

Title: PSI 2016/2017 Relativity - Lecture 7

Date: Sep 14, 2016 09:00 AM

URL: <http://pirsa.org/16090007>

Abstract:

reminder

$$\nabla_\mu T^\alpha_\beta = T^\alpha_{\beta;\mu} = T^\alpha_{\beta,\mu} + \Gamma^\alpha_{\gamma\mu} T^\gamma_\beta - \Gamma^\gamma_{\beta\mu} T^\alpha_\gamma$$

$$\partial_\mu T^\alpha_\beta = T^\alpha_{\beta,\mu}$$

$$\Gamma_{\beta\mu}^{\alpha} T^{\mu}_{\gamma}$$

$$\frac{d}{d\tau} \frac{dx^{\alpha}}{d\tau} + \Gamma_{\mu\nu}^{\alpha} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = 0 \quad \text{geodesic eqn}$$

$$= \frac{D}{D\tau} \left(\frac{dx^{\alpha}}{d\tau} \right) = \frac{D}{D\tau} u^{\alpha} = 4\text{-vec.}$$

$$\frac{du^{\alpha}}{d\tau} \neq 4\text{-vec}$$

$$\frac{D}{D} u^{\alpha} = 4\text{-vec}$$

$$\Gamma_{\mu\gamma}^{\alpha}$$

$$\frac{d}{d\tau} \frac{dx^{\mu}}{d\tau} + \Gamma_{\mu\nu}^{\alpha} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = 0 \quad \text{geodesic eqn}$$

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$$\frac{d}{d\tau} \frac{dx^{\alpha}}{d\tau} + \Gamma_{\mu\nu}^{\alpha} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = 0 \quad \text{geodesic eqn}$$

$$= \frac{D}{D\tau} \left(\frac{dx^{\alpha}}{d\tau} \right) = \frac{D}{D\tau} u^{\alpha} = 4\text{-vec.}$$

$$u^{\alpha} = \frac{dx^{\alpha}}{d\tau}$$

$$\frac{du^{\alpha}}{d\tau} \neq \text{4-vec}$$

$$\frac{D}{D\tau} u^{\alpha} = 4\text{-vec}$$

$$\frac{D}{D\tau} u^{\alpha} = \frac{du^{\alpha}}{d\tau} + \Gamma_{\mu\nu}^{\alpha} u^{\mu} u^{\nu}$$

$$\Gamma_{\mu\gamma}^{\alpha}$$

$$\frac{d}{d\tau} \frac{dx^{\alpha}}{d\tau} + \Gamma_{\mu\nu}^{\alpha} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = 0$$

geodesic
eqⁿ

$$\frac{dx^{\alpha}}{d\tau} = u^{\alpha}$$

$$= \frac{D}{D\tau} \left(\frac{dx^{\alpha}}{d\tau} \right) = \frac{D}{D\tau} u^{\alpha} = 4\text{-vec.}$$

$$u^{\alpha} = \frac{dx^{\alpha}}{d\tau}$$

$$\frac{du^{\alpha}}{d\tau} \neq \text{4-vec}$$

$$\frac{D}{D\tau} u^{\alpha} = 4\text{-vec}$$

$$\frac{Du^{\alpha}}{D\tau} = \frac{du^{\alpha}}{d\tau} + \Gamma_{\mu\nu}^{\alpha} u^{\mu} u^{\nu}$$

$$\frac{DS^{\alpha}}{D\tau} = \frac{dS^{\alpha}}{d\tau} + \Gamma_{\mu\nu}^{\alpha} u^{\mu} S^{\nu}$$

$$\Gamma_{\mu\nu}^{\lambda}$$

$$\frac{d}{d\tau} \frac{dx^{\lambda}}{d\tau} + \Gamma_{\mu\nu}^{\lambda} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = 0$$

geodesic eqⁿ

$$\frac{dx^{\lambda}}{d\tau} = u^{\lambda}$$

$$= \frac{D}{D\tau} \left(\frac{dx^{\lambda}}{d\tau} \right) = \frac{D}{D\tau} u^{\lambda} = 4\text{-vec.}$$

$$u^{\lambda} = \frac{dx^{\lambda}}{d\tau}$$

$$\frac{du^{\lambda}}{d\tau} \neq \text{a 4-vec}$$

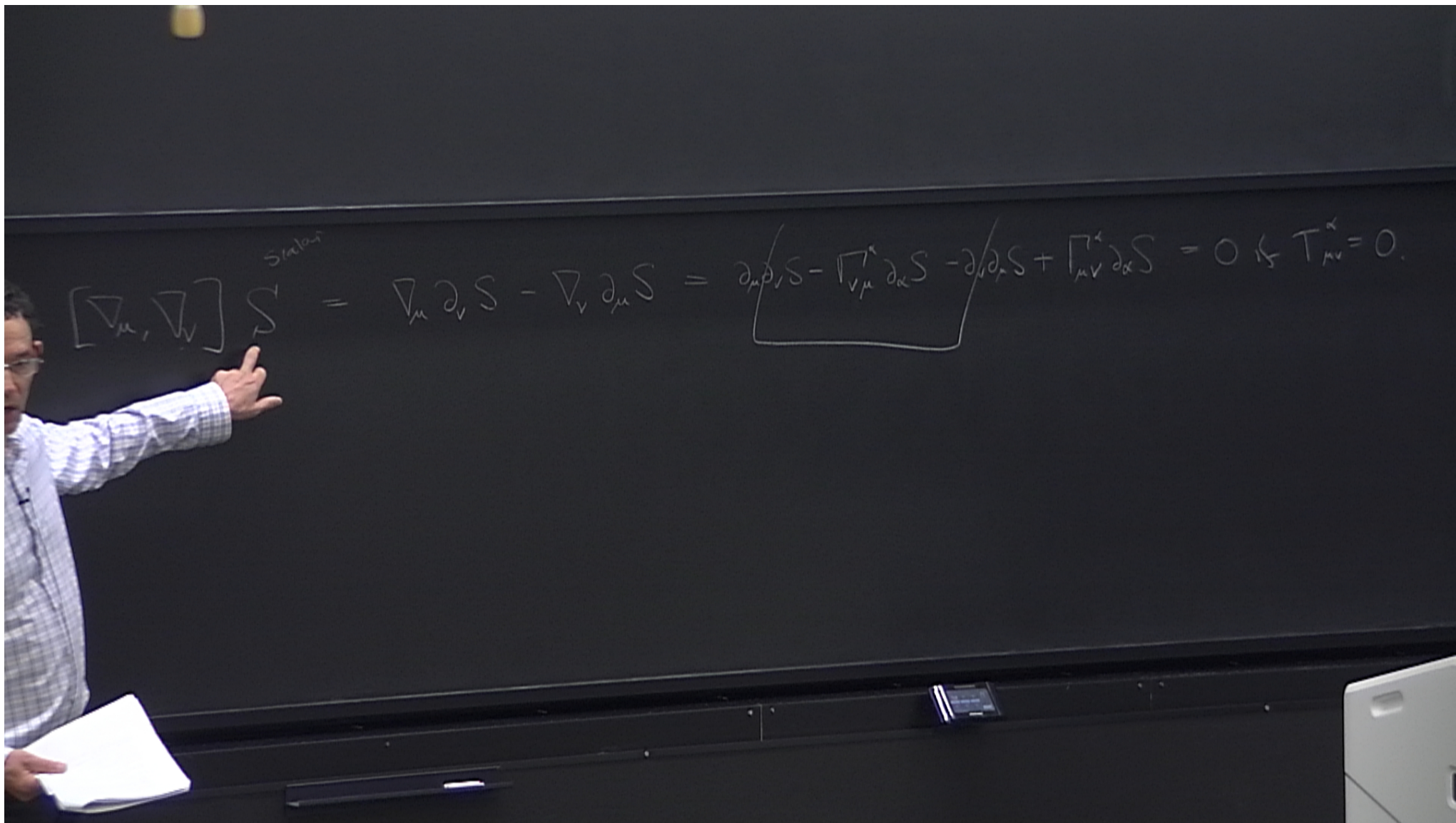
$$\frac{D u^{\lambda}}{D\tau} = 4\text{-vec}$$

$$\frac{D u^{\lambda}}{D\tau} = \frac{du^{\lambda}}{d\tau} + \Gamma_{\mu\nu}^{\lambda} u^{\mu} u^{\nu}$$

$$\frac{D S^{\lambda}}{D\tau} = \frac{dS^{\lambda}}{d\tau} + \Gamma_{\mu\nu}^{\lambda} u^{\mu} S^{\nu}$$

$$u^{\mu} \nabla_{\mu} S^{\lambda}(x) = \frac{dS^{\lambda}}{d\tau} + \Gamma_{\mu\nu}^{\lambda} S^{\mu} u^{\nu} \Big|_{x=x(\tau)}$$

$$\frac{d^{\mu}}{d\tau} \frac{\partial}{\partial x^{\mu}} S(x) = \frac{dS}{d\tau}$$

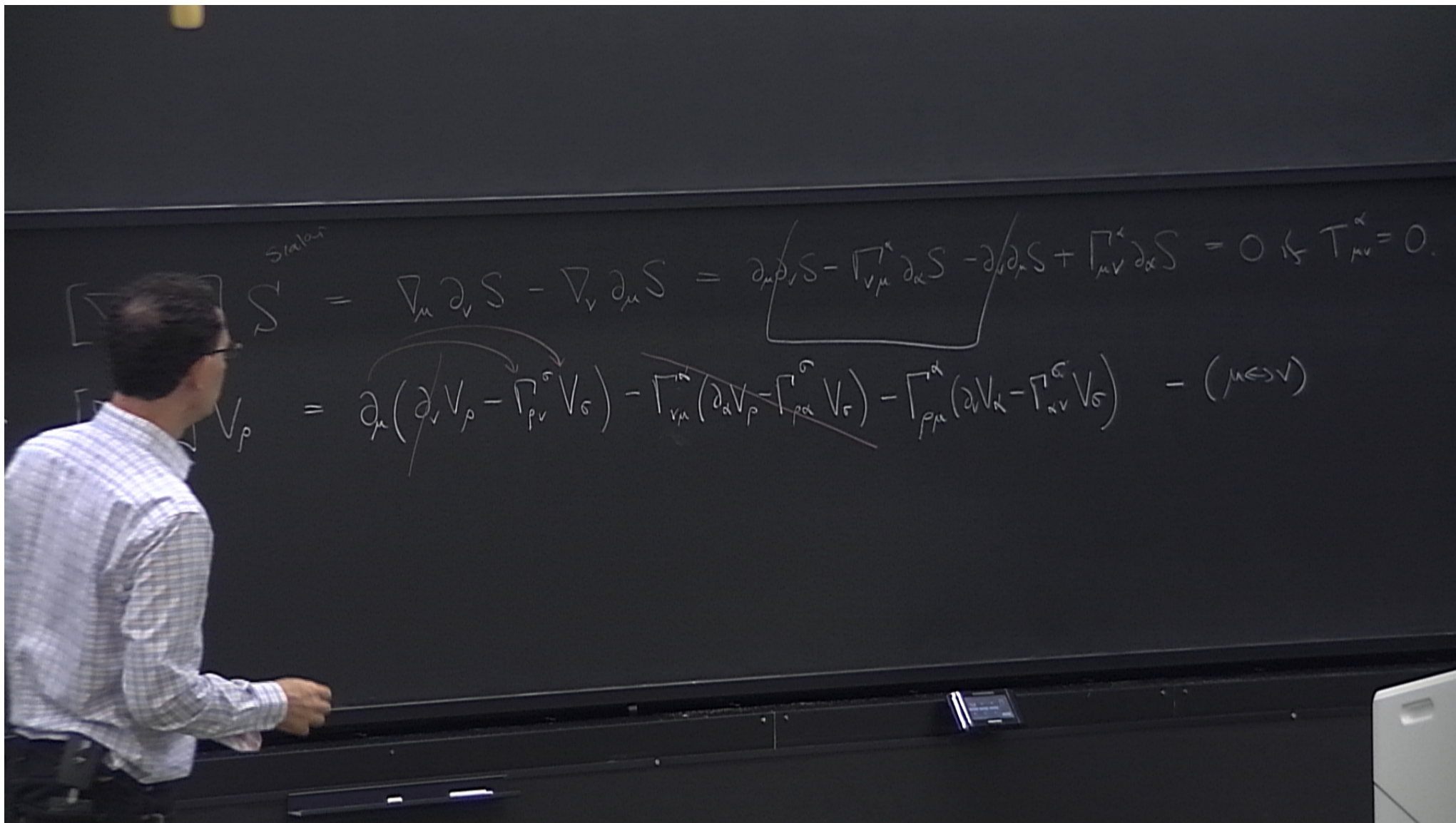


$$[\nabla_\mu, \nabla_\nu] \overset{\text{Scalar}}{S} = \nabla_\mu \partial_\nu S - \nabla_\nu \partial_\mu S = \partial_\mu \partial_\nu S - \Gamma^\alpha_{\nu\mu} \partial_\alpha S - \partial_\nu \partial_\mu S + \Gamma^\alpha_{\mu\nu} \partial_\alpha S = 0 \Leftrightarrow T^\alpha_{\mu\nu} = 0.$$

$$[\nabla_\mu, \nabla_\nu] S^{\text{scalar}} = \nabla_\mu \partial_\nu S - \nabla_\nu \partial_\mu S = \partial_\mu \partial_\nu S - \Gamma_{\nu\mu}^\alpha \partial_\alpha S - \partial_\nu \partial_\mu S + \Gamma_{\mu\nu}^\alpha \partial_\alpha S = 0 \text{ if } T_{\mu\nu}^\alpha = 0.$$

$$[\nabla_\mu, \nabla_\nu] V_\rho = \partial_\mu (\partial_\nu V_\rho - \Gamma_{\rho\nu}^\sigma V_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha V_\rho - \Gamma_{\rho\alpha}^\sigma V_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\nu V_\alpha - \Gamma_{\alpha\nu}^\sigma V_\sigma)$$

$$\begin{aligned}
 [\nabla_\mu, \nabla_\nu] S &= \nabla_\mu \partial_\nu S - \nabla_\nu \partial_\mu S = \cancel{\partial_\mu \partial_\nu S - \Gamma_{\nu\mu}^\alpha \partial_\alpha S} - \cancel{\partial_\nu \partial_\mu S + \Gamma_{\mu\nu}^\alpha \partial_\alpha S} = 0 \quad \text{if } T_{\mu\nu}^\alpha = 0. \\
 [\nabla_\mu, \nabla_\nu] V_\rho &= \partial_\mu (\partial_\nu V_\rho - \Gamma_{\rho\nu}^\sigma V_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha V_\rho - \Gamma_{\rho\alpha}^\sigma V_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\nu V_\alpha - \Gamma_{\alpha\nu}^\sigma V_\sigma) - (\mu \leftrightarrow \nu)
 \end{aligned}$$



Scalar

$$[\nabla_\mu, \nabla_\nu] S = \nabla_\mu \partial_\nu S - \nabla_\nu \partial_\mu S = \partial_\mu \partial_\nu S - \Gamma_{\nu\mu}^\alpha \partial_\alpha S - \partial_\nu \partial_\mu S + \Gamma_{\mu\nu}^\alpha \partial_\alpha S = 0 \quad \text{if } T_{\mu\nu}^\alpha = 0.$$

Symmetric under $\mu \leftrightarrow \nu \Rightarrow$ Cancels

$$[\nabla_\mu, \nabla_\nu] V_\rho = \partial_\mu (\partial_\nu V_\rho - \Gamma_{\rho\nu}^\sigma V_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha V_\rho - \Gamma_{\rho\alpha}^\sigma V_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\nu V_\alpha - \Gamma_{\alpha\nu}^\sigma V_\sigma) - (\mu \leftrightarrow \nu)$$

$$= (-\partial_\mu \Gamma_{\rho\nu}^\sigma + \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma) V_\sigma - (\mu \leftrightarrow \nu)$$

$$\equiv -R_{\rho\mu\nu}^\sigma V_\sigma$$

$$\Rightarrow R^{\sigma}_{\rho\mu\nu} = \partial_{\mu}\Gamma^{\sigma}_{\rho\nu} - \partial_{\nu}\Gamma^{\sigma}_{\rho\mu} + \Gamma^{\alpha}_{\rho\mu}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\rho\nu}\Gamma^{\sigma}_{\alpha\mu} \quad \text{is a tensor, called the Riemann tensor.}$$

$$\Rightarrow R^{\sigma}_{\rho\mu\nu} = \partial_{\mu}\Gamma^{\sigma}_{\rho\nu} - \partial_{\nu}\Gamma^{\sigma}_{\rho\mu} + \Gamma^{\lambda}_{\rho\mu}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu}\Gamma^{\sigma}_{\lambda\mu} \quad \text{is a tensor, called the Riemann tensor}$$

$$[\nabla_\mu, \nabla_\nu] S = \nabla_\mu \partial_\nu S - \nabla_\nu \partial_\mu S = \partial_\mu \partial_\nu S - \Gamma_{\nu\mu}^\alpha \partial_\alpha S - \partial_\nu \partial_\mu S + \Gamma_{\mu\nu}^\alpha \partial_\alpha S = 0 \text{ if } \Gamma_{\mu\nu}^\alpha = 0.$$

Consider $[\nabla_\mu, \nabla_\nu] V_\rho = \partial_\mu (\partial_\nu V_\rho - \Gamma_{\rho\nu}^\sigma V_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha V_\rho - \Gamma_{\rho\alpha}^\sigma V_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\nu V_\alpha - \Gamma_{\alpha\nu}^\sigma V_\sigma) - (\mu \leftrightarrow \nu)$

$$\Gamma^\sigma = \frac{1}{2} g^{\sigma\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$

trans
f v

$$= (-\partial_\mu \Gamma_{\rho\nu}^\sigma + \Gamma_{\rho\mu}^\alpha \Gamma_{\alpha\nu}^\sigma) V_\sigma - (\mu \leftrightarrow \nu)$$

$$\equiv -R_{\rho\mu\nu}^\sigma V_\sigma$$

$$\Gamma_{\rho\nu}^{\lambda} \Gamma_{\lambda\mu}^{\sigma}$$

by quotient th.
is a tensor, called the Riemann tensor.

$$[\nabla_{\mu}, \nabla_{\nu}] S = \nabla_{\mu} \partial_{\nu} S - \nabla_{\nu} \partial_{\mu} S$$

Consider $[\nabla_{\mu}, \nabla_{\nu}] V_{\rho} = \partial_{\mu} (\partial_{\nu} V_{\rho} - \Gamma_{\rho\nu}^{\sigma} V_{\sigma}) - \partial_{\nu} (\partial_{\mu} V_{\rho} - \Gamma_{\rho\mu}^{\sigma} V_{\sigma})$

$$\begin{aligned} & \Gamma_{\rho\nu}^{\sigma} \partial_{\mu} V_{\sigma} - \Gamma_{\rho\mu}^{\sigma} \partial_{\nu} V_{\sigma} \\ & \text{tensor} \times V \\ & = (-\partial_{\mu} \Gamma_{\rho\nu}^{\sigma} + \Gamma_{\rho\mu}^{\sigma} \partial_{\nu}) V_{\sigma} \\ & \equiv -R_{\rho\mu\nu}^{\sigma} V_{\sigma} \end{aligned}$$

$$\Rightarrow R^{\sigma}_{\rho\mu\nu} = \partial_{\mu}\Gamma^{\sigma}_{\rho\nu} - \partial_{\nu}\Gamma^{\sigma}_{\rho\mu} + \Gamma^{\alpha}_{\rho\mu}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\rho\nu}\Gamma^{\sigma}_{\alpha\mu}$$

/ by 9
is a tensor, cal

Symmetries: choose coords (nhdⁱⁿ of a point) s.t. $\partial_{\alpha}g_{\mu\beta} = 0$ at that point.

$$\Rightarrow R^{\sigma}_{\rho\mu\nu} = \partial_{\mu}\Gamma^{\sigma}_{\rho\nu} - \partial_{\nu}\Gamma^{\sigma}_{\rho\mu} + \Gamma^{\alpha}_{\rho\mu}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\rho\nu}\Gamma^{\sigma}_{\alpha\mu}$$

by quotient th.
is a tensor, called the R.

Symmetries: choose coords in hnd of a point, s.t. $\partial_{\mu}g_{\alpha\beta} = 0$ at that point. $\rightarrow \Gamma^{\sigma}_{\rho\nu} = 0$ at the point.

$$\Gamma^{\sigma}_{\rho\nu} = \frac{1}{2}g^{\sigma\lambda}(g_{\rho\lambda,\nu} + g_{\nu\lambda,\rho} - g_{\rho\nu,\lambda})$$

$$\Rightarrow R^{\sigma}_{\rho\mu\nu} = \partial_{\mu}\Gamma^{\sigma}_{\rho\nu} - \partial_{\nu}\Gamma^{\sigma}_{\rho\mu} + \Gamma^{\alpha}_{\rho\mu}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\rho\nu}\Gamma^{\sigma}_{\alpha\mu}$$

by quotient th. is a tensor, called the Riemann tensor.

tries: choose coords. in nhd of a point, st. $\partial_{\alpha}g_{\mu\beta} = 0$ at that point. $\rightarrow \Gamma^{\lambda}_{\mu\nu} = 0$ at the point.

$$\Gamma^{\sigma}_{\rho\nu} = \frac{1}{2}g^{\sigma\lambda}(g_{\rho\lambda,\nu} + g_{\nu\lambda,\rho} - g_{\rho\nu,\lambda})$$

$$\partial_{\mu}g^{\alpha\beta} = -g^{\alpha\gamma}(\partial_{\mu}g_{\gamma\delta})g^{\delta\beta}$$

$$\partial_{\lambda}g_{\mu\beta}$$

$$g^{\alpha\beta}g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$$

$$\partial_{\lambda}g^{\alpha\beta} + g^{\alpha\gamma}\partial_{\lambda}g_{\gamma\delta}g^{\delta\beta} = 0$$

$$\Rightarrow R^\sigma_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\sigma_{\rho\mu} + \Gamma^\alpha_{\rho\mu} \Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\rho\nu} \Gamma^\sigma_{\alpha\mu} \quad \text{by quotient th. is a tensor, called the Riemann tensor}$$

inertial: choose coords. in nhd of a point, s.t. $\partial_\alpha g_{\lambda\beta} = 0$ at that point. $\Rightarrow \Gamma^\sigma_{\rho\nu} = 0$ at the point.

$$\Gamma^\sigma_{\rho\nu} = \frac{1}{2} g^{\sigma\lambda} (g_{\rho\lambda,\nu} + g_{\nu\lambda,\rho} - g_{\rho\nu,\lambda})$$

in these coords, $\partial_\mu \Gamma^\sigma_{\rho\nu} = \frac{1}{2} g^{\sigma\lambda} (g_{\rho\lambda,\mu\nu} + g_{\nu\lambda,\mu\rho} - g_{\rho\nu,\lambda\mu})$

$$\partial_\nu \Gamma^\sigma_{\rho\mu} = \frac{1}{2} g^{\sigma\lambda} (g_{\rho\lambda,\mu\nu} + g_{\mu\lambda,\rho\nu} - g_{\rho\mu,\lambda\nu}) \Rightarrow R^\sigma_{\rho\mu\nu} = \frac{1}{2} g^{\sigma\lambda} (g_{\nu\lambda,\mu\rho} - g_{\rho\lambda,\mu\nu} - g_{\mu\lambda,\rho\nu} + g_{\rho\mu,\lambda\nu})$$

$$\partial_\mu g^{\alpha\beta} ?$$

$$= -g^{\alpha\gamma} (\partial_\mu g_{\gamma\delta}) g^{\delta\beta}$$

$$\partial_\alpha g_{\lambda\beta}$$

$$g^{\alpha\delta} g_{\beta\delta} = \delta^\alpha_\beta$$

$$\partial_\gamma g^{\alpha\delta} g_{\beta\delta} + g^{\alpha\delta} \partial_\gamma g_{\beta\delta} = 0$$

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these covariants, $\partial_\mu \Gamma_{\rho\nu}^\sigma = \frac{1}{2} g^{\sigma\lambda} (g_{\rho\lambda,\nu\mu} + g_{\nu\lambda,\rho\mu} - g_{\rho\nu,\lambda\mu})$ $= -g^{\sigma\lambda} (\partial_\mu g_{\rho\lambda}) g_{\nu\sigma}$

$\partial_\nu \Gamma_{\rho\mu}^\sigma = \frac{1}{2} g^{\sigma\lambda} (g_{\rho\lambda,\mu\nu} + g_{\mu\lambda,\rho\nu} - g_{\rho\mu,\lambda\nu}) \Rightarrow R^\sigma_{\rho\mu\nu} = \frac{1}{2} g^{\sigma\lambda} (g_{\nu\lambda,\rho\mu} - g_{\rho\lambda,\mu\nu} - g_{\mu\lambda,\rho\nu} + g_{\rho\mu,\lambda\nu})$

or, lowering σ index, $R_{\sigma\rho\mu\nu} = \frac{1}{2} (g_{\nu\sigma,\rho\mu} - g_{\rho\sigma,\mu\nu} - g_{\mu\sigma,\rho\nu} + g_{\rho\mu,\sigma\nu})$

these coords, $\partial_\mu \Gamma_{\rho\nu}^\sigma = \frac{1}{2} g^{\sigma\lambda} (g_{\rho\lambda,\nu\mu} + g_{\nu\lambda,\rho\mu} - g_{\rho\nu,\lambda\mu})$ $= -g^{\sigma\lambda} (\partial_\mu g_{\rho\lambda}) g_{\nu\sigma}$

$\partial_\nu \Gamma_{\rho\mu}^\sigma = \frac{1}{2} g^{\sigma\lambda} (g_{\rho\lambda,\mu\nu} + g_{\mu\lambda,\rho\nu} - g_{\rho\mu,\lambda\nu}) \Rightarrow R^\sigma_{\rho\mu\nu} = \frac{1}{2} g^{\sigma\lambda} (g_{\nu\lambda,\rho\mu} - g_{\rho\lambda,\nu\mu} - g_{\mu\lambda,\rho\nu} + g_{\rho\mu,\lambda\nu})$

or, lowering σ index, $R_{\sigma\rho\mu\nu} = \frac{1}{2} (g_{\nu\sigma,\rho\mu} - g_{\rho\sigma,\nu\mu} - g_{\mu\sigma,\rho\nu} + g_{\rho\mu,\sigma\nu})$

- Symmetries.
1. Antisymmetric $\mu \leftrightarrow \nu$ or $\sigma \leftrightarrow \rho$ under
 2. Symmetric under $\sigma\rho \leftrightarrow \mu\nu$
 3. Cyclic on last 3 indices $R_{\sigma\rho\mu\nu} + R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} = 0$
(result of the Jacobi identity) ∇

these covars, $\partial_\mu \Gamma_{\rho\nu}^\sigma = \frac{1}{2} g^{\sigma\lambda} (g_{\rho\lambda,\nu\mu} + g_{\nu\lambda,\rho\mu} - g_{\rho\nu,\lambda\mu})$ $= -g^{\sigma\lambda} (\partial_\mu g_{\rho\sigma}) g_{\lambda\nu}$

$\partial_\nu \Gamma_{\rho\mu}^\sigma = \frac{1}{2} g^{\sigma\lambda} (g_{\rho\lambda,\mu\nu} + g_{\mu\lambda,\rho\nu} - g_{\rho\mu,\lambda\nu}) \Rightarrow R^\sigma_{\rho\rho\nu} = \frac{1}{2} g^{\sigma\lambda} (g_{\nu\lambda,\rho\mu} - g_{\rho\lambda,\nu\mu} - g_{\mu\lambda,\rho\nu} + g_{\rho\mu,\lambda\nu})$

or, lowering σ index, $R_{\sigma\rho\mu\nu} = \frac{1}{2} (g_{\nu\sigma,\rho\mu} - g_{\rho\nu,\sigma\mu} - g_{\mu\sigma,\rho\nu} + g_{\rho\mu,\sigma\nu})$

- Symmetries.
1. Antisymmetric $\overset{\text{under}}{\mu \leftrightarrow \nu}$ or $\sigma \leftrightarrow \rho$ $[\nabla_\mu, [\nabla_\rho, \nabla_\nu]] + \text{cycles} = 0$
 2. Symmetric under $\sigma\rho \leftrightarrow \mu\nu$
 3. Cyclic on last 3 indices $R_{\sigma\rho\mu\nu} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0$

these coords, $\partial_\mu \Gamma_{\rho\nu}^\sigma = \frac{1}{2} g^{\sigma\lambda} (g_{\rho\lambda,\nu\mu} + g_{\nu\lambda,\rho\mu} - g_{\rho\nu,\lambda\mu})$ $= -g^{\sigma\lambda} (\partial_\mu g_{\rho\lambda}) g_{\nu\sigma}$

$\partial_\nu \Gamma_{\rho\mu}^\sigma = \frac{1}{2} g^{\sigma\lambda} (g_{\rho\lambda,\mu\nu} + g_{\mu\lambda,\rho\nu} - g_{\rho\mu,\lambda\nu}) \Rightarrow R^\sigma_{\rho\mu\nu} = \frac{1}{2} g^{\sigma\lambda} (g_{\nu\lambda,\rho\mu} - g_{\rho\lambda,\nu\mu} - g_{\mu\lambda,\rho\nu} + g_{\rho\mu,\lambda\nu})$

or, lower σ index, $R_{\sigma\rho\mu\nu} = \frac{1}{2} (g_{\nu\sigma,\rho\mu} - g_{\rho\sigma,\nu\mu} - g_{\mu\sigma,\rho\nu} + g_{\rho\mu,\sigma\nu})$

symmetries.

1. Antisymmetric $\mu \leftrightarrow \nu$ or $\sigma \leftrightarrow \rho$

2. Symmetric under $\sigma\rho \leftrightarrow \mu\nu$

3. Cyclic on last 3 indices

$$[\nabla_\mu, [\nabla_\nu, \nabla_\rho]] + \text{cycles} = 0$$

$$R_{\sigma\rho\mu\nu} + R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} = 0$$

Qⁿ: How many independent components does $R_{\sigma\rho\mu\nu}$ have?

2. Symmetric under $\mu \leftrightarrow \nu$

3. Cyclic on last 3 indices $R_{\rho\mu\nu} + R_{\mu\nu\rho} + R_{\nu\rho\mu} = 0$

Qⁿ: How many independent components does $R_{\rho\mu\nu}$ have?

regard first and second pair of indices as indices by 1. and 2., $R \sim$ symmetric matrix with $\frac{D(D-1)}{2}$ independent components.

dimension

$$\frac{D(D-1)}{2}$$

$$\Rightarrow \frac{1}{2} \cdot \frac{D(D-1)}{2} \cdot \left(\frac{D(D-1)}{2} + 1 \right)$$

2. Symmetric under exchange of any two indices

3. Cyclic on last 3 indices $R_{\alpha\beta\mu\nu} + R_{\alpha\nu\mu\beta} + R_{\alpha\mu\nu\beta} = 0$

Qⁿ: How many independent components does $R_{\alpha\beta\mu\nu}$ have?

regard first and second pair of indices as indices by 1 and 2, $R \sim$ symmetric matrix with $\frac{D(D-1)}{2}$ independent components.

dimension

$\frac{D(D-1)}{2}$

Cyclic equation is totally antisymmetric under exchange of any two indices

$$\begin{array}{ccc} R_{\alpha\beta\mu\nu} + R_{\alpha\nu\mu\beta} + R_{\alpha\mu\nu\beta} \\ \parallel & \text{---} & \parallel \\ -R_{\alpha\beta\mu\nu} & -R_{\alpha\beta\mu\nu} & -R_{\alpha\beta\mu\nu} \end{array}$$

two indices

$$\begin{array}{rcl}
 R_{\rho\sigma\mu\nu} & + & R_{\rho\mu\nu\sigma} & + & R_{\rho\nu\sigma\mu} \\
 \parallel & & \underbrace{\hspace{1cm}}_2 & & \parallel 2 \\
 -R_{\sigma\rho\mu\nu} & & R_{\nu\sigma\rho\mu} & & R_{\sigma\mu\rho\nu} \\
 & & \parallel & & \parallel \\
 & & -R_{\sigma\nu\rho\mu} & & -R_{\sigma\mu\nu\rho}
 \end{array}$$

regard first and second pair of indices as indices by 1. and 2., $R \sim \text{sym}$
 $\Rightarrow \frac{1}{2} \cdot \frac{D(D-1)}{2} \cdot$

Cyclic equation is totally antisymmetric under exchange of any two indices
 $\Rightarrow$ provides only $\frac{1}{4!} D(D-1)(D-2)(D-3)$

$R_{\rho\sigma\mu}$
 $\parallel$
 $-R_{\rho\sigma\mu}$

Cyclic on last 3 indices $R_{\rho\mu\nu} + R_{\mu\nu\rho} + R_{\nu\rho\mu} = 0$

Qⁿ: How many independent components does $R_{\rho\mu\nu}$ have?

regard first and second pair of indices as indices by 1 and 2, $R \sim$ symmetric matrix with $\frac{D(D-1)}{2}$

$\Rightarrow \frac{1}{2} \cdot \frac{D(D-1)}{2} \cdot \left(\frac{D(D-1)}{2} + 1 \right)$ independent

Cyclic equation is totally antisymmetric under exchange of any two indices

$\Rightarrow$ provides only $\frac{1}{4!} D(D-1)(D-2)(D-3)$ equations

$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} + R_{\sigma\rho\mu\nu} + R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} + R_{\mu\rho\nu\sigma} + R_{\mu\nu\sigma\rho} + R_{\mu\sigma\rho\nu} + R_{\nu\rho\sigma\mu} + R_{\nu\sigma\mu\rho} + R_{\nu\mu\rho\sigma} = 0$

independent components of $R_{\mu\nu\rho\sigma}$ taking 1, 2 and 3 into account are:

$\frac{1}{8} D(D-1)(D^2-D+2) - \frac{1}{4!} D(D-1)(D-2)(D-3) = \frac{D^2}{12}(D^2-1)$

How many independent components does $R_{\alpha\beta\gamma\delta}$ have?

regard first and second pair of indices as indices by 1. and 2., $R \sim$ symmetric matrix

$$\Rightarrow \frac{1}{2} \cdot \frac{D(D-1)}{2} \cdot \left(\frac{D(D-1)}{2} + 1 \right) \text{ independent components.}$$

totally antisymmetric under exchange of any two indices

only $\frac{1}{4!} D(D-1)(D-2)(D-3)$ equations

$$\frac{1}{8} D(D-1)(D^2-D+2) - \frac{1}{4!} D(D-1)(D-2)(D-3) = \frac{D^2}{12} (D^2-1)$$

are:

$$\begin{array}{ccc} R_{\alpha\beta\gamma\delta} & + & R_{\alpha\mu\nu\sigma} & + & R_{\rho\sigma\mu\nu} \\ \parallel & & \parallel & & \parallel \\ -R_{\alpha\beta\mu\nu} & & -R_{\alpha\mu\beta\nu} & & -R_{\alpha\mu\nu\beta} \end{array}$$

i.e. exactly the number of second derivatives of $g_{\alpha\beta}$ which cannot be set zero by a coordinate transformation.

$$R_{\rho\mu\nu} = 2 \left(\frac{1}{2} g_{\sigma\rho} \frac{\partial g_{\mu\nu}}{\partial x^\sigma} - \frac{1}{2} g_{\rho\mu} \frac{\partial g_{\nu\sigma}}{\partial x^\sigma} + \frac{1}{2} g_{\mu\sigma} \frac{\partial g_{\rho\nu}}{\partial x^\sigma} - \frac{1}{2} g_{\nu\sigma} \frac{\partial g_{\rho\mu}}{\partial x^\sigma} \right)$$

1. Antisymmetric under $\mu \leftrightarrow \nu$ or $\sigma \leftrightarrow \rho$

2. Symmetric under $\sigma \rho \leftrightarrow \mu \nu$

3. Cyclic on last 3 indices $(R_{\rho\mu\nu} + R_{\rho\nu\mu} + R_{\rho\mu\nu}) = 0$

$$([\nabla_\mu, [\nabla_\mu, \nabla_\nu]] + \text{cycles}) S' = 0 \quad \forall S'$$

many independent components does $R_{\rho\mu\nu}$ have?

and first and second pair of indices as indices by 1. and 2., $R \sim$ symmetric matrix with $\frac{D(D-1)}{2}$ dimension

$$\Rightarrow \frac{1}{2} \cdot \frac{D(D-1)}{2} \cdot \left(\frac{D(D-1)}{2} + 1 \right) \text{ independent components.}$$

0 Japs which cannot be set zero by a coordinate transformation.

$$R_{\mu\nu} = \frac{1}{2} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma})$$

1. Antisymmetric $\mu \leftrightarrow \nu$ or $\sigma \leftrightarrow \rho$
2. Symmetric under $\sigma \rho \leftrightarrow \mu \nu$
3. Cyclic on last 3 indices $(R_{\sigma\mu\nu} + R_{\sigma\nu\mu} + R_{\sigma\mu\mu}) = 0$

Jacobi Identity

$$([\nabla_\mu, [\nabla_\nu, \nabla_\rho]] + \text{cycles}) S' = 0 \quad \forall S'$$

How many independent components does $R_{\sigma\mu\nu}$ have?
 regard first and second pair of indices as indices by 1. and 2., $R \sim$ symmetric matrix

$\Rightarrow \frac{1}{2} \cdot \frac{D(D-1)}{2} \cdot \left(\frac{D(D-1)}{2} + 1 \right)$ independent components.

Consider
$$[\nabla_\mu, \nabla_\nu] V_\rho = \partial_\mu (\partial_\nu V_\rho - \Gamma_{\rho\nu}^\sigma V_\sigma) - \Gamma_{\nu\mu}^\alpha (\partial_\alpha V_\rho - \Gamma_{\rho\alpha}^\sigma V_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\nu V_\alpha - \Gamma_{\alpha\nu}^\sigma V_\sigma) - (\mu \leftrightarrow \nu)$$

Relatives of Riemann

$$R_{\nu\rho} = R^\sigma_{\nu\rho\sigma}$$

Consider
$$[\nabla_\mu, \nabla_\nu] V_\rho = \partial_\mu (\partial_\nu V_\rho - \Gamma_{\rho\nu}^\sigma V_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\alpha V_\rho - \Gamma_{\rho\alpha}^\sigma V_\sigma) - \Gamma_{\rho\mu}^\alpha (\partial_\nu V_\alpha - \Gamma_{\alpha\nu}^\sigma V_\sigma) - (\mu \leftrightarrow \nu)$$

Relatives of Riemann

$$R_{\nu\rho} = R_{\rho\nu}^{\sigma\sigma} \quad \text{Ricci tensor (symm under } \nu \leftrightarrow \rho)$$

$$R = g^{\nu\rho} R_{\nu\rho} \quad \text{Ricci scalar}$$

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In fact, R is the unique scalar which involves only 2 derivatives of the metric

Consider
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Relatives of Riemann

$$R_{\nu\rho} = R_{\rho\nu}^{\sigma} \quad \text{Ricci tensor (symm under } \nu \leftrightarrow \rho)$$

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In fact, R is the unique scalar which involves only 2 derivatives of the metric
and is used to form the action of general relativity $S \propto \int d^4x$

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Relatives of Riemann

$R_{\nu\rho} = R_{\rho\nu}^{\sigma\sigma}$ Ricci tensor (symm under $\nu \leftrightarrow \rho$)

$R = g^{\nu\rho} R_{\nu\rho}$ Ricci scalar

In fact, R is the unique scalar which involves only 2 derivatives of the metric and is used to form the action of general relativity

$$S \propto \int d^4x \sqrt{-g} R$$

Invariant under coord transformations

Relatives of Riemann

$$R^{\sigma}_{\rho\sigma\mu}$$

$$R_{\nu\rho} = R^{\sigma}_{\nu\sigma\rho} \quad \text{Ricci tensor (symm under } \nu \leftrightarrow \rho \text{)}$$

$$R = g^{\nu\rho} R_{\nu\rho} \quad \text{Ricci scalar}$$

In fact, R is the unique scalar which involves only 2 derivatives of the metric

2nd order
in derivs $\propto \int \dot{x}^2$

and is used to form the action of general relativity

$$S \propto \int d^4x \sqrt{-g} R$$

Invariant under coord transformations

derivatives of the metric

2nd order in derivs c.f. $\int \ddot{x}^2$

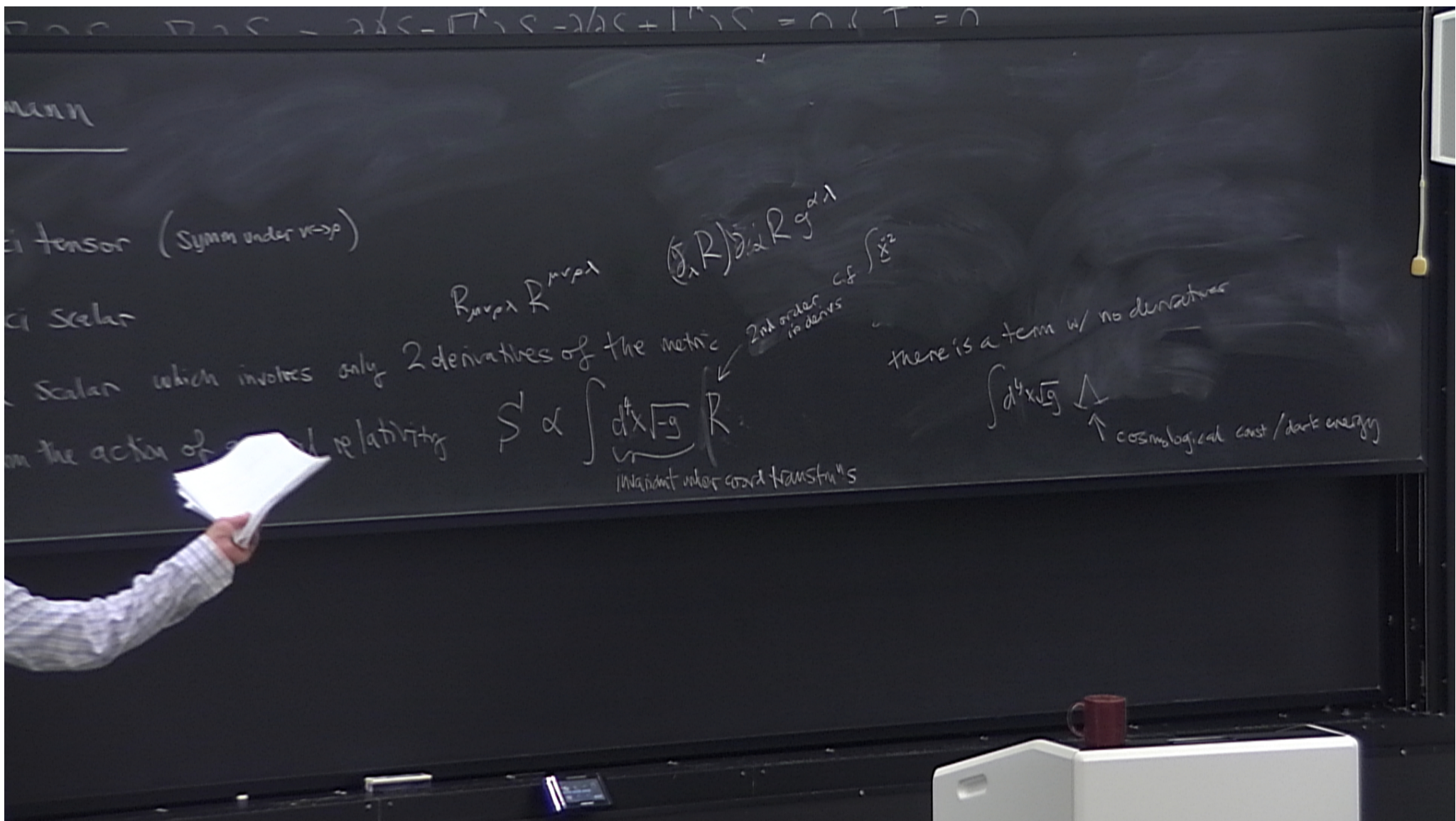
$$S \propto \int d^4x \sqrt{-g} R$$

invariant under coord transfms

there is a term w/ no derivatives

$\int d^4x \sqrt{g} \Lambda$

↑ cosmological const / dark energy



Ex: prove that $\sqrt{-g} d^4x$ is coordinate invariant.
 $g \equiv \det(g_{\mu\nu})$

and is used to form the action of general relativity

$\int \sqrt{-g} d^4x$
invariant under coordinate transformations

Ex: prove that $\sqrt{-g} d^4x$ is coordinate invariant.
 $g \equiv \det(g_{\mu\nu})$

Weyl tensor $C_{\mu\nu\rho\lambda} = R_{\mu\nu\rho\lambda} + \frac{1}{(D-2)} (g_{\mu\lambda} R_{\nu\rho} + g_{\nu\rho} R_{\mu\lambda} - g_{\nu\lambda} R_{\rho\mu} - g_{\mu\rho} R_{\nu\lambda}) + \frac{1}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\lambda} - g_{\nu\rho} g_{\mu\lambda}) R$

and is used to form the action of general relativity

$\int \sqrt{-g} d^4x$
invariant under coordinate transformations

Ex: prove that $\sqrt{-g} d^4x$ is coordinate invariant.
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$$g^{\mu\nu} g_{\nu\mu} = D$$

Weyl tensor

$$C_{\mu\nu\rho\lambda} = R_{\mu\nu\rho\lambda} + \frac{1}{(D-2)} (g_{\mu\lambda} R_{\nu\rho} + g_{\nu\rho} R_{\mu\lambda} - g_{\mu\rho} R_{\nu\lambda} - g_{\nu\mu} R_{\rho\lambda}) + \frac{1}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\lambda} - g_{\nu\rho} g_{\mu\lambda}) R$$

designed to ensure that all contractions of $C_{\mu\nu\rho\lambda}$ vanish

and is used to form the action of general relativity

$\int \sqrt{-g} d^4x$
invariant under coordinate transformations

Ex: prove that $\sqrt{-g} d^4x$ is coordinate invariant.
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has same symmetries as Riemann,
no contractions

designed to ensure
that all contractions of $C_{\mu\nu\rho\lambda}$ vanish

and is used to form the action of general relativity

invariant under coord transformations

that $\sqrt{-g} d^4x$ is coordinate invariant.

$$g \equiv \det(g_{\mu\nu})$$

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$$

$$C_{\mu\nu\rho\lambda} = R_{\mu\nu\rho\lambda} + \frac{1}{(D-2)} (g_{\mu\lambda} R_{\nu\rho} + g_{\nu\rho} R_{\mu\lambda} - g_{\nu\lambda} R_{\mu\rho} - g_{\mu\rho} R_{\nu\lambda}) + \frac{1}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\lambda} - g_{\nu\rho} g_{\mu\lambda}) R$$

symmetries as Riemann,

contractions

designed to ensure
that all contractions of $C_{\mu\nu\rho\lambda}$ vanish

As we shall see, for pure gravity, Einstein field equation reads $R_{\mu\nu} = 0$
(no matter or Λ)

from the action of general relativity $\int \sqrt{-g} R$
 invariant under coord transformations

is coordinate invariant.

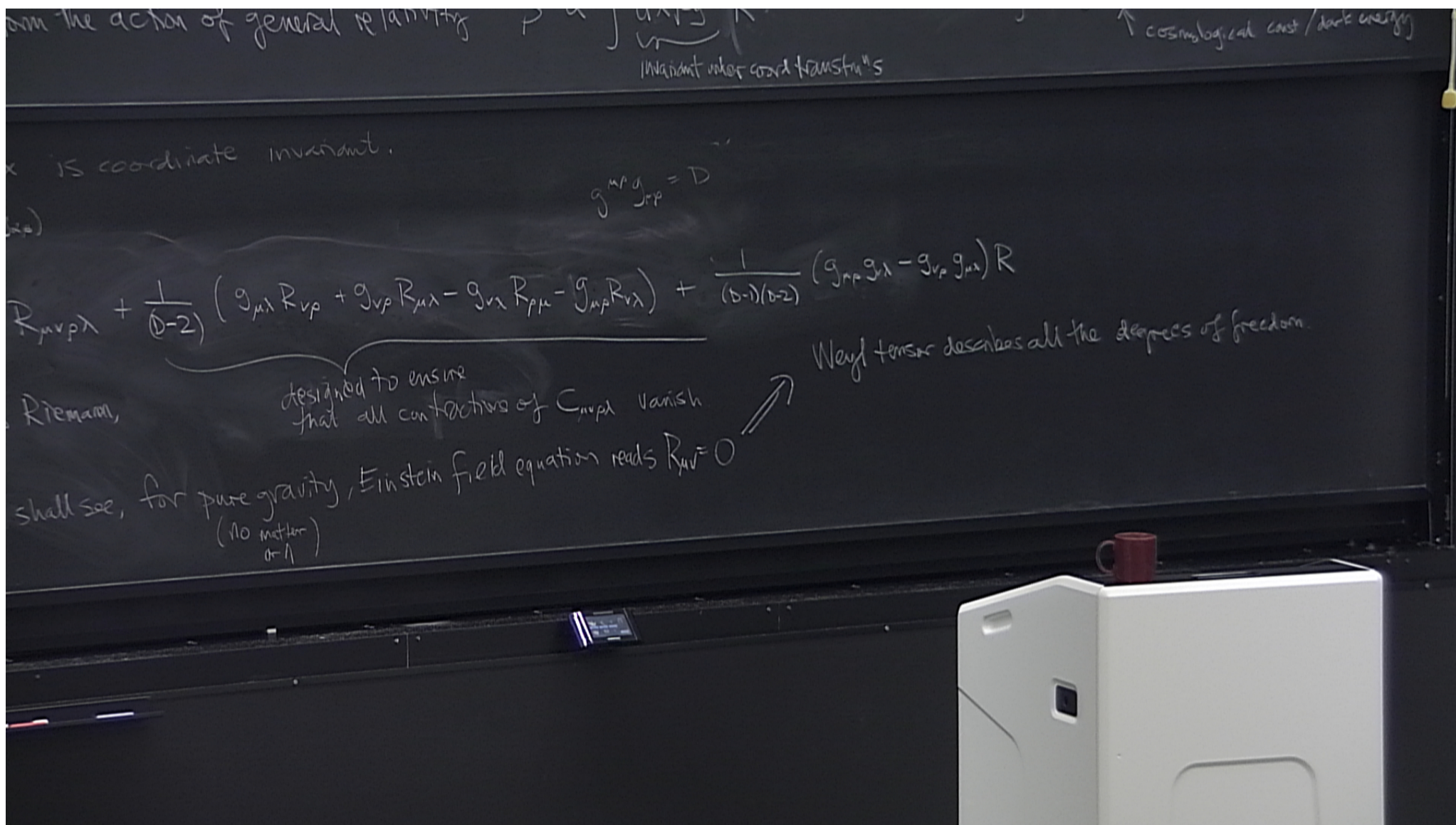
$g^{\mu\rho} g_{\rho\mu} = D$

$$R_{\mu\nu\rho\lambda} + \frac{1}{(D-2)} (g_{\mu\lambda} R_{\nu\rho} + g_{\nu\rho} R_{\mu\lambda} - g_{\nu\lambda} R_{\rho\mu} - g_{\mu\rho} R_{\nu\lambda}) + \frac{1}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\lambda} - g_{\nu\rho} g_{\mu\lambda}) R$$

Riemann, designed to ensure that all contractions of $C_{\mu\nu\rho\lambda}$ vanish

shall see, for pure gravity, Einstein field equation reads $R_{\mu\nu} = 0$
 (no matter or Λ)

Weyl tensor describes all the degrees of freedom.



Riemann,

designed to ensure
that all contractions of $C_{\mu\nu\rho\sigma}$ vanish

shall see, for pure gravity, Einstein field equation reads $R_{\mu\nu} = 0$
(no matter)
 $C^{\lambda}{}_{\mu\lambda\nu} = 0$

Weyl tensor describes all the degrees of freedom.

indept components

$$\frac{1}{12} D(D-1) - \frac{1}{2} D(D+1) = \frac{1}{12} D(D+1)(D+2)(D-3)$$

$$(\nabla_\mu R_{\rho\mu} - g_{\mu\rho} R_{\mu\lambda}) + \frac{1}{(D-1)(D-2)} (g_{\mu\rho} g_{\nu\lambda} - g_{\nu\rho} g_{\mu\lambda}) R$$

most of $C_{\mu\rho\lambda}$ vanish

Einstein equation reads $R_{\mu\nu} = 0$

$$C^{\lambda}_{\mu\lambda\nu} = 0$$

Weyl tensor describes all the degrees of freedom.
indept components

$$\frac{1}{12} D^2(D^2-1) - \frac{1}{2} D(D+1) = \frac{1}{12} D(D+1)(D+2)(D-3)$$

$$\begin{aligned} D=1 & R_{\mu\nu\rho\lambda} = 0 \\ D=2 & C_{\mu\nu\rho\lambda} = 0 \end{aligned}$$

$$(\nabla_\mu R_{\nu\lambda} - \nabla_\nu R_{\mu\lambda}) + \frac{1}{(D-1)(D-2)} (g_{\mu\lambda} g_{\nu\lambda} - g_{\nu\mu} g_{\mu\lambda}) R$$

most of $C_{\mu\nu\rho\lambda}$ vanish.

field equation reads $R_{\mu\nu} = 0$

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Weyl tensor describes all the degrees of freedom.
indept components

$$\frac{1}{12} D^2(D^2-1) - \frac{1}{2} D(D+1) = \frac{1}{12} D(D+1)(D+2)(D-3)$$

$D=1$ $R_{\mu\nu\rho\lambda} = 0$
 $D=2$ curvature can be expressed in terms of R

$$-g_{\mu\lambda}R_{\rho\mu} - g_{\mu\rho}R_{\nu\lambda}) + \frac{(D-1)(D-2)}{2} (g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho})R$$

the
traces of $C_{\mu\nu\rho\lambda}$ vanish

field equation reads $R_{\mu\nu} = 0$

$$C^{\lambda}_{\mu\lambda\nu} = 0$$

Weyl tensor describes all the degrees of freedom.

indept components

$$\frac{1}{12} D^2(D^2-1) - \frac{1}{2} D(D+1) = \frac{1}{12} D(D+1)(D+2)(D-3)$$

$D=1$ $R_{\mu\nu\rho\lambda} = 0$
 $D=2$ curvature can be expressed in terms of R

$D=3$ $C=0$, curvature can be expressed in $R_{\mu\nu}$

As we shall see, for pure gravity, Einstein field equation reads $R_{\mu\nu} = 0$
(no matter or Λ)

$$C^{\lambda}_{\mu\lambda\nu} = 0$$

$\exists$ coords in which

Interesting theorems: $R_{\mu\nu\rho\sigma} = 0 \iff g_{\mu\nu} = \eta_{\mu\nu}$

no contractions

As we shall see, for pure gravity, Einstein field equation reads $K_{\mu\nu} = 0$
(no matter or Λ)

$$\frac{1}{2} D^2(D-1)$$

$$C^{\lambda}_{\mu\lambda\nu} = 0$$

Interesting theorems:

$$R_{\mu\nu\rho\sigma} = 0 \iff \exists \text{ coords in which } g_{\mu\nu} = \eta_{\mu\nu} \quad \swarrow \text{conformal factor}$$

$$C_{\mu\nu\rho\sigma} = 0 \iff g_{\mu\nu} = \Omega^2(x) \eta_{\mu\nu} \quad \text{ie metric is conformally flat.}$$

no contractions

As we shall see, for pure gravity, Einstein field equation reads $R_{\mu\nu} = 0$
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$$R_{\mu\nu\rho\sigma} = 0 \iff$$

$$g_{\mu\nu} = \eta_{\mu\nu}$$

conformal factor

$$C_{\mu\nu\rho\sigma} = 0 \iff$$

$$g_{\mu\nu} = \Omega^2(x) \eta_{\mu\nu} \text{ i.e. metric is conformally flat.}$$

fact: the early universe was nearly conformally flat.

no contractions

As we shall see, for pure gravity, Einstein field equation reads $R_{\mu\nu} = 0$
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Interesting theorems: $R_{\mu\nu\rho\sigma} = 0 \iff \exists$ coords in which $g_{\mu\nu} = \eta_{\mu\nu}$ conformal factor

$$C_{\mu\nu\rho\sigma} = 0 \iff g_{\mu\nu} = \Omega^2(x) \eta_{\mu\nu} \quad \text{is}$$

$C=0$

big bang

fact: the early universe was nearly conformally flat

to contractions

As we shall see, for pure gravity, Einstein field equation reads $R_{\mu\nu} = 0$
(no matter or Λ)

$$\frac{1}{2} D^2(D^2 - 1)$$
$$C^{\lambda}{}_{\mu\lambda\nu} = 0$$

Interesting theorems:

$\exists$ coords in which

$$R_{\mu\nu\rho\sigma} = 0 \iff g_{\mu\nu} = \eta_{\mu\nu} \quad \swarrow \text{conformal factor}$$

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big bang

fact: the early universe was nearly conformally flat.

FRW metric is conf. flat.