Title: Front End - Functions, "Functions", etc. - 5

Date: Aug 29, 2016 09:00 AM

URL: http://pirsa.org/16080079

Abstract:

Orthogonal Functions + Green's Functions

Sturm-Liouville Theory Revisted

$$\Omega u_\lambda(x) + \lambda w(x) u_\lambda(x) = 0 \quad \text{on } [a,b]$$

$$\Omega = \frac{d}{dx}\left[s(x)\frac{d}{dx}\right] + t(x) \qquad \begin{matrix} s(x) > 0 \\ t(x) > 0 \end{matrix}$$

multiply by u_ρ

$$\int_a^b u_\rho (\Omega u_\lambda)\, dx = -\lambda \int_a^b u_\rho u_\lambda w\, dx$$

Similarly

$$\int_a^b (\Omega u_\rho) u_\lambda\, dx = -\rho \int_a^b u_\rho u_\lambda w\, dx$$

$$\langle u_\rho, \Omega u_\lambda \rangle - \langle \Omega u_\rho, u_\lambda \rangle = -(\lambda - \rho) \langle u_\rho, u_\lambda \rangle_w$$

multiply by u_ρ

$$\int_a^b u_\rho(\Omega u_\lambda)\,dx = -\lambda \int_a^b u_\rho u_\lambda w\,dx$$

Similarly

$$\int_a^b (\Omega u_\rho) u_\lambda\,dx = -\rho \int_a^b u_\rho u_\lambda w\,dx$$

$$\langle u_\rho, \Omega u_\lambda\rangle - \langle \Omega u_\rho, u_\lambda\rangle = -(\lambda-\rho)\langle u_\rho, u_\lambda\rangle_w$$

If Ω is truly self-adjoint, then LHS vanishes $\Rightarrow$ if $\lambda \neq \rho$ then $\langle u_\rho, u_\lambda\rangle_w = 0$

homogeneous b.c.s make Ω self-adjoint:

$$\alpha_a y(a) + \beta_a y'(a) = 0 \qquad \text{with } \alpha_a^2 + \beta_a^2 > 0$$

$$\alpha_b y(b) + \beta_b y'(b) = 0 \qquad \alpha_b^2 + \beta_b^2 > 0$$

the $Q|_a^b = 0$

↑ from Lagrange's identity

with these b.cs

- eigenvalues are real + distinct $\quad -\infty < \lambda_0 < \lambda_1 < \cdots$

- u_{λ_n} form an orthonormal basis of $L^2[a,b]_w$

$$\int_a^b u_{\lambda_n}(x)\, u_{\lambda_m}(x)\, w(x)\, dx = \delta_{nm}$$

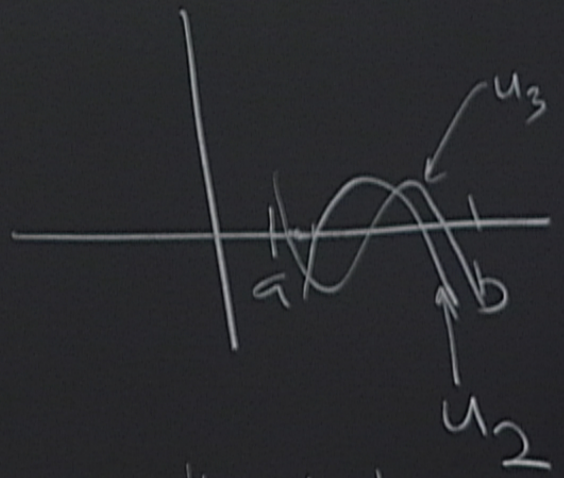
- zeros of $u_{\lambda_{n+1}}$ are in between the zeros of u_{λ_n}

$\lambda_0 < \lambda_1 < \cdots$

of $L^2[a,b]_w$

$= \delta_{nm}$

the zeros of $u_{\lambda n}$



the u_3 has 3 zeros

with these b.cs

- eigenvalues are real + distinct $\quad -\infty < \lambda_0 < \lambda_1 < \cdots$

- u_{λ_n} form an orthonormal basis of $L^2[a,b]_w$

$$\int_a^b u_{\lambda_n}(x)\, u_{\lambda_m}(x)\, w(x)\, dx = \delta_{nm}$$

- zeros of $u_{\lambda_{n+1}}$ are in between the zeros of u_{λ_n}

- u_{λ_n} has exactly n distinct zeros in (a,b)

First 3 points hold even if

- $s(x) = 0$ at $x = a$ or b

- a or b infinite

If a or b is infinite

demanding that u_x is in $L^2[a,b]_w$

- u_x is finite at a or b

- $\sqrt{w}\, u \to 0$ as $x \to \pm\infty$

is enough

mple: Hermite's equation

$(a,b) = (-\infty,\infty)$

$H_\lambda'' - 2xH_\lambda' = -2\lambda H_\lambda$ mult by $g(x)$

$gH_\lambda'' - 2xgH_\lambda' = -2\lambda g H_\lambda$ compare w/ $sH'' + s'H' + tH = -\lambda wH$

$s(x) = g(x) = \frac{1}{2}w(x)$

$-2xg(x) = s'(x) = g'(x)$

$g(x) = e^{-x^2} = \frac{1}{2}w(x)$

$\Rightarrow \int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx = c_n \delta_{nm}$ (normalization constant)

Constructing Green's Functions from Eigenfunctions

L has a complete set of orthonormal eigenfunction ϕ_n with eigenvalues λ_n

$$Ly = f$$

we can expand

$$y(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x)$$

$$f(x) = \sum_{n=1}^{\infty} \beta_n \phi(x)$$

reen's Functions from Eigenfunctions

a complete set of orthormal eigenfunction ϕ_n

alues λ_n

expand $y(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x)$

$f(x) = \sum_{n=1}^{\infty} \beta_n \phi_n(x)$

$L \sum_{n=1}^{\infty} \alpha_n$

$\sum_{n=1}^{\infty} \alpha_n$

α

If all of

$$L \sum_{n=1}^{\infty} \alpha_n \phi_n(x) = \sum_{n=1}^{\infty} \beta_n \phi_n(x)$$

$$\sum_{n=1}^{\infty} \alpha_n \lambda_n \phi_n(x) = \sum_{n=1}^{\infty} \beta_n \phi_n(x)$$

$$\alpha_n \lambda_n = \beta_n$$

If all of the $\lambda_n \neq 0$ then we can use $\beta_n = \langle \phi_n, f \rangle$

$$y(x) = \sum_{n=1}^{\infty} \frac{\beta_n}{\lambda_n} \phi_n(x)$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n \int \phi_n^{\dagger}(x') f(x') dx'$$

$$y(x) = \int \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n(x) \phi_n^*(x') f(x') dx'$$

$$G(x,x') = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n(x) \phi_n^*(x')$$

Orthogonal Polynomials

polynomials form a complete basis of $L^2[a,b]_w$

Gram-Schmidt:

start with $P_0(x) = \frac{1}{\|1\|_w}$

$$P_{n+1} = \frac{xP_n - \sum_{i=0}^{n} P_i \langle P_i, xP_n \rangle_w}{\alpha_{n+1}}$$

α_{n+1} ← norm of numerator

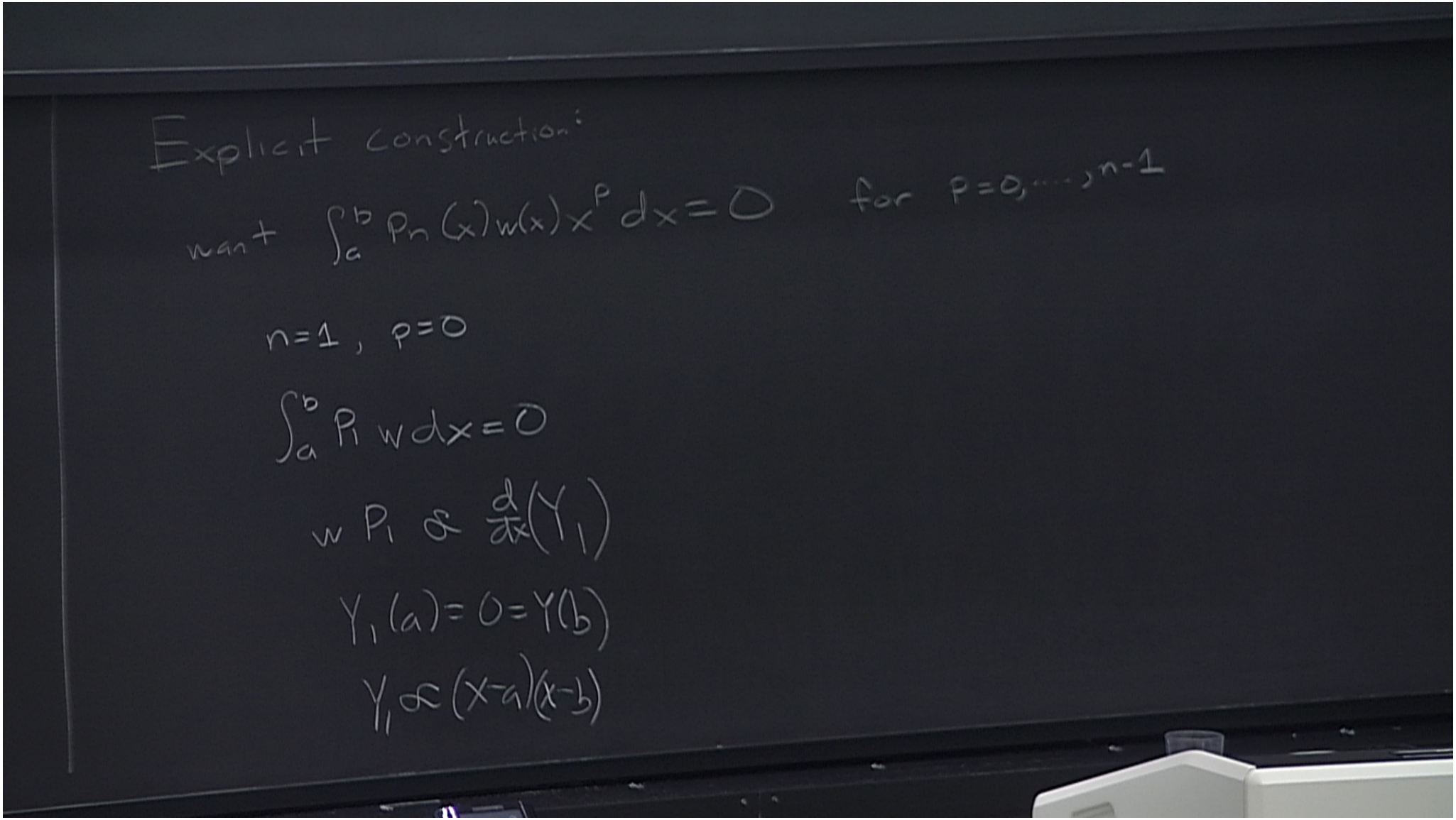
$$\langle P_i, x P_n \rangle_w = \int_a^b dx\, w\, P_i (x P_n)$$

$$= \int_a^b dx\, w\, (x P_i) P_n$$

$$= \langle x P_i, P_n \rangle_w$$

$$= 0 \quad \text{for } i < n-1 \quad \text{by construction}$$

$$\Rightarrow P_{n+1} = a_n x P_n + b_n P_n + c_n P_{n-1}$$


Explicit construction:
want $\int_a^b P_n(x) w(x) x^p dx = 0$ for $p = 0, \ldots, n-1$
$n=1$, $p=0$
$\int_a^b P_1 w dx = 0$
$w P_1 \propto \frac{d}{dx}(Y_1)$
$Y_1(a) = 0 = Y(b)$
$Y_1 \propto (x-a)(x-b)$

$n=2 \quad p=1$

$$\int_a^b P_2 \, w \, x \, dx = 0$$

$$w P_2 \propto \frac{d^2}{dx^2}(Y_2)$$

$$Y_2 \propto (x-a)^2(x-b)^2$$

$$\int_a^b \left(\frac{d^2}{dx^2} Y_2\right) x \, dx = \frac{d}{dx} Y_2 x \Big|_a^b - \int_a^b \frac{d}{dx} Y_2 \, dx$$

In general if

$$w\, P_n \propto \frac{d^n}{dx^n} Y_n$$

$$Y_n \propto (x-a)^n (x-b)^n$$

then $\int_a^b \frac{d^n}{dx^n} Y_n \, x^p \, dx = 0$ for $\alpha \le p < n$

Suppose $a = -\infty$, b finite

$$Y_n = (x-b)^n f_n(x)$$ such that $Y_n \to 0$ at $-\infty$

Similarly if $b = \infty$, a finite

or $a = -\infty, b = \infty$

$$Y_n \propto s^n(x) f_n(x)$$

$$s(x) = \begin{cases} (x-a)(x-b) & a, b \text{ finite} \\ x-a & b \text{ infinite} \\ x-b & a \text{ infinite} \\ 1 & a, b \text{ infinite} \end{cases}$$

Choose $f_n(x) = w(x)$ normalization

Rodrigues' Formula

$$P_n(x) = \frac{c_n}{w(x)} \frac{d^n}{dx^n}\left(w s^n\right)$$

When are these polynomials?

3 cases to check

$$(a,b) = \begin{cases} (-\infty,\infty) \\ [0,\infty) \\ [0,1] \end{cases}$$

for any other interval $P_n(cx+d)$

will be a polynomial on one of these intervals

Case 1: $(a,b) = (-\infty, \infty)$

make a linear transformation so

$$P_1(x) = \pm 2x$$

$$\frac{1}{w}\frac{dw}{dx} = \pm 2x$$

$$w(x) = k \exp(\pm x^2)$$

must choose $-$ so that $Y_n \to 0$ at $\pm\infty$

$$P_n(x) = C_n \exp(x^2) \frac{d^n}{dx^n}\left(\exp(-x^2)\right)$$

$C_n = (-1)^n$

Hermite polynomials

Cases 2+3 in tutorial

Best approximation

$f \approx \sum_{n=1}^{N} a_n u_n$ where $a_n = \langle u_n, f \rangle$ and u_n are orthonormal

minimizes $\Delta_N = \| f - \sum_{n=1}^{N} a_n u_n \|^2$

min value is $\Delta_N = \|f\|^2 - \sum_{n=1}^{N} |a_n|^2$

Parseval's Theorem

If a function space is complete then $f(x) = \sum_{n=1}^{\infty} a_n u_n$

converges for every f

In other words if $\|f\|^2 < \infty$ then $\Delta_N \to 0$ as $N \to \infty$

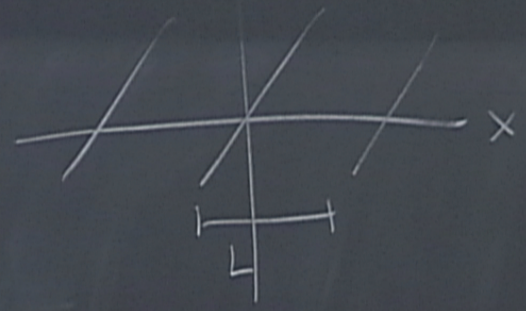
or $\|f\|^2 = \sum_{n=1}^{\infty} |a_n|^2$

Example: Consider a wave with amplitude $f(x)$ and period L

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x / L}$$

$$\text{intensity} = \frac{\text{energy}}{\text{length}} \propto U = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |a_n|^2$$

$f(x) = x$



$$U = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2\,dx$$

$$= \frac{1}{L}\,\frac{x^3}{3}\Big|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{L^2}{12}$$

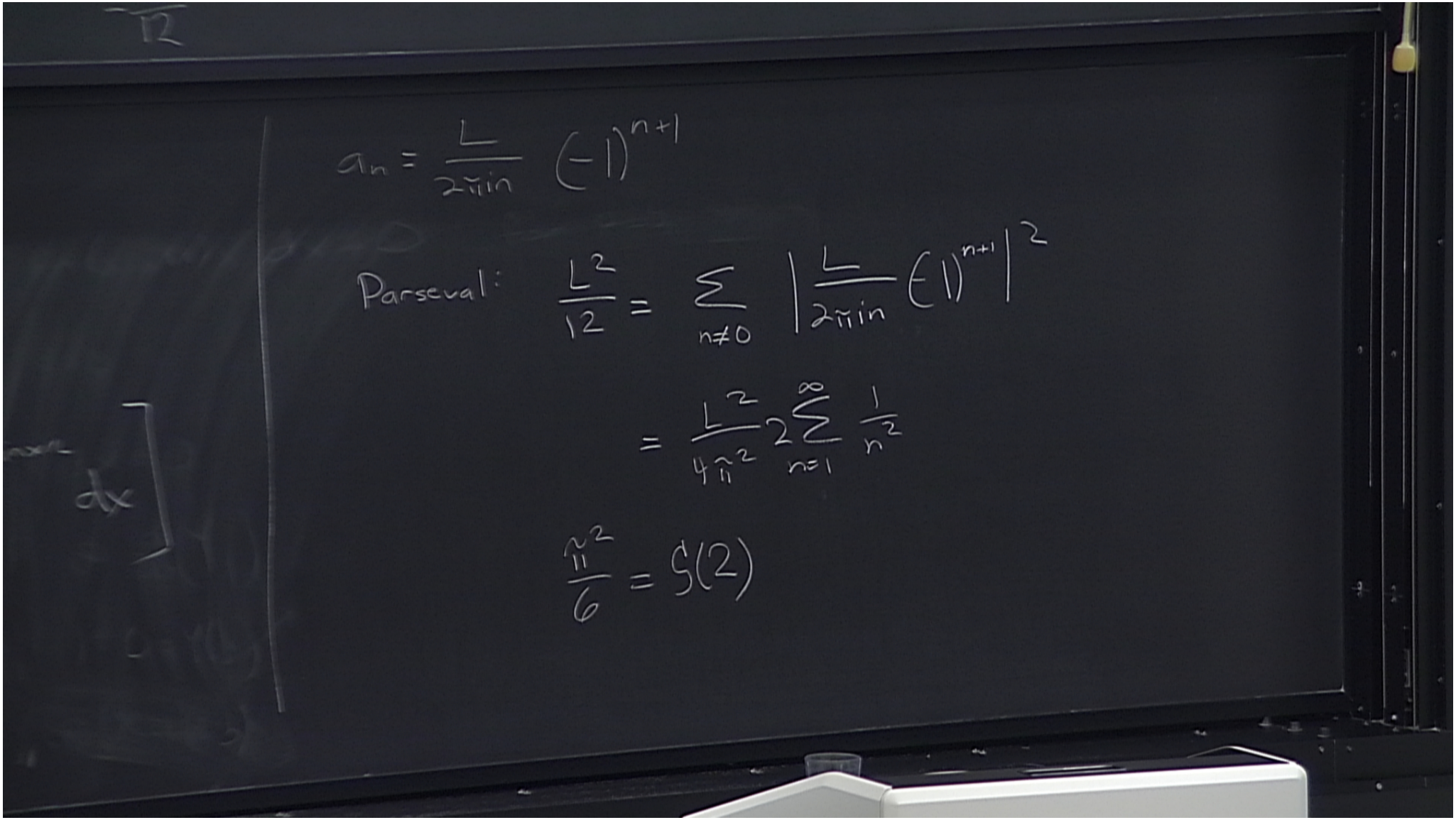
$a_0 = 0$ by symmetry

For $n \neq 0$ $\quad a_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} x e^{-2\pi i n x/L}\, dx$

$$= -\frac{1}{2\pi i n}\int_{-\frac{L}{2}}^{\frac{L}{2}} x \frac{d}{dx} e^{-2\pi i n x/L}\, dx$$

$$= -\frac{1}{2\pi i n}\left[x e^{-2\pi i n x/L}\Big|_{-\frac{L}{2}}^{\frac{L}{2}} - \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-2\pi i n x/L}\, dx \right]$$

$$= -\frac{1}{2\pi i n}\left[\frac{L}{2} e^{-\pi i n} + \frac{L}{2} e^{\pi i n} + 0 \right]$$


$a_n = \frac{L}{2\pi i n}(-1)^{n+1}$
Parseval: $\frac{L^2}{12} = \sum_{n\neq 0} \left|\frac{L}{2\pi i n}(-1)^{n+1}\right|^2$
$= \frac{L^2}{4\pi^2} 2\sum_{n=1}^{\infty} \frac{1}{n^2}$
$\frac{\pi^2}{6} = \zeta(2)$
dx

$$\Lambda(s) = \Lambda(1-s)$$

$$\Lambda(s) = -\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$\zeta(2)\,\Gamma(1)\,\pi^{-1} = \zeta(-1)\,\underbrace{\Gamma\left(-\tfrac{1}{2}\right)}_{-2\sqrt{\pi}}\,\pi^{1/2}$$

$$\sum_{n=1}^{\infty} n = \zeta(-1) = -\frac{1}{12}$$