Title: Front End - Functions, "Functions", etc. - 6

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Abstract:

# Asymptotic Analysis

Motivation: Often want to understand functions or solns to diff. eq if a parameter is large or small

QFT:

$$Z = \int \mathcal{D}[\phi]\, e^{\frac{i}{\hbar}\int d^d x \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right]}$$

$$Z(\lambda) = \sum_{n=0} (-1) \, \lambda \, \frac{\Gamma(2n+\frac{1}{2})}{n!}$$

For large $n$ $\quad \Gamma(2n+\frac{1}{2}) \approx (2n)! > (n!)^2$

sum does not converge!

$$Z(\lambda) = \int_{-\infty}^{\infty} e^{-x^2 - \lambda x^4}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2} \sum_{n=0}^{\infty} (-1)^n \lambda^n \frac{x^{4n}}{n!}\, dx$$

$$\overset{?}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \lambda^n \int_{-\infty}^{\infty} x^{4n} e^{-x^2}\, dx \qquad u = x^2$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n}{n!} \underbrace{\int_0^{\infty} u^{2n-\frac{1}{2}} e^{-u}\, du}_{\Gamma(2n+\frac{1}{2})}$$

Definitions:

2. $f(x) \sim g(x) \qquad x \to x_0$

if $\quad f(x) - g(x) \ll g(x) \qquad x \to x_0$

or $\quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$

Example: $\quad \cos x \sim -1 \qquad x \to \pi$

$x^2 \not\sim 0 \qquad x \to 0$

$x^2 \not\sim x \qquad x \to 0$

3.

3. $$y(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad x \to x_0$$

if $$y(x) - \sum_{n=0}^{N} a_n (x-x_0)^n \ll a_N (x-x_0)^N \quad x \to x_0 \quad \text{for all } N$$

Comments: • convergent series become better approximations as we take more terms

asymptotic series become better approximations as $x \to x_0$ with a fixed # of terms

- If $F(\lambda) \sim \sum_{n=0}^{\infty} a_n \lambda^n \quad \lambda \to 0$

then so is $F(\lambda) + e^{-\frac{1}{\lambda^2}} \sim \sum_{n=0}^{\infty} a_n \lambda^n \quad \lambda \to 0$

non-perturbative function

- $f(z) \sim g(z) \quad z \to z_0$

generally only valid $a < \arg(z - z_0) < b$

Stokes' phenomenon

Most useful method for finding an asymptotic series requires an integral

## Example: Stirling's Approximation

want $n! = \Gamma(n+1)$ for large $n$

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt$$

integrand only large in small region

$t = wx$

$$\Gamma(x+1) = x^{x+1} \int_0^\infty e^{-x\underbrace{(w - \ln w)}_{f(w)}} \, dw$$

$$f'(w) = 1 - \frac{1}{w} = 0 \quad \text{at } w = 1$$

$$f(w) = 1 - \frac{1}{2}(w-1)^2 + \cdots$$

$$\Gamma(x+1) = x^{x+1} e^{-x} \int_0^\infty e^{-\frac{x}{2}(w-1)^2 + \cdots} \, dw$$

$$\simeq x^{x+1} e^{-x} \int_{-\infty}^{\infty} e^{-\frac{x}{2}(w-1)^2} \, dw$$

$\uparrow$ extend

$$= x^{x+1} e^{-x} \sqrt{\frac{2\pi}{x}}$$

$f(w)$

$$\frac{\Gamma(x+1)}{\sqrt{2\pi}\, x^{x+\frac{1}{2}}\, e^{-x}} \sim 1 + \underbrace{\frac{1}{12x} + \cdots}_{\text{from} + \cdots} \qquad x \to \infty$$

## Method of Steepest Descents / Saddle-Point Method

$$I(z) = \int_C g(w)\, e^{z f(w)}\, dw \quad \text{for large real } z$$

- might expect dominant contributions to come from near a point $w_0$ at which $u \equiv \mathrm{Re}(f(w))$ is a local max

- must also demand $v = \mathrm{Im}(f(w)) \simeq$ constant

$\hat{t}$ = tangent vector to curve at $w_0$ $\quad$ $w = x + iy$

① $\nabla u \cdot \hat{t} = 0$ for $w_0$ to be local max

② $\nabla v \cdot \hat{t} = 0$ for phase to be constant

③ $\nabla u \cdot \nabla v = 0$ by Cauchy-Riemann equations

3 $\perp$ vectors in 2d space

If $\nabla u = 0$ then $\dfrac{\partial u}{\partial x} = 0 = \dfrac{\partial v}{\partial y}$ so $\nabla v = 0$

$$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$$

$f'(w_0) = 0$

Near $w_0$ we still want ② (+③) to hold

$\nabla u \parallel \hat{t}$ near $w_0$

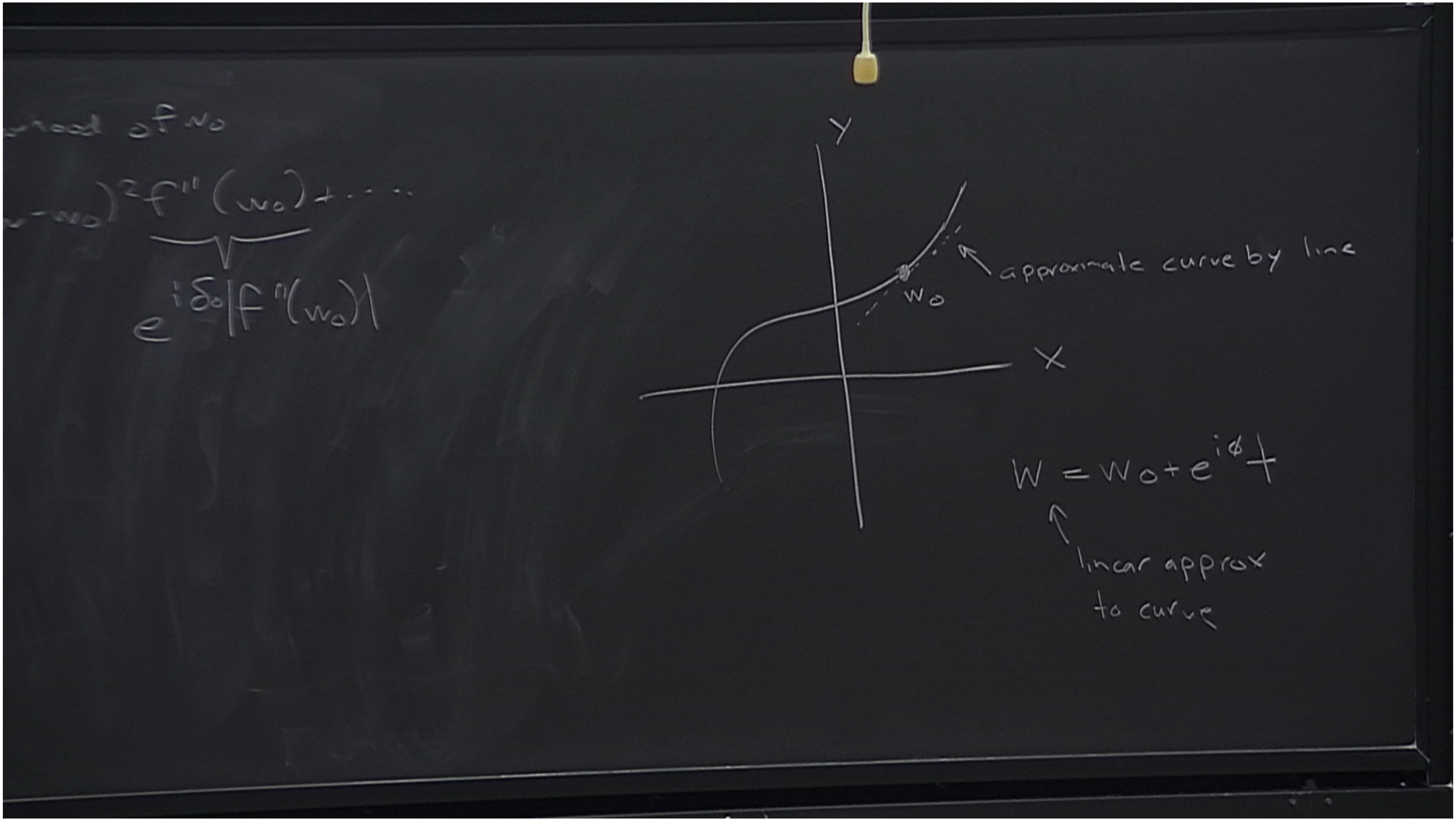
$u$ has no max

① holds at $w_0$

②+③ hold in a region around $w_0$

Expand $f(w)$ in a neighborhood of $w_0$

$$f(w) = f(w_0) + \frac{1}{2}(w-w_0)^2 \underbrace{f''(w_0)}_{e^{i\delta_0}|f''(w_0)|} + \ldots$$



of $w_0$
$)^2 f''(w_0)+\dots$
$e^{i\delta_0}|f''(w_0)|$
y
x
$w_0$
approximate curve by line
$W = w_0 + e^{i\phi} t$
linear approx
to curve

Expand $f(w)$ in a neighborhood of $w_0$

$$f(w) = f(w_0) + \frac{1}{2}(w-w_0)^2 \underbrace{f''(w_0)}_{e^{i\delta_0}|f''(w_0)|} + \cdots$$

want $f(w) = f(w_0) - \frac{1}{2} t^2 |f''(w_0)|$

$$(w-w_0)^2 e^{i\delta_0} = e^{2i\phi} t^2 e^{i\delta_0} = -t^2$$

$$e^{2i\phi} e^{i\delta_0} = e^{\pm i\pi}$$

$$I(z) \approx e^{z f(w_0)} \int_{-t_1}^{t_1} g(w_0 + e^{i\phi} t)\, e^{-\frac{z}{2} t^2 |f''(w_0)|} \frac{dw}{dt}$$

range where contour is linear

$$\approx e^{z f(w_0)} \int_{-\infty}^{\infty} \left[ g(w_0) + \frac{g''(w_0)}{2!} e^{2i\phi} t^2 + \dots \right] e^{-\frac{z}{\dots}}$$

$$I(z) \sim \sqrt{\frac{2\pi}{z}} \frac{e^{z f(w_0)}}{\sqrt{-f''(w_0)}} \left[ g(w_0) - \frac{1}{z} \frac{g''(w_0)}{1!\, 2 f''(w_0)} + \dots \right]$$

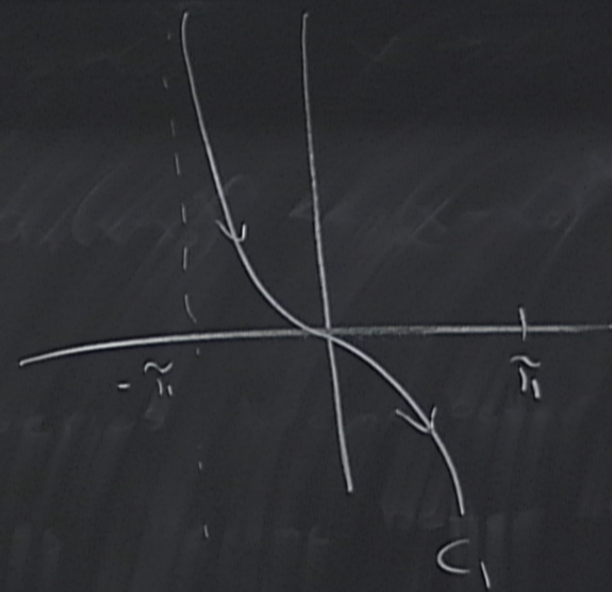
$$) \; e^{-\frac{z}{2} t^2 |f''(w_0)|} \left.\frac{dw}{dt}\right|_{t=0} dt$$

$$\frac{{}''(w_0)}{2!} e^{2i\phi} t^2 + \dots \Big] e^{-\frac{z}{2} t^2 |f''(w_0)|} e^{i\phi} \, dt$$

$$) - \frac{1}{z} \frac{g''(w_0)}{1!\,2f''(w_0)} + \dots \Big]$$

phases cancel using $|f''(w_0)| = e^{2i\phi} f''(w_0)$

$$\int_{C_1} e^{iz\cos w} e^{i\nu w}\, dw$$



$\sqrt{-f''(w_0)}$

xample: $H_\nu(z) = \frac{e}{\pi} \int_{C_1} e \qquad e \, dw$

$g(w) = e^{i\nu w}$

$f(w) = i\cos w$

$f'(w) = -i\sin w \qquad w = 0$ is on contour

$f''(w) = -i\cos w$

$$H^1_\nu(z) \underset{z\to\infty}{\sim} \frac{e^{-i\pi\nu/2}}{\pi} \sqrt{\frac{2\pi}{z}}\, e^{iz}\, e^{-i\pi/4} \left[1 + \frac{\nu^2 i}{2z} + \cdots\right]$$