

Title: PSI 2016/2017 Relativity - Lecture 3

Date: Sep 07, 2016 09:00 AM

URL: <http://pirsa.org/16090003>

Abstract:

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \quad (1)$$

$$\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0 \quad (2)$$

Note $\partial_\nu (1) \Rightarrow \partial_\nu \partial_\mu F^{\mu\nu} = -\mu_0 \partial_\nu J^\nu$

} variational principle?

$$S = \int d^4x$$

I: Set $\bar{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu =$

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Note $\partial_\nu (1) \Rightarrow \partial_\nu \partial_\mu F^{\mu\nu} = -\mu_0 \partial_\nu J^\nu$

"0"

} variational

} variational principle? I: Set $\bar{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow \text{antisym}$

$$S = \int \underbrace{d^4x}_{cdt d^3\vec{x}} \left(-\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + J^\alpha A_\alpha \right)$$

I: Set $\vec{E} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow$ automatically satisfy (2).

$$-\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + J^\alpha A_\alpha$$

$$d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x = \underbrace{\det \Lambda}_{+1} d^4x$$

$$x' = \Lambda x$$

$$\frac{\partial x'}{\partial x} = \Lambda$$

principle?

I: Set $T_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$S = \int \underbrace{d^4x}_{cdt d^3\vec{x}} \left(-\frac{1}{4\mu_0} \underbrace{F_{\mu\nu}}_{(\partial_\mu A_\nu - \partial_\nu A_\mu)} \underbrace{F^{\mu\nu}}_{\eta^{\mu\alpha} \eta^{\nu\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)} + J^\alpha A_\alpha \right)$$

variational principle?

I: Set $\bar{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow$ automatically satisfy

$$S = \int \underbrace{d^4x}_{cdt d^3\vec{x}} \left(-\frac{1}{4\mu_0} \underbrace{F_{\mu\nu} F^{\mu\nu}}_{(\partial_\mu A_\nu - \partial_\nu A_\mu) \eta^{\mu\alpha} \eta^{\nu\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)} + J^\alpha A_\alpha \right)$$

$$d^4x' =$$

$$x' = \Lambda x$$

$$\frac{\partial x'}{\partial x} = \Lambda$$

vary w.r.t. A_μ

$$\delta S = S(A_\mu + \delta A_\mu) - S(A_\mu) \text{ to 1st order in } \delta A_\mu.$$

principle? I: Set $\bar{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow$ automatically satisfy (2).

$$S = \int \underbrace{d^4x}_{cdt d^3\vec{x}} \left(-\frac{1}{4\mu_0} \underbrace{F_{\mu\nu} F^{\mu\nu}}_{(\partial_\mu A_\nu - \partial_\nu A_\mu) \eta^{\mu\alpha} \eta^{\nu\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)} + J^\alpha A_\alpha \right)$$

$$d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x$$

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$$\delta S = S(A_\mu + \delta A_\mu) - S(A_\mu) \quad \text{to 1st order in } \delta A_\mu$$

$$\delta S = \int d^4x \left(\underbrace{-\frac{1}{\mu_0} \partial_\mu \delta A_\nu F^{\mu\nu}}_{\text{int by parts}} + J^\alpha \delta A_\alpha \right) = \int d^4x \left(\frac{1}{\mu_0} \partial_\mu F^{\mu\nu} + J^\nu \right) \delta A_\nu$$

I: Set $\bar{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow$ automatically satisfy (2)

$$-\frac{1}{4\mu_0} \underbrace{F_{\mu\nu} F^{\mu\nu}} + J^\alpha A_\alpha$$

$$(\partial_\mu A_\nu - \partial_\nu A_\mu) \eta^{\mu\alpha} \eta^{\nu\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

$$d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x = \underbrace{\det \Lambda}_{+1} d^4x$$

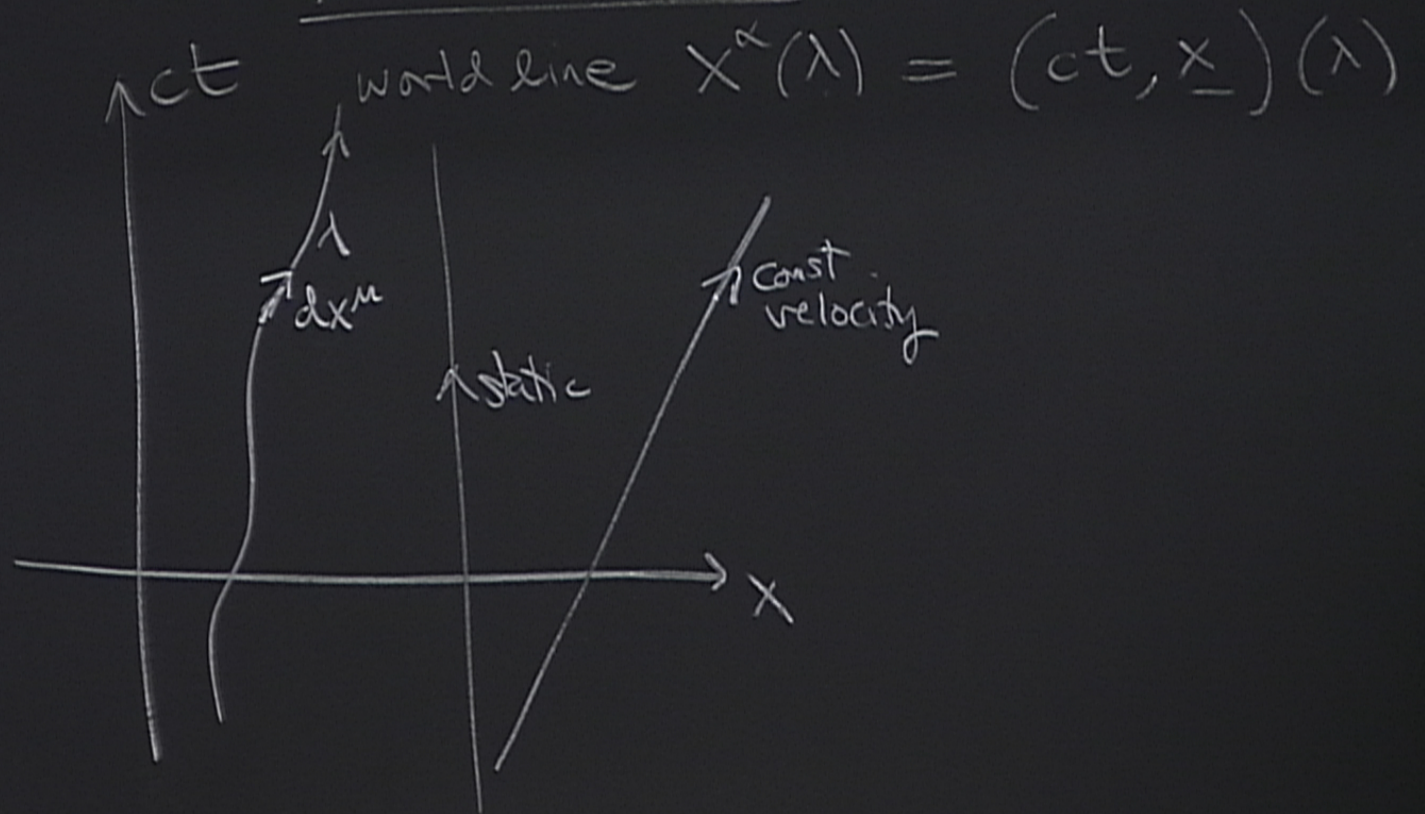
$$x' = \Lambda x$$

$$\frac{\partial x'}{\partial x} = \Lambda$$

$A_\mu) - S(A_\mu)$ to 1st order in δA_μ

$$\underbrace{\frac{1}{\mu_0} \partial_\mu \delta A_\nu F^{\mu\nu}}_{\text{int by parts}} + J^\alpha \delta A_\alpha = \int d^4x \underbrace{\left(\frac{1}{\mu_0} \partial_\mu F^{\mu\nu} + J^\nu \right)}_{\Rightarrow = 0} \delta A_\nu = 0 \quad \forall \delta A_\nu$$

Particles



$$\int_{\text{by parts}} + \frac{1}{\mu_0} \oint A_\nu d_\mu F^{\mu\nu} \Rightarrow = 0.$$

Lorentz invariant length of dx^μ

$$\eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + d\vec{x}^2$$

$$< 0$$

curve is
timelike (at that point)

$$= 0$$

null

$$> 0$$

spacelike.

Lorentz invariant length of dx^μ

$$\eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + d\vec{x}^2$$

$$< 0$$

curve is
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(at th

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$$> 0$$

spacelike.

- Massive particles follow timelike worldlines

- massless particles " null "

- "tachyons" (which don't exist) would follow spacelike world lines

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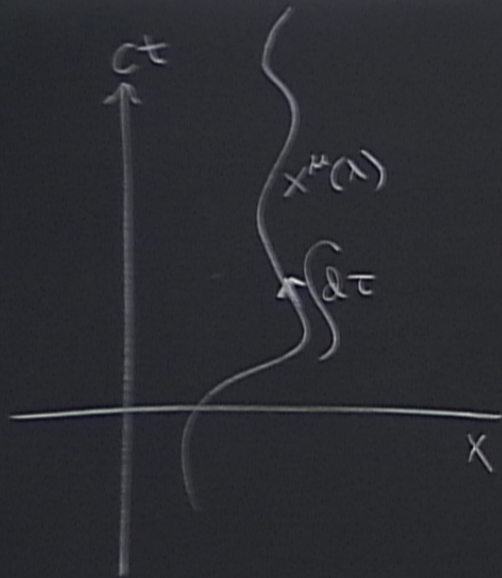
For
massive
particles,

Define τ = proper-time along the world line

$$\text{via } d\tau = \sqrt{dt^2 - \frac{d\vec{x}^2}{c^2}} = \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} \quad \text{Lorentz invariant.}$$

So $d\tau = dt$ in the Lorentz frame where the particle is static ($d\vec{x} = 0$)

$$\begin{aligned} \text{in a general frame } d\tau &= dt \sqrt{1 - \frac{\vec{v}^2}{c^2}} \quad \text{where } \vec{v} = \frac{d\vec{x}}{dt} \text{ is the velocity of the particle.} \\ &\equiv \frac{dt}{\gamma} \quad \gamma \equiv \frac{1}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \end{aligned}$$



Total proper time along pticle world l

4-velocity of pticle $u^\mu = \frac{dx^\mu}{d\tau} = \gamma$

4-momentum $p^\mu = m_0 u^\mu$
 $\uparrow$ "rest mass"

Action for pticle $S' = \int \frac{1}{c} \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$

the world line $\tau = \int d\tau = \int \frac{dt}{\gamma}$

$$= \frac{dx^\mu}{d\tau} = \gamma(c, \underline{v})$$

u^μ
"rest mass"

$$\int \frac{1}{c} \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

dim's for an action

$$\frac{i}{\hbar} \left(\int dt \overset{\frac{1}{2}mv^2}{E} \int dx \overset{mv}{p} \right)$$

Total proper time along pticle world line $\tau =$

4-velocity of pticle $u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \underline{v})$

4-momentum $p^\mu = \underbrace{m_0}_{\substack{\uparrow \text{"rest mass"} \\ \downarrow \text{justify soon}}} u^\mu$

Action for pticle

$$S' = -m_0 c \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda = -m_0 c \int dt$$

-reparameterization
invariant

$$\lambda \rightarrow \lambda'(\lambda) \quad \frac{d\lambda'}{d\lambda} \neq 0$$

$$= \int d\tau = \int \frac{dt}{\gamma}$$

dim's for an action

$$|\vec{v}|/c \ll 1$$

$$\frac{i}{\hbar} \left(\int dt E \quad \int dx p \right)$$

$\frac{1}{2} m v^2$ $m v$

$$c \int dt \sqrt{1 - \frac{1}{c^2} \left(\frac{d\vec{x}}{dt} \right)^2} \approx \int dt \left(-m_0 c^2 + \frac{1}{2} m_0 \left(\frac{d\vec{x}}{dt} \right)^2 + \dots \right)$$

a line $\tau = \int d\tau = \int \frac{dt}{\gamma}$

$\gamma(c, \underline{v})$

dim's for an action

$|\frac{\vec{v}}{c}| \ll 1$

$\frac{i}{\hbar} \left(\int dt E \quad \int dx p \right)$
 $\frac{1}{2} m \vec{v}^2$ mv

$\left(\frac{d\vec{x}}{dt} \right)^2 + \left(\frac{d\vec{x}}{dt} \right)^2$

$\int dt = -m_0 c \int dt \sqrt{1 - \frac{1}{c^2} \left(\frac{d\vec{x}}{dt} \right)^2} \approx \int dt \left(\underbrace{-m_0 c^2}_{\text{P.E.} = m_0 c^2} + \frac{1}{2} m_0 \underbrace{\left(\frac{d\vec{x}}{dt} \right)^2}_{\text{Newtonian K.E.}} + \dots \right)$

action
 $\frac{d\lambda'}{d\lambda} \neq 0$

(global)
ous symmetry of an action (up to surface terms)
ervation law.

al version

$$q^a(t) \rightarrow q'^a(t) = q^a(t) + \epsilon^{ab} X_b(q, \dot{q}, t)$$

parameters (small)

generators

her theorem : for every continuous symmetry of ^(global)
there exists a conservation law.

Any continuous symmetry has an infinitesimal version $\downarrow$
 $q^a(t)$

$$S = \int dt L(q, \dot{q}, t)$$

$$\lambda \rightarrow \lambda'(\lambda) \quad \frac{d\lambda'}{d\lambda} \neq 0$$

theorem : for every continuous symmetry of an action (up to surface terms) (global)
there exists a conservation law.

Any continuous symmetry has an infinitesimal version $q^a(t) \rightarrow q'^a(t) = q^a(t) + \epsilon^a X_b^a(q)$

$S' = \int dt L(q, \dot{q}, t)$ (*) is a symmetry if $\delta S' = 0$ for constant ϵ .

$$\delta S' = S' - S = \int dt$$

(global)

continuous symmetry of an action (up to surface terms)
conservation law.

infesimal version

$$q^a(t) \rightarrow q'^a(t) = q^a(t) + \epsilon^{ab} X_b(q, \dot{q}, t)$$

parameters (small)
(*)
↑ generators

(*) is a symmetry if $\delta S = 0$ (up to s.t.s) for constant ϵ .

This is
property
explained by

Noether theorem : for every
there exists

Noether's trick :
let ϵ^{ab} be a
f.n. of t .

Any continuous symmetry has an

$$S' = \int dt L$$

$$\delta S' = S' - S = \int dt$$

theorem : for every continuous symmetry ^(global) δ
there exists a conservation law.

Any continuous symmetry has an infinitesimal version

$$S' = \int dt L(q, \dot{q}, t) \quad (*) \text{ is a symmetry}$$

$$\delta S' = S' - S = \int dt \left(\epsilon^{ab} J_{ab}(q, \dot{q}, t) + \epsilon^{ab} \frac{d}{dt} K_{ab}(q, \dot{q}, t) \right)$$

global)
 symmetry of an action (up to surface terms)
 action law.

version

$$q^a(t) \rightarrow q'^a(t) = q^a(t) + \epsilon^{ab} X_b(q, \dot{q}, t)$$

parameters (small)
 (*)
 ↑ generators

(*) is a symmetry if $\delta S = 0$ for constant ϵ .
 (up to s.t's)

these are the only possible terms consistent
 with (*) being a global symm. of the action.

$\epsilon^{ab} \frac{d}{dt} K_{ab}(q, \dot{q}, t)$

f^a of t.

$$\delta S = S' - S = \int dt \left(\epsilon^{ab} J_{ab}(q, \dot{q}, t) + \epsilon^{ab} \frac{d}{dt} K_{ab}(q, \dot{q}, t) \right)$$

For any $\epsilon^{ab}(t)$,

(*) is a legitimate variation, so.

$$\delta S = 0 \Leftrightarrow \begin{array}{l} \text{eq.s of} \\ \text{motion} \\ \text{are satisfied.} \end{array}$$

$$= \int dt \epsilon^{ab} \frac{d}{dt} (K_{ab} - J_{ab}) + \text{s.t.'s}$$

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$\Rightarrow$ If e.o.m. are satisfied, then $\frac{d}{dt} (K_{ab} - J_{ab}) = 0$

e.g. ^{Newtonian} free particle

$$\int \frac{1}{2} m \dot{\vec{x}}^2 dt$$

$$\vec{x} \rightarrow \vec{x} + \vec{\epsilon}$$

$$\delta S = \int m \dot{\vec{x}}$$

$$(K_{ab} - J_{ab}) + \text{s.t.'s}$$

, then $\frac{d}{dt}(K_{ab} - J_{ab}) = 0$ conservation law.

$$\vec{x} \rightarrow \vec{x} + \vec{\epsilon}(t) \quad \delta \vec{x} = \vec{\epsilon}(t) \quad \delta \dot{\vec{x}} = \dot{\vec{\epsilon}}(t)$$

$$S = \int m \dot{\vec{x}} \delta \dot{\vec{x}} = \int m \dot{\vec{x}} \dot{\vec{\epsilon}} = - \int (\dot{m \dot{\vec{x}}}) \vec{\epsilon} = 0 \quad \forall \vec{\epsilon}(t) \Rightarrow \frac{d}{dt}(m \dot{\vec{x}}) = 0$$

For any $\epsilon^{ab}(t)$,
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 variation, so.

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e.g. ^{Newtonian} free particle

$$\int \frac{1}{2} m \dot{x}^2 dt$$

$$\delta S =$$

$$\frac{d}{dt} (K_{ab} - J_{ab}) + \text{s.t.'s}$$

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$$\vec{x} \rightarrow \vec{x} + \vec{\epsilon}(t) \quad \delta \vec{x} = \vec{\epsilon}(t) \quad \delta \dot{\vec{x}} = \dot{\vec{\epsilon}}(t)$$

$$\delta S = \int m \dot{\vec{x}} \delta \dot{\vec{x}} = \int m \dot{\vec{x}} \dot{\vec{\epsilon}} = - \int (\dot{m \dot{\vec{x}}}) \vec{\epsilon} = 0 \quad \forall \vec{\epsilon}(t) \Rightarrow \frac{d}{dt} (m \dot{\vec{x}}) = 0$$

particle in a potential. $S' = \int \left(\frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}) \right) dt$

time translⁿ: $\vec{x}(t) \rightarrow \vec{x}(t+\epsilon) \approx \vec{x}(t) + \epsilon \dot{\vec{x}}(t) \Rightarrow \begin{aligned} \delta \vec{x} &= \epsilon \dot{\vec{x}}(t) \\ \delta \dot{\vec{x}} &= (\epsilon \dot{\vec{x}})' \end{aligned}$

$$\delta S = \int m \dot{\vec{x}} (\epsilon \dot{\vec{x}})' - \vec{\nabla} V \epsilon \dot{\vec{x}}$$

particle in a potential. $S' = \int \left(\frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}) \right) dt$

time translⁿ. $\vec{x}(t) \rightarrow \vec{x}(t+\epsilon) \approx \vec{x}(t) + \epsilon \dot{\vec{x}}(t) \Rightarrow \begin{aligned} \delta \vec{x} &= \epsilon \dot{\vec{x}}(t) \\ \delta \dot{\vec{x}} &= (\epsilon \dot{\vec{x}})' \end{aligned}$

$$\delta S' = \int m \dot{\vec{x}} (\epsilon \dot{\vec{x}})' - \vec{\nabla} V \epsilon \dot{\vec{x}}$$

$$= \int - (m \dot{\vec{x}})' \epsilon \dot{\vec{x}} - \vec{\nabla} V \epsilon \dot{\vec{x}}$$

$$= - \int \epsilon \frac{d}{dt} \left(\frac{m \dot{\vec{x}}^2}{2} + V(\vec{x}) \right) \Rightarrow \frac{d}{dt} \left(\frac{1}{2} m \dot{\vec{x}}^2 + V(\vec{x}) \right) = 0.$$

More generally, $S = \int dt L(q, \dot{q}, t)$

If S is translation invariant, then $q^a \rightarrow q^a + \epsilon^a$ is a symmetry

$$\delta S = \int \left(\underbrace{\frac{\partial L}{\partial q^a}}_{\text{by transl. invariance}} \epsilon^a + \frac{\partial L}{\partial \dot{q}^a} \dot{\epsilon}^a \right) \Rightarrow p_a = \frac{\partial L}{\partial \dot{q}^a} \text{ is conserved.}$$

time translations $\left(\frac{\partial L}{\partial t} = 0 \right)$

$$q^a \rightarrow q^a + \epsilon \dot{q}^a$$

$$\dot{q}^a \rightarrow \dot{q}^a + (\epsilon \dot{q}^a)'$$

$$\delta S' = \int \left(\frac{\partial L}{\partial q} \epsilon \dot{q}^a + \frac{\partial L}{\partial \dot{q}^a} (\epsilon \dot{q}^a)' \right)$$

$$= \int \epsilon \left[\frac{\partial L}{\partial q^a} \dot{q}^a - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^a} \right) \dot{q}^a \right] =$$

$$\frac{d}{dt} \left[L - \frac{\partial L}{\partial \dot{q}^a} \dot{q}^a \right] = \frac{\partial L}{\partial q^a} \dot{q}^a + \frac{\partial L}{\partial \dot{q}^a} \ddot{q}^a$$

$$- \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^a} \right) \dot{q}^a - \frac{\partial L}{\partial \dot{q}^a} \ddot{q}^a$$

$$L - p_a \dot{q}^a = -H$$

$$\left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^a} \right) \dot{q}^a \right] = \int \epsilon \frac{d}{dt} \left(L - \overbrace{\frac{\partial L}{\partial \dot{q}^a} \dot{q}^a}^{= p_a} \right) \Rightarrow \frac{d}{dt} \left(\underbrace{p_a \dot{q}^a - L}_H \right) = 0$$