Title: PSI 2016/2017 Relativity - Lecture 6

Date: Sep 13, 2016 09:00 AM

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Abstract:

$\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x) \xrightarrow{\text{choose coords}} \eta_{\mu\nu} + O(x^2)$

Tensors $A'^{\mu\nu\rho}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \frac{\partial x'^{\rho}}{\partial x^{\gamma}} A^{\alpha\beta\gamma}(x)$ etc.

$A^{\mu\nu} B_{\rho\lambda}$ ✓ $A^{\mu\nu} B_{\nu\lambda}$ ✓

$A^{\mu\nu} + B^{\mu\nu}$

Quotient theorem: If $A^{\mu\nu\lambda} V_\lambda$ is a tensor $\forall V_\lambda$, then $A^{\mu\nu\lambda}$ is

$\eta_{\mu\nu} + o(x^2)$

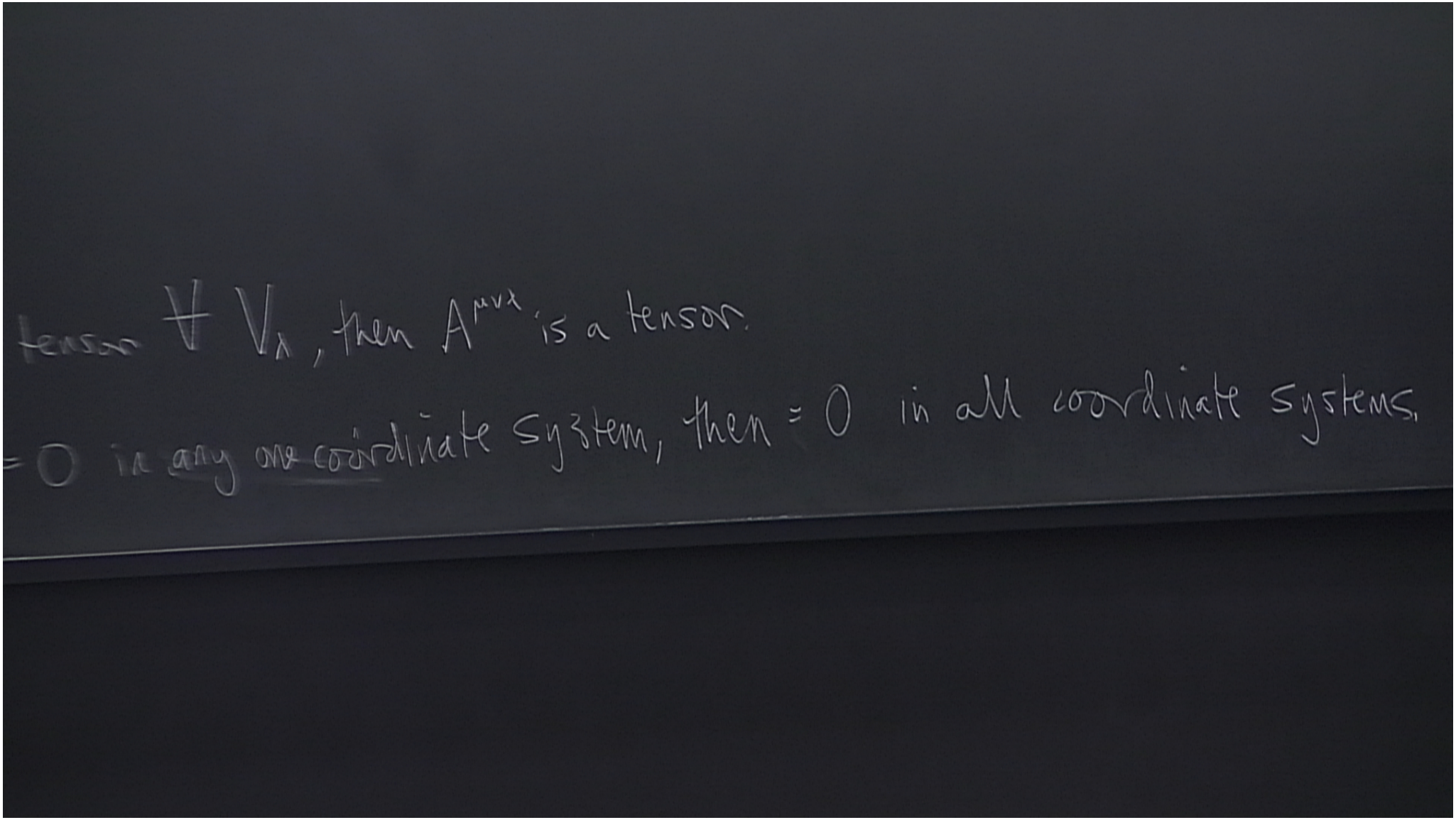
$A'^{\mu\nu\rho}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \frac{\partial x'^{\rho}}{\partial x^{\gamma}} A^{\alpha\beta\gamma}(x)$ etc.

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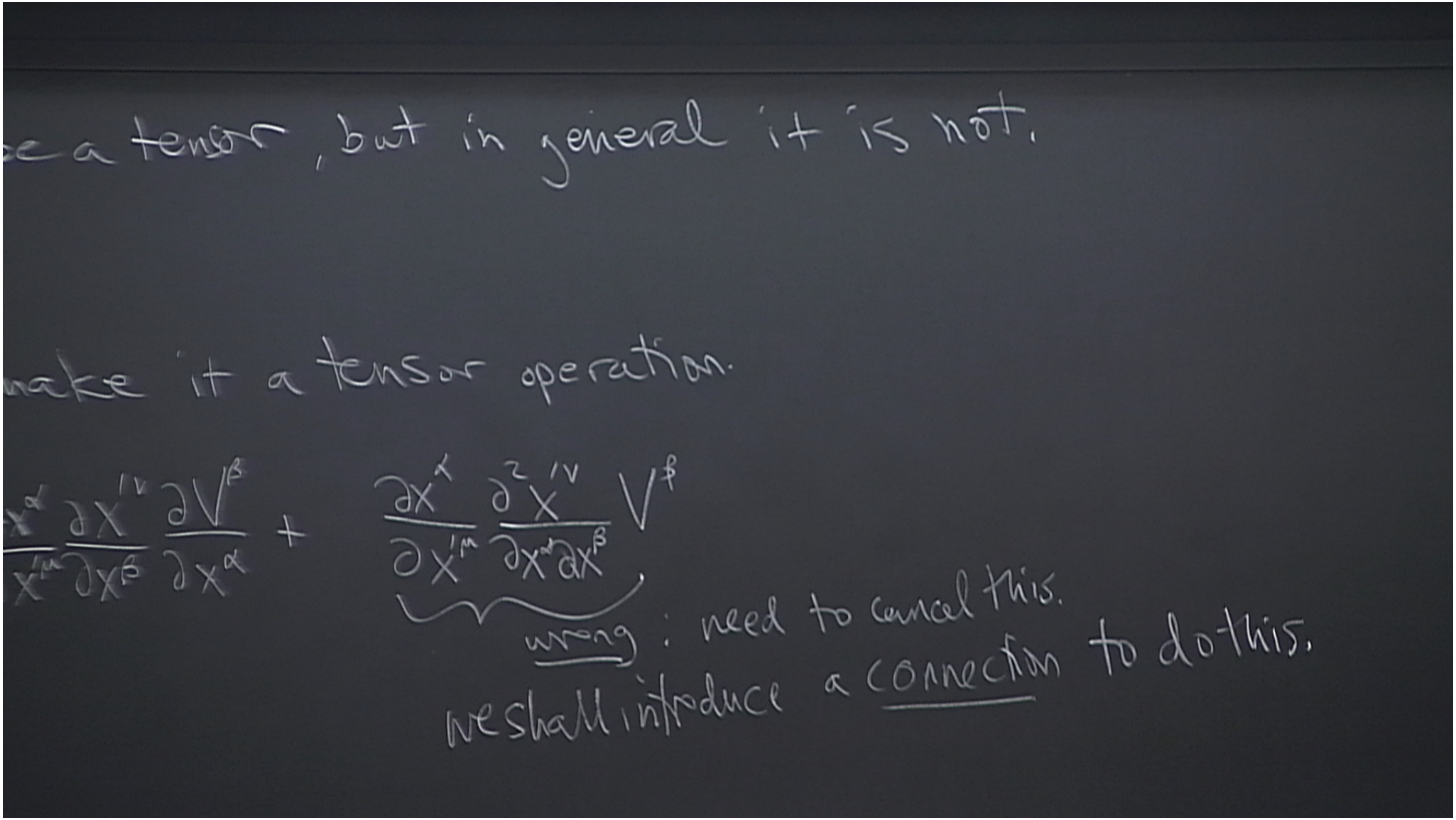
Zero theorem: If a tensor $A^{\mu\nu\lambda} = 0$ in any one coordinate sy


tensor $\forall V_\lambda$, then $A^{\mu\nu\ldots}$ is a tensor.
$=0$ in any one coordinate system, then $=0$ in all coordinate systems.

Naively, you might expect $\partial_\lambda A^{\alpha\beta} = \frac{\partial}{\partial x^\lambda} A^{\alpha\beta}$ to be a tensor, but in general it is not.

Derivative needs "correcting" in order to make it a tensor operation.

$$\frac{\partial}{\partial x'^\mu} V'^\nu(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial}{\partial x^\alpha}\left(\frac{\partial x'^\nu}{\partial x^\beta} V^\beta\right) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\beta}\frac{\partial V^\beta}{\partial x^\alpha} + \underbrace{\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\alpha \partial x^\beta} V^\beta}_{\text{wrong}}$$


a tensor, but in general it is not.
make it a tensor operation.
wrong: need to cancel this.
we shall introduce a connection to do this.

a tensor operation.

$$+ \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\alpha \partial x^\beta} V^\beta$$

wrong: need to cancel this.

we shall introduce a connection to do this.

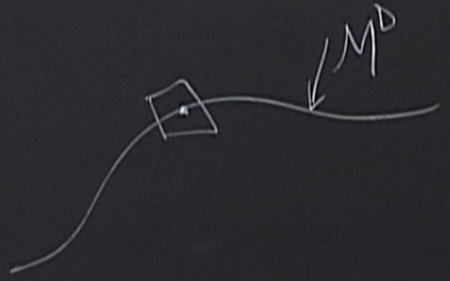
$$\Gamma'^{\mu}_{\nu\alpha} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\delta}{\partial x'^\nu}\frac{\partial x^\epsilon}{\partial x'^\alpha}\Gamma^\rho_{\delta\epsilon} + \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\nu \partial x'^\alpha}$$

Let us try to see how Γ arises naturally in two contexts:

1) if we do embed our curved space in a flat space

2) Consider variational problem (action) for a particle on a curved space

1)

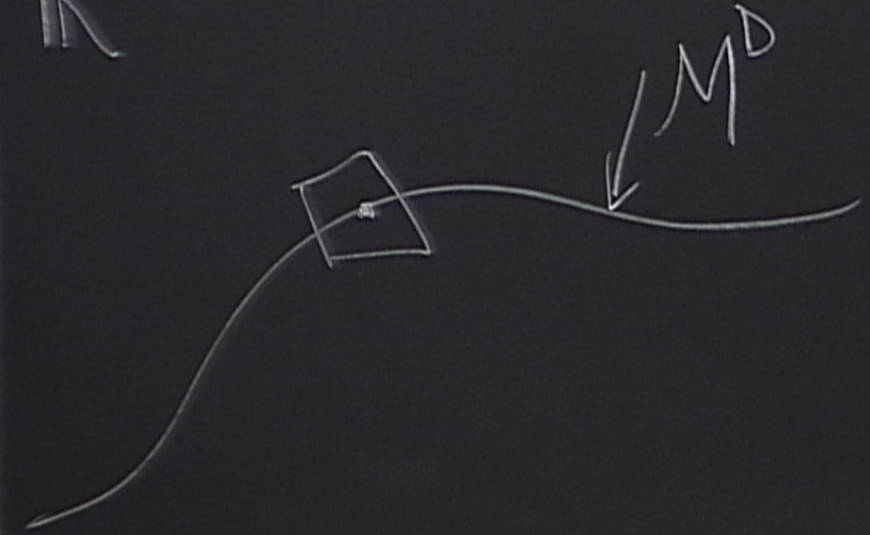


Let us try to see how Γ arises n

R^N ← dim n embedding space

1) if we do embed our curved

2) Consider variational problem (



John Nash

arises naturally in two contexts:

our curved space in a flat space

final problem (action) for a particle on a curved space

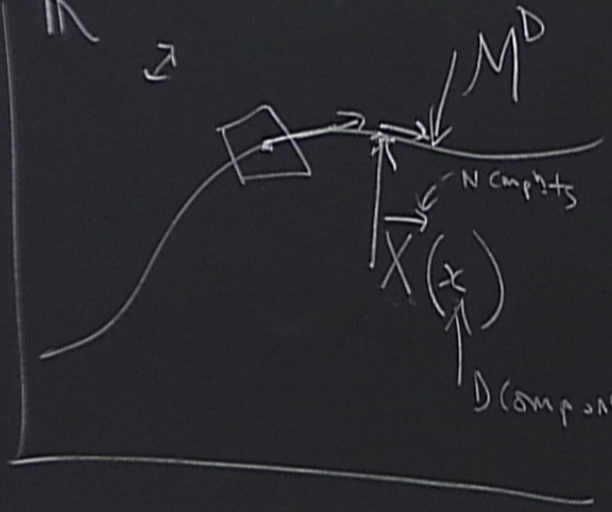
Nash proved that any M^D can be embedded (isometrically) in $\mathbb{R}^{\frac{D(3D+1)}{2}}$

Let us try to see how Γ arises naturally in tw

1) if we do embed our curved space in a flat

2) Consider variational problem (action) for a par

R^N ← dim N of embedding space



John Nash proved that any M^D c

Tangent vectors $\partial_\alpha X^I$ ← $I = 1 \ldots N$

$= \frac{\partial}{\partial x^\alpha}$

$\alpha = 1 \ldots D$

w Γ arises naturally in two contexts:

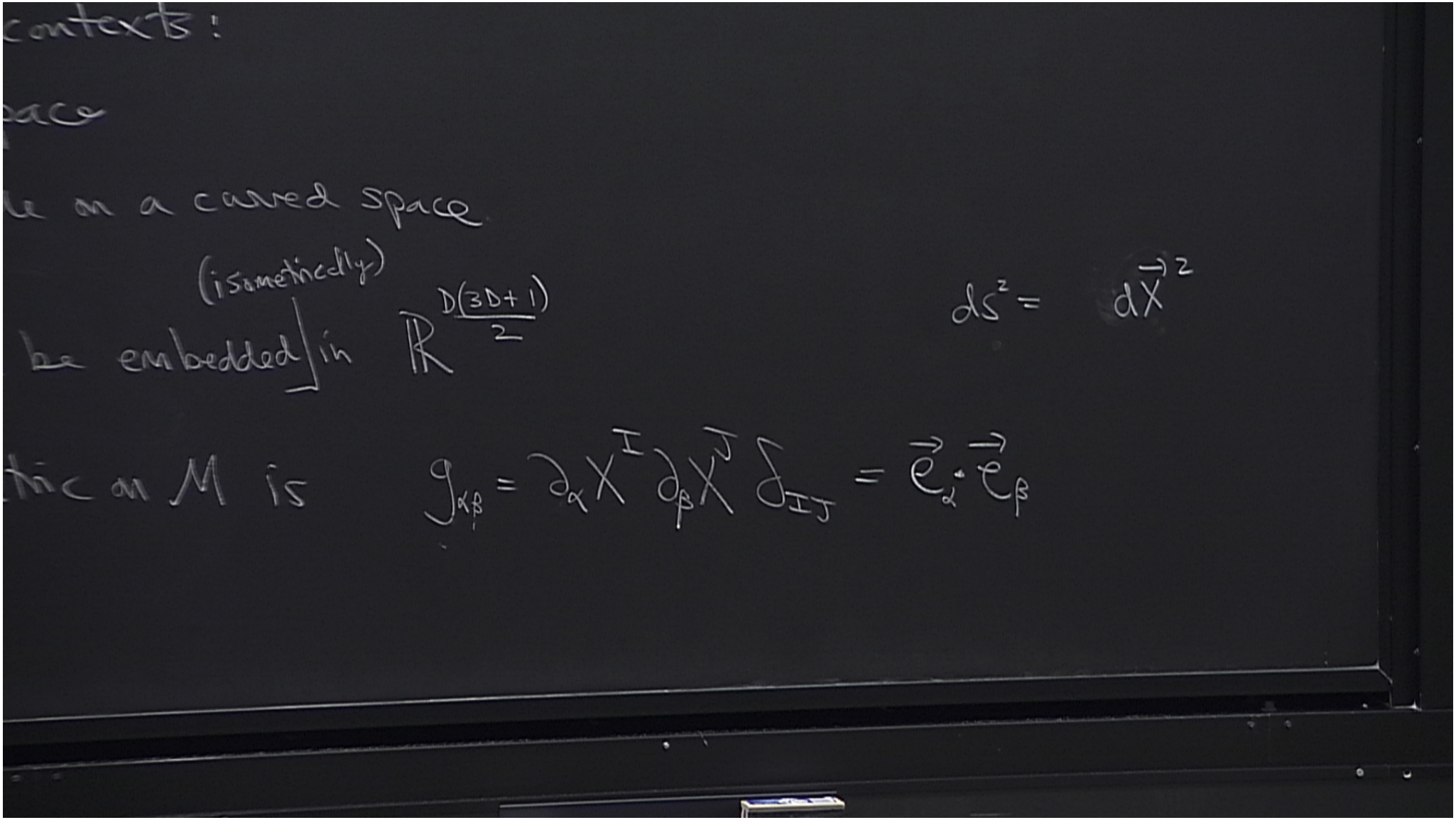
mbed our curved space in a flat space

variational problem (action) for a particle on a curved space

John Nash proved that any M^D can be embedded (isometrically) in $\mathbb{R}^{\frac{D(3D+1)}{2}}$

Tanget vectors $\partial_\alpha X^I = \frac{\partial}{\partial x^\alpha} X^I$ $I = 1 \ldots N$; induced metric on M is $g_{\alpha\beta} = \partial_\alpha X^I \partial_\beta X^J \delta_{IJ}$

$\alpha = 1 \ldots D$


contexts:
pace
le on a curved space
(isometrically)
be embedded in $\mathbb{R}^{\frac{D(3D+1)}{2}}$
$ds^2 = d\vec{X}^2$
tric on M is $g_{\alpha\beta} = \partial_\alpha X^I \partial_\beta X^J \delta_{IJ} = \vec{e}_\alpha \cdot \vec{e}_\beta$

...ssee naturally in two contexts:

curved space in a flat space

problem (action) for a particle on a curved space

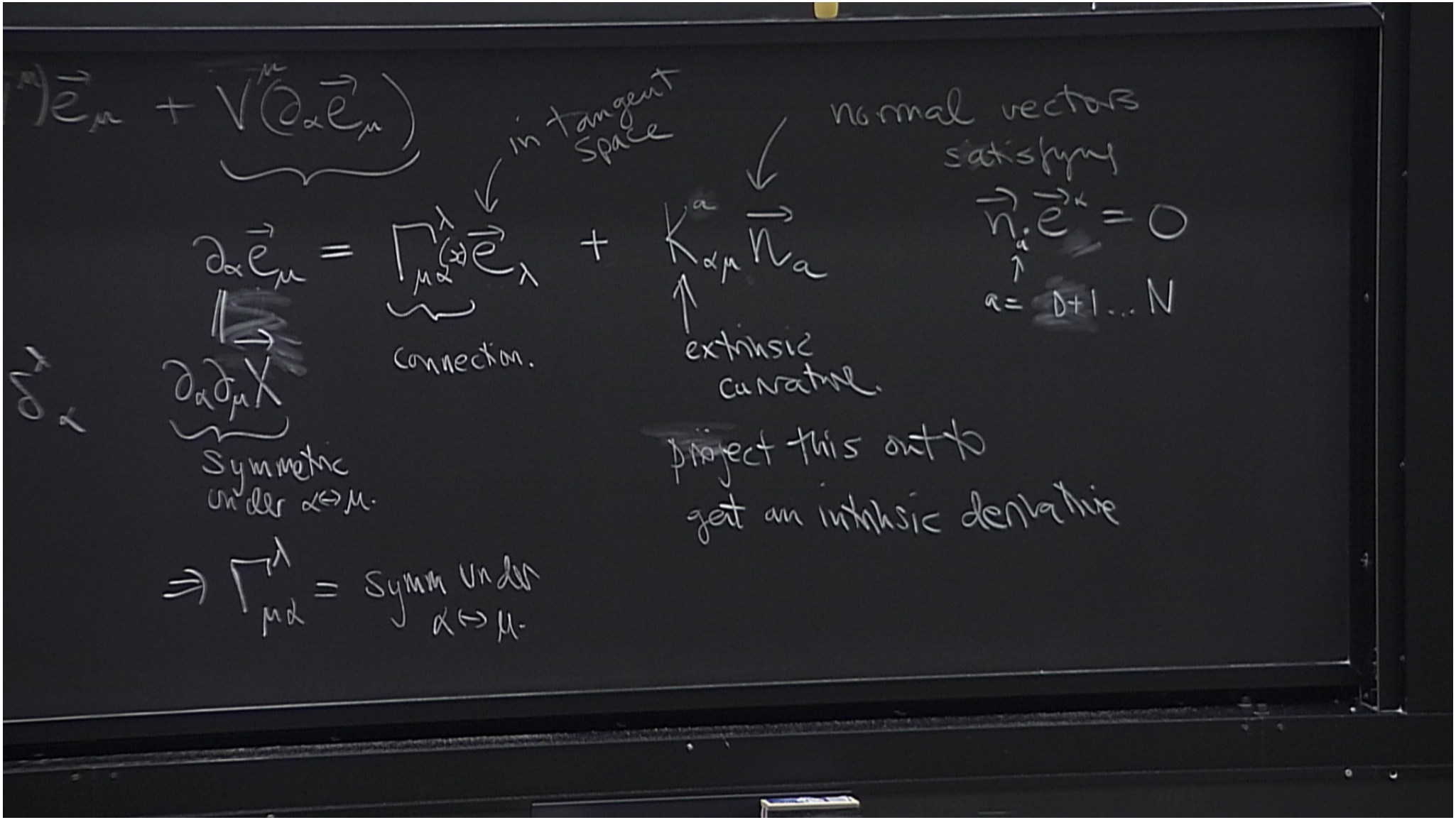
proved that any M^D (isometrically) can be embedded in $\mathbb{R}^{\frac{D(3D+1)}{2}}$

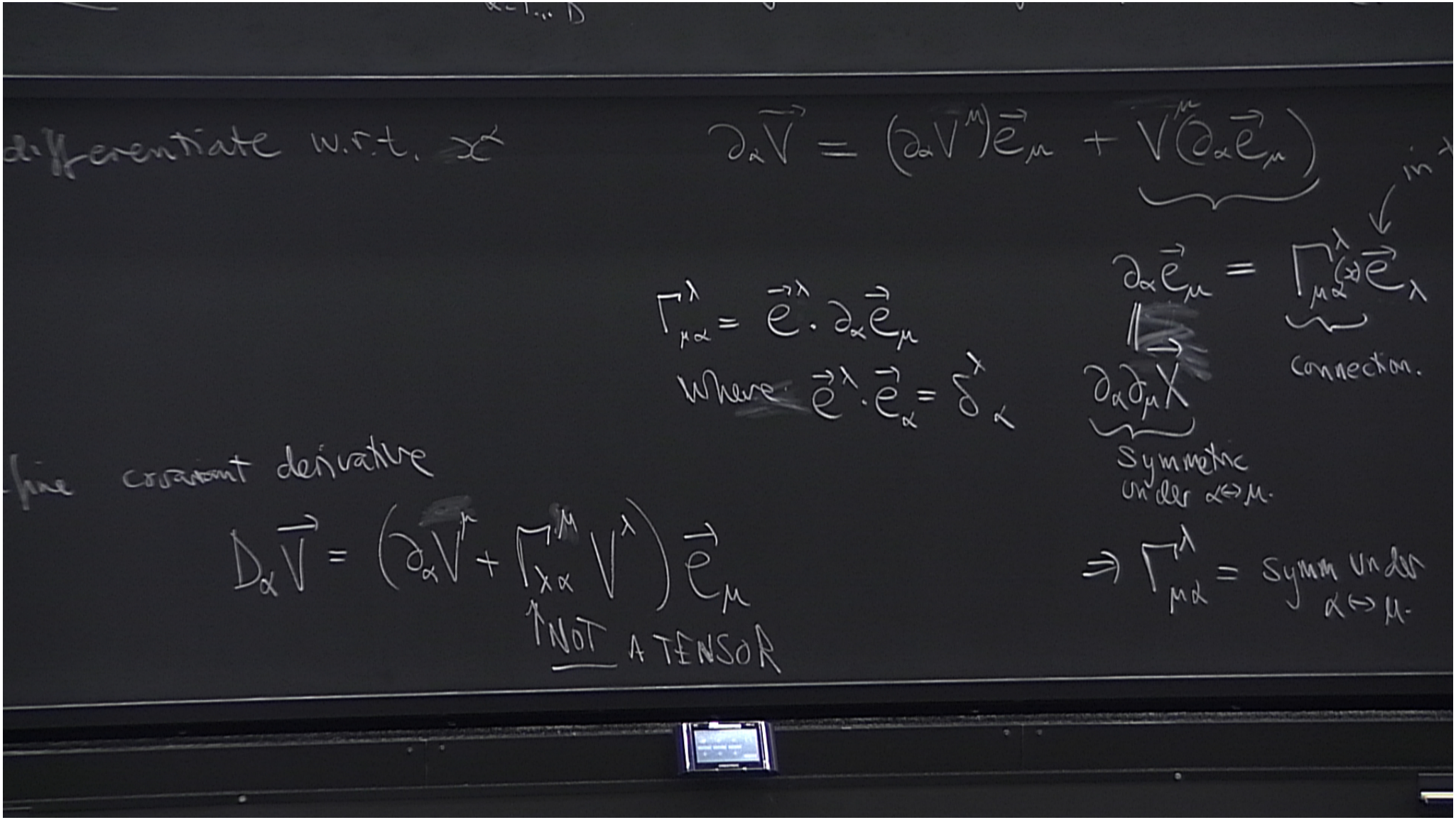
$ds^2 = d\vec{X}$

vectors $\partial_\alpha X^I \vec{e}_I$ ($I = 1 \dots N$): induced metric on M is $g_{\alpha\beta} = \partial_\alpha X^I \partial_\beta X^J \delta_{IJ} = \vec{e}_\alpha \cdot \vec{e}_\beta$

$\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ ($\alpha = 1 \dots D$)

Any vector in tangent space to M can be expressed as $\vec{V} = V^\mu \vec{e}_\mu$


$\vec{e}_\mu + V^\mu(\partial_\alpha \vec{e}_\mu)$
in tangent space
normal vectors satisfying
$\partial_\alpha \vec{e}_\mu = \Gamma^\lambda_{\mu\alpha} \vec{e}_\lambda + K^a_{\alpha\mu} \vec{n}_a$
$\vec{n}_a \cdot \vec{e}^\alpha = 0$
$a = D+1 \ldots N$
connection.
extrinsic curvature.
$\partial_\alpha \partial_\mu X$
Symmetric under $\alpha \leftrightarrow \mu$.
project this out to get an intrinsic derivative
$\Rightarrow \Gamma^\lambda_{\mu\alpha}$ = Symm under $\alpha \leftrightarrow \mu$.

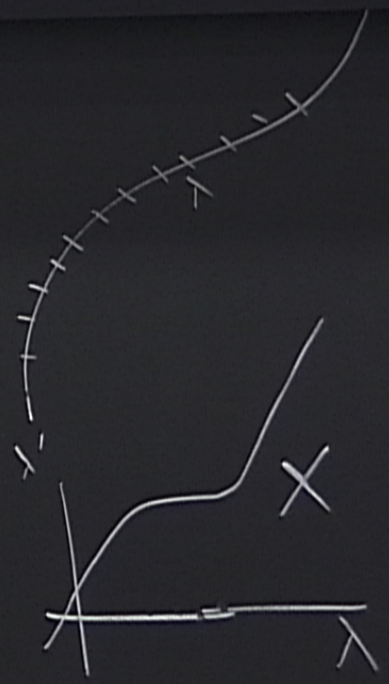

differentiate w.r.t. x^α
$\partial_\alpha \vec{V} = (\partial_\alpha V^\mu)\vec{e}_\mu + V^\mu(\partial_\alpha \vec{e}_\mu)$
$\Gamma^\lambda_{\mu\alpha} = \vec{e}^\lambda \cdot \partial_\alpha \vec{e}_\mu$
where $\vec{e}^\lambda \cdot \vec{e}_\alpha = \delta^\lambda_\alpha$
$\partial_\alpha \vec{e}_\mu = \Gamma^\lambda_{\mu\alpha} \vec{e}_\lambda$
connection.
$\partial_\alpha \partial_\mu X$
symmetric under $\alpha \leftrightarrow \mu$.
line covariant derivative
$D_\alpha \vec{V} = (\partial_\alpha V^\lambda + \Gamma^\lambda_{\kappa\alpha} V^\kappa)\vec{e}_\mu$
NOT A TENSOR
$\Rightarrow \Gamma^\lambda_{\mu\alpha}$ = Symm under $\alpha \leftrightarrow \mu$.

$$S \propto \int d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

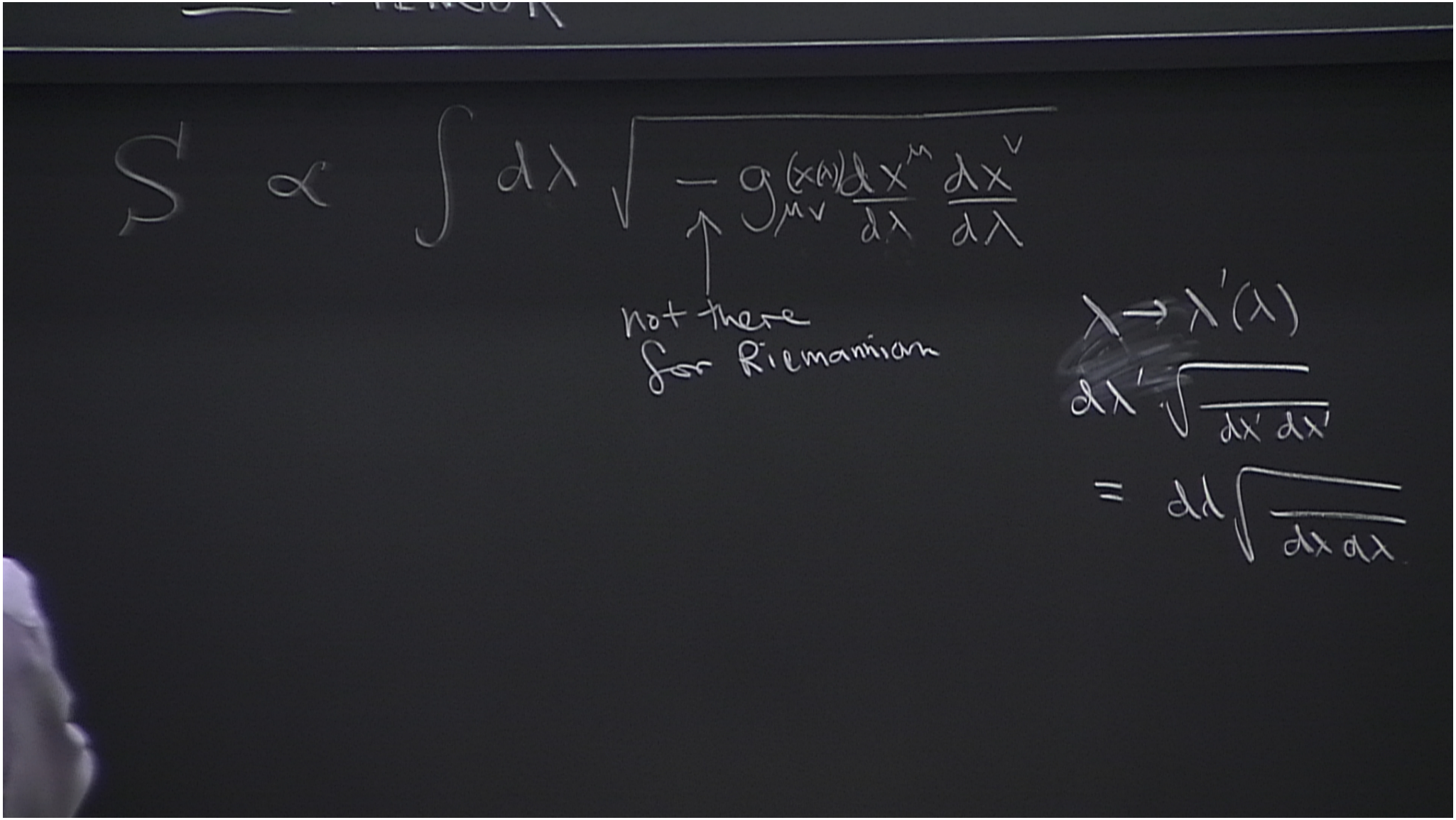
not there for Riemannian

$\lambda \to \lambda'(\lambda)$

$$d\lambda' \sqrt{\frac{dx' dx'}{}} = d\lambda \sqrt{\frac{dx\, dx}{}}$$



insist $\left|\frac{d\lambda'}{d\lambda}\right|$


$S \propto \int d\lambda \sqrt{-g^{(x(\lambda))}_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$
not there for Riemannian
$\lambda \to \lambda'(\lambda)$
$d\lambda' \sqrt{\frac{}{dx' dx'}}$
$= d\lambda \sqrt{\frac{}{dx\, d\lambda}}$

TENSOR

particle moving on M^D

$$S \propto \int d\lambda \sqrt{-g_{\mu\nu}(x(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

not there for Riemannian

$\lambda \to \lambda'$

$d\lambda' \sqrt{\quad}$ = $d\lambda$

$$\delta S = \int \frac{d\lambda}{2\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}} \left(- g_{\mu\nu,\alpha} \delta x^\alpha(\lambda) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - 2 g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{d\delta x^\nu}{d\lambda} \right)$$

$$\delta S' = \alpha \int \frac{d\lambda}{2\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}} \left(- g_{\mu\nu,\alpha}\, \delta x^\alpha(\lambda) \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - 2 g_{\mu\nu}\frac{dx^\mu}{d\lambda}\right.$$

$$\frac{d\tau}{d\lambda}$$

$$\frac{d}{d\tau} =$$

$$= \frac{1}{2}\int d\tau \left(- g_{\mu\nu,\alpha}\delta x^\alpha \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - 2 g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}\right) =$$

not there for Riemannian

$\lambda \to \lambda'(\lambda)$

$d\lambda' \sqrt{\frac{}{dx' dx'}} = d\lambda \sqrt{\frac{}{dx\, d\lambda}}$



insist $\left|\frac{d\lambda'}{d\lambda}\right| > 0$

$2 g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{d\delta x^\nu}{d\lambda}$)

$\frac{d}{d\tau} =$ •

$= \frac{1}{2} \int d\tau \left(g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu \delta x^\alpha - 2 g_{\mu\nu,\alpha} \dot{x}^\alpha \dot{x}^\mu \delta x^\nu - 2 g_{\mu\nu} \ddot{x}^\mu \delta x^\nu \right)$

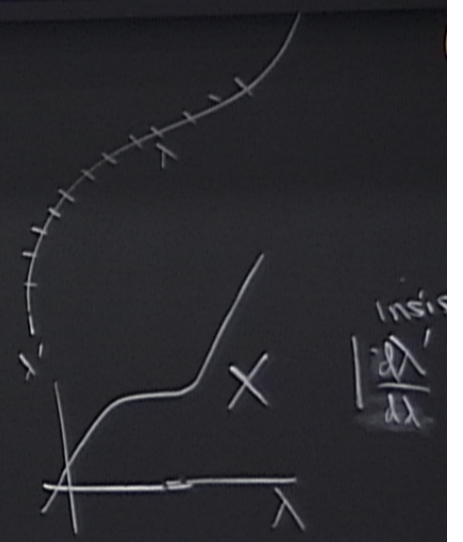
NOT A TENSOR

moving on M^D $\quad S \propto \int d\lambda \sqrt{-g_{\mu\nu}(x(\lambda)) \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$

not there for Riemannian

$\lambda \to \lambda'(\lambda)$

$d\lambda' \sqrt{\frac{dx' dx'}{}} = d\lambda \sqrt{\frac{dx\, dx}{}}$

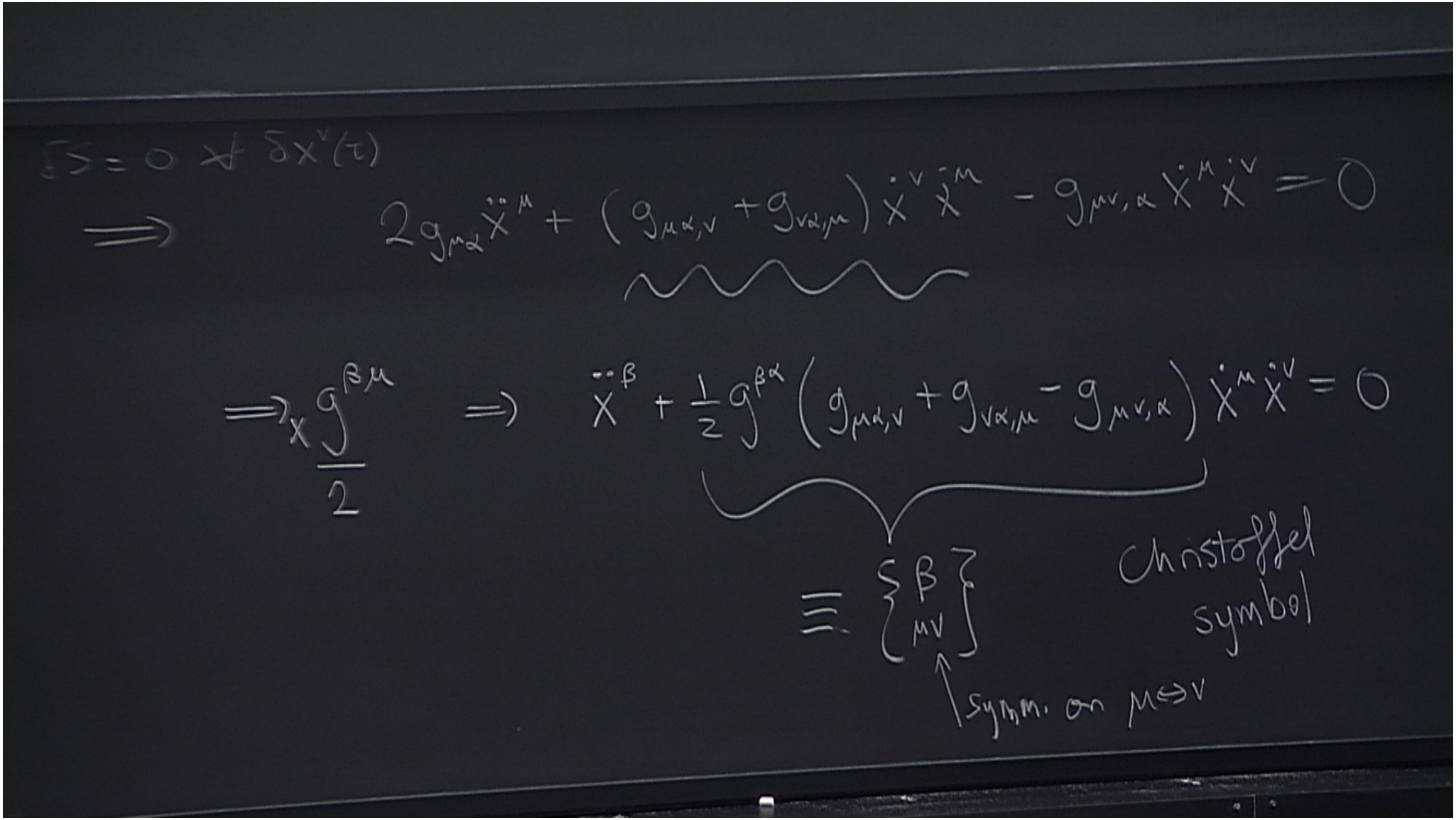


$$\int \frac{d\lambda}{2\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}} \left(-g_{\mu\nu,\alpha}\delta x^\alpha(\lambda)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - 2g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda} \right)$$

$\frac{d\tau}{d\lambda}$

$\frac{d}{d\tau} = \cdot$

$$= \frac{1}{2}\int d\tau \left(-g_{\mu\nu,\alpha}\delta x^\alpha \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - 2g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau} \right) = \frac{1}{2}\int d\tau \left(-g_{\mu\nu,\alpha}\dot{x}^\mu\dot{x}^\nu\delta x^\alpha + 2g_{\mu\nu,\alpha}\dot{x}^\alpha\dot{x}^\mu\delta x^\nu + 2g_{\mu\nu}\ddot{x}^\mu\delta x^\nu \right)$$


$\delta S = 0 \;\forall\; \delta x^\nu(\tau)$
$\Longrightarrow \quad 2 g_{\mu\alpha}\ddot{x}^\mu + (g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu})\dot{x}^\nu \dot{x}^\mu - g_{\mu\nu,\alpha}\dot{x}^\mu \dot{x}^\nu = 0$
$\Longrightarrow \times \frac{g^{\beta\mu}}{2} \quad \Longrightarrow \quad \ddot{x}^\beta + \frac{1}{2} g^{\beta\alpha}\left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)\dot{x}^\mu \dot{x}^\nu = 0$
$\equiv \left\{ {\beta \atop \mu\nu} \right\}$
Christoffel symbol
Symm. on $\mu \leftrightarrow \nu$

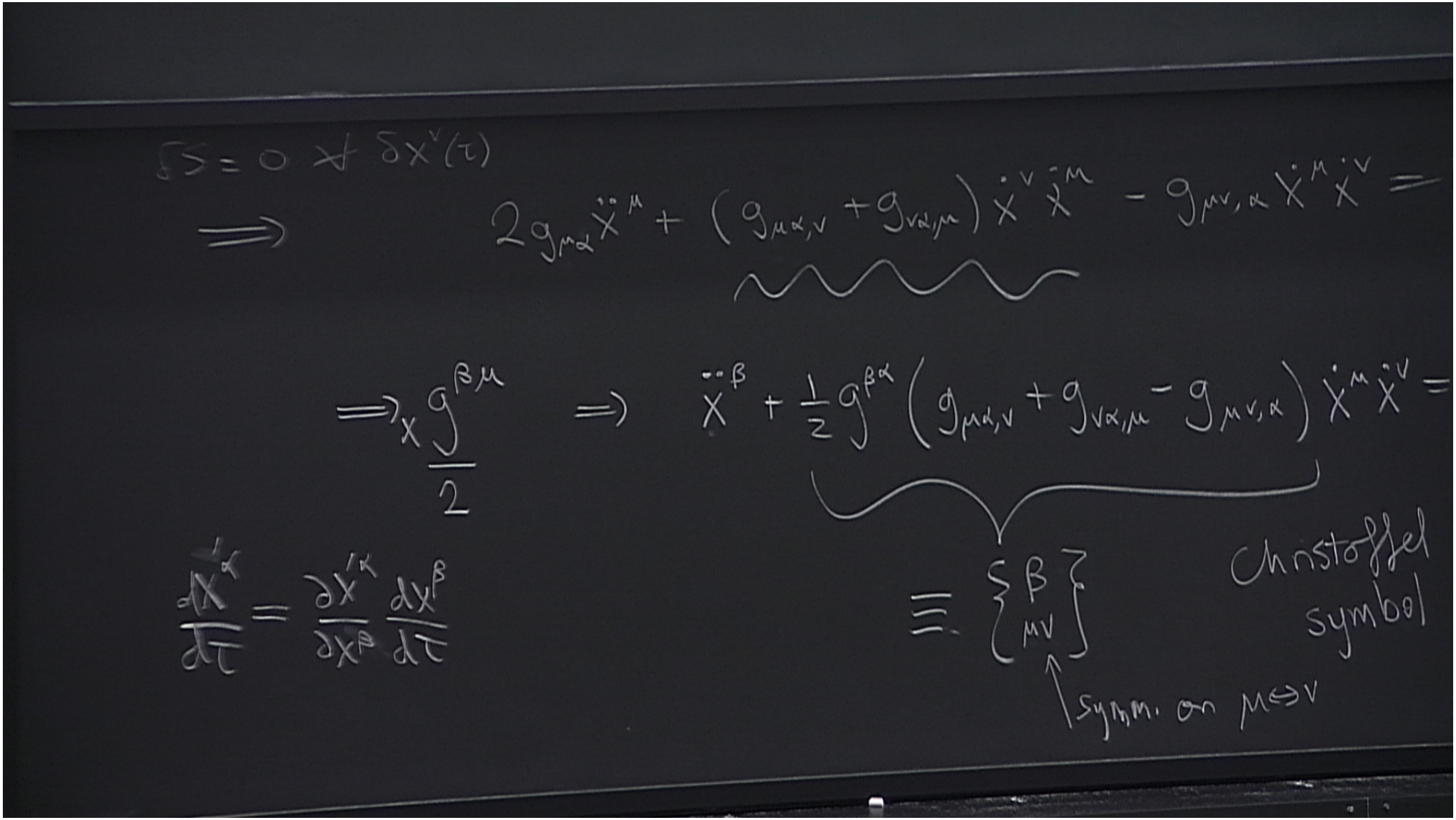
$$(g_{\alpha\nu,\mu} + g_{\nu\alpha,\mu})\dot{x}^\nu \dot{x}^\mu - g_{\mu\nu,\alpha}\dot{x}^\mu \dot{x}^\nu = 0$$

$$-\frac{1}{2}g^{\beta\alpha}\left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)\dot{x}^\mu \dot{x}^\nu = 0$$

$$\equiv \left\{ {\beta \atop \mu\nu} \right\}$$

Christoffel symbol ; simplest case of a connection.

↑ Sym. on $\mu \leftrightarrow \nu$


$\delta S = 0 \;\forall\; \delta x^\nu(\tau)$
$\Longrightarrow$
$2 g_{\mu\alpha}\ddot{x}^\mu + (g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu})\dot{x}^\nu \dot{x}^\mu - g_{\mu\nu,\alpha}\dot{x}^\mu \dot{x}^\nu =$
$\Longrightarrow \times \frac{g^{\beta\mu}}{2}$
$\Longrightarrow$
$\ddot{x}^\beta + \frac{1}{2} g^{\beta\alpha}\left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)\dot{x}^\mu \dot{x}^\nu =$
$\frac{d\dot{x}^\alpha}{d\tau} = \frac{\partial x'^\alpha}{\partial x^\beta}\frac{dx^\beta}{d\tau}$
$\equiv \left\{ {\beta \atop \mu\nu} \right\}$
Christoffel symbol
Symm. on $\mu \leftrightarrow \nu$

$\delta S = 0 \;\forall\; \delta x^\nu(\tau)$

$$\Longrightarrow \quad 2 g_{\mu\alpha}\ddot{x}^\mu + \left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu}\right)\dot{x}^\nu \dot{x}^\mu - g_{\mu\nu,\alpha}\dot{x}^\mu \dot{x}^\nu = 0$$

$$\Longrightarrow \times \frac{g^{\beta\mu}}{2} \quad \Longrightarrow \quad \ddot{x}^\beta + \frac{1}{2} g^{\beta\alpha}\left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)\dot{x}^\mu \dot{x}^\nu = 0$$

$$\frac{d}{d\tau}\frac{dx'^\alpha}{d\tau} = \frac{d}{d\tau}\left(\frac{\partial x'^\alpha}{\partial x^\beta}\frac{dx^\beta}{d\tau}\right) = \frac{\partial x'^\alpha}{\partial x^\beta}\frac{d^2x^\beta}{d\tau^2} + \frac{\partial^2 x'^\alpha}{\partial x^\beta \partial x^\gamma}\frac{dx^\gamma}{d\tau}\cdot\frac{dx^\beta}{d\tau}$$

WRONG for x^β to be a tensor

$$\equiv \left\{ {\beta \atop \mu\nu} \right\}$$

Christoffel symbol

↑ Symm. on $\mu \leftrightarrow \nu$

now that $\left\{ {\mu \atop \nu\alpha} \right\}' = \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\epsilon}{\partial x'^\alpha}\frac{\partial x'^\mu}{\partial x^\rho}\left\{ {\rho \atop \sigma\epsilon} \right\} + \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\nu \partial x'^\alpha}$

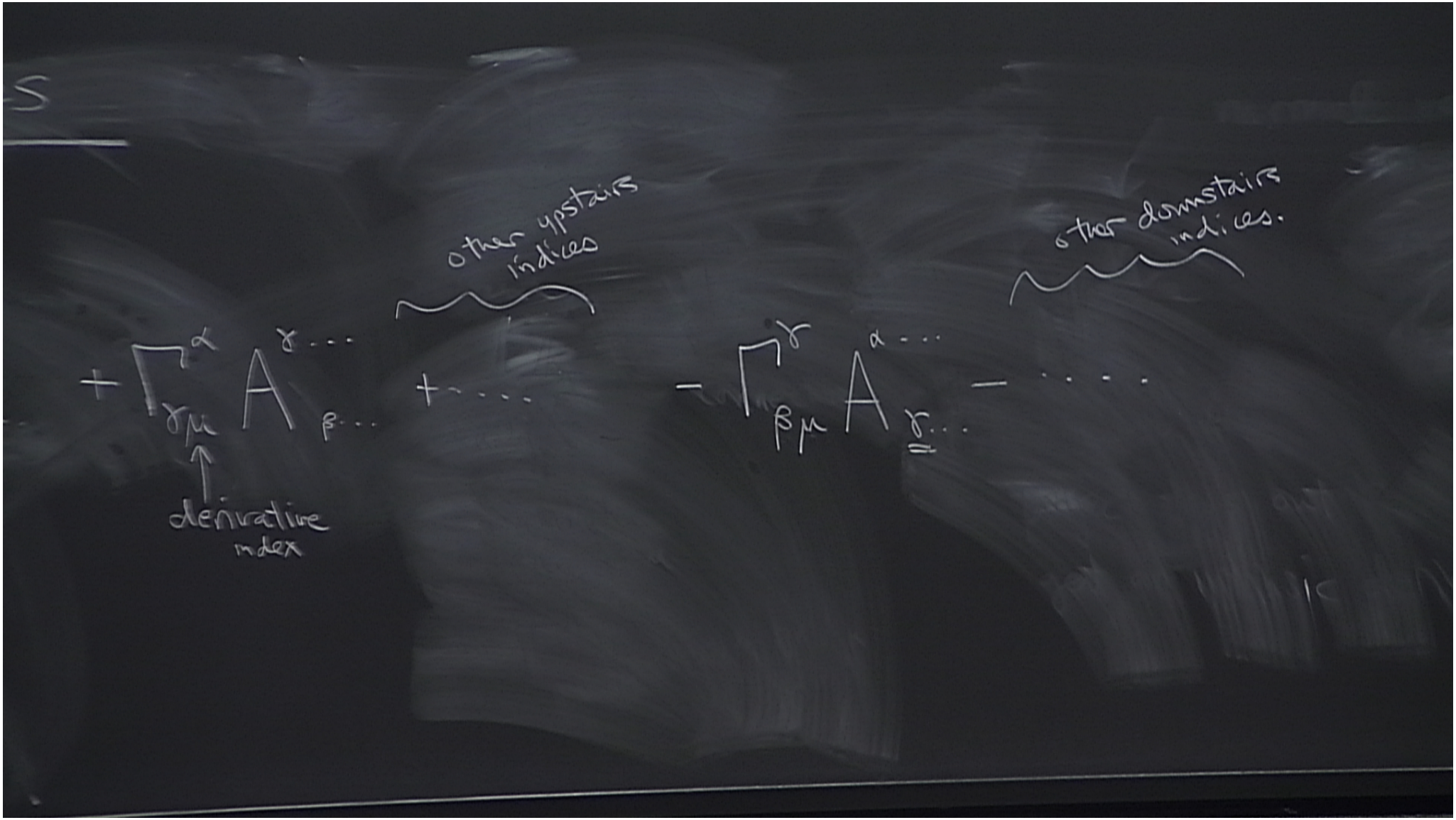
and that $\frac{D}{D\tau}\frac{dx^\mu}{d\tau} \equiv \frac{d^2x^\mu}{d\tau^2} + \left\{ {\mu \atop \alpha\beta} \right\}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$ is a 4-vector.

or

entiate w.r.t. x^α $\partial_\alpha \vec{V} = (\partial_\alpha V^\mu)\vec{e}_\mu + V^\mu(\partial_\alpha \vec{e}_\mu)$

$$\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} = \frac{d\tau}{d\lambda}$$

$$\left(\frac{d\tau}{d\lambda}\right) \equiv \sqrt{-g^{(x)}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$


other upstairs indices
other downstairs indices.
derivative index

Covariant Derivatives

In general, we define

$$\nabla_\mu A^{\alpha\ldots}_{\beta\ldots} = \partial_\mu A^{\alpha\ldots}_{\beta\ldots} + \Gamma^{\alpha}_{\gamma\mu} A^{\gamma\ldots}_{\beta\ldots} + \ldots$$

derivative index

- this transforms as a tensor.

$\frac{\partial x^\mu}{\partial x'^\nu \partial x'^\rho}$ wrong: need to cancel this.

We shall introduce a connection to do this.

$$\Gamma'^{\mu}_{\nu\alpha} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\delta}{\partial x'^\nu}\frac{\partial x^\epsilon}{\partial x'^\alpha}\Gamma^\rho_{\delta\epsilon} + \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\nu \partial x'^\alpha} \qquad (*)$$

Notice that the "bad" term in the transformation of Γ; given in (

$$\Rightarrow \quad T^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} - \Gamma^{\mu}_{\lambda\nu} \quad \underline{\text{is}} \text{ a tensor!}$$

This is the torsion.

It is an independent assumption that the torsion vanishes. (In gener

(In general it does not: if introduce fermions in simplest, "first order" formalism, then even standard model gives torsion - but very small).

$$T_{\mu\cdot\nu\lambda} = g_{\mu\alpha} T^{\alpha}_{\nu\lambda} \propto \overbrace{\bar{\psi}\, \gamma_{[\mu}\gamma_{\nu}\gamma_{\lambda]}\psi}^{\text{spin}} \propto \text{density of fermions}$$

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It is an independent assumption that the torsion $T^{\mu}_{\nu\lambda}$ vanishes. (In genera

"first ord

$T_{\mu\cdot\nu\lambda}$

mation of Γ; given in (*), is symmetrical in lower two indices

is a tensor!

$T^{\alpha}_{\mu\nu}$

torsion vanishes. (In general it does not; if introduce fermions in simplest, "first order" formalism, then even standard model gives torsion - but very small).

$T_{\mu,\nu\lambda} = g_{\mu\alpha} T^{\alpha}_{\nu\lambda} \propto \overline{\psi}\, \gamma_{[\mu}\gamma_{\nu}\gamma_{\lambda]}\psi$ ∝ density of fermions

spin

Notice that the "bad" term in the transformation of Γ; gi

$$\Rightarrow \quad T^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} - \Gamma^{\mu}_{\lambda\nu} \quad \underline{\text{is}} \text{ a tensor!}$$

This is the torsion.

It is an independent assumption that the torsion $T^{\mu}_{\nu\lambda}$ vanishes. (I

If the torsion is totally antisymmetric $T_{\mu\nu\lambda} = -T_{\nu\mu\lambda}$ etc.

and impose metricity $\nabla_\mu g_{\alpha\beta} = 0$, then $\Gamma^{\mu}_{(\alpha\beta)} = \left\{ {\mu \atop \alpha\beta} \right\}$.

If $T^{\lambda}_{\nu\lambda} = 0$

then $\nabla_{\mu} g_{\alpha\beta} = 0$

$\Rightarrow \Gamma^{\lambda}_{\nu\lambda} = \left\{ {\lambda \atop \nu\lambda} \right\}$.

It is a

If the torsion is

and impose metri

Already saw that we can always choose coords so that $g_{\mu\nu}$

Can we set $\nabla_\alpha g_{\beta\gamma} = 0$ (*) at all points simultaneous

i.e. metric is covariantly constant

– "metricity".

Can solve (*) for Γ:

$$g_{\beta\sigma,\alpha} - \Gamma^\rho_{\beta\alpha} g_{\rho\sigma} - \Gamma^\rho_{\sigma\alpha} g_{\rho\beta} = 0 \quad \Rightarrow$$

$+\ \alpha \leftrightarrow \beta \qquad + g_{\gamma\sigma,\beta} - \Gamma^\rho_{\gamma\beta} g_{\rho\alpha} - \Gamma^\rho_{\alpha\beta} g_{\rho\sigma}$

$- \alpha \leftrightarrow \gamma \qquad - g_{\alpha\beta,\gamma} + \Gamma^\rho_{\alpha\sigma} g_{\rho\beta} + \Gamma^\rho_{\beta\sigma} g_{\rho\alpha}$

that $g_{\mu\nu,\alpha} = 0$ at a particular point.

multaneously?

$$\Rightarrow \quad g_{\rho\sigma}\Gamma^{\rho}_{\alpha\beta} = \frac{1}{2}\left(g_{\beta\sigma,\alpha} + g_{\alpha\sigma,\beta} - g_{\alpha\beta,\sigma}\right)$$

$$\Gamma^{\rho}_{\alpha\beta} = \frac{1}{2} g^{\rho\sigma}\left(g_{\beta\sigma,\alpha} + g_{\alpha\sigma,\beta} - g_{\alpha\beta,\sigma}\right) = \left\{ {\rho \atop \alpha\beta} \right\}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -i\left[\underbrace{\partial_\mu + iA_\mu}_{\text{cov. deriv.}}, \partial_\nu + iA_\nu\right]$$

$$R_{\mu\nu\rho\sigma} V^\rho \sim \left[\nabla_\mu, \nabla_\nu\right] V_\rho$$

simplest case of a connection.