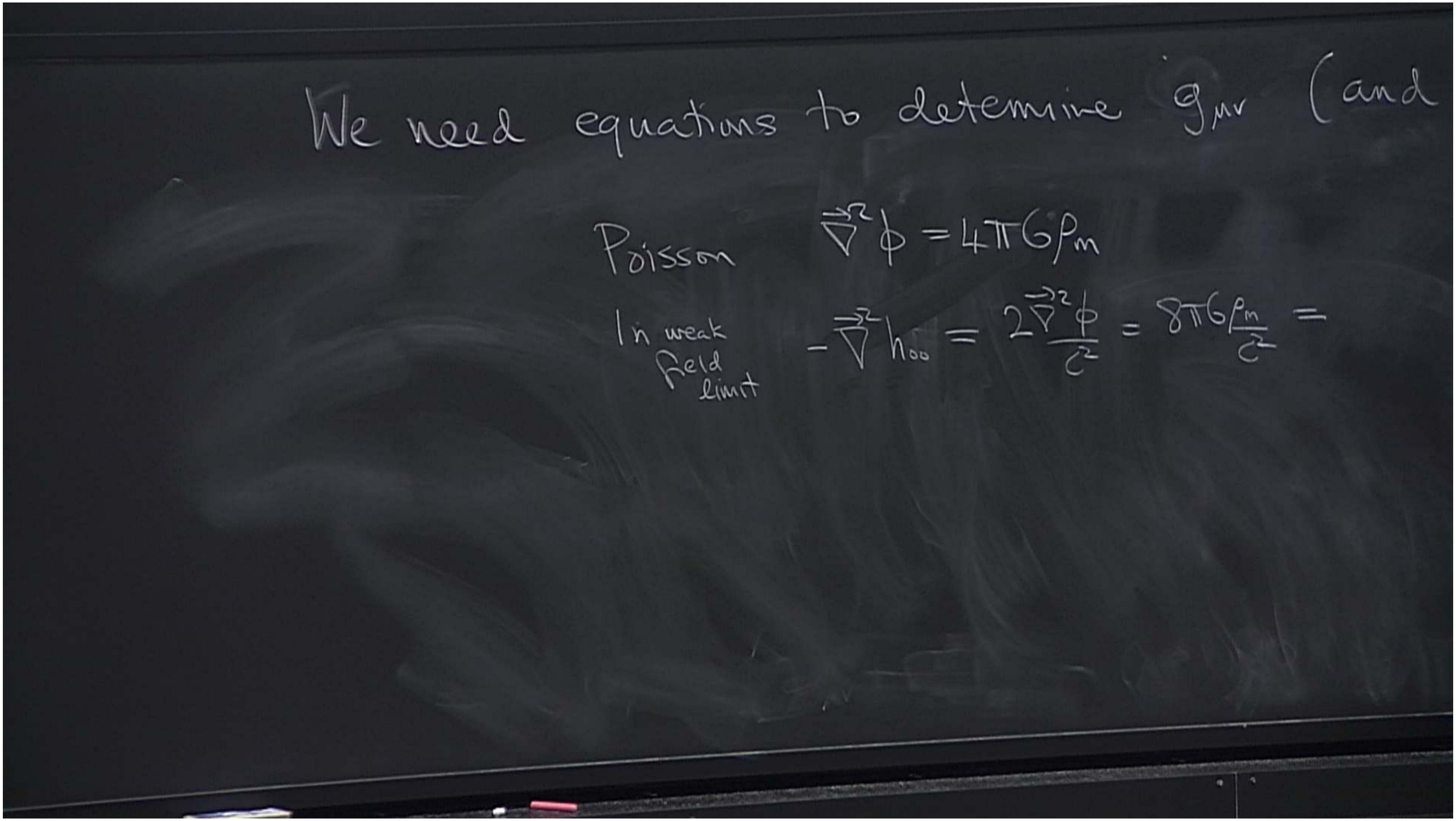
⇒ $h_{00} \approx +\frac{2GM}{rc^2}$
h_{00}
at surface of Earth, 10^{-9}
Sun 10^{-6}
White Dwarf 10^{-4}
Neutron Star 10^{-1}
Black Hole ≈ 1.

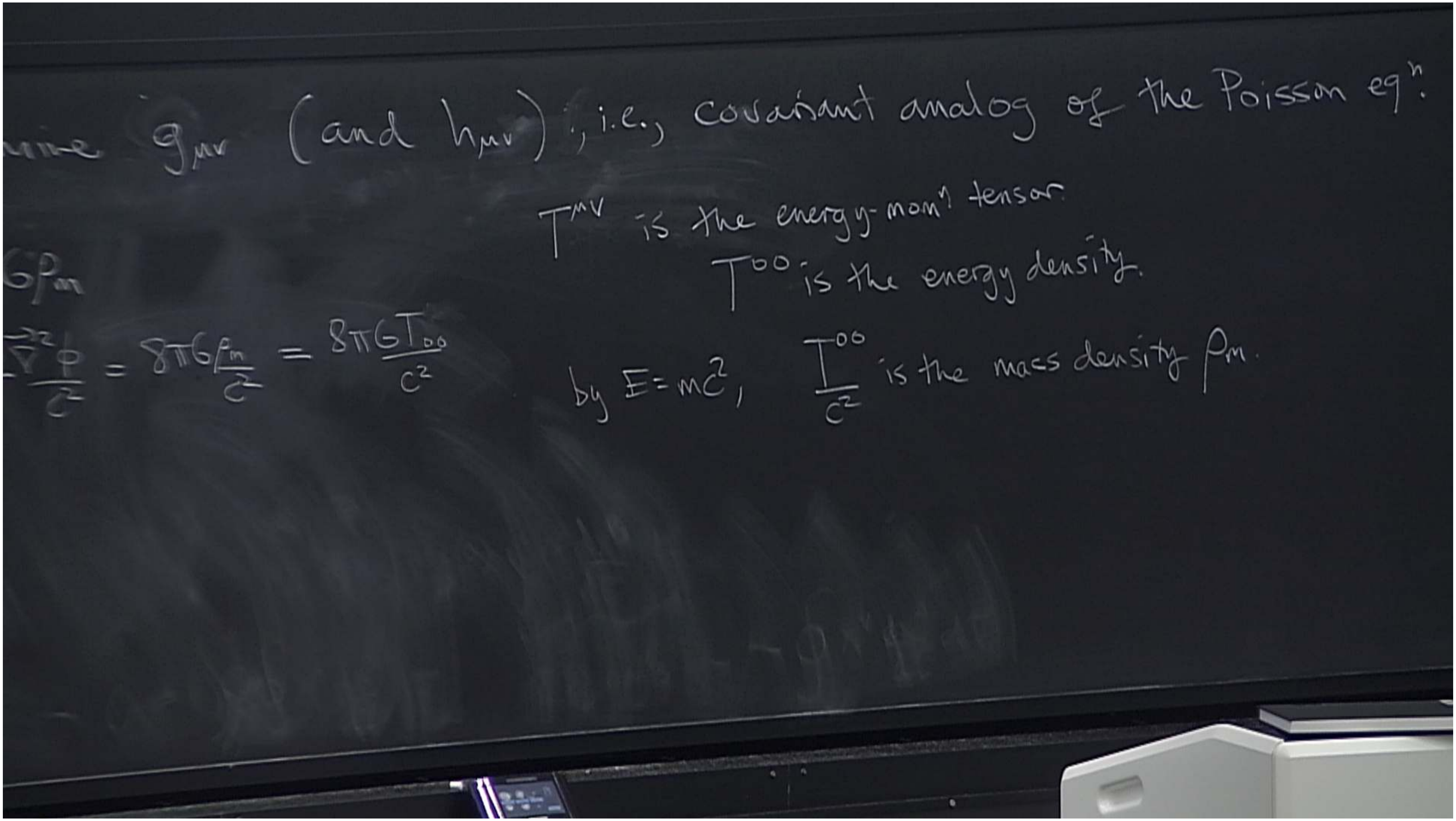
cf Newton $\quad m\dfrac{d^2\vec{x}}{dt^2} = \vec{F}_g = -m\underline{\nabla}\phi \quad ; \; \phi$ dete

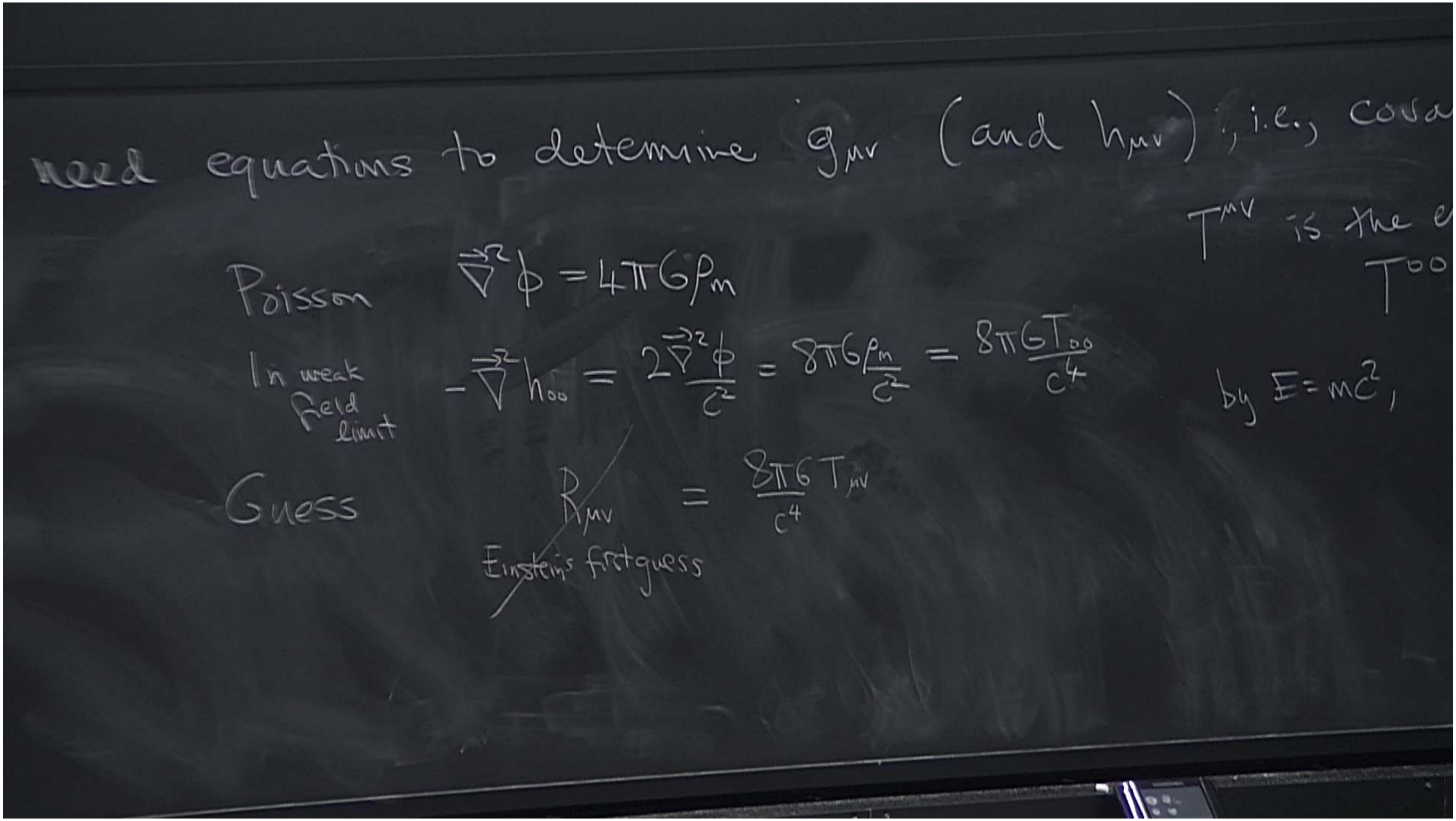
$\Longrightarrow$ for weak, static fields, $\quad h_{00} = -\dfrac{2\phi}{c^2}$

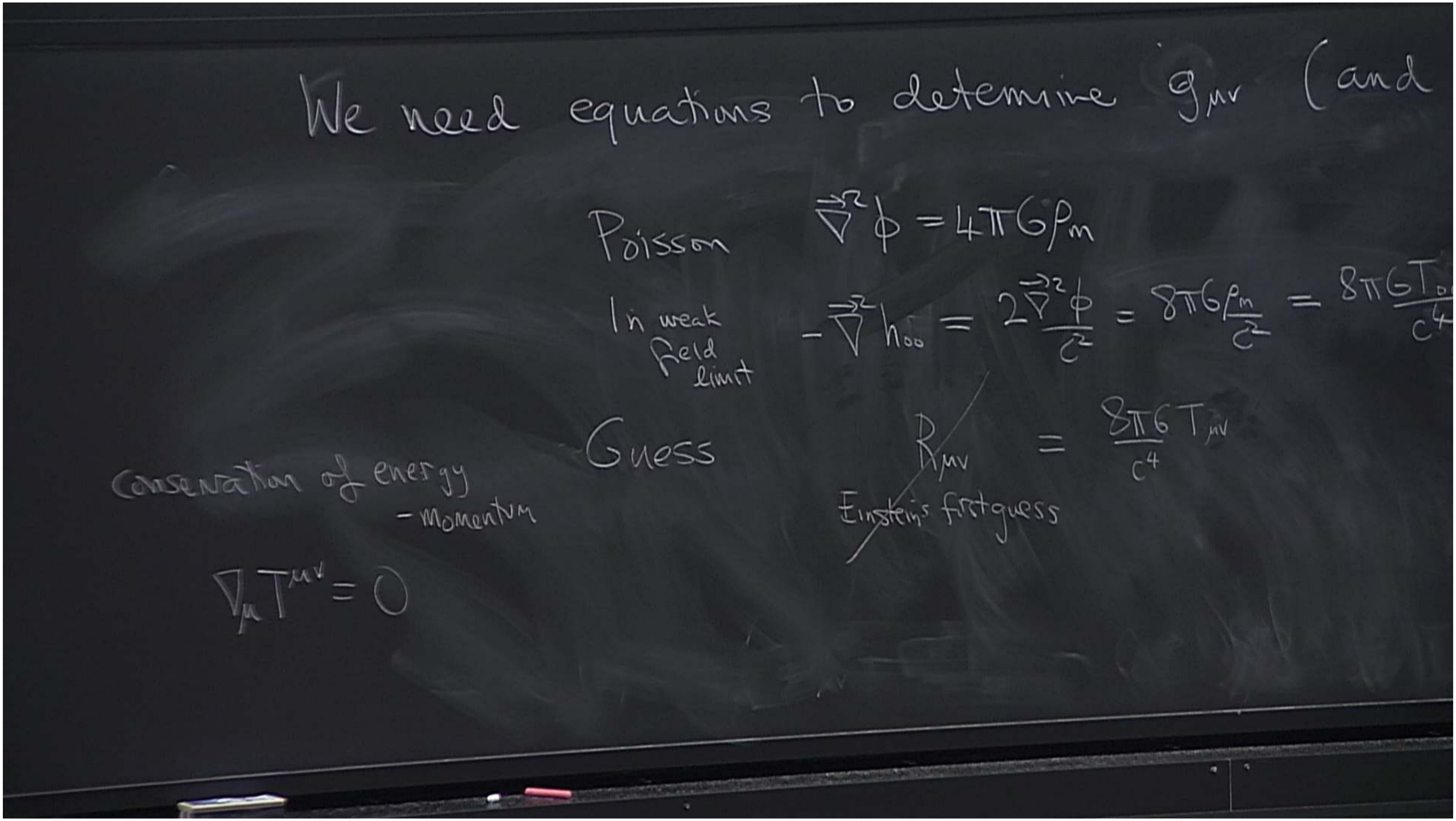
e.g. around a spherical mass M, $\quad \phi = -\dfrac{GM}{r} \quad \Longrightarrow$

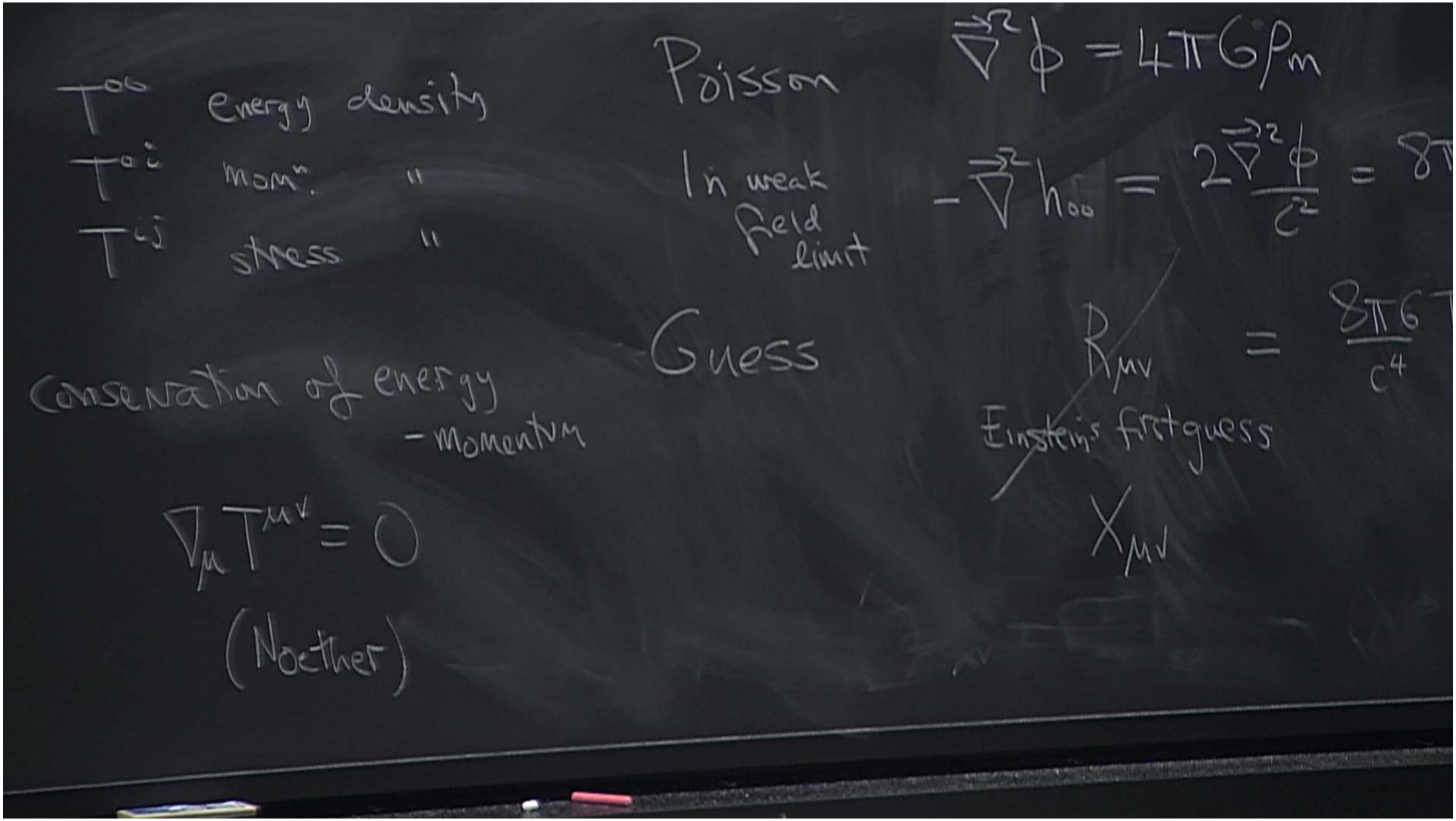
$$g_{00} = -1 + h_{00} = -\left(1 - \frac{2GM}{rc^2}\right) \; :$$

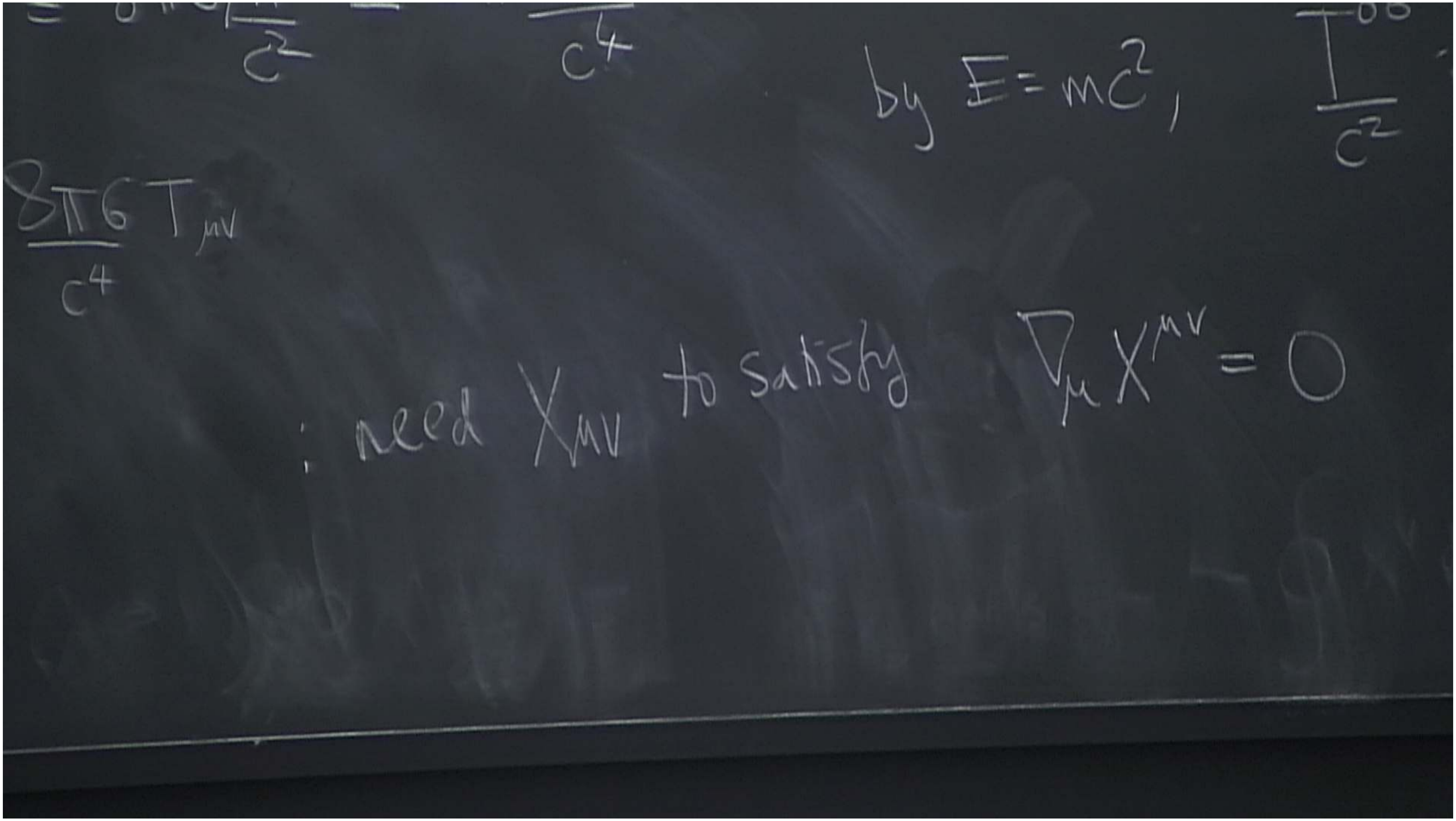

We need equations to determine $g_{\mu\nu}$ (and
Poisson $\vec{\nabla}^2 \phi = 4\pi G \rho_m$
In weak field limit $-\vec{\nabla}^2 h_{00} = 2\frac{\vec{\nabla}^2 \phi}{c^2} = 8\pi G \frac{\rho_m}{c^2} =$

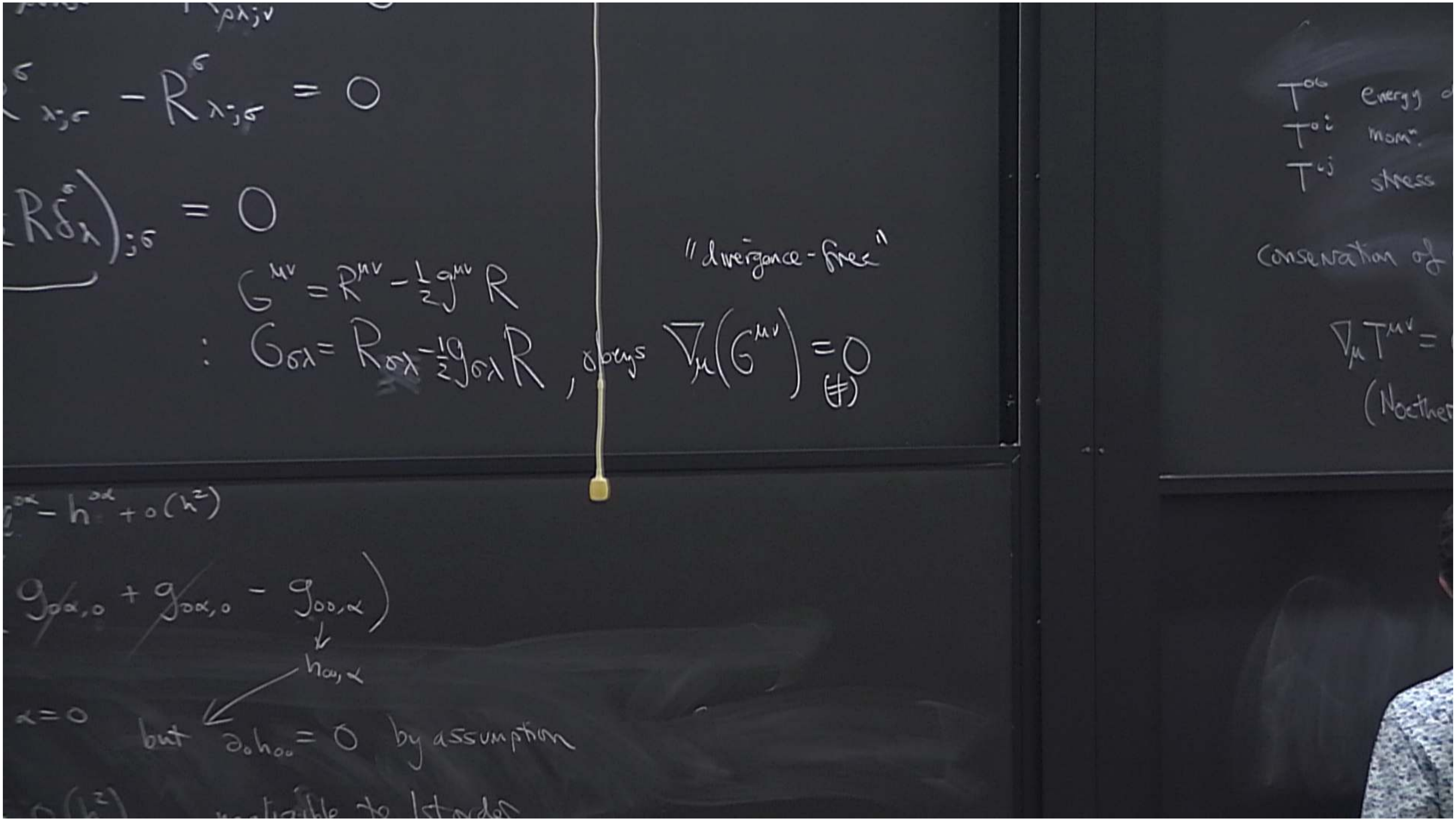

g_μν (and h_μν); i.e., covariant analog of the Poisson eqn:
T^μν is the energy-mom tensor
T^00 is the energy density.
∇²φ/c² = 8πGρm/c² = 8πGT00/c²
by E = mc², T^00/c² is the mass density ρm.


need equations to determine $g_{\mu\nu}$ (and $h_{\mu\nu}$); i.e., cova
$T^{\mu\nu}$ is the e
T^{00}
Poisson $\vec{\nabla}^2\phi = 4\pi G\rho_m$
In weak field limit
$-\vec{\nabla}^2 h_{00} = \frac{2\vec{\nabla}^2\phi}{c^2} = \frac{8\pi G\rho_m}{c^2} = \frac{8\pi G T_{00}}{c^4}$
by $E = mc^2$,
Guess $R_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$
Einstein's first guess


We need equations to determine $g_{\mu\nu}$ (and
Poisson $\vec{\nabla}^2 \phi = 4\pi G \rho_m$
In weak field limit
$-\vec{\nabla}^2 h_{00} = \frac{2\vec{\nabla}^2 \phi}{c^2} = \frac{8\pi G \rho_m}{c^2} = \frac{8\pi G T_{00}}{c^4}$
Guess
$R_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$
Einstein's first guess
Conservation of energy - momentum
$\nabla_\mu T^{\mu\nu} = 0$


T^00 energy density
T^0i mom^n. "
T^ij stress "
Poisson
∇²φ = 4πGρ_m
In weak field limit
−∇² h_00 = 2∇²φ / c² = 8π
Guess
R_μν = 8πG / c⁴ T
Einstein's first guess
X_μν
Conservation of energy
- momentum
∇_μ T^μν = 0
(Noether)


by E = mc²,
8πG T_μν / c⁴
; need X_μν to satisfy ∇_μ X^μν = 0

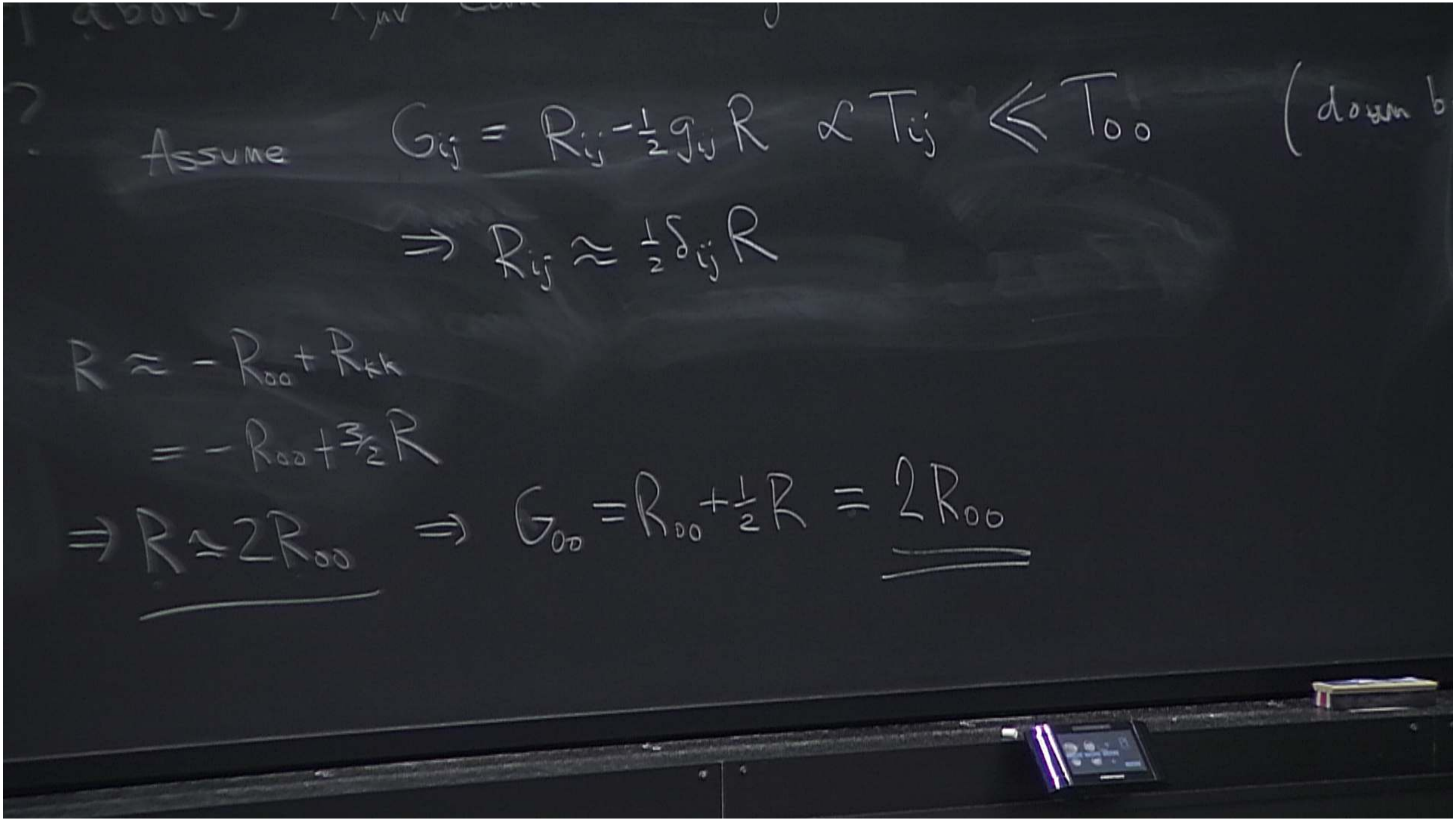

"divergence-free"
$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$
: $G_{\sigma\lambda} = R_{\sigma\lambda} - \frac{1}{2} g_{\sigma\lambda} R$, obeys $\nabla_\mu (G^{\mu\nu}) = 0$
T^{00} energy
T^{0i} mom^n.
T^{ij} stress
Conservation of
$\nabla_\mu T^{\mu\nu} =$
(Noether
$- h^{\sigma\alpha} + o(h^2)$
$g_{0\alpha,0} + g_{0\alpha,0} - g_{00,\alpha}$
$h_{00,\alpha}$
$\alpha = 0$
but $\partial_0 h_{00} = 0$ by assumption

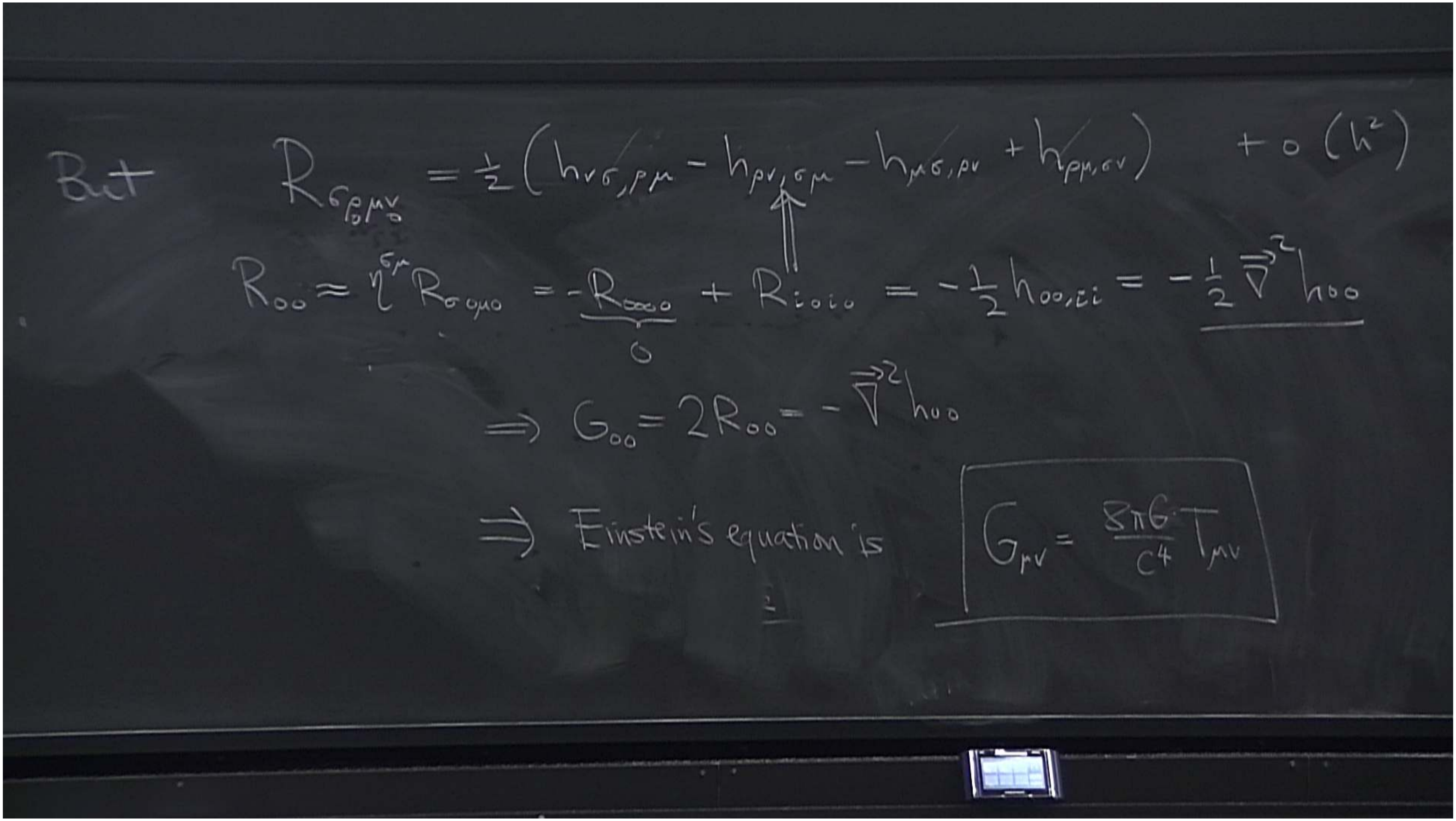
...n (‡) above, $X_{\mu\nu}$ can be any constant times $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$.

...$g_{00} R$? Assume $G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R \propto T_{ij} \ll T_{00}$ (down by v^2/c^2)

$$\Rightarrow R_{ij} \approx \frac{1}{2} \delta_{ij} R$$

$$R \approx -R_{00} + R_{kk}$$


Assume $G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R \propto T_{ij} \ll T_{00}$
(down b
$\Rightarrow R_{ij} \approx \frac{1}{2} \delta_{ij} R$
$R \approx -R_{00} + R_{kk}$
$= -R_{00} + \frac{3}{2} R$
$\Rightarrow R \approx 2R_{00} \Rightarrow G_{00} = R_{00} + \frac{1}{2} R = 2R_{00}$


But $R_{\sigma\rho\mu\nu} = \frac{1}{2}\left(h_{\nu\sigma,\rho\mu} - h_{\rho\nu,\sigma\mu} - h_{\mu\sigma,\rho\nu} + h_{\rho\mu,\sigma\nu}\right) + o(h^2)$
$R_{00} \approx \eta^{\sigma\mu} R_{\sigma 0 \mu 0} = -R_{0000} + R_{i0i0} = -\frac{1}{2} h_{00,ii} = -\frac{1}{2}\vec{\nabla}^2 h_{00}$
$\Rightarrow G_{00} = 2R_{00} = -\vec{\nabla}^2 h_{00}$
$\Rightarrow$ Einstein's equation is $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$