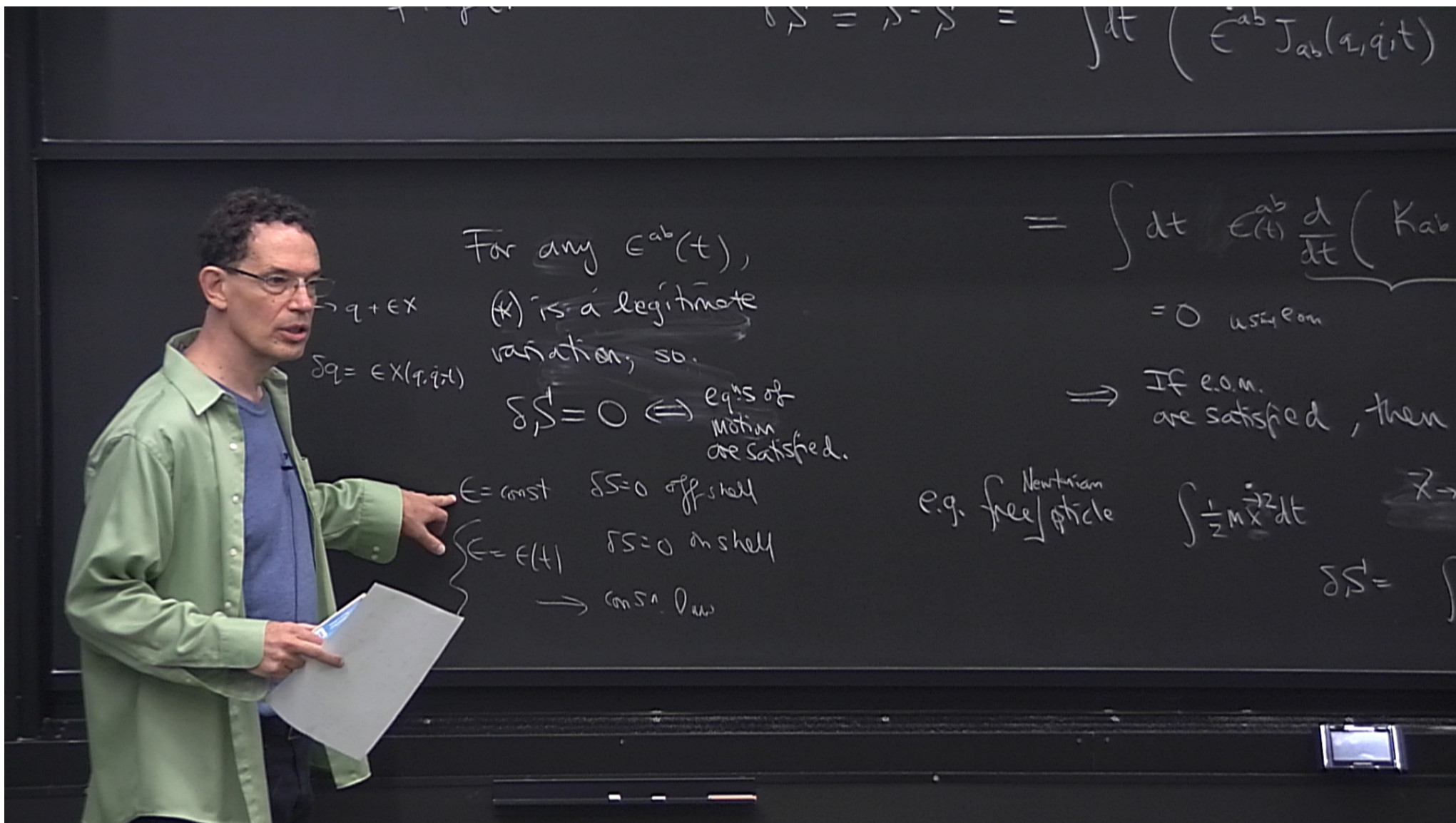


Title: PSI 2016/2017 Relativity - Lecture 4

Date: Sep 07, 2016 10:45 AM

URL: <http://pirsa.org/16090004>

Abstract:



$$\delta S = S - S = \int dt \left(\epsilon^{ab} J_{ab}(q, \dot{q}, t) \right)$$

For any $\epsilon^{ab}(t)$,
 $q \rightarrow q + \epsilon X$ (*) is a legitimate
 $\delta q = \epsilon X(q, \dot{q}, t)$ variation, so.
 $\delta S = 0 \Leftrightarrow$ eq's of
 motion
 are satisfied.

$\epsilon = \text{const}$ $\delta S = 0$ off shell

$\left\{ \begin{array}{l} \epsilon = \epsilon(t) \\ \epsilon = \epsilon(t, x) \end{array} \right.$ $\delta S = 0$ on shell
 $\rightarrow$ cons. laws

$$= \int dt \epsilon^{ab}(t) d(K_{ab})$$

$\Rightarrow$ If e.o.m.
are satisfied

e.g. ^{Newtonian} free particle

→ $\text{const. } 0_{\text{ew}}$

$$\delta S = \int m \dot{\vec{x}} \delta \vec{x} = \int m \dot{\vec{x}} \vec{\epsilon} = - \int (\dot{m \vec{x}})^T \vec{\epsilon} =$$

$$S' = -m_0 c^2 \int dt \sqrt{1 - \frac{1}{c^2} \left(\frac{d\vec{x}}{dt} \right)^2}$$

$$L = -m_0 c^2 \sqrt{1 - \frac{1}{c^2} \left(\frac{d\vec{x}}{dt} \right)^2}$$

transl. invariant
 $\vec{x} \rightarrow \vec{x} + \vec{\epsilon}$

Noether $\rightarrow \vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = \frac{m_0 c^2 \dot{\vec{x}}}{c^2 \sqrt{1 - \dot{\vec{x}}^2/c^2}} = \frac{m \dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}}^2/c^2}}$

Cons. Law

$$\delta S = \int m \dot{\vec{x}} \delta \vec{x} = \int m \dot{\vec{x}} \vec{\epsilon} = - \int (\dot{m \vec{x}}) \vec{\epsilon} = 0$$

$$S' = -m_0 c^2 \int dt \sqrt{1 - \frac{1}{c^2} \left(\frac{d\vec{x}}{dt} \right)^2}$$

$$L = -m_0 c^2 \sqrt{1 - \frac{1}{c^2} \left(\frac{d\vec{x}}{dt} \right)^2}$$

invariant
 $\rightarrow \vec{x} + \vec{\epsilon}$

Noether $\rightarrow \vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m_0 c^2 \frac{\dot{\vec{x}}}{c^2 \sqrt{1 - \dot{\vec{x}}^2/c^2}} = \frac{m \dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}}^2/c^2}} = \gamma m \dot{\vec{x}} = m \vec{u}$

$$\delta S = \int m \dot{\vec{x}} \delta \vec{x} = \int m \dot{\vec{x}} \vec{\epsilon} = - \int (\dot{m \vec{x}}) \vec{\epsilon} = 0 \quad \forall \vec{\epsilon}(t)$$

$$S' = -m_0 c^2 \int dt \sqrt{1 - \frac{1}{c^2} \left(\frac{d\vec{x}}{dt} \right)^2}$$

$$L = -m_0 c^2 \sqrt{1 - \frac{1}{c^2} \left(\frac{d\vec{x}}{dt} \right)^2}$$

transl: invariant
 $\vec{x} \rightarrow \vec{x} + \vec{\epsilon}$

$$\text{Noether} \rightarrow \vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m_0 c^2 \frac{\dot{\vec{x}}}{c^2 \sqrt{1 - \dot{\vec{x}}^2/c^2}} = \frac{m_0 \dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}}^2/c^2}} = \gamma m_0 \dot{\vec{x}} = m_0 \vec{u}$$

time transl:

$$\text{Noether: } E = H = \vec{p} \dot{\vec{x}} - L = \frac{m_0 \dot{\vec{x}}^2}{\sqrt{1 - \dot{\vec{x}}^2/c^2}} + m_0 c^2 \sqrt{1 - \dot{\vec{x}}^2/c^2} =$$

$$\delta S = \int m \dot{\vec{x}} \delta \dot{\vec{x}} = \int m \dot{\vec{x}} \dot{\vec{\epsilon}} = - \int (\dot{m \vec{x}}) \dot{\vec{\epsilon}} = 0 \quad \forall \vec{\epsilon}(t) \Rightarrow \frac{d}{dt}(m \dot{\vec{x}}) = 0$$

$$u^\mu = \gamma(c, \dot{\vec{x}})$$

$$\frac{\vec{x}}{c \sqrt{1 - \dot{\vec{x}}^2/c^2}} = \frac{m_0 \dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}}^2/c^2}} = \gamma m_0 \dot{\vec{x}} = m_0 \vec{u}$$

$$\vec{p} \dot{\vec{x}} - L = \frac{m_0 \dot{\vec{x}}^2}{\sqrt{1 - \dot{\vec{x}}^2/c^2}} + m_0 c^2 \sqrt{1 - \dot{\vec{x}}^2/c^2} = \frac{m_0 c^2}{\sqrt{1 - \dot{\vec{x}}^2/c^2}} = m_0 c u^0$$

$$\left. \begin{aligned} p^\mu &= m_0 u^\mu \\ &= \left(\frac{E}{c}, \vec{p} \right) \end{aligned} \right\}$$

$$c^2 d\tau^2 = - dx^\mu dx_\mu$$

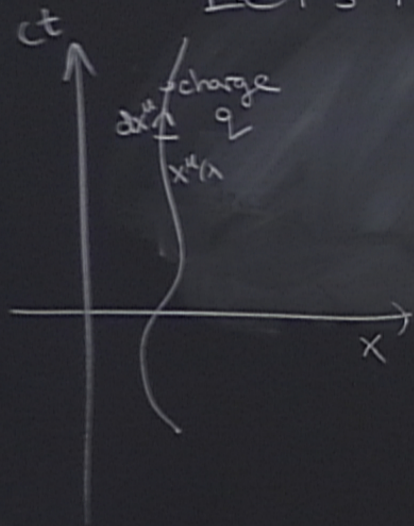
$$\Rightarrow u^\mu = \frac{dx^\mu}{d\tau} \text{ sats. } u^\mu u_\mu = -c^2$$

$$\Rightarrow p^\mu = m_0 u^\mu \text{ sats. } \left. \begin{aligned} p^\mu p_\mu &= -m_0^2 c^2 \\ &= -\frac{E^2}{c^2} + \vec{p}^2 \end{aligned} \right\} \begin{aligned} E^2 &= c^2 (\vec{p}^2 + m_0^2) \\ E &= c \sqrt{\vec{p}^2 + m_0^2} \end{aligned}$$

$$\begin{aligned}
 & \left. \begin{aligned} & -m_0^2 c^2 \\ & -\frac{E^2}{c^2} + \vec{p}^2 \end{aligned} \right\} E^2 = c^2 (\vec{p}^2 + m_0^2 c^2) \quad \vec{p}^2 \ll m_0^2 c^2 \\
 & E = c \sqrt{\vec{p}^2 + m_0^2 c^2} \approx m_0 c^2 + \frac{\vec{p}^2}{2m_0}
 \end{aligned}$$

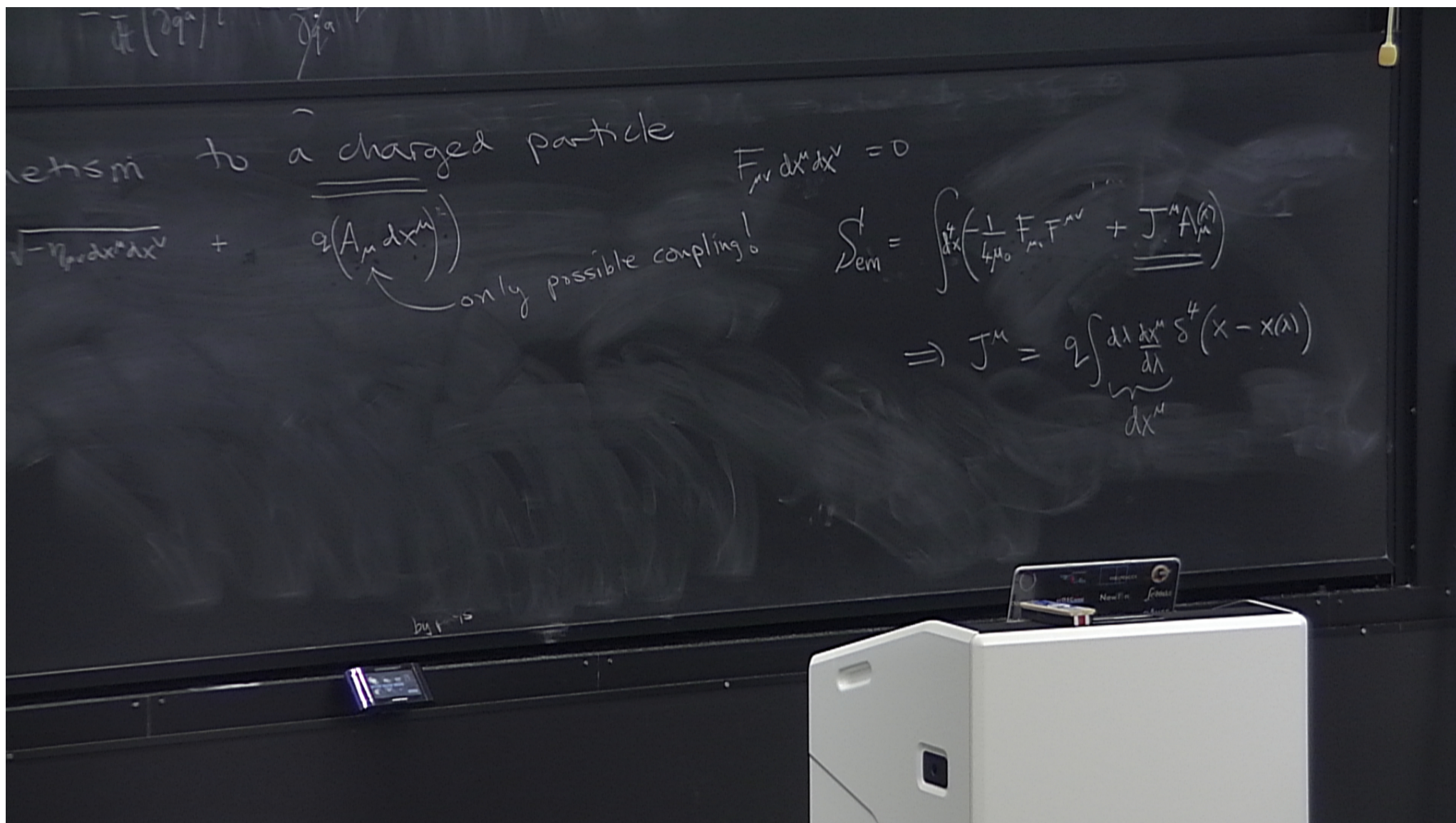
Note also that $\vec{p} = \frac{m_0 \vec{v}}{\sqrt{1 - v^2/c^2}} \Rightarrow \vec{v} = \frac{d\vec{x}}{dt} = \frac{\vec{p} c}{\sqrt{\vec{p}^2 + m_0^2 c^2}}$

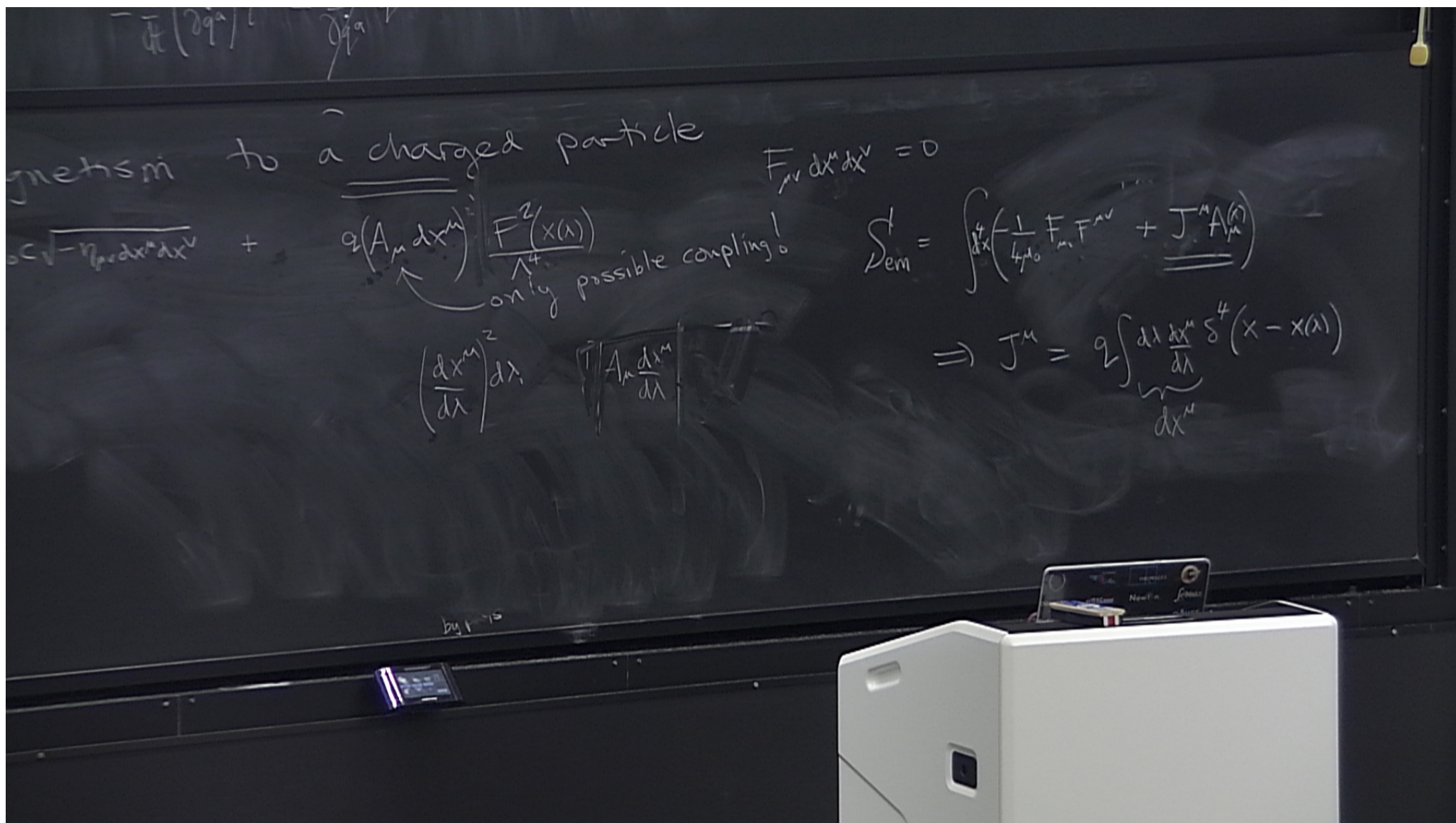
$|\vec{v}| < c$
 $|\vec{v}| = c$ for $|\vec{p}| \rightarrow \infty$



Lets now couple electromagnetism to a charged pa

$$S_{\text{particle}} = \int \left(-m_0 c \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} + \right.$$





$\frac{1}{2} (\partial_\mu a)^2$

magnetism to a charged particle

$\frac{1}{2} m_0 c^2 \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} + q(A_\mu dx^\mu)$

$A_\mu A^\mu dx^\mu dx_\mu$

simplest & only possible coupling!

$F_{\mu\nu} dx^\mu dx^\nu = 0$

which

1) introduces no additional scale

$\Rightarrow J^\mu = q \int \frac{dx^\mu}{d\lambda} \delta^4(x - x(\lambda))$

$\int d^4x \left(-\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + \underline{J^\mu A_\mu} \right)$

$\frac{A_\mu A^\mu}{M_0^2}$

by 1-15

$\bar{\psi}(\partial \not{a}) \psi$

magnetism to a charged particle

$$\left(-m_0 c \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} + q(A_\mu dx^\mu) \right)$$

$A_\mu A^\mu dx^\mu dx_\mu$

simplest & only possible coupling!

$F_{\mu\nu} dx^\mu dx^\nu = 0$

$$S_{\text{em}} = \int d^4x \left(-\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + \underline{\underline{J^\mu A_\mu^{(A)}}} \right)$$

$$\Rightarrow J^\mu = q \int d\lambda \frac{dx^\mu}{d\lambda} \delta^4(x - x(\lambda))$$

$dx^\mu A_\mu \left(\frac{A_\mu A^\mu}{m_0^2} \right)$

which

1) introduces no additional scale

$A_\mu \rightarrow A_\mu + \partial_\mu \chi$

$\int \partial_\mu \chi dx^\mu = \text{surface term.}$

by 15

Lets now couple electromagnetism to a charged particle

$$S_{\text{particle}} = \int \left(-m_0 c \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} + q(A_\mu dx^\mu) \right)$$

$$\delta S = \int \left(+m_0 c \frac{\eta_{\mu\nu} dx^\mu d\delta x^\nu}{\sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}} + q(\partial_\nu A_\mu) \delta x^\nu dx^\mu + q A_\mu d\delta x^\mu \right)$$

$$\int d\tau \left(m_0 \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d\delta x^\nu}{d\tau} + q \partial_\nu A_\mu \frac{dx^\mu}{d\tau} \delta x^\nu + q A_\mu \frac{d\delta x^\mu}{d\tau} \right)$$

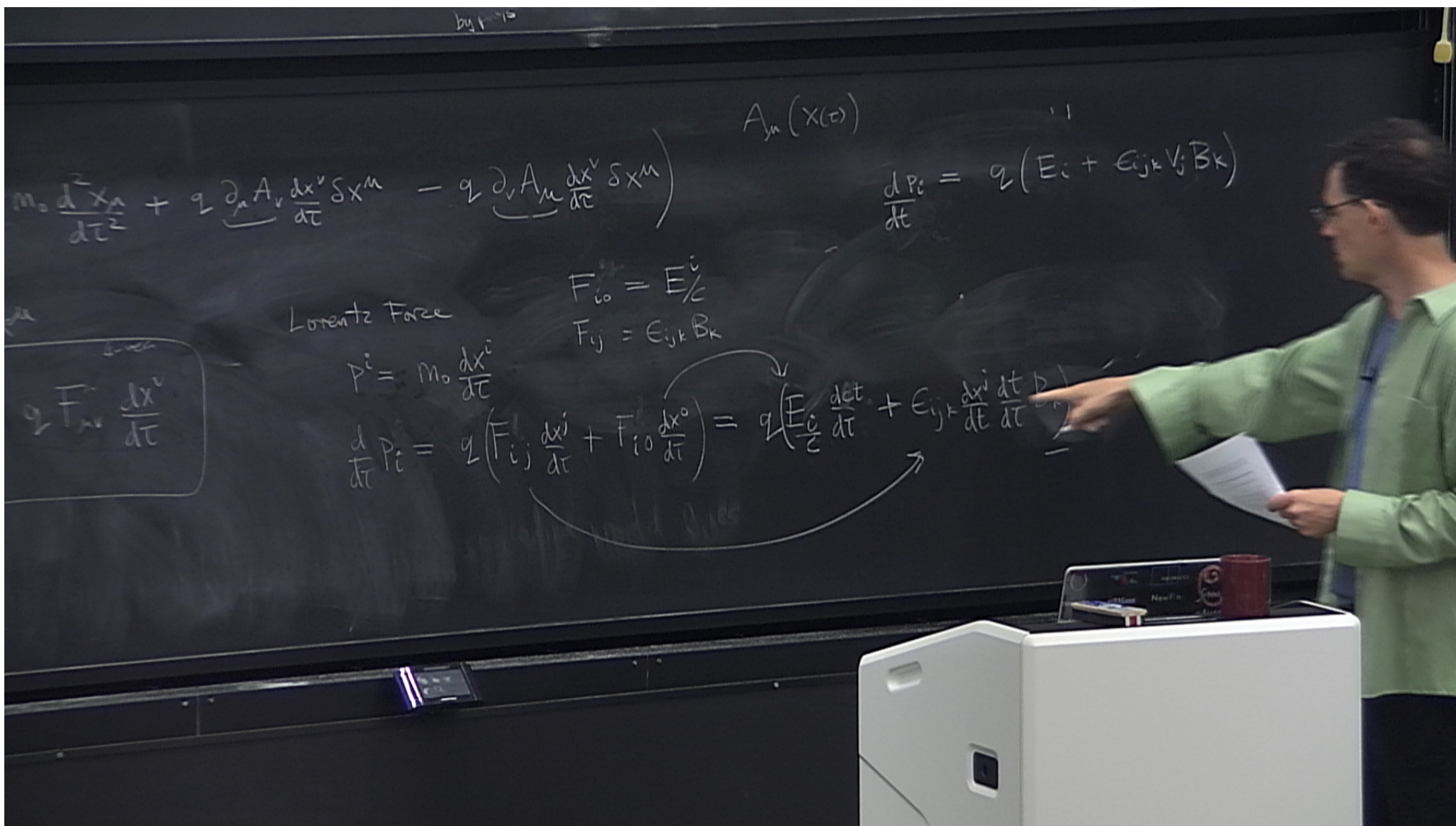
$A_\mu(x(\tau))$

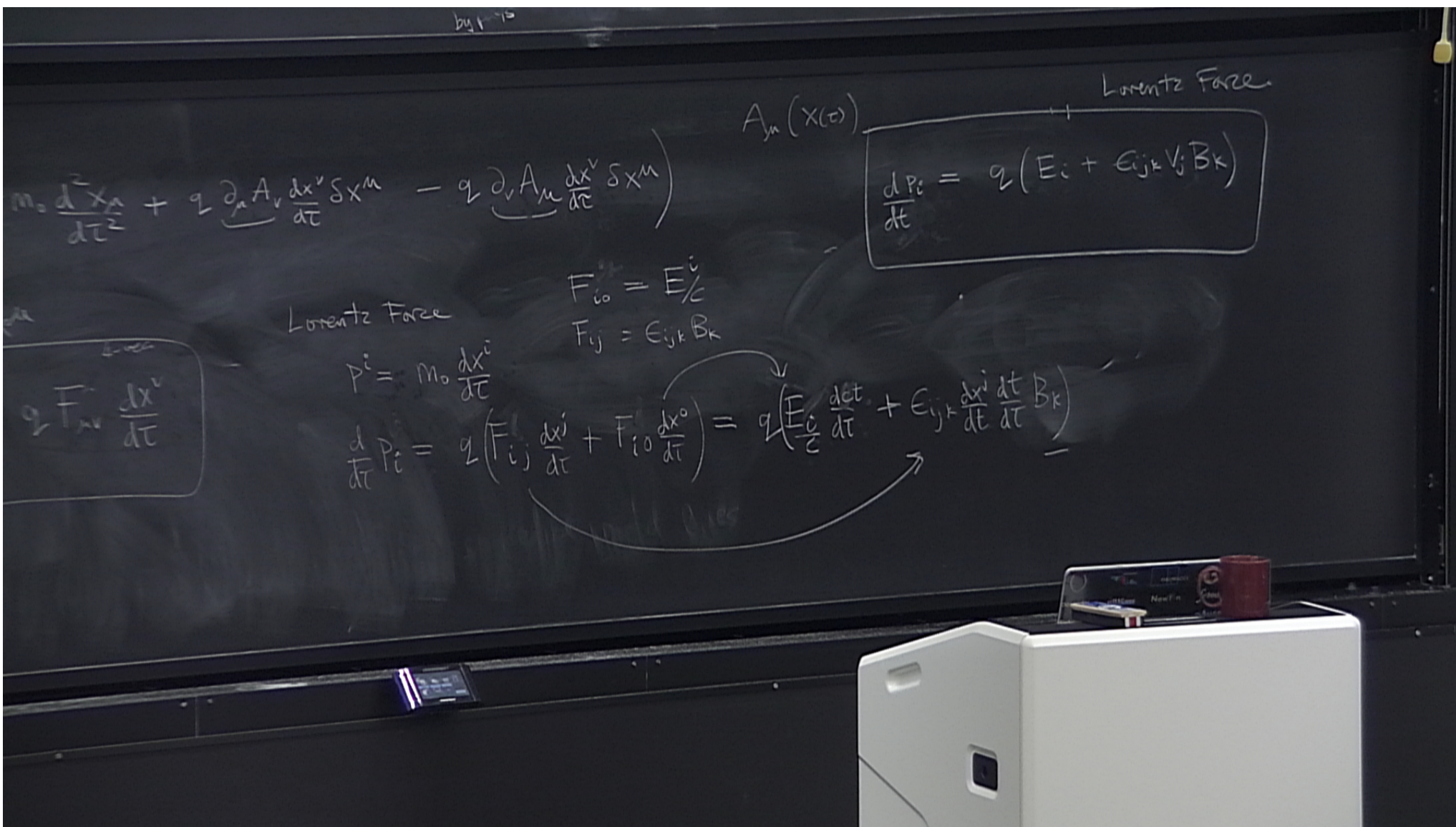
$$= \int d\tau \left(-\delta x^\mu m_0 \frac{d^2 x_\mu}{d\tau^2} + q \underbrace{\partial_\mu A_\nu}_{4\text{-vec}} \frac{dx^\nu}{d\tau} \delta x^\mu - q \underbrace{\partial_\nu A_\mu}_{4\text{-vec}} \frac{dx^\nu}{d\tau} \delta x^\mu \right)$$

$$= 0 \quad \forall \delta x^\mu$$

Lorentz Force

$$\Rightarrow \boxed{m \frac{d^2 x^\mu}{d\tau^2} = q F_{\mu\nu} \frac{dx^\nu}{d\tau}}$$

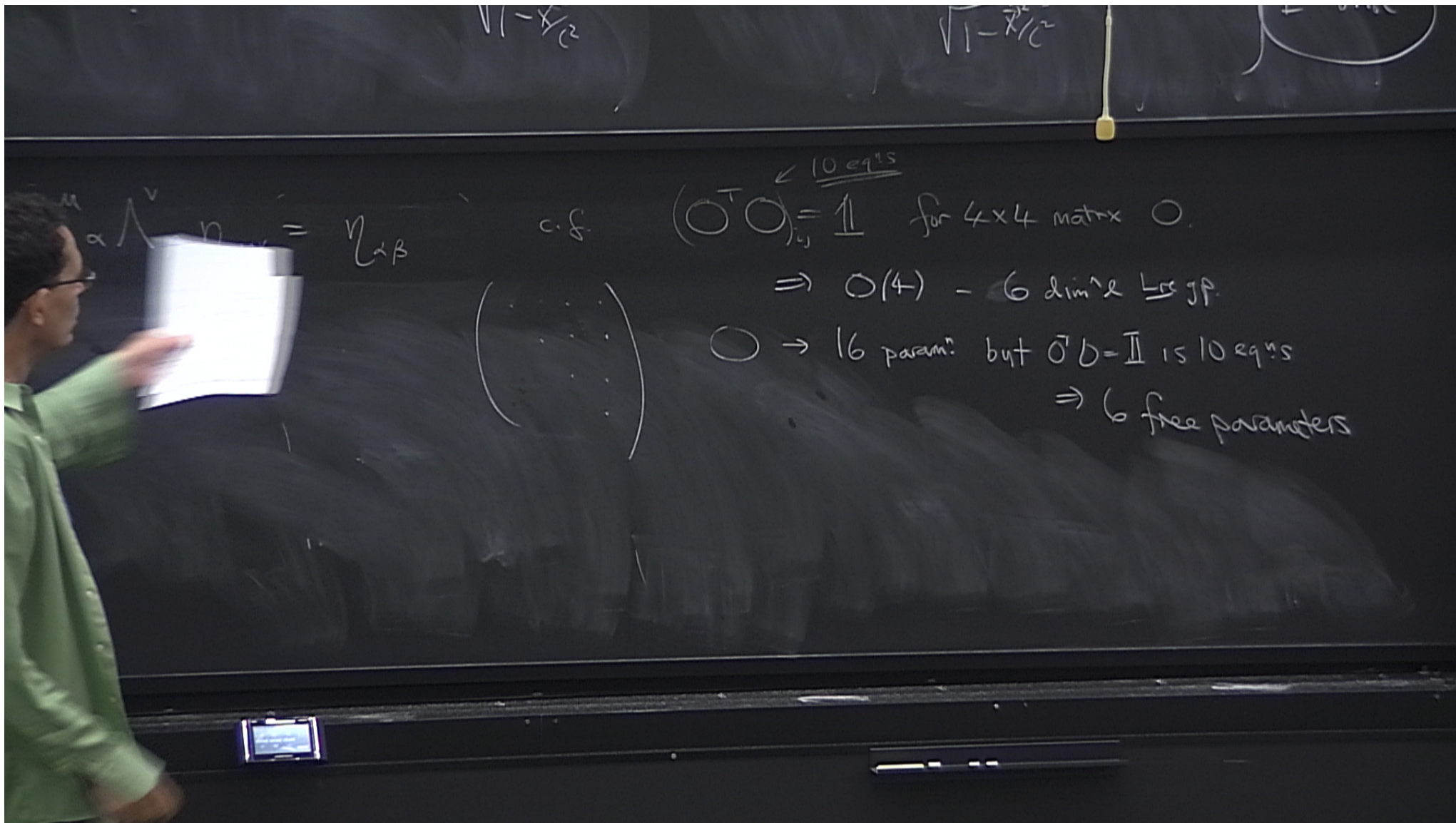




recall: LTs defined to obey $\Lambda^\mu_\alpha \Lambda^\nu_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$ c.f.

$$\sqrt{1 - v^2/c^2}$$

$$O^T O$$



Λ is defined to obey $\Lambda^\mu_\alpha \Lambda^\nu_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$ c.f. $\left(\overset{\leftarrow 10 \text{ eq's}}{\Lambda^T \Lambda} \right)_{ij} = \mathbb{1}$ for 4×4
 $\Lambda \cdot \eta \cdot \Lambda^T = \eta$
 $\Rightarrow O(3,1)$ 6 parameters $\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\Rightarrow O(4) - O$
 $O \rightarrow 16 \text{ param. but}$

recall: LTs defined to obey $\Lambda^\mu_\alpha \Lambda^\nu_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$ c.f.

$$\Lambda \cdot \eta \cdot \Lambda^T = \eta$$

$$\Rightarrow O(3,1) \quad \underline{6 \text{ parameters}}$$

We have seen that electrodynamics of fields + particles is invariant under

$$x^\mu \rightarrow x'^\mu = \underbrace{\Lambda^\mu_\nu x^\nu}_{\uparrow \text{Lorentz}} + \underbrace{a^\mu}_{\uparrow \text{transl}}$$

Poincare group - 10-dim.

us consider a general coordinate transformation $x^\mu \rightarrow x'^\mu(x)$

$$dx'^\mu = \left(\frac{\partial x'^\mu}{\partial x^\nu} \right) dx^\nu$$

us consider a general coordinate transformation

$$x^\mu \rightarrow x'^\mu(x)$$

$$dx'^\mu = \left(\frac{\partial x'^\mu}{\partial x^\nu} \right) dx^\nu \quad \left(\text{instead of } \Lambda^\mu_\nu dx^\nu \right)$$

$$\frac{\partial}{\partial x'^\mu} = \left(\frac{\partial x^\nu}{\partial x'^\mu} \right) \frac{\partial}{\partial x^\nu}$$

$$\text{Note } \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\alpha} = \frac{\partial x'^\mu}{\partial x'^\alpha} = \delta^\mu_\alpha$$

us consider a general coordinate transformation

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show that invariance of $\eta \Rightarrow$ Poincare.

$$\eta_{\alpha\beta} =$$

let us consider a general coordinate transformation $x^\mu \rightarrow x'^\mu(x)$

$$dx'^\mu = \left(\frac{\partial x'^\mu}{\partial x^\nu} \right) dx^\nu \quad \left(\text{instead of } \Lambda^\mu_\nu dx^\nu \right)$$

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can show that invariance of $\eta \Rightarrow$ Poincare.

$$\eta_{\alpha\beta} \frac{\partial x'^\alpha}{\partial x^\epsilon} \frac{\partial x'^\beta}{\partial x^\delta} = \eta_{\epsilon\delta}$$

$$\sqrt{1 - \frac{v^2}{c^2}}$$

$$\sqrt{1 - \frac{v^2}{c^2}}$$

let us consider a general coordinate transformation $x^\mu \rightarrow x'^\mu(x)$

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