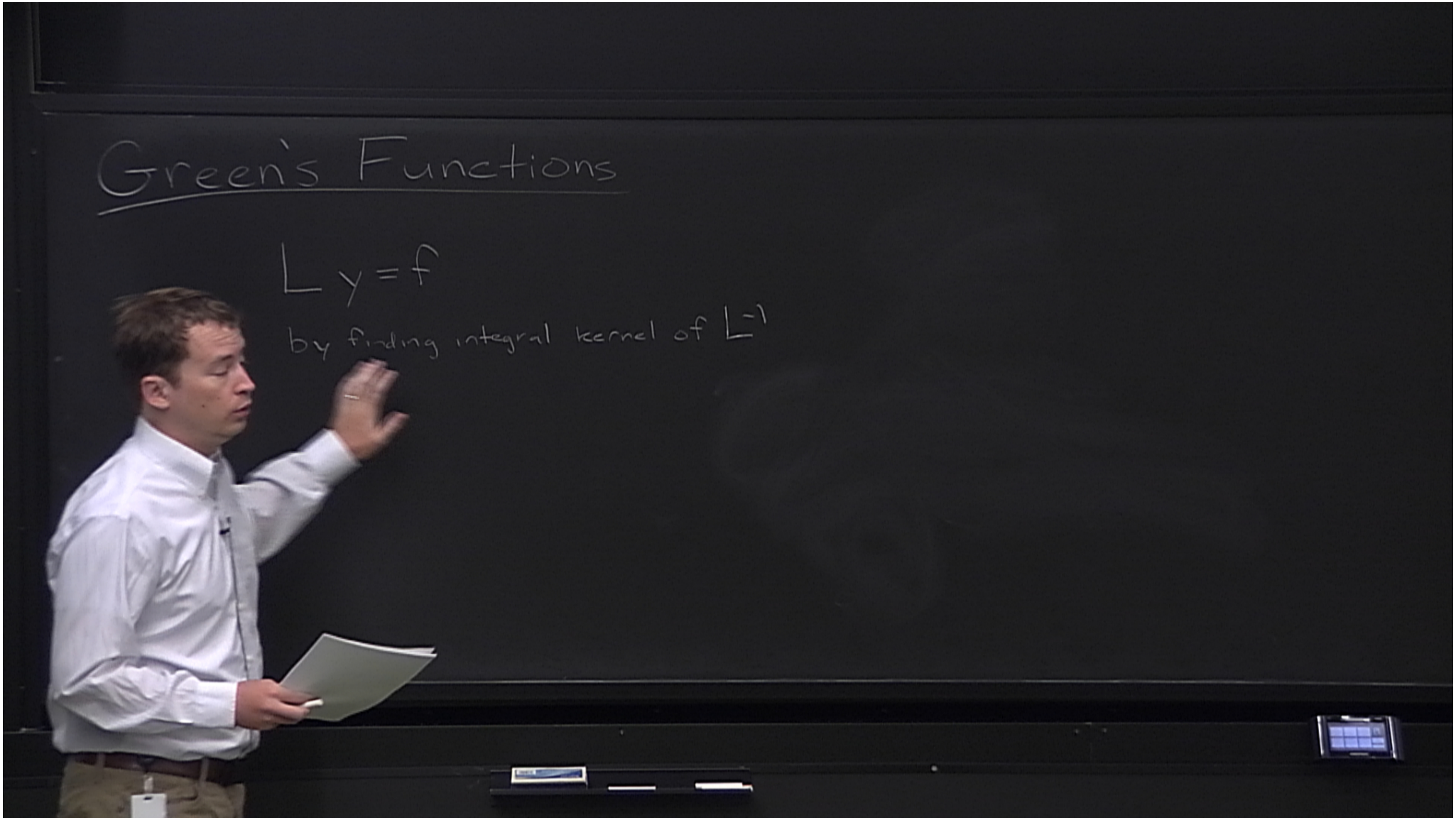
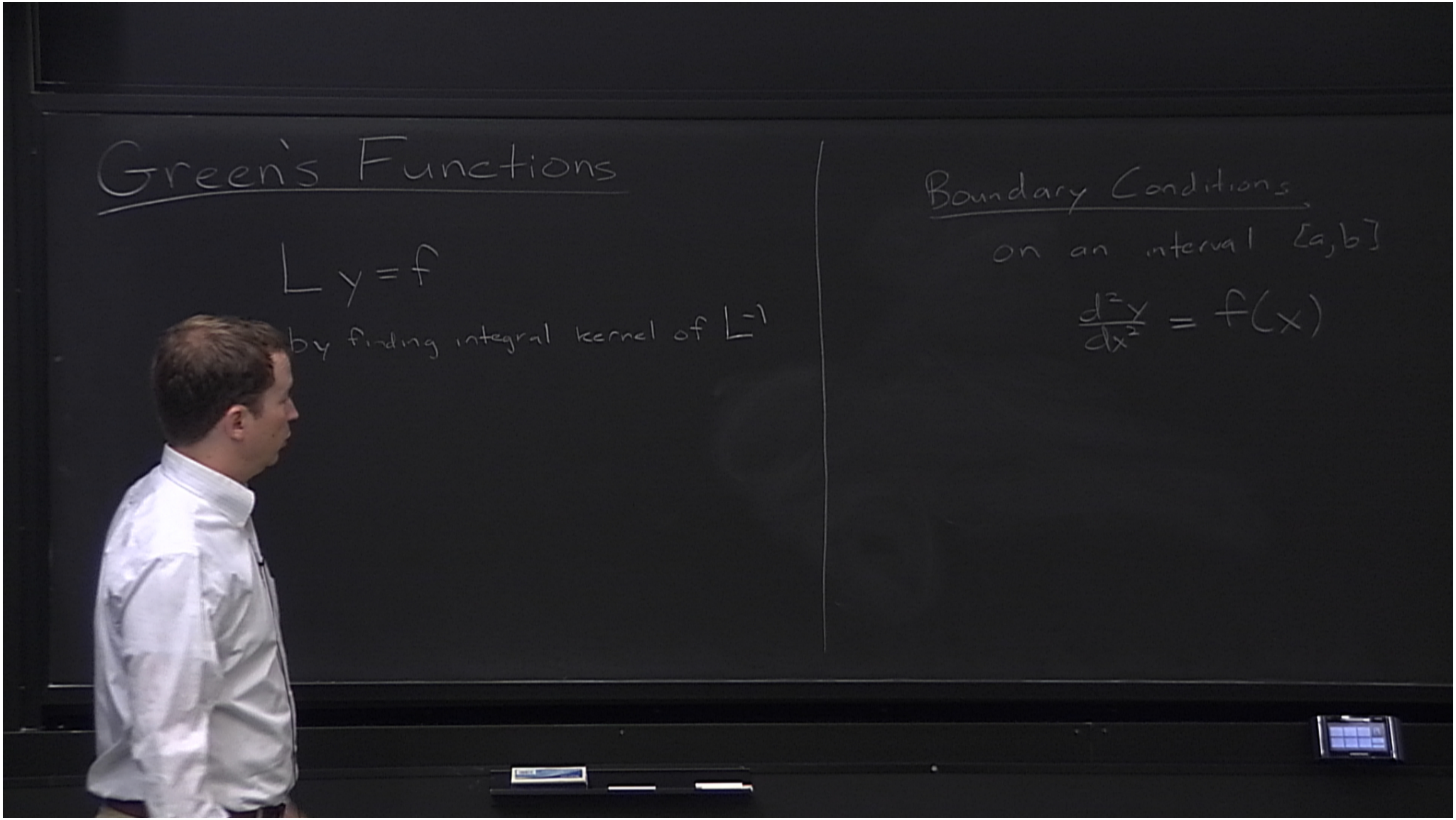
Title: Front End - Functions, "Functions", etc. - 4

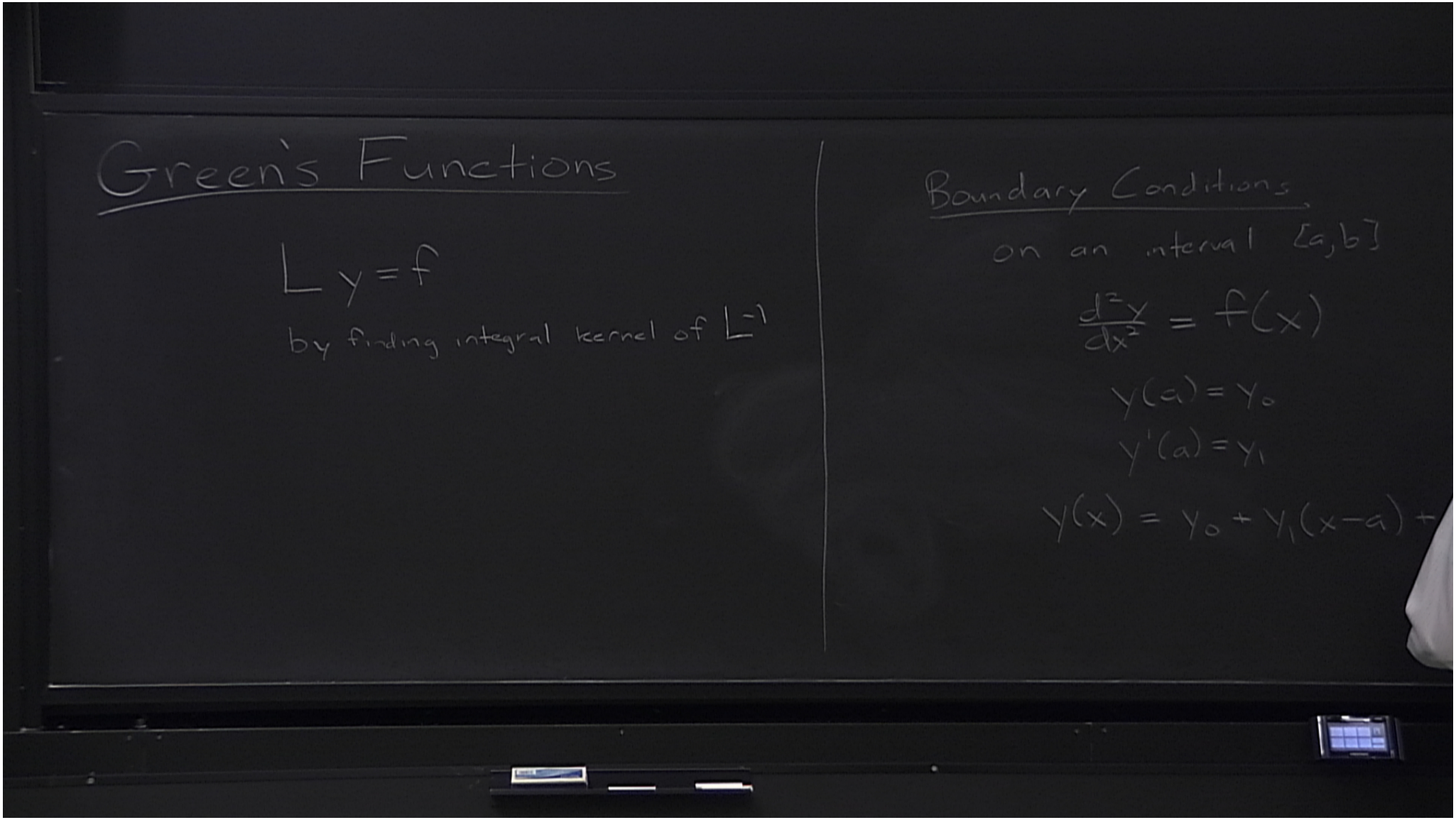
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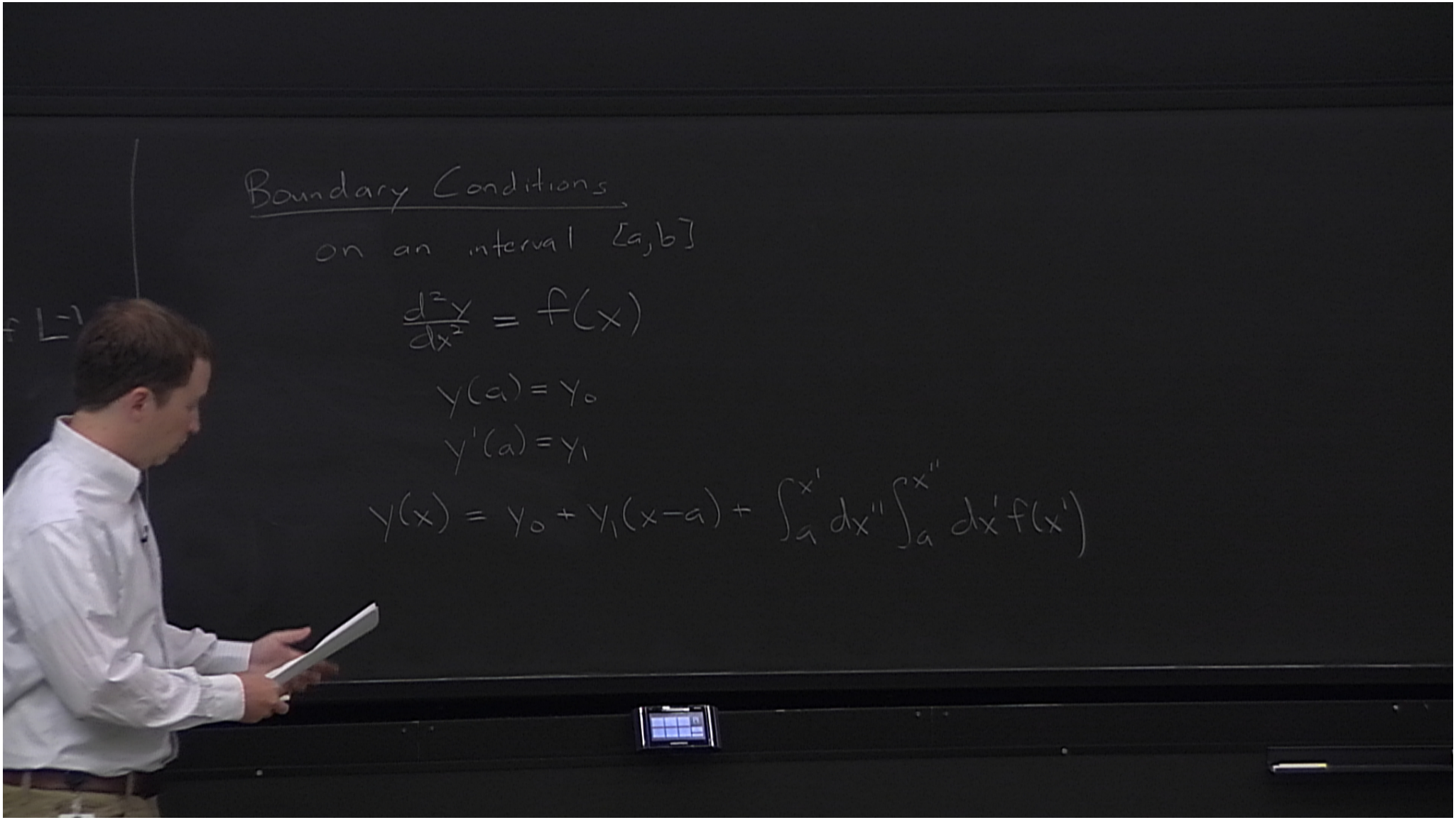
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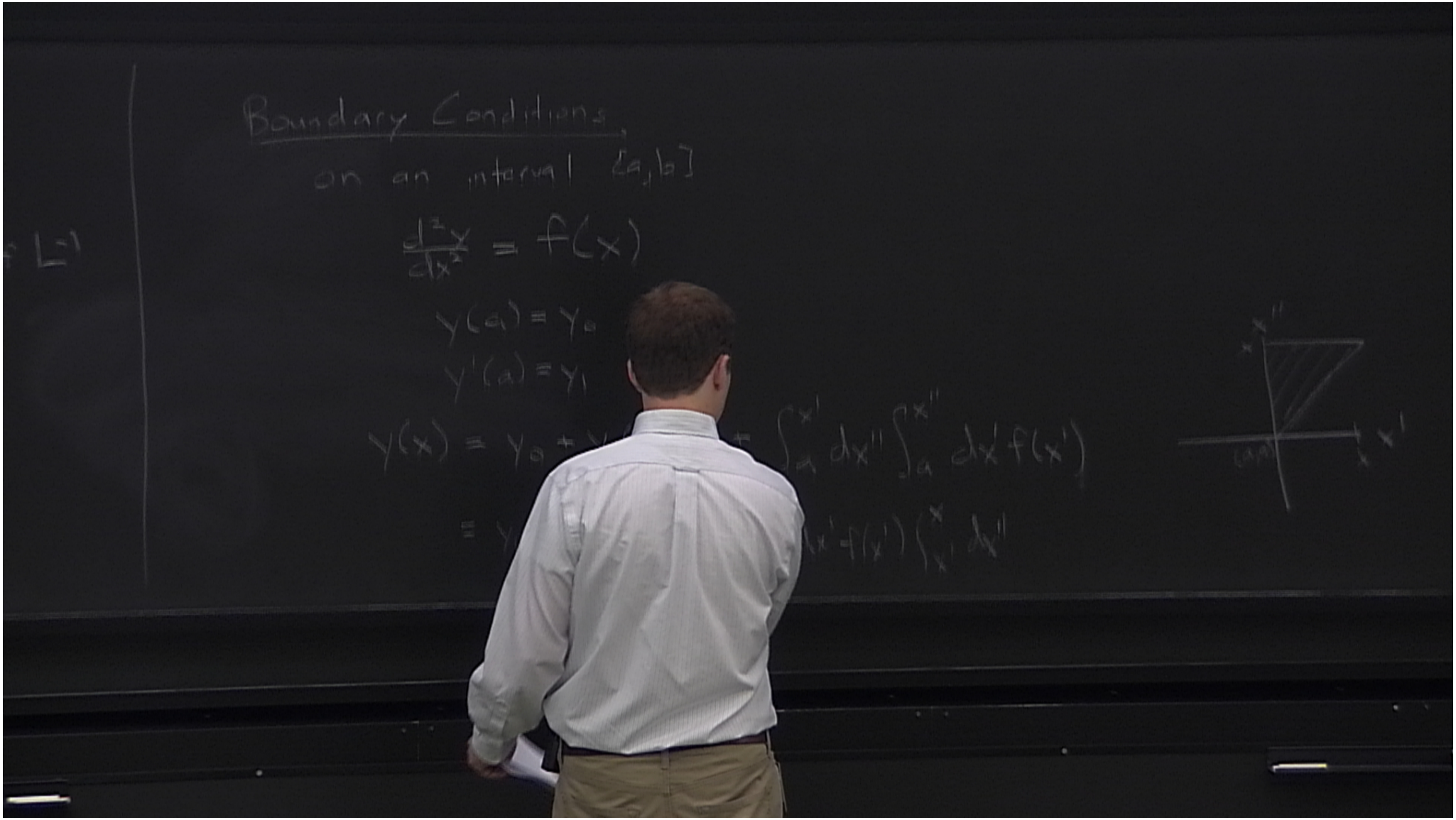
Abstract:


Green's Functions
$Ly = f$
by finding integral kernel of L^{-1}


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on an interval $[a,b]$
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Boundary Conditions

on an interval $[a,b]$

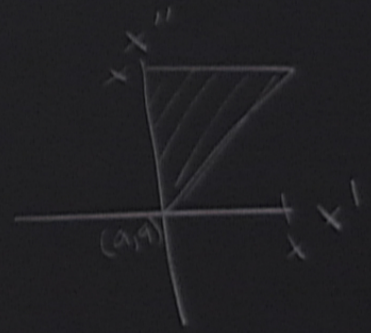
$$\frac{d^2y}{dx^2} = f(x)$$

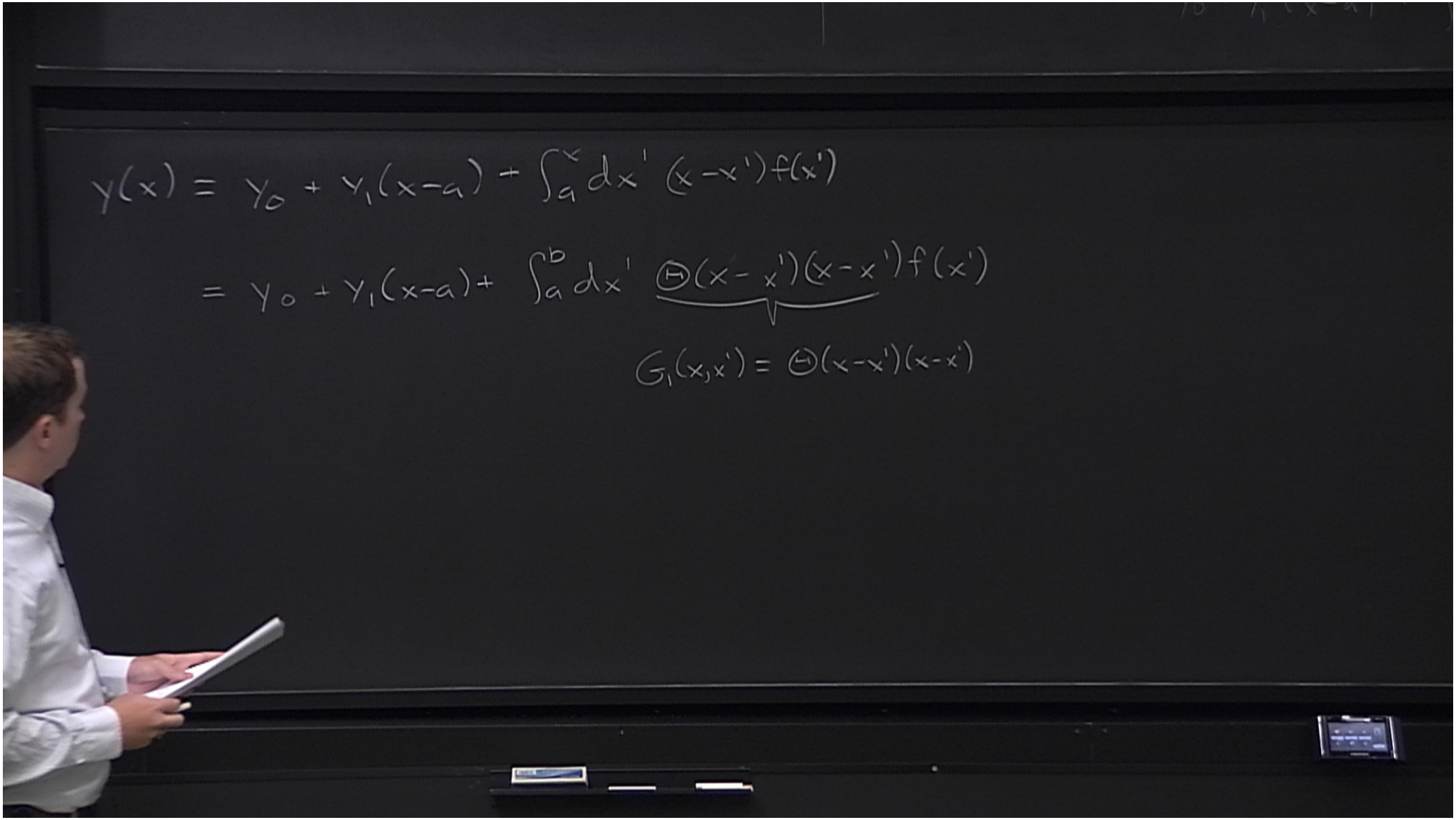
$$y(a) = y_0$$

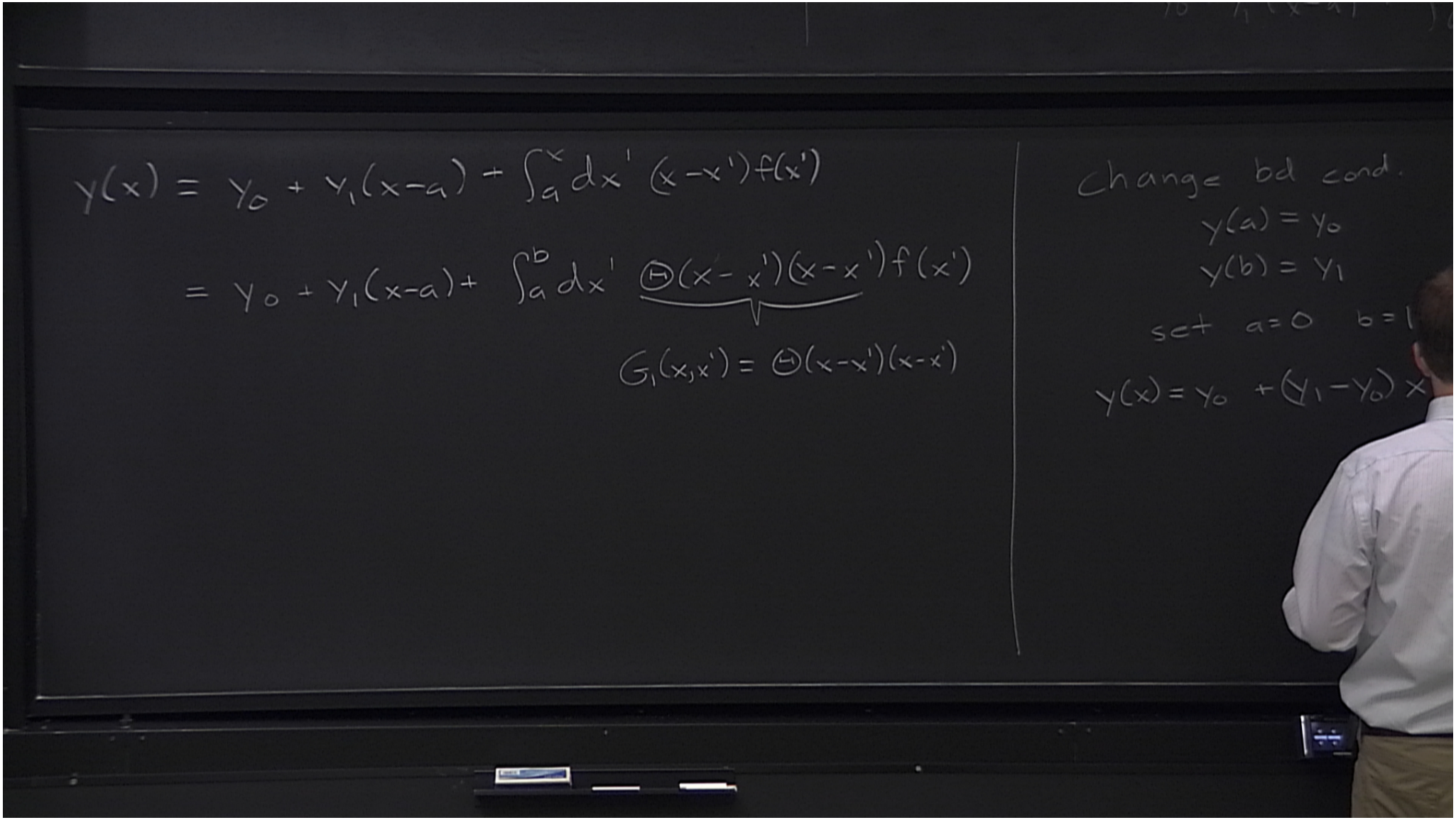
$$y'(a) = y_1$$

$$y(x) = y_0 + y_1(x-a) + \int_a^{x'} dx'' \int_a^{x''} dx' f(x')$$

$$= y_0 + y_1(x-a) + \int_a^{x} dx' f(x') \int_{x'}^{x} dx''$$




$y(x) = y_0 + y_1(x-a) + \int_a^x dx' (x-x') f(x')$
$= y_0 + y_1(x-a) + \int_a^b dx' \, \Theta(x-x')(x-x') f(x')$
$G_1(x,x') = \Theta(x-x')(x-x')$


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$= y_0 + y_1(x-a) + \int_a^b dx' \Theta(x-x')(x-x') f(x')$
$G_1(x,x') = \Theta(x-x')(x-x')$
change bd cond.
$y(a) = y_0$
$y(b) = y_1$
set $a=0$ $b=1$
$y(x) = y_0 + (y_1 - y_0) x$

Change bd cond.

$$y(a) = y_0$$

$$y(b) = y_1$$

set $a = 0$ $b = 1$

$$y(x) = y_0 + (y_1 - y_0)x - \int_0^1 x(1-x')f'(x')dx' + \int_0^1 dx'(x-x')\Theta(x-x')f(x')$$

$$G_2(x,x') = x(1-x') + (x-x')\Theta(x-x')$$

linear homogeneous bd cond:

If y_1 and y_2 obey linear homogeneous bd conditions

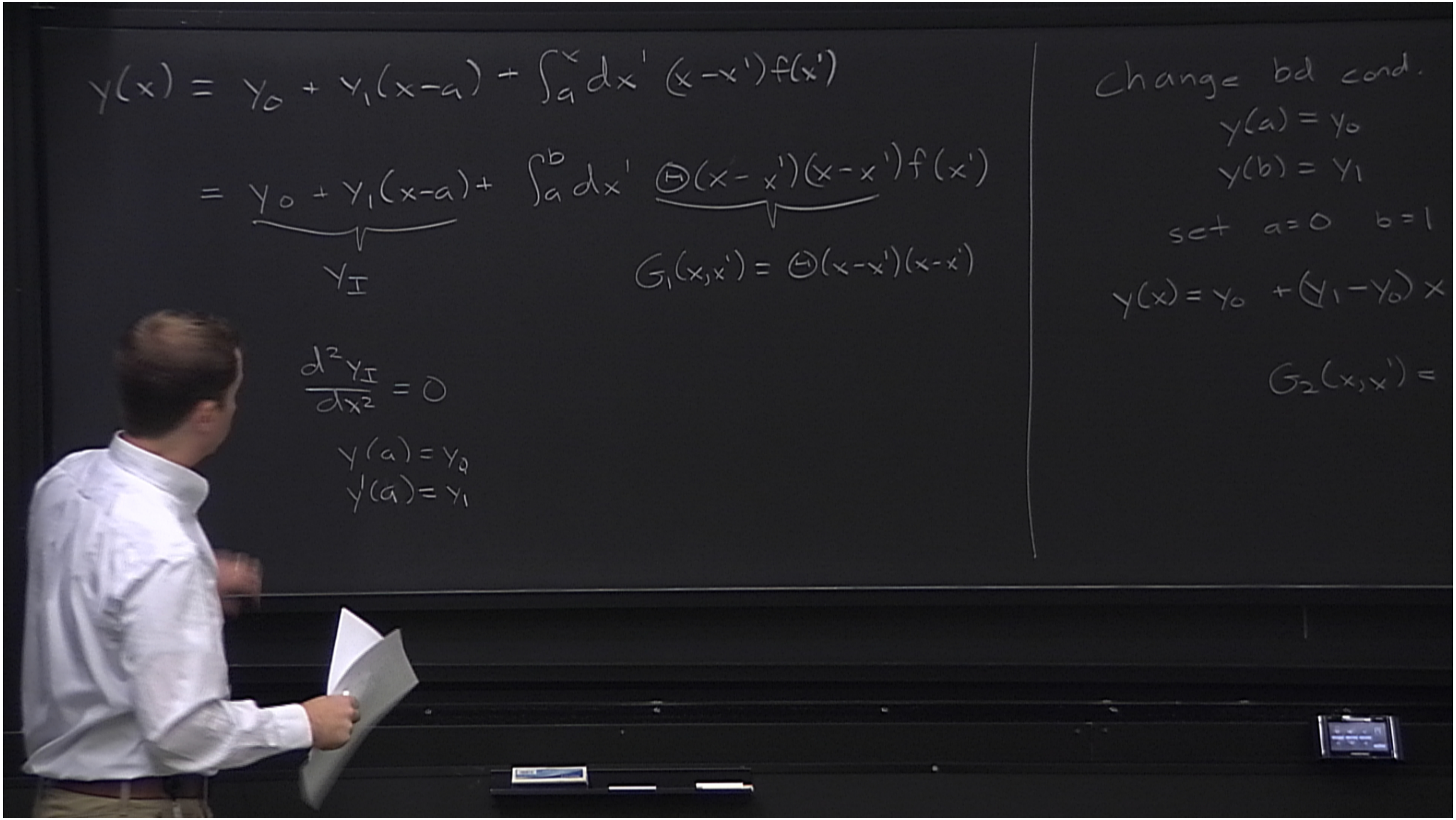
then so does $\alpha y_1 + \beta y_2$

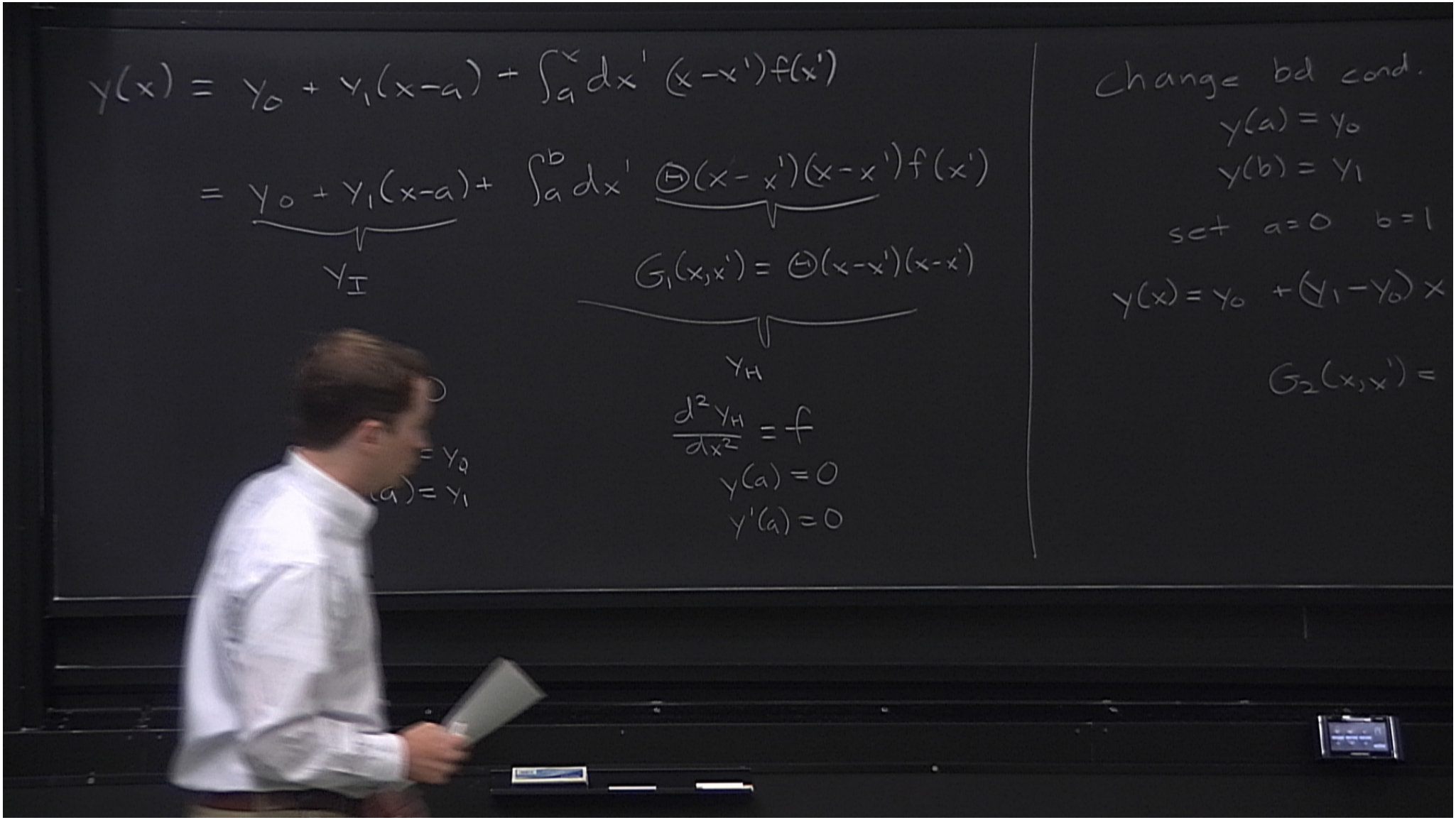
Solutions were of the form

$$y = y_I + y_H$$

y_I solution to $\frac{d^2 y_I}{dx^2} = 0$ and inhomogeneous bd cond

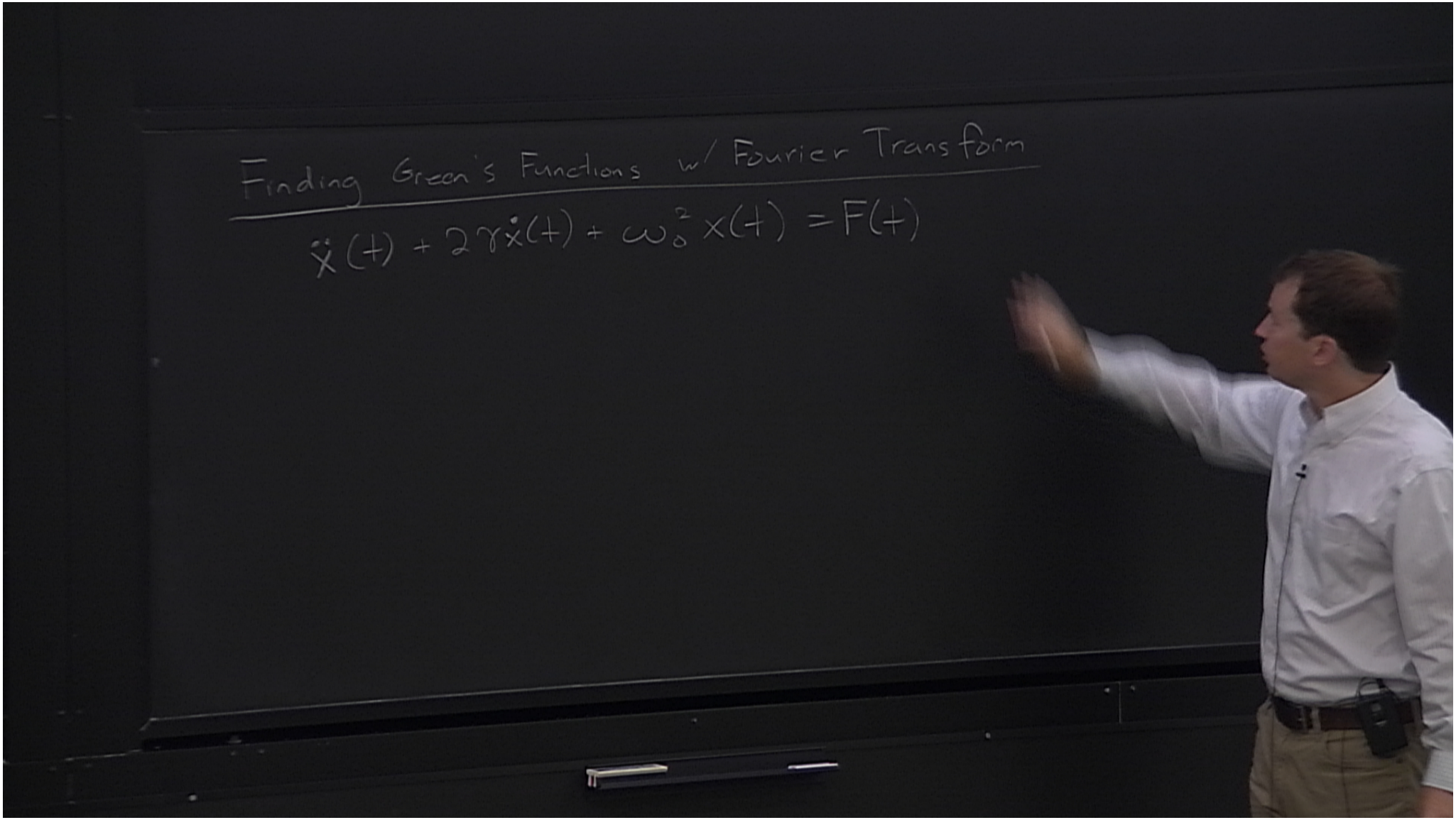
y_H so

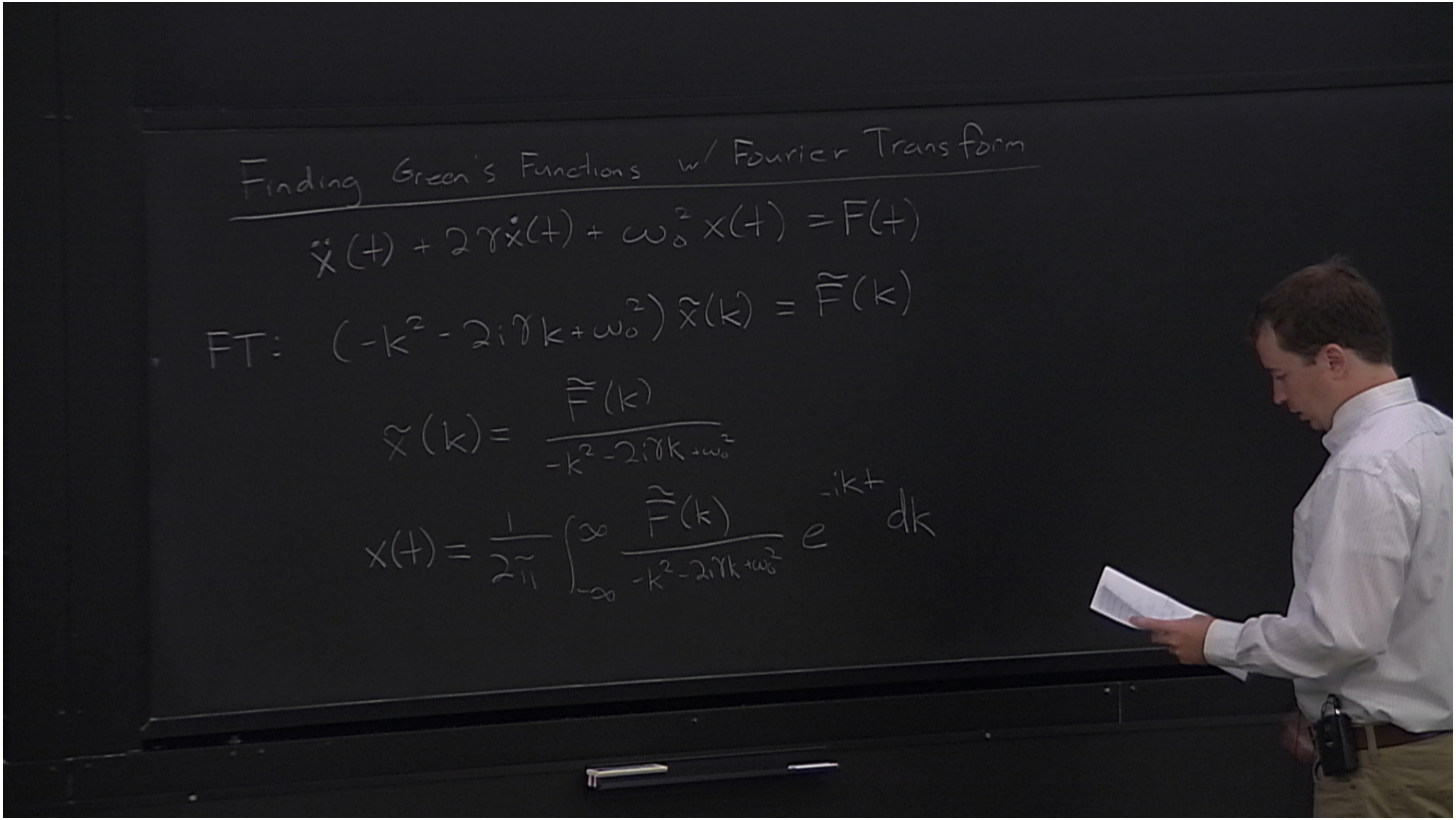

$y(x) = y_0 + y_1(x-a) + \int_a^x dx'\,(x-x')f(x')$
$= y_0 + y_1(x-a) + \int_a^b dx'\,\Theta(x-x')(x-x')f(x')$
y_I
$G_1(x,x') = \Theta(x-x')(x-x')$
$\frac{d^2 y_I}{dx^2} = 0$
$y(a) = y_0$
$y'(a) = y_1$
Change bd cond.
$y(a) = y_0$
$y(b) = y_1$
set $a=0$ $b=1$
$y(x) = y_0 + (y_1 - y_0)x$
$G_2(x,x') =$

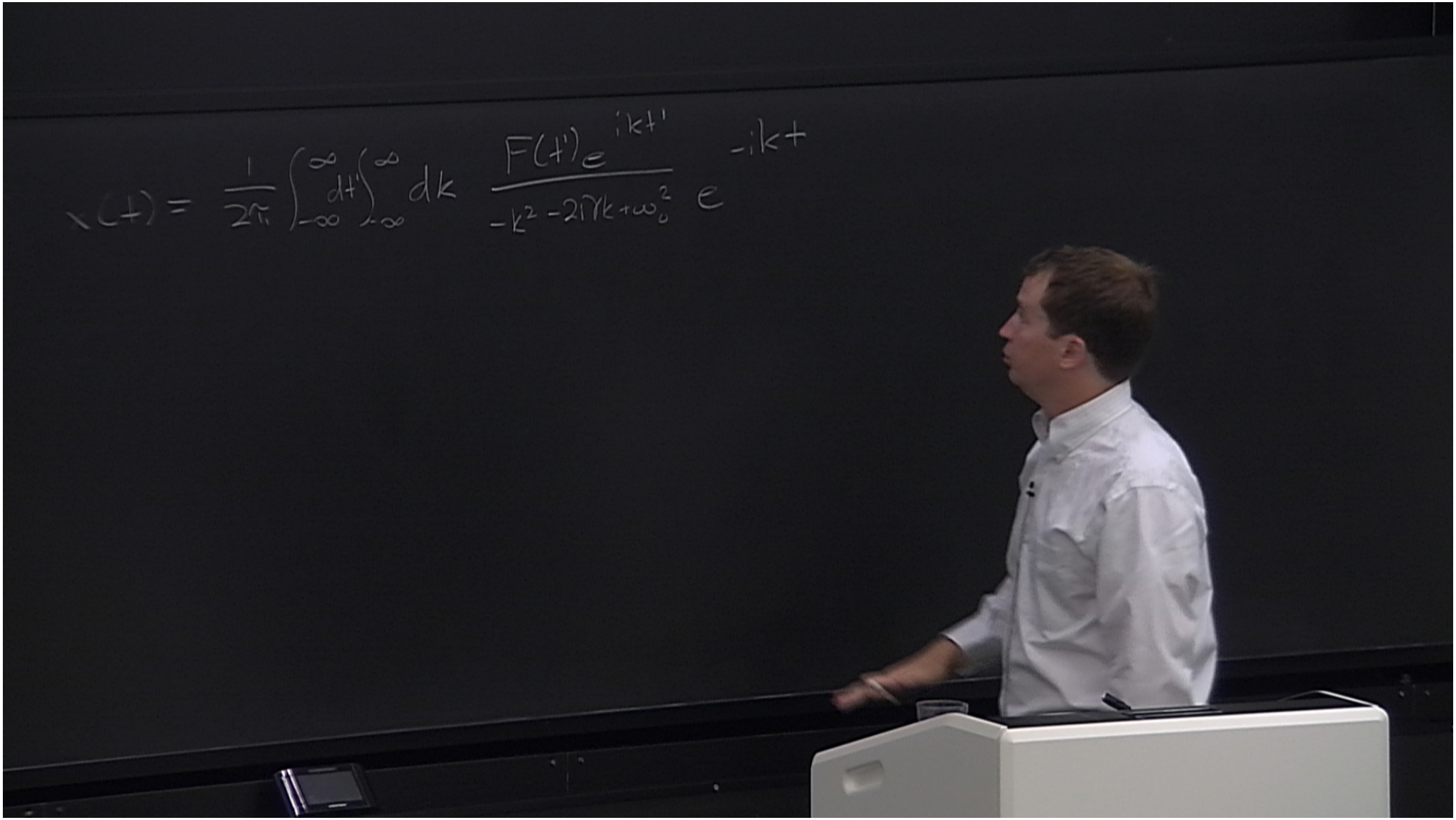

$y(x) = y_0 + y_1(x-a) + \int_a^x dx'\,(x-x')f(x')$
$= \underbrace{y_0 + y_1(x-a)}_{y_I} + \int_a^b dx'\,\underbrace{\Theta(x-x')(x-x')}_{G_1(x,x') = \Theta(x-x')(x-x')} f(x')$
y_H
$\frac{d^2 y_H}{dx^2} = f$
$y(a) = 0$
$y'(a) = 0$
Change bd cond.
$y(a) = y_0$
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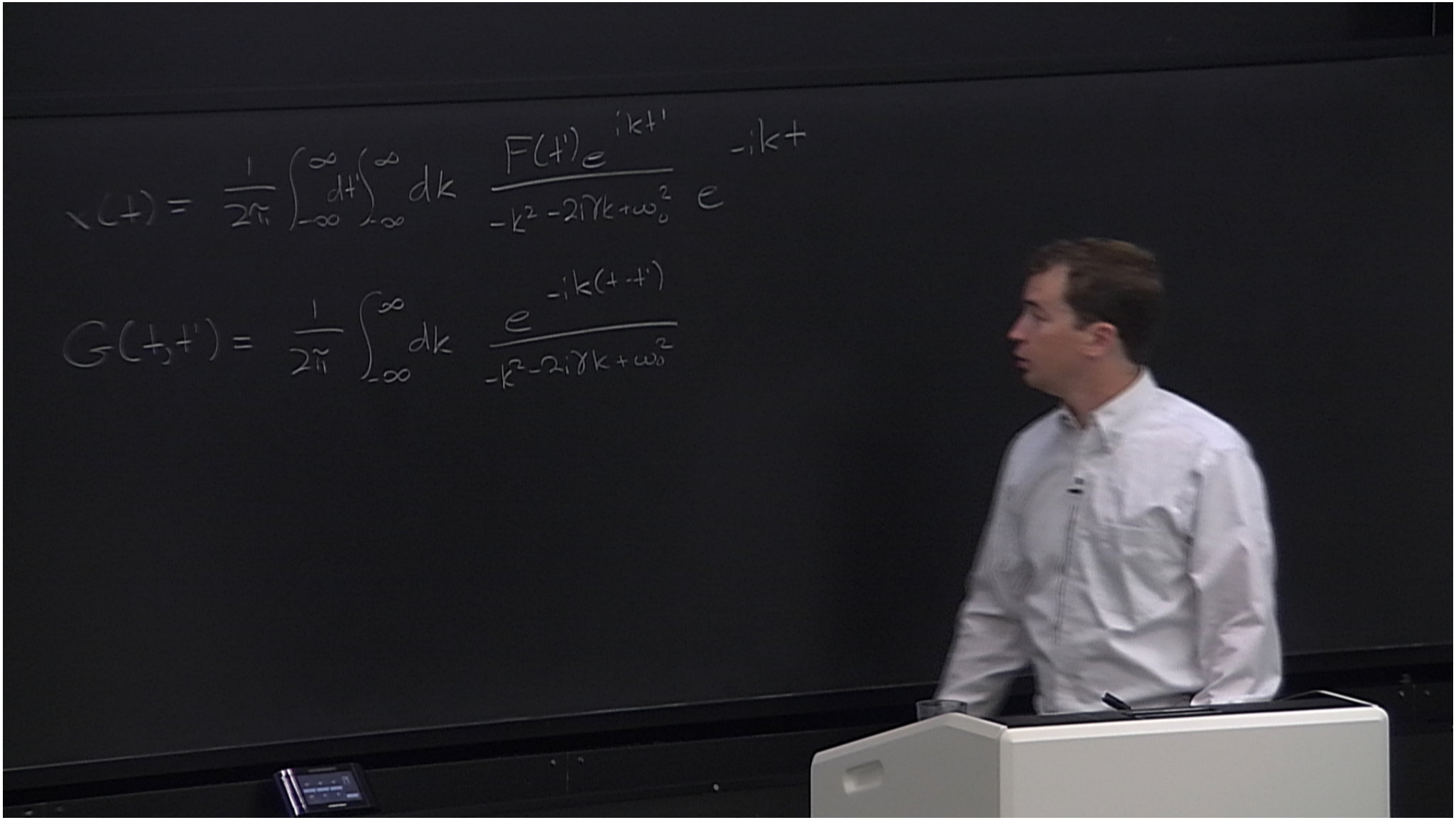
y_H solved $\frac{d^2 y_{H1}}{dx^2} = f$ w/ homogeneous bd cond

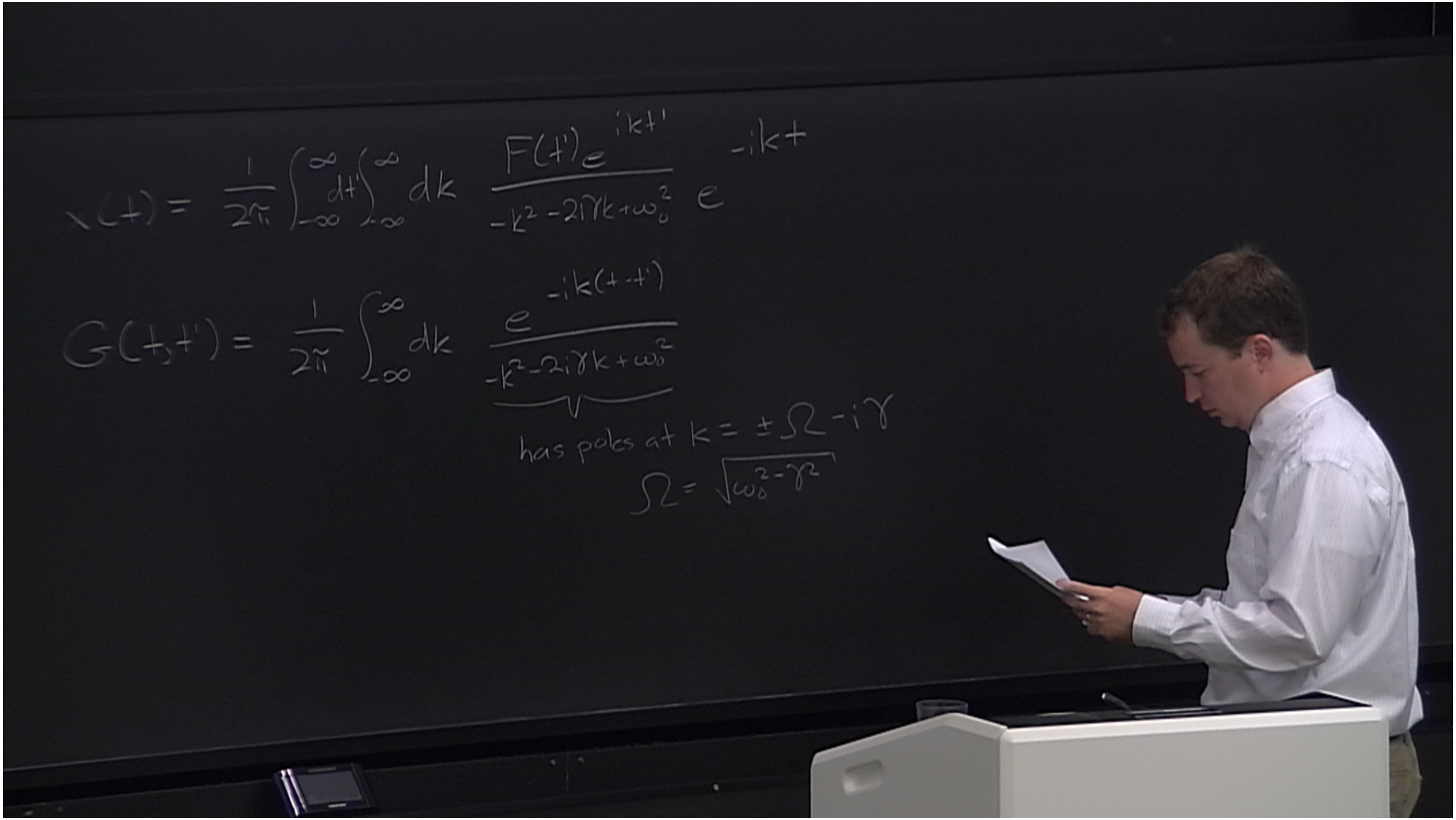
y_H solved $\frac{d^2 y_H}{dx^2} = f$ w/ homogeneous bd cond

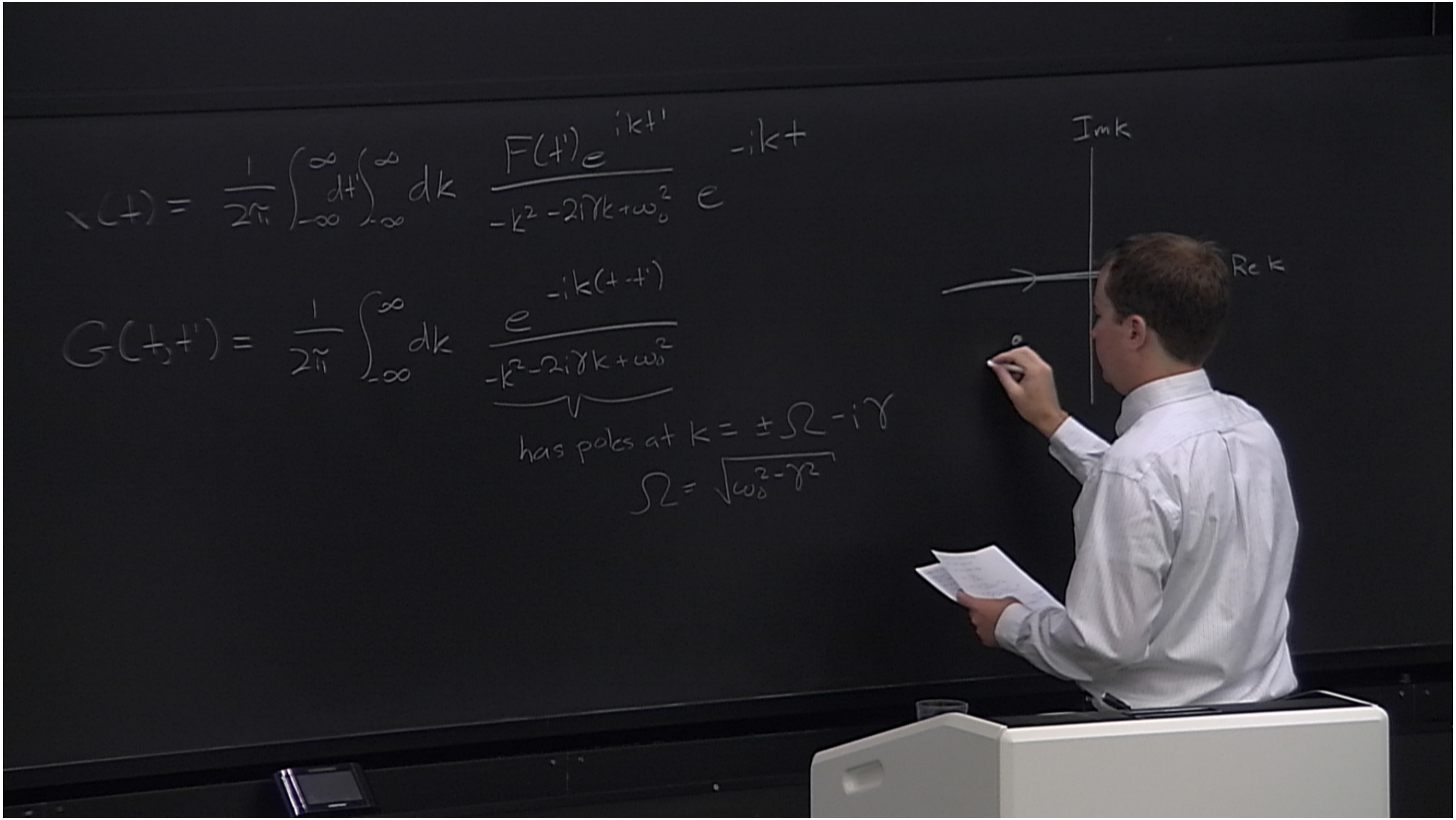

Finding Green's Functions w/ Fourier Transform
$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = F(t)$


Finding Green's Functions w/ Fourier Transform
$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = F(t)$
FT: $(-k^2 - 2i\gamma k + \omega_0^2)\,\tilde{x}(k) = \tilde{F}(k)$
$\tilde{x}(k) = \frac{\tilde{F}(k)}{-k^2 - 2i\gamma k + \omega_0^2}$
$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\tilde{F}(k)}{-k^2 - 2i\gamma k + \omega_0^2} e^{-ikt}\,dk$


$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dk \; \frac{F(t') e^{ikt'}}{-k^2 - 2\Gamma k + \omega_0^2} e^{-ikt}$


$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dk \; \frac{F(t') e^{ikt'}}{-k^2 - 2i\gamma k + \omega_0^2} e^{-ikt}$
$G(t,t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; \frac{e^{-ik(t-t')}}{-k^2 - 2i\gamma k + \omega_0^2}$


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has poles at $k = \pm\Omega - i\gamma$
$\Omega = \sqrt{\omega_0^2 - \gamma^2}$

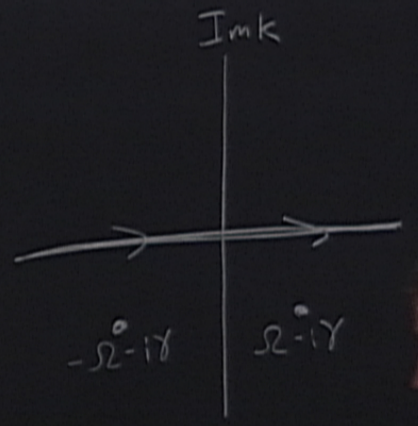

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Im k
Re k

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dk \; \frac{F(t')e^{ikt'}}{-k^2 - 2i\gamma k + \omega_0^2} e^{-ikt}$$

$$G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \; \frac{e^{-ik(t-t')}}{-k^2 - 2i\gamma k + \omega_0^2}$$

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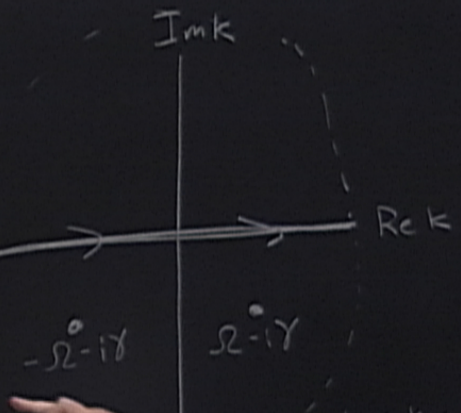


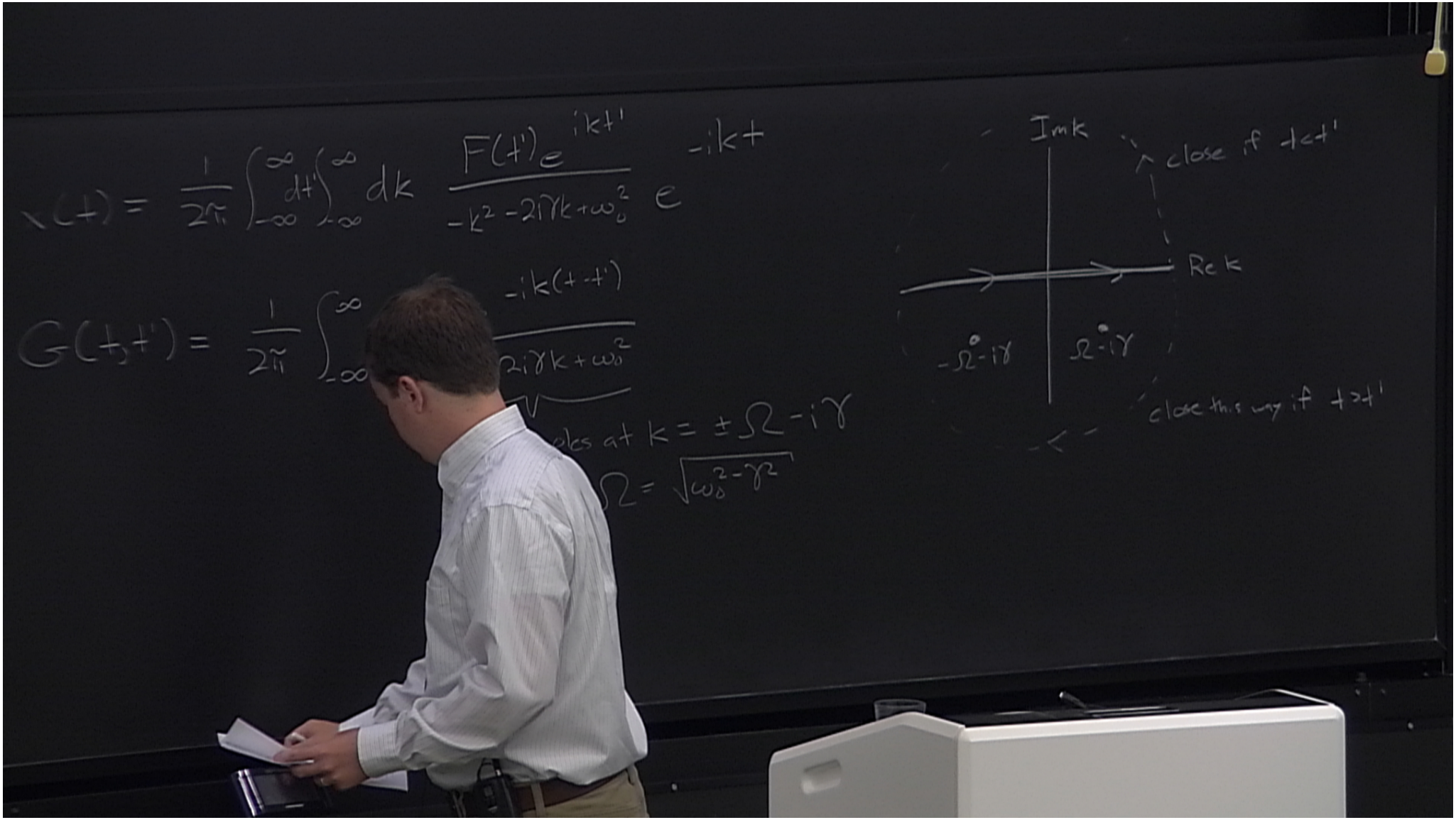
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dk \, \frac{F(t')e^{ikt'}}{-k^2 - 2i\gamma k + \omega_0^2} e^{-ikt}$$

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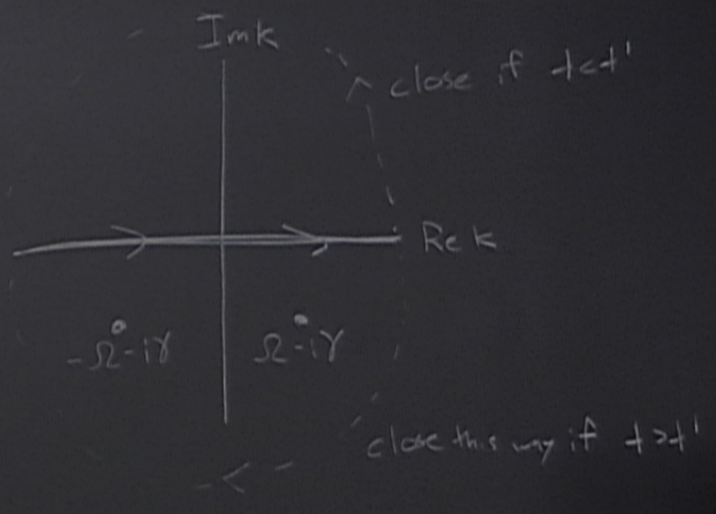

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$G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}$
$-ik(t-t')$
$2i\gamma k + \omega_0^2$
poles at $k = \pm\Omega - i\gamma$
$\Omega = \sqrt{\omega_0^2 - \gamma^2}$
Im k
Re k
close if $t<t'$
close this way if $t>t'$
$-\Omega - i\gamma$
$\Omega - i\gamma$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dk \; \frac{F(t')\, e^{ikt'}}{-k^2 - 2i\gamma k + \omega_0^2}\, e^{-ikt}$$

$$G(t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \; \frac{e^{-ik(t-t')}}{-k^2 - 2i\gamma k + \omega_0^2}$$

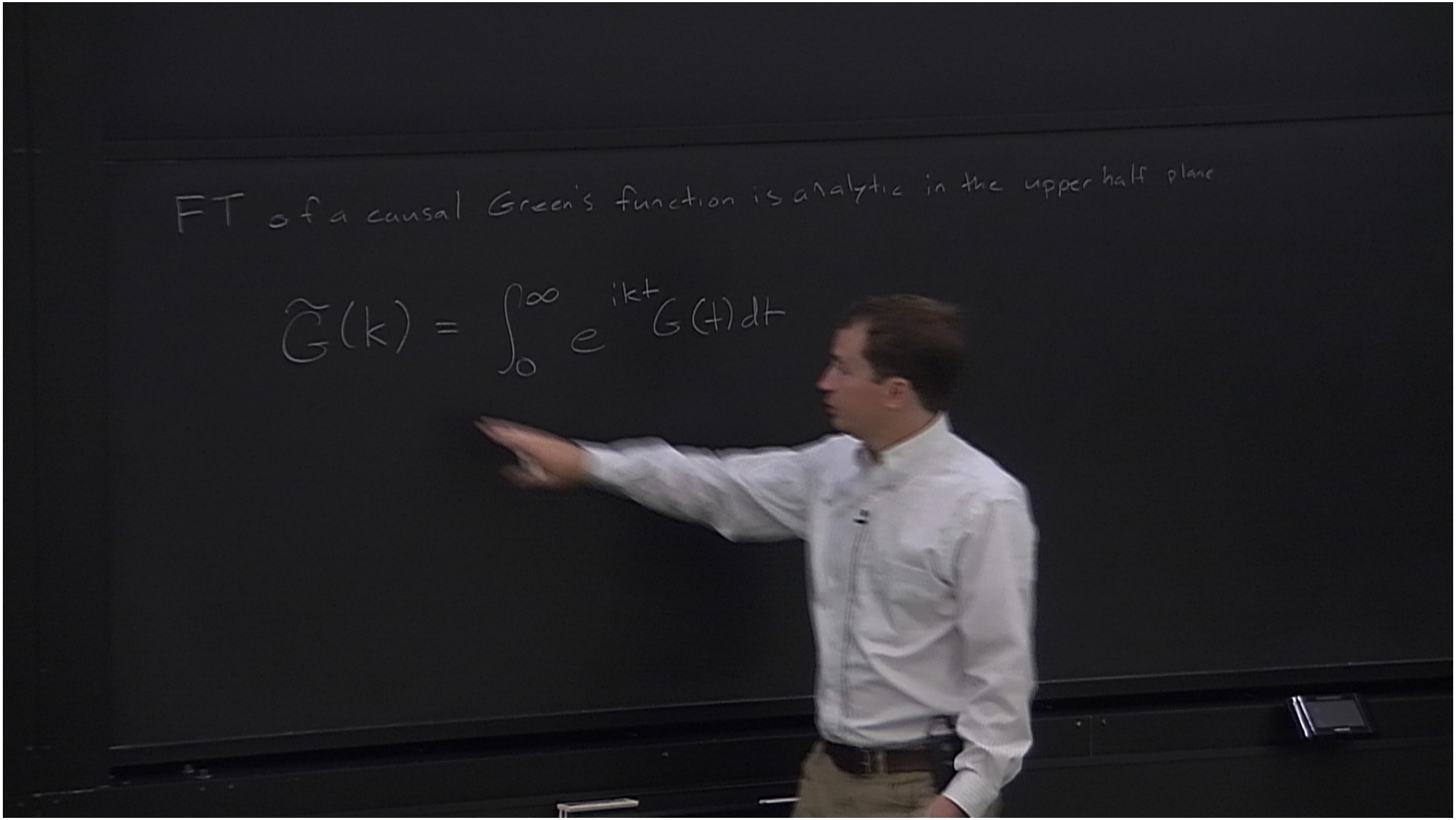
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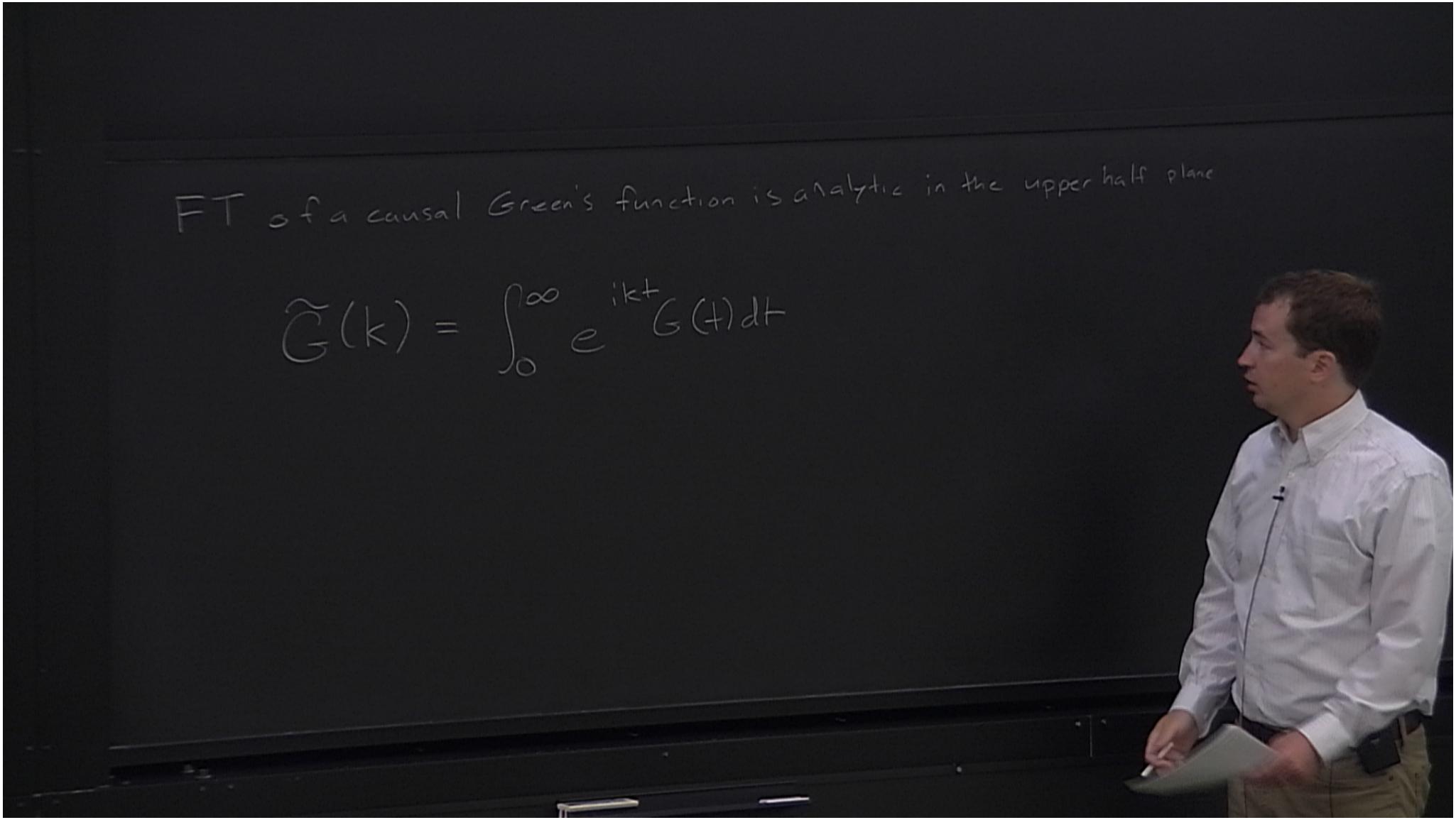
$$\Omega = \sqrt{\omega_0^2 - \gamma^2}$$

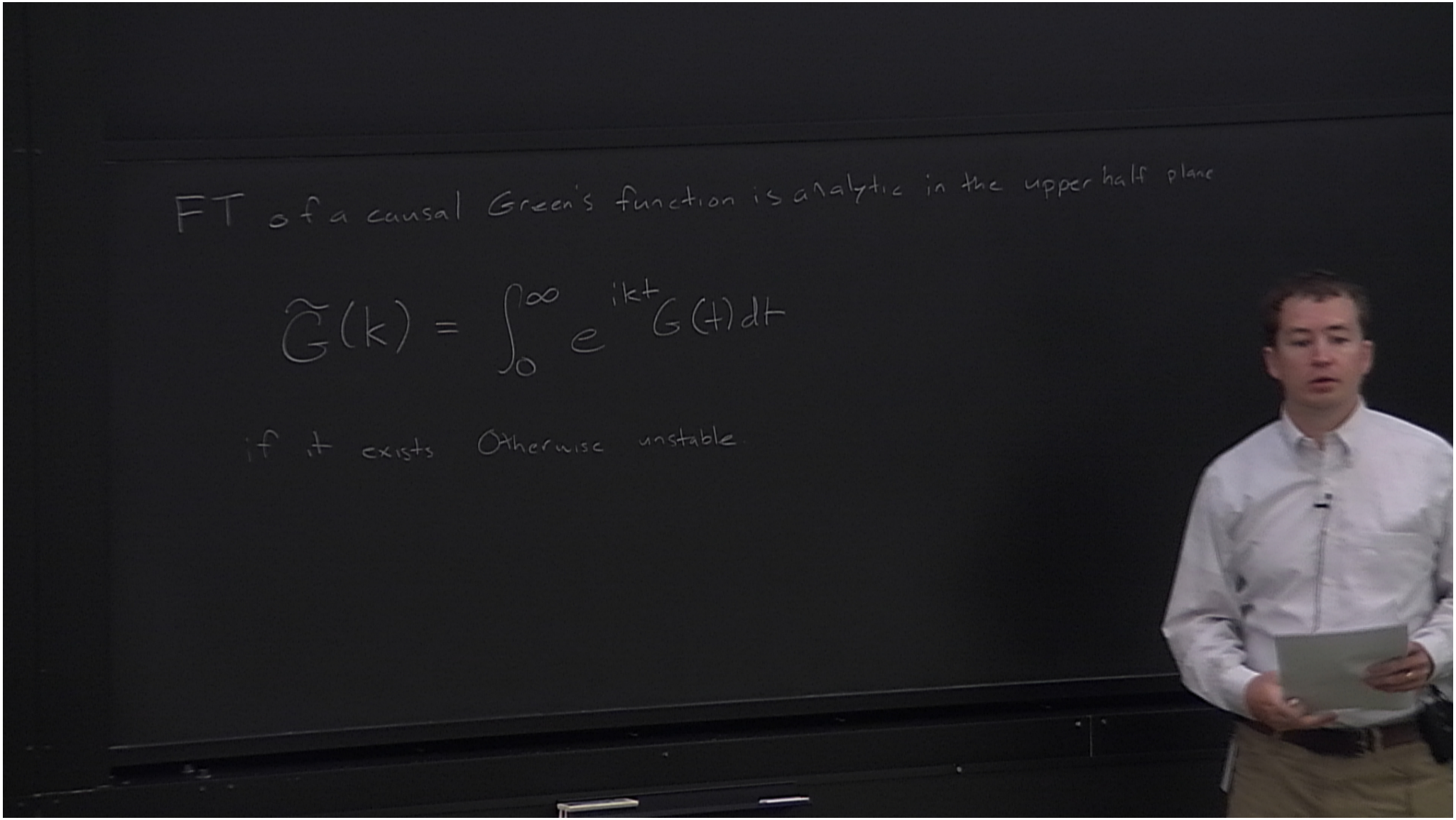


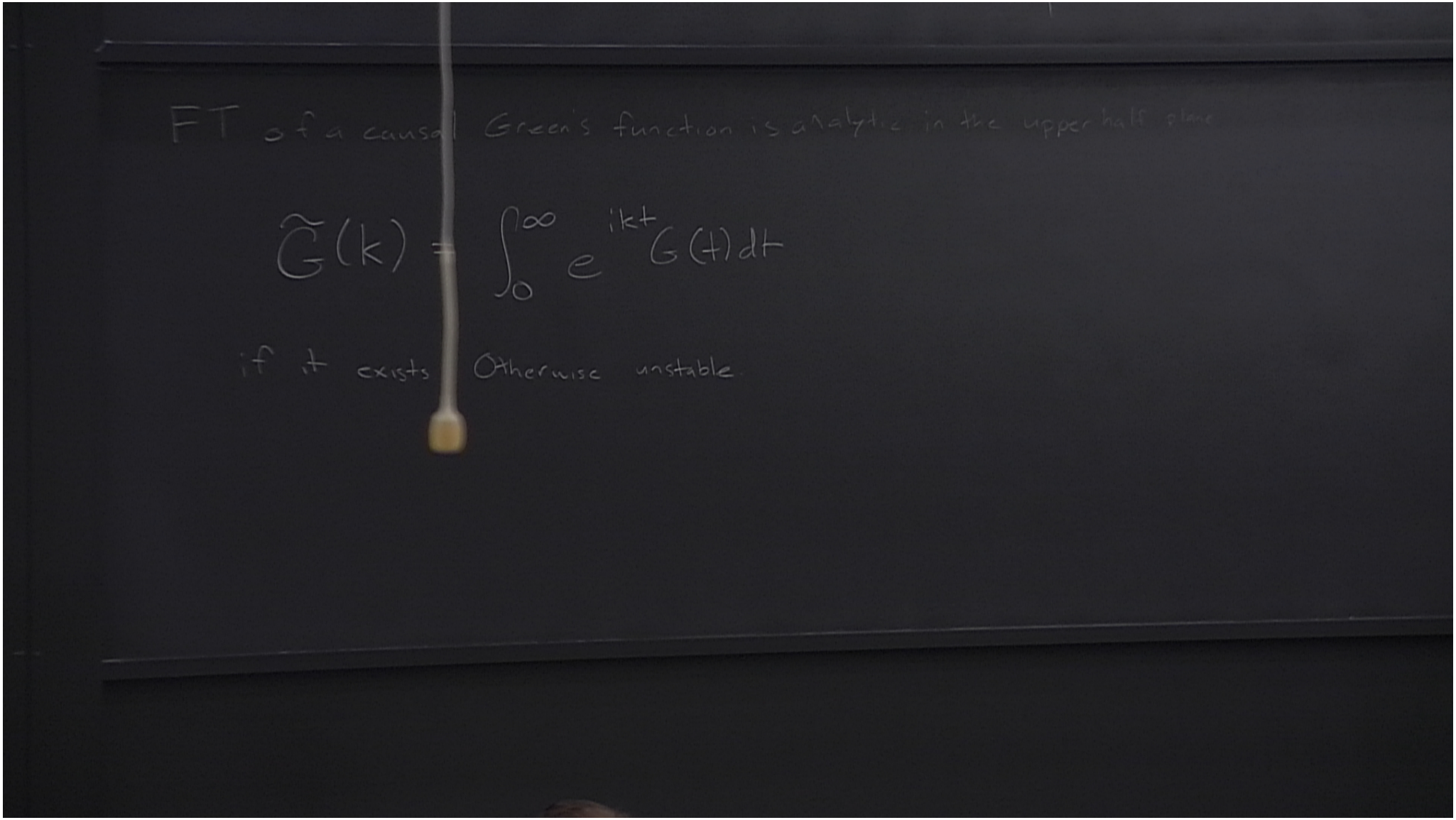
$$G(t,t') = \frac{1}{\Omega} e^{-i\gamma(t-t')} \sin\left(\Omega(t-t')\right) \Theta(t-t')$$

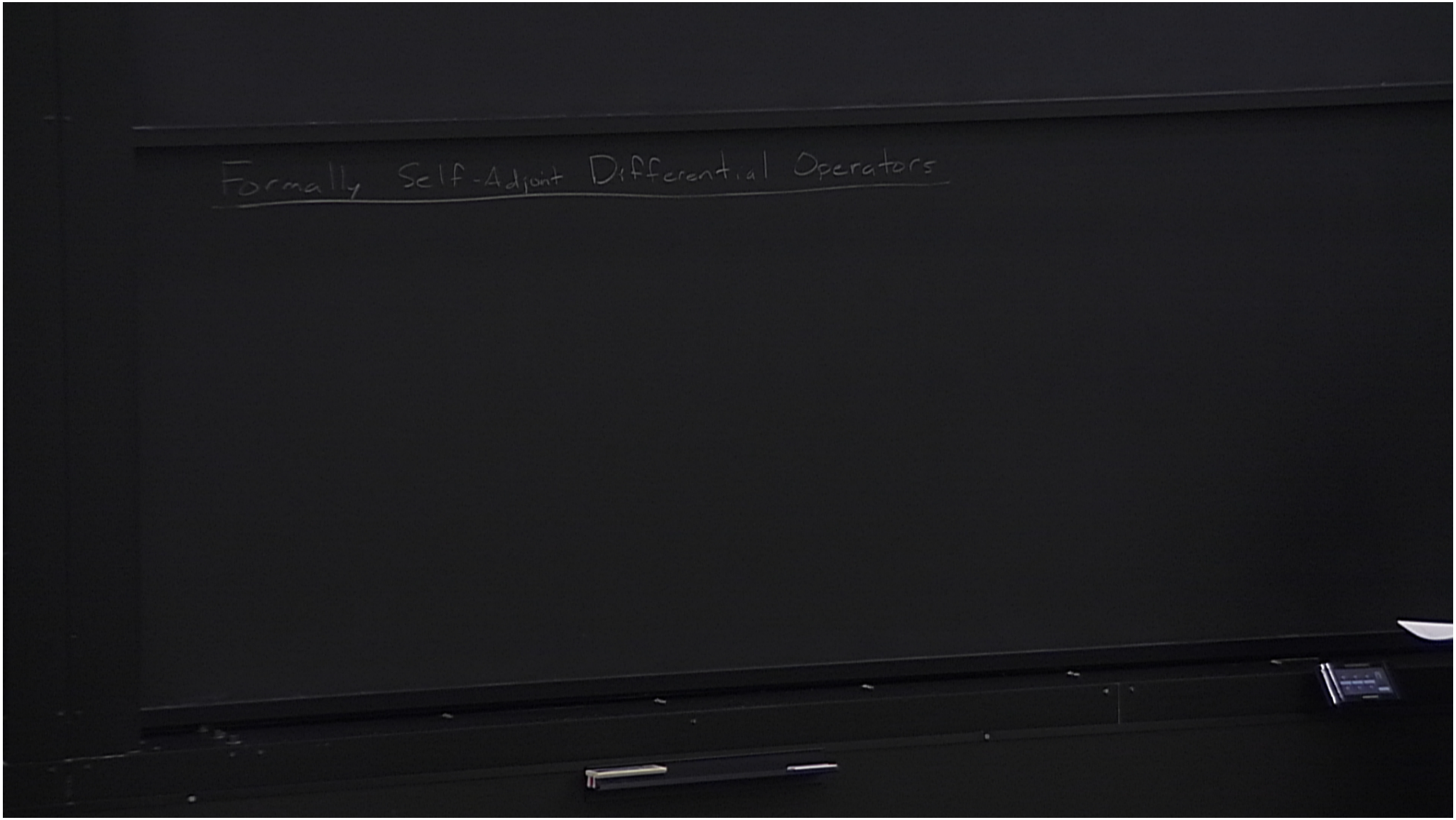
vanishes for $t<t'$

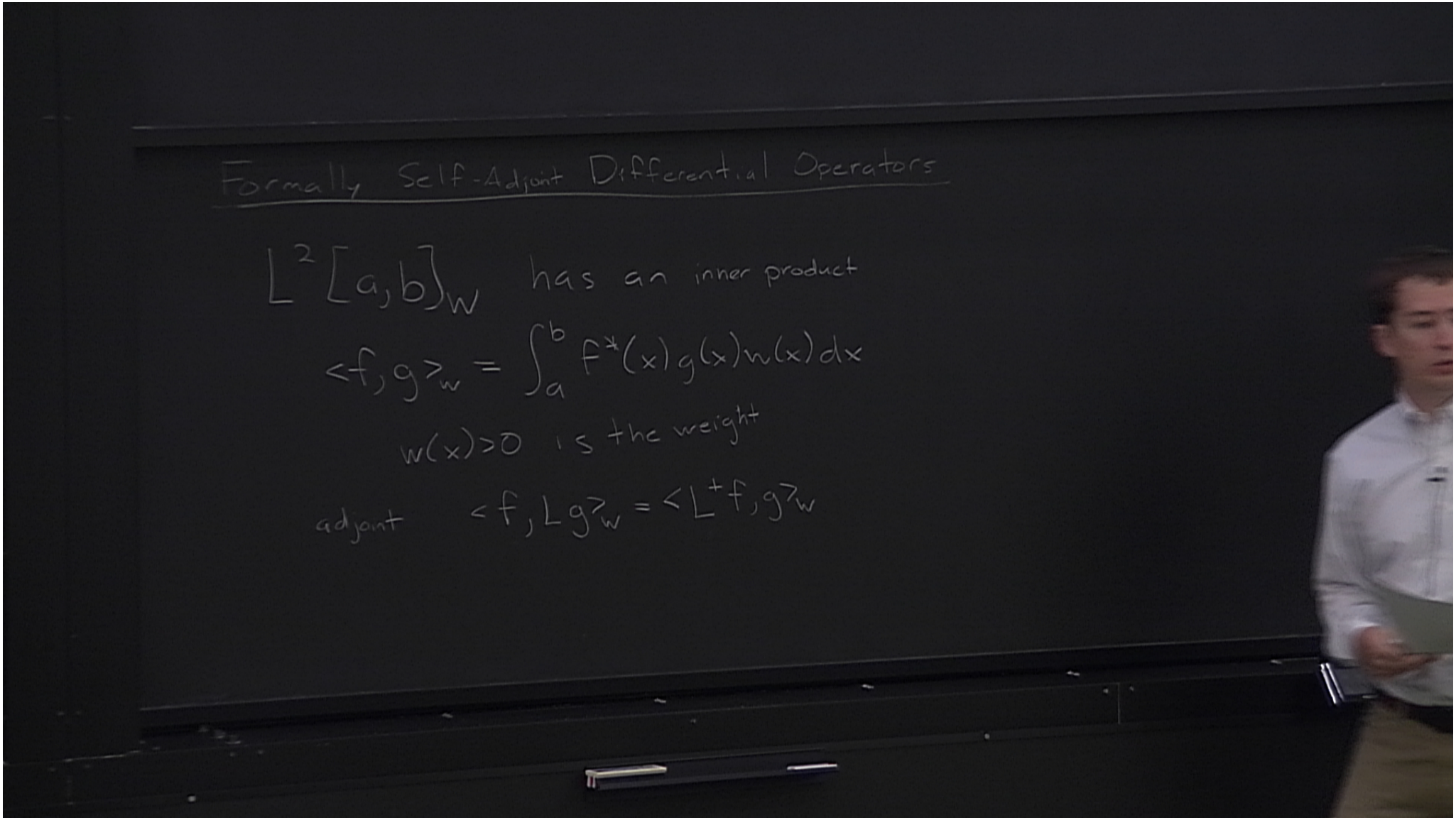

FT of a causal Green's function is analytic in the upper half plane
$\tilde{G}(k) = \int_0^\infty e^{ikt} G(t)\,dt$

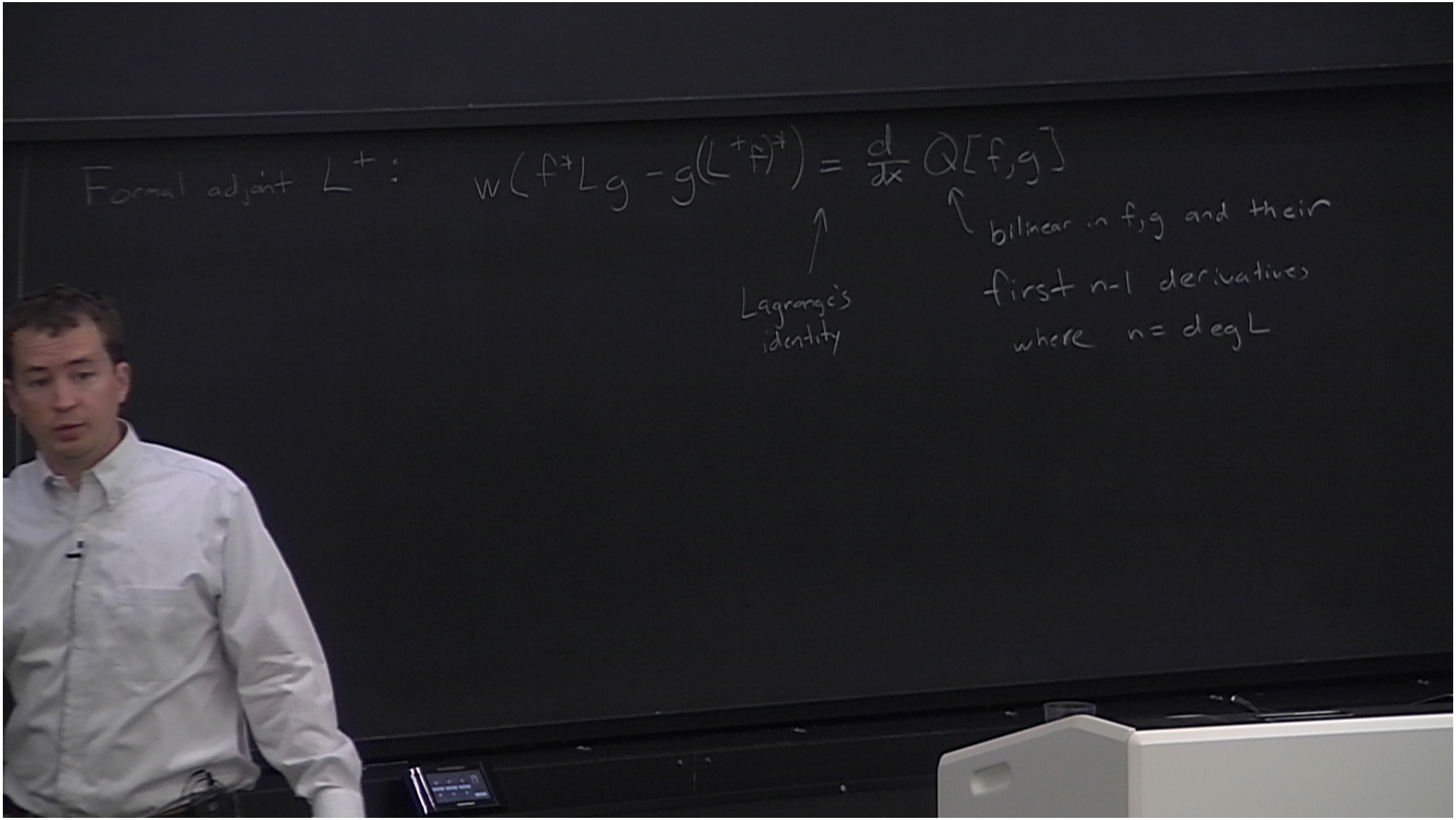

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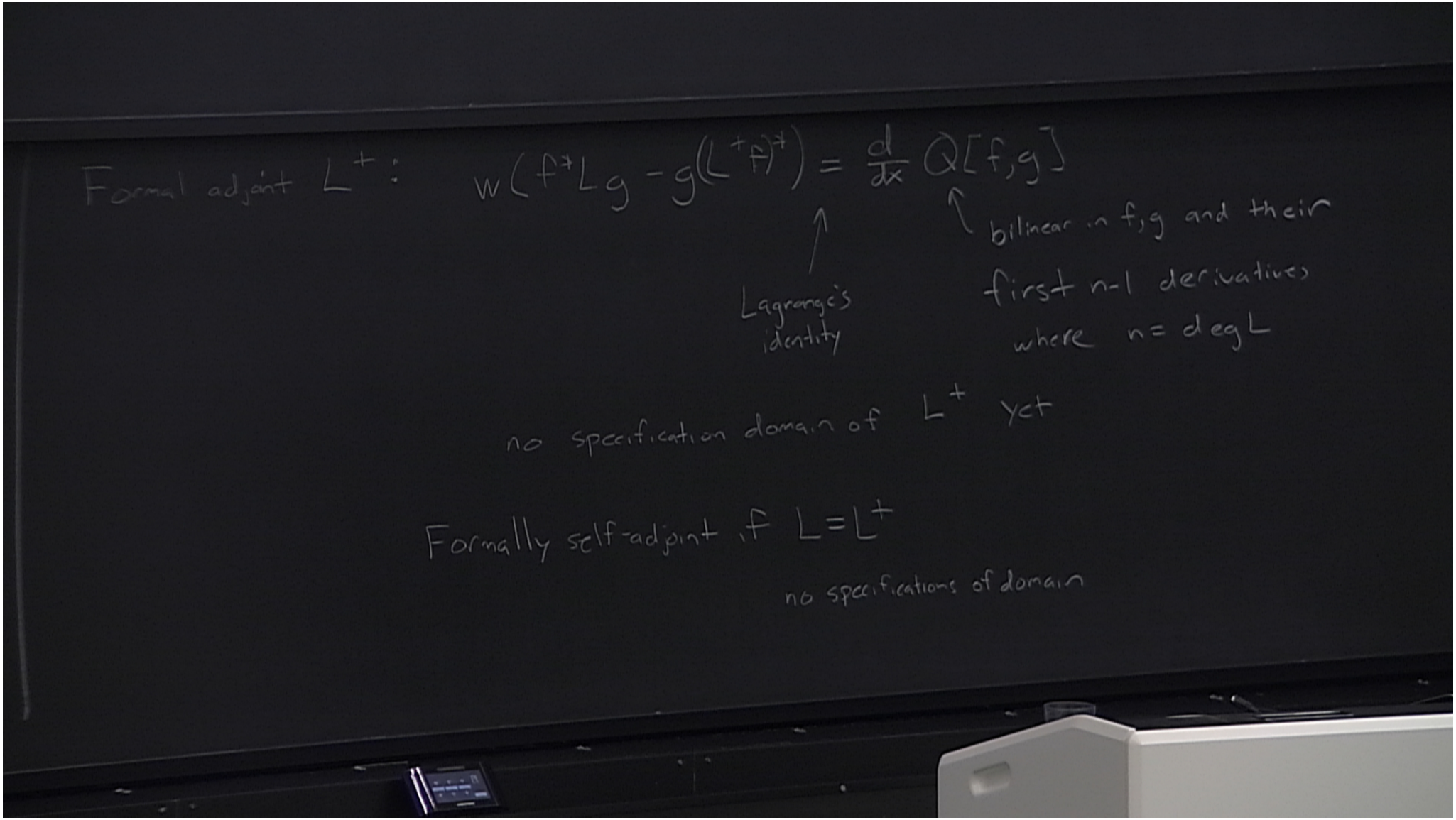

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if it exists Otherwise unstable.

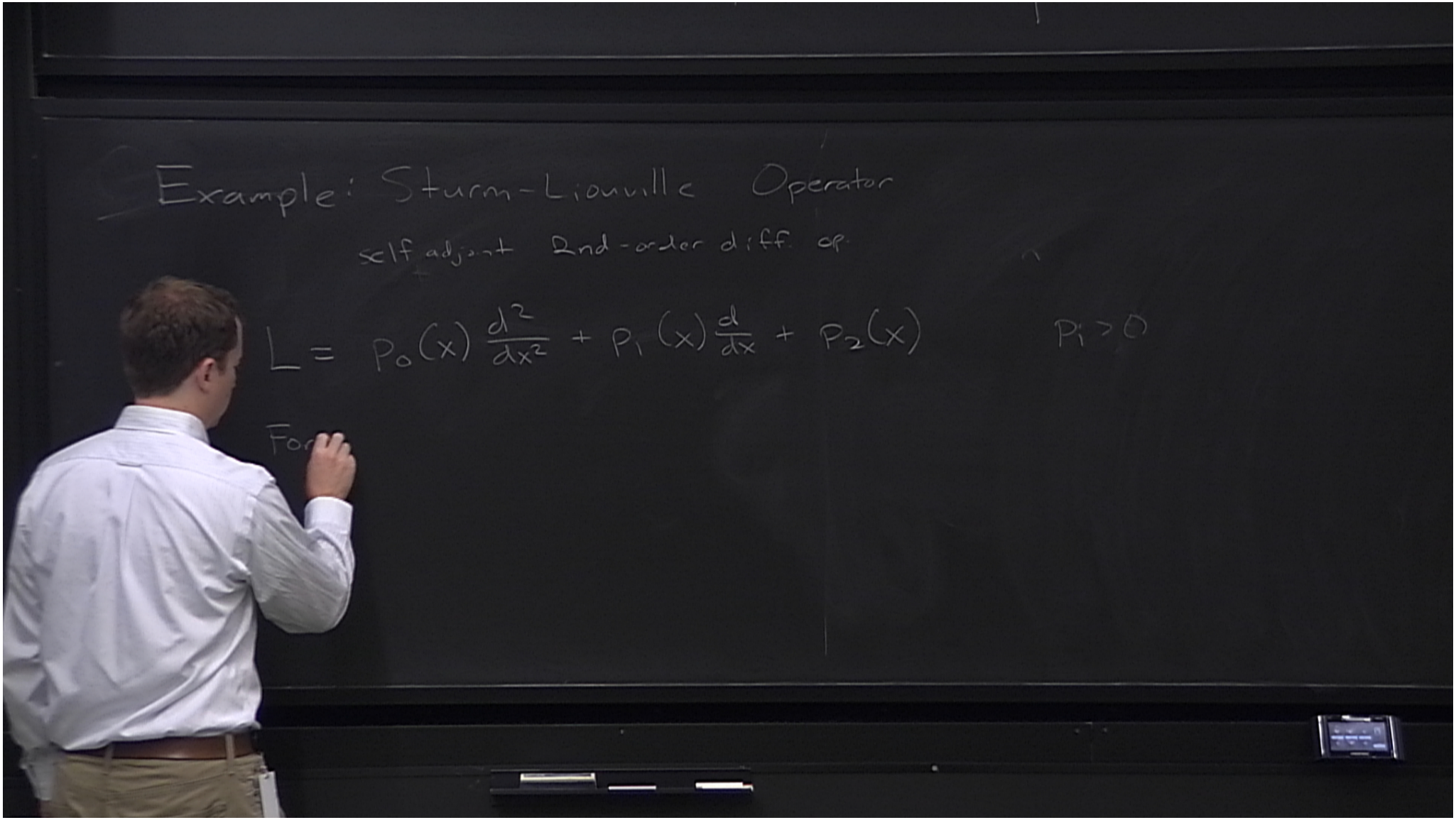

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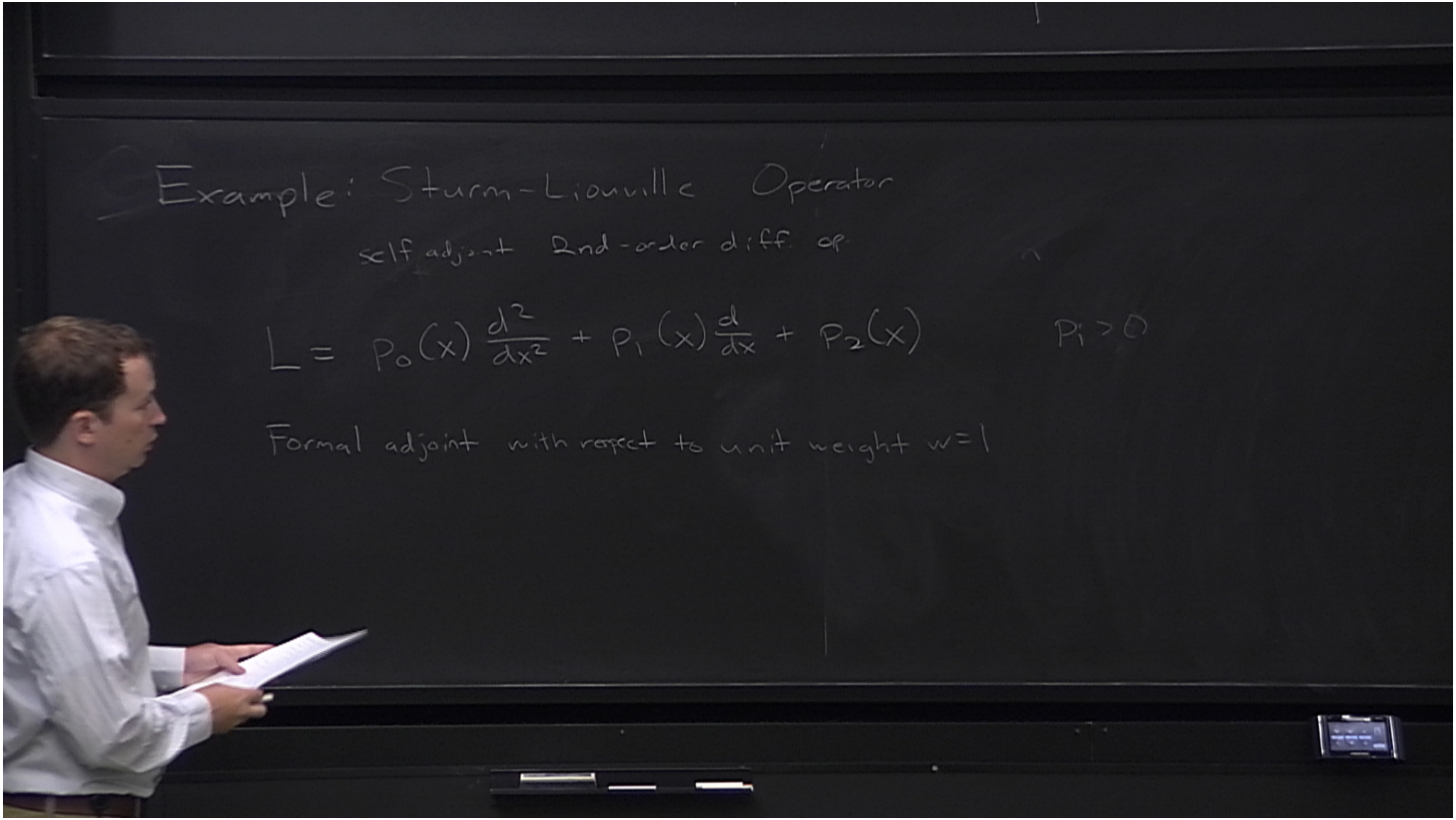

Formally Self-Adjoint Differential Operators

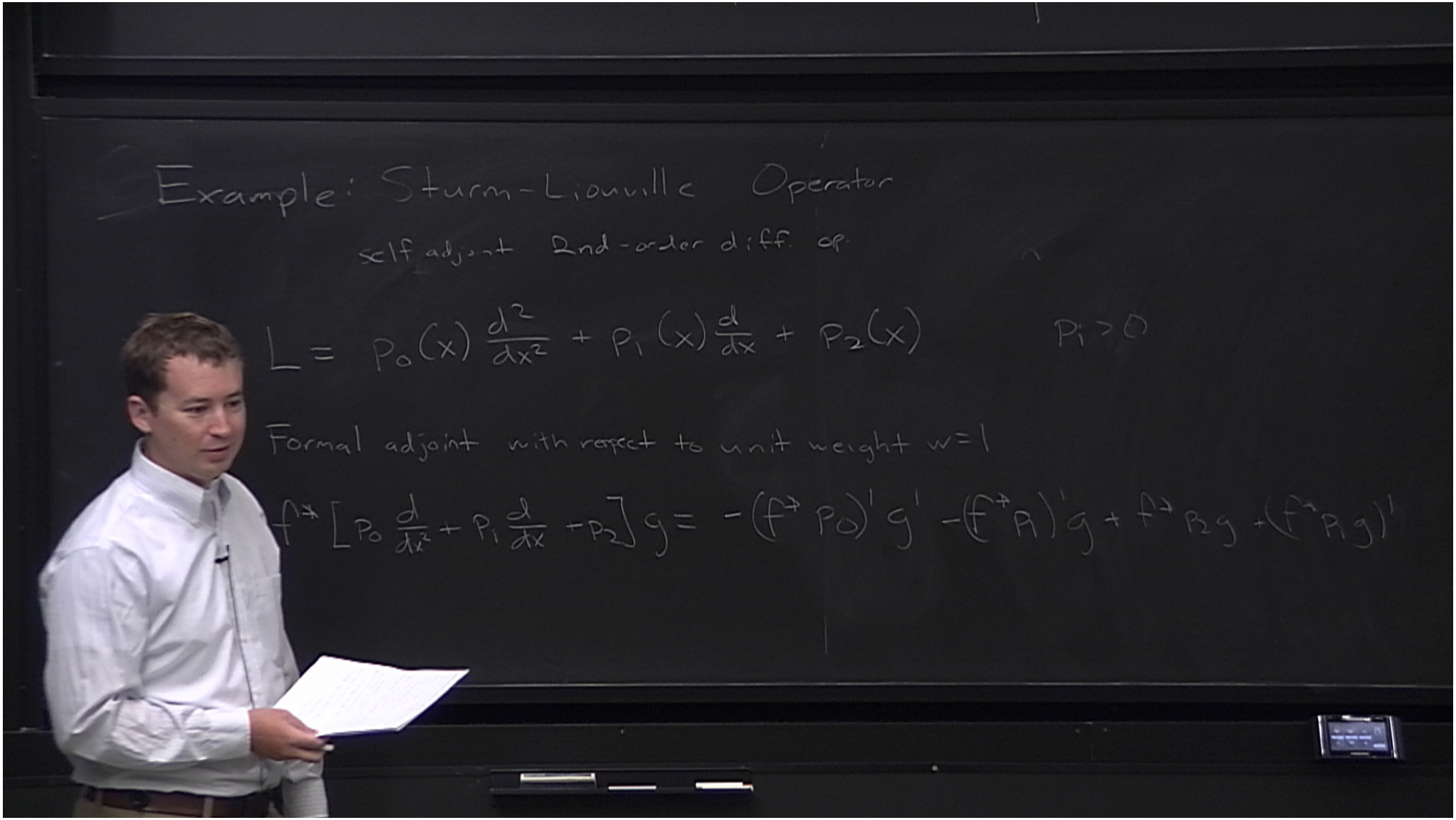

Formally Self-Adjoint Differential Operators
$L^2[a,b]_w$ has an inner product
$\langle f,g\rangle_w = \int_a^b f^*(x)g(x)w(x)dx$
$w(x)>0$ is the weight
adjoint $\langle f, Lg\rangle_w = \langle L^+ f, g\rangle_w$


Formal adjoint L^+: $w(f^* Lg - g(L^+ f)^*) = \frac{d}{dx} Q[f,g]$
Lagrange's identity
bilinear in f,g and their first n-1 derivatives where n = deg L


Formal adjoint L^+: $w(f^*Lg - g(L^+f)^*) = \frac{d}{dx}Q[f,g]$
Lagrange's identity
bilinear in f,g and their first n-1 derivatives where n = deg L
no specification domain of L^+ yet
Formally self-adjoint if $L = L^+$
no specifications of domain


Example: Sturm-Liouville Operator
self adjoint 2nd-order diff. op.
$L = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$
$p_i > 0$
For


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Formal adjoint with respect to unit weight $w=1$


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$f^*[p_0\frac{d}{dx^2} + p_1\frac{d}{dx} + p_2]g = -(f^* p_0)' g' - (f^* p_1)' g + f^* p_2 g + (f^* p_1 g)'$

$y'(a) = 0$

ample: Sturm-Liouville Operator

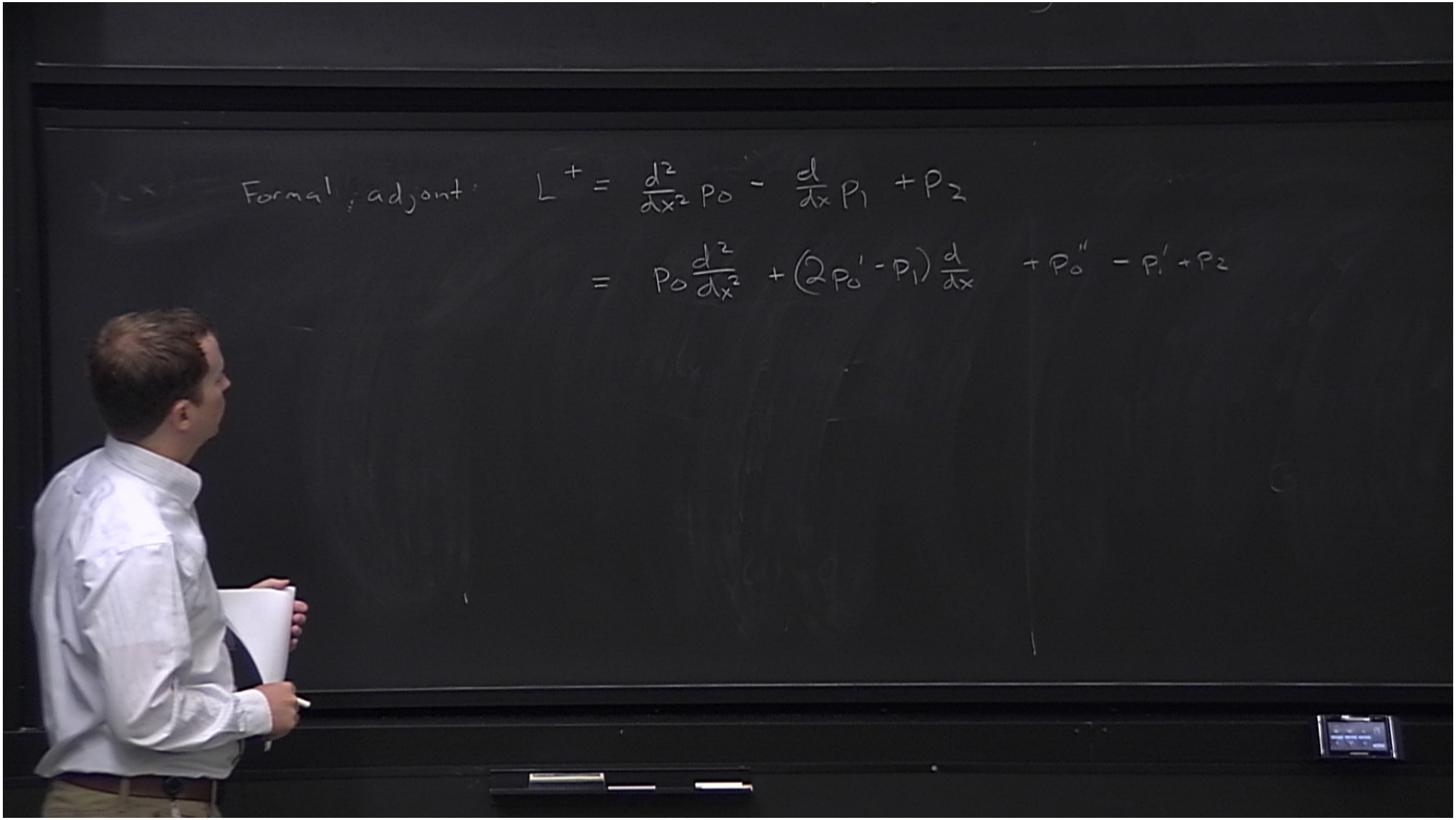
self adjoint 2nd-order diff. op

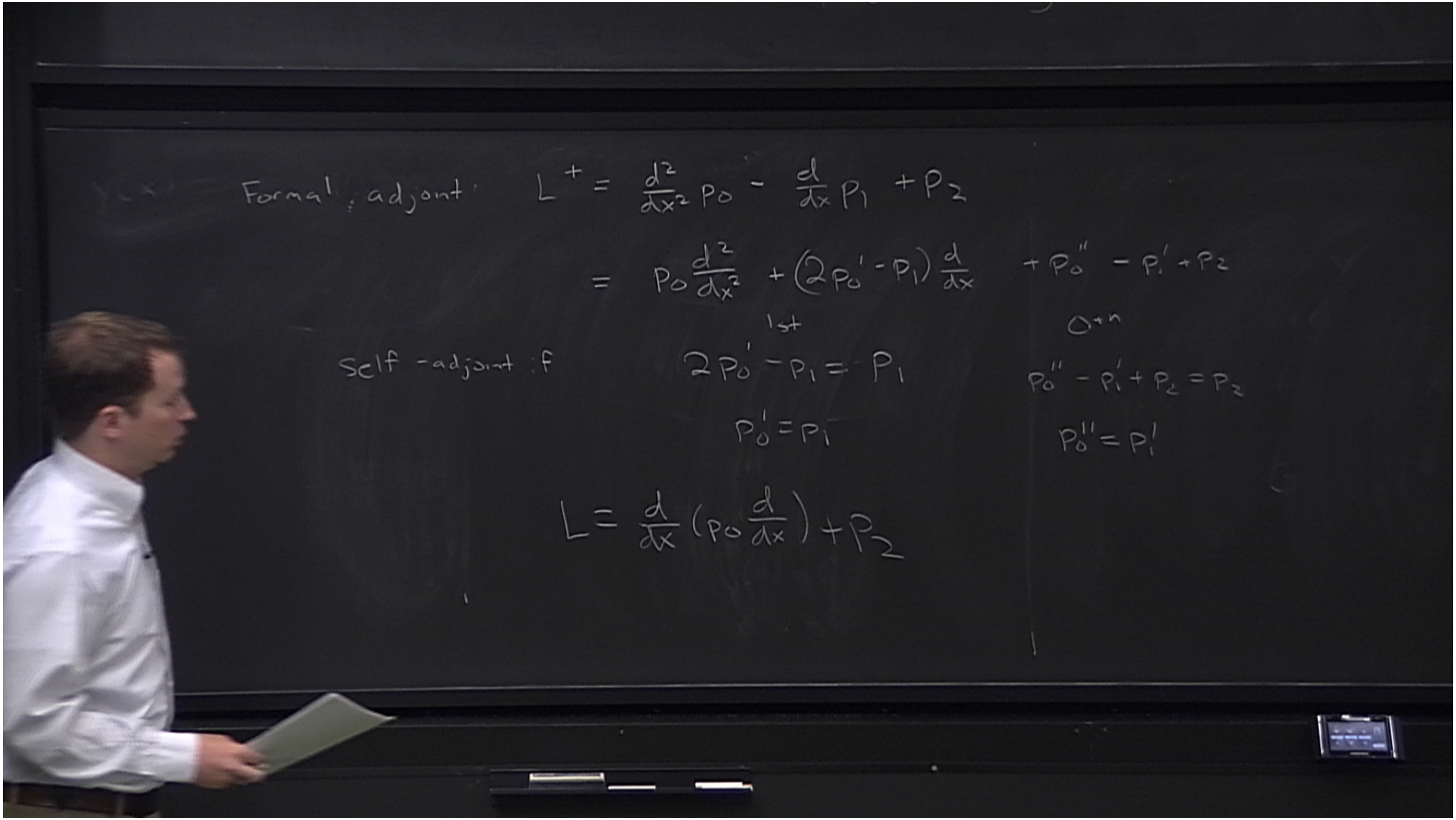
$$L = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x) \qquad p_i > 0$$

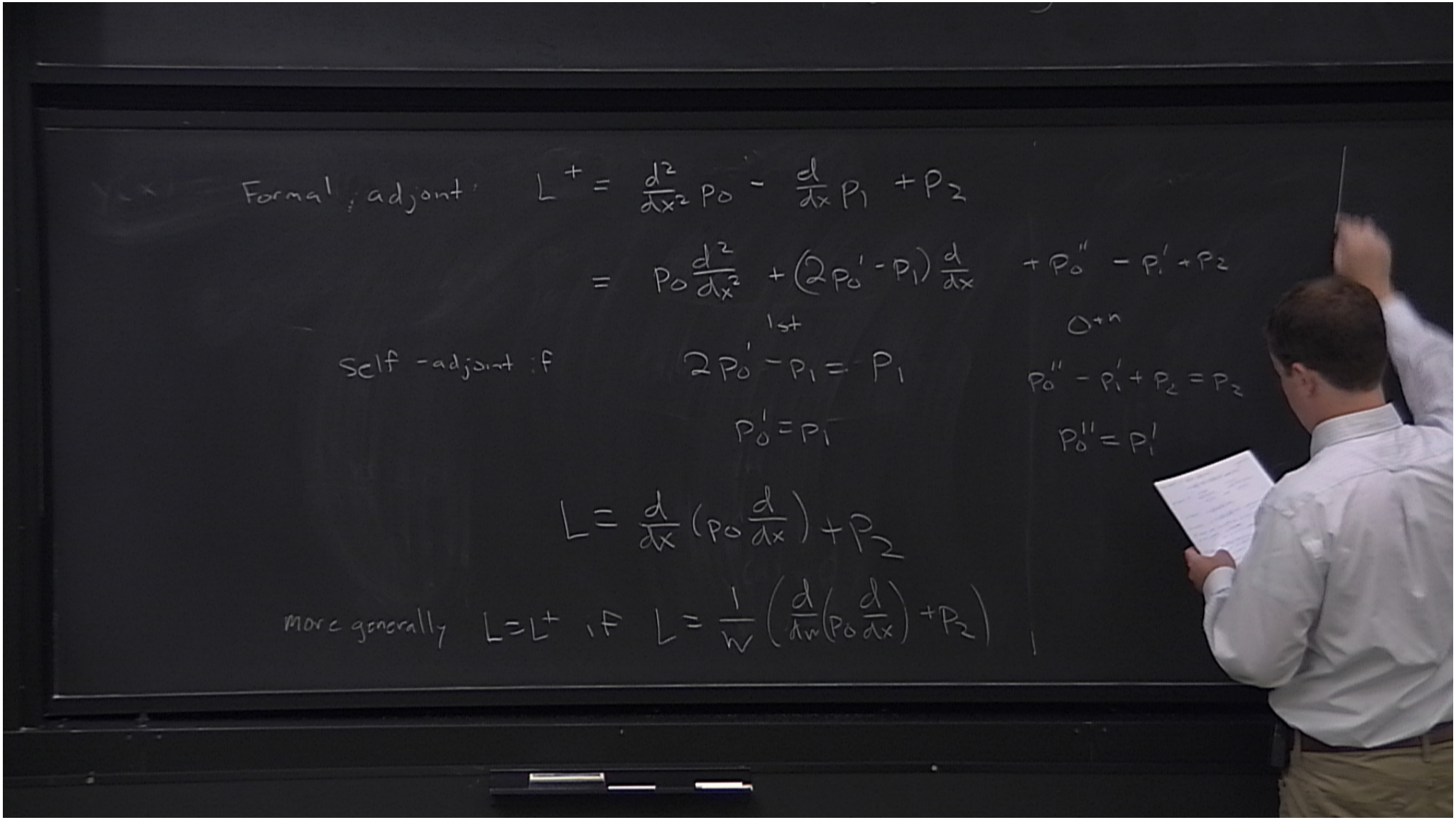
Formal adjoint with respect to unit weight $w = 1$

$$f^*\left[p_0\frac{d}{dx^2} + p_1\frac{d}{dx} + p_2\right]g = -(f^* p_0)' g' - (f^* p_1)' g + f^* p_2 g + (f^* p_1 g)' + (f^* p_0 g')'$$

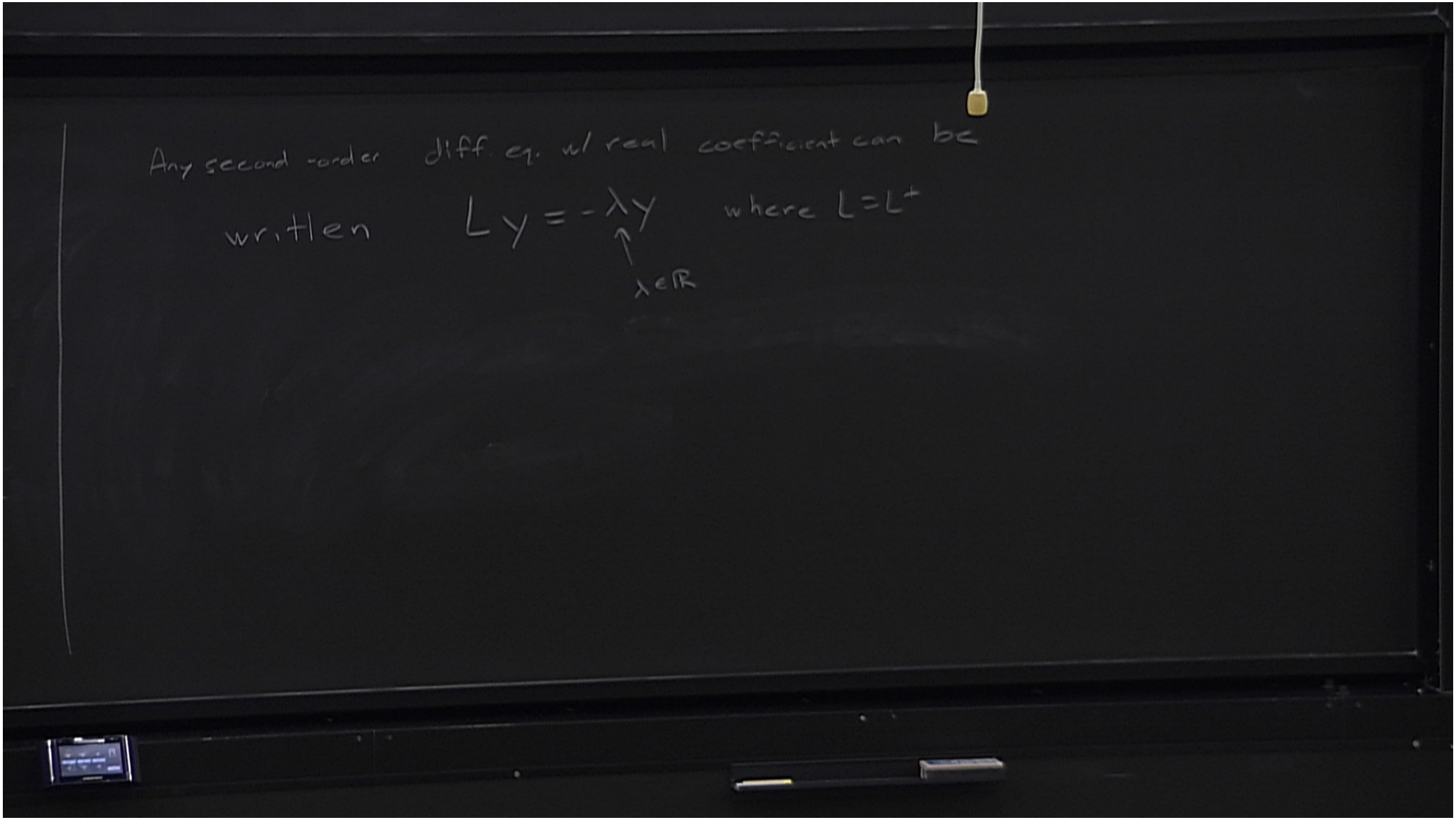
$$= (f^* p_0)'' g - (f^* p_1)' g + f^* p_2 g - (f^* p_0' g)' + (f^* g p_0)' + (f^* p_1 g)'$$

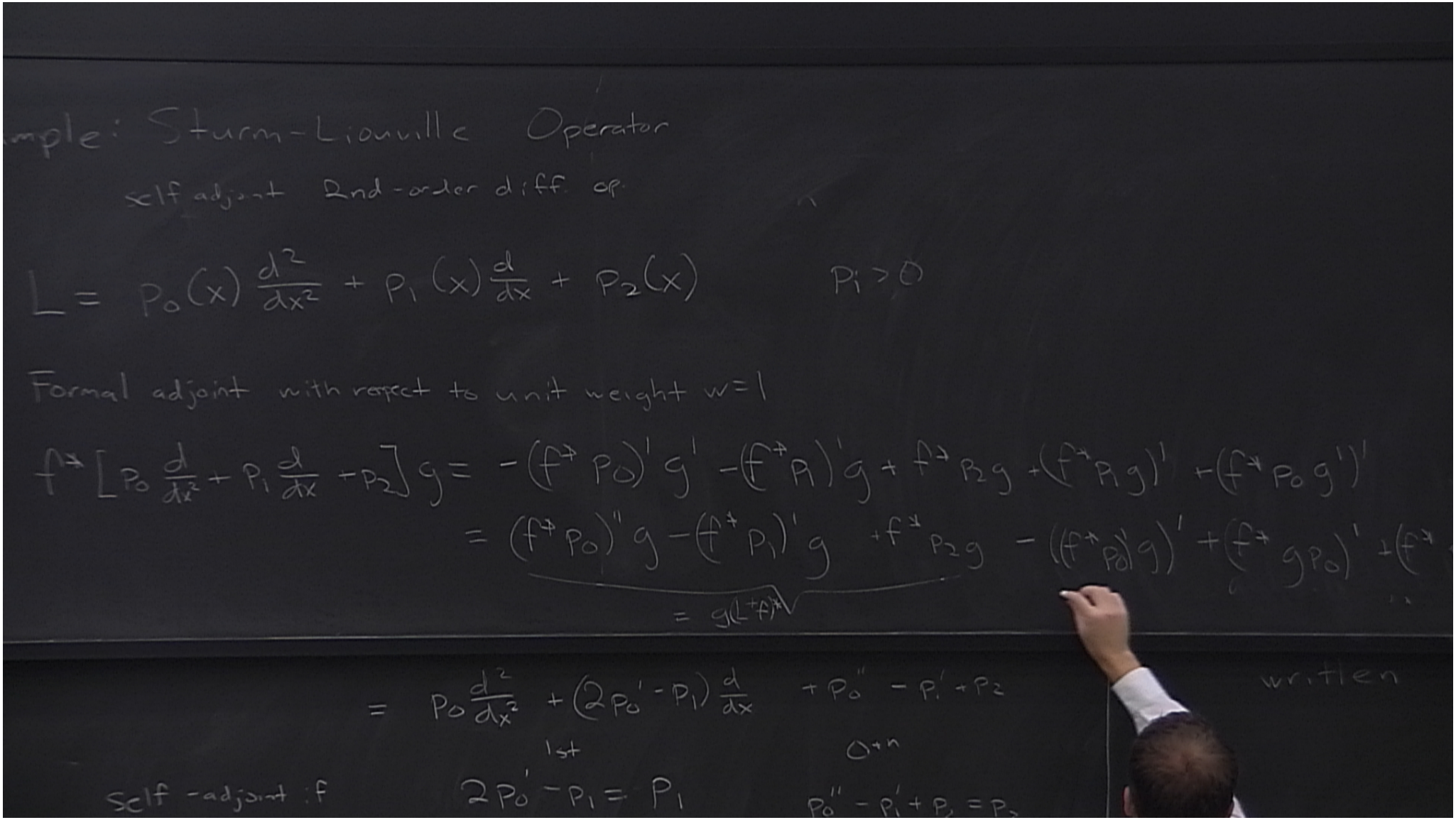

Formal adjoint: $L^+ = \frac{d^2}{dx^2} p_0 - \frac{d}{dx} p_1 + p_2$
$= p_0 \frac{d^2}{dx^2} + (2p_0' - p_1) \frac{d}{dx} + p_0'' - p_1' + p_2$

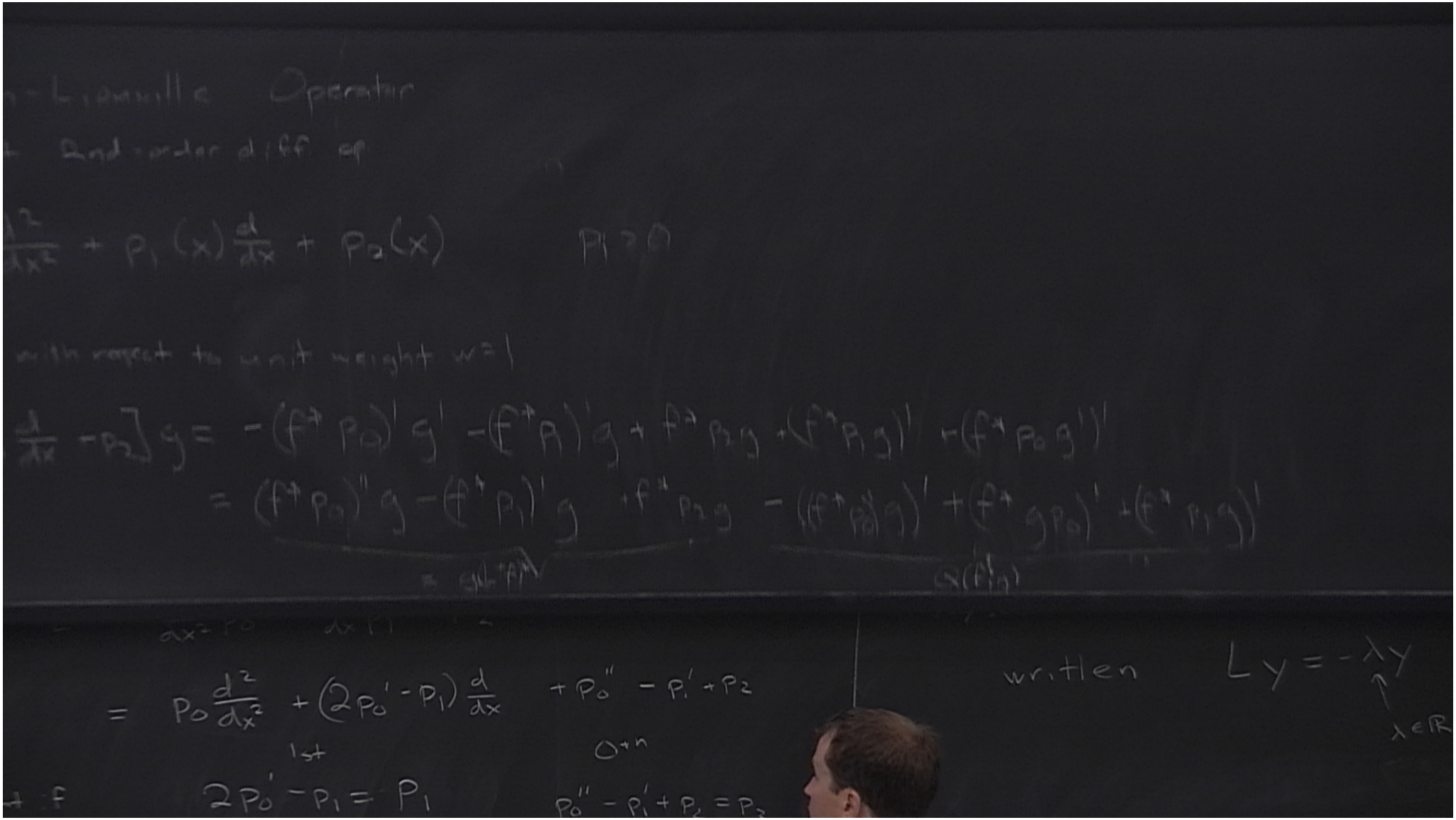

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Self-adjoint if
1st
$2p_0' - p_1 = p_1$
$p_0' = p_1$
0th
$p_0'' - p_1' + p_2 = p_2$
$p_0'' = p_1'$
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more generally $L = L^+$ if $L = \frac{1}{w}\left(\frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_2\right)$

Any second-order diff. eq. w/ real coefficient can be written $Ly = -\lambda y$ where $L = L^{+}$


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mple: Sturm-Liouville Operator
self adjoint 2nd-order diff. op.
$L = p_0(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$ $p_i > 0$
Formal adjoint with respect to unit weight w=1
$f^*[p_0\frac{d}{dx^2} + p_1\frac{d}{dx} - p_2]g = -(f^*p_0)'g' - (f^*p_1)'g + f^*p_2 g + (f^*p_1 g)' + (f^*p_0 g')'$
$= (f^*p_0)''g - (f^*p_1)'g + f^*p_2 g - ((f^*p_0)'g)' + (f^*gp_0)' + (f^*$
$= g(L^*f)^*$
written
$= p_0\frac{d^2}{dx^2} + (2p_0' - p_1)\frac{d}{dx} + p_0'' - p_1' + p_2$
1st
0th
self-adjoint: f
$2p_0' - p_1 = p_1$
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Liouville Operator
2nd order diff op
with respect to unit weight w=1
written
$Ly = -\lambda y$
$\lambda \in \mathbb{R}$
1st
0th
$2p_0' - p_1 = p_1$
$p_0'' - p_1' + p_2 = p_3$

2nd-order diff. op.

$$\frac{d^2}{dx^2} + P_1(x)\frac{d}{dx} + P_2(x) \qquad P_1 > 0$$

with respect to unit weight $w=1$

$$\left[\frac{d}{dx} - P_2\right] g = -(f^* P_0)' g' - (f^* P_1)' g + f^* P_2 g + (f^* P_1 g)' + (f^* P_0 g')'$$

$$= \underbrace{(f^* P_0)'' g - (f^* P_1)' g + f^* P_2 g}_{= g(L^+ f)^*} - \underbrace{(f^* P_0' g)' + (f^* g P_0)' + (f^* P_1 g)'}_{Q(f,g)}$$

$$L^+ = \frac{d^2}{dx^2} P_0 - \frac{d}{dx} P_1 + P_2$$

$$P_0 \frac{d^2}{dx^2} + (2P_0' - P_1)\frac{d}{dx} + P_0'' - P_1' + P_2$$

1st

0th

$$P_0'' - P_1' + P_2 = P_2$$

Any second-order diff eq. w/ real

written $Ly = -\lambda y$

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More generally $L = L^+$ if $L = \frac{1}{w}\left(\frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_2\right)$

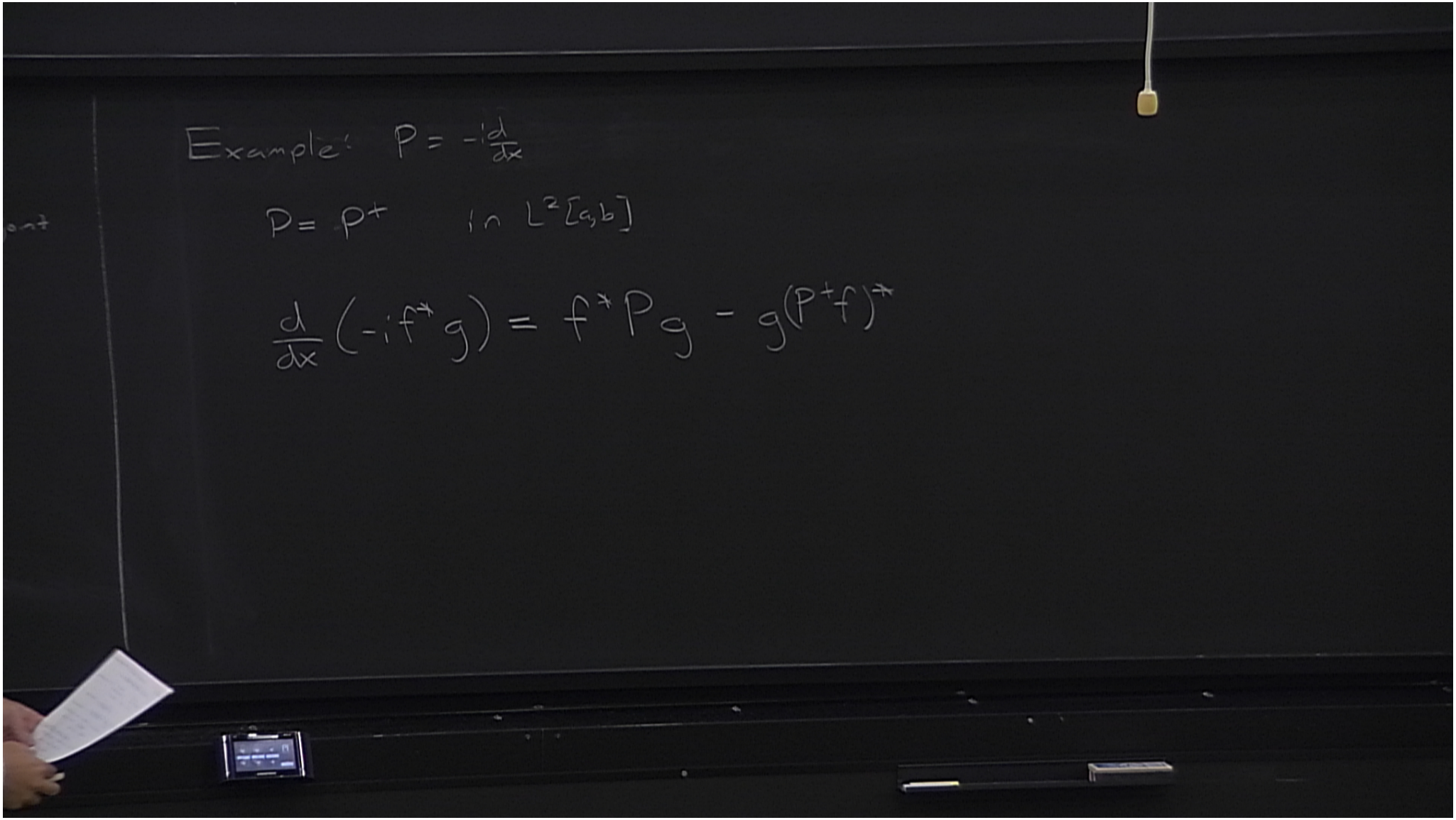
Self-Adjoint Boundary Conditions

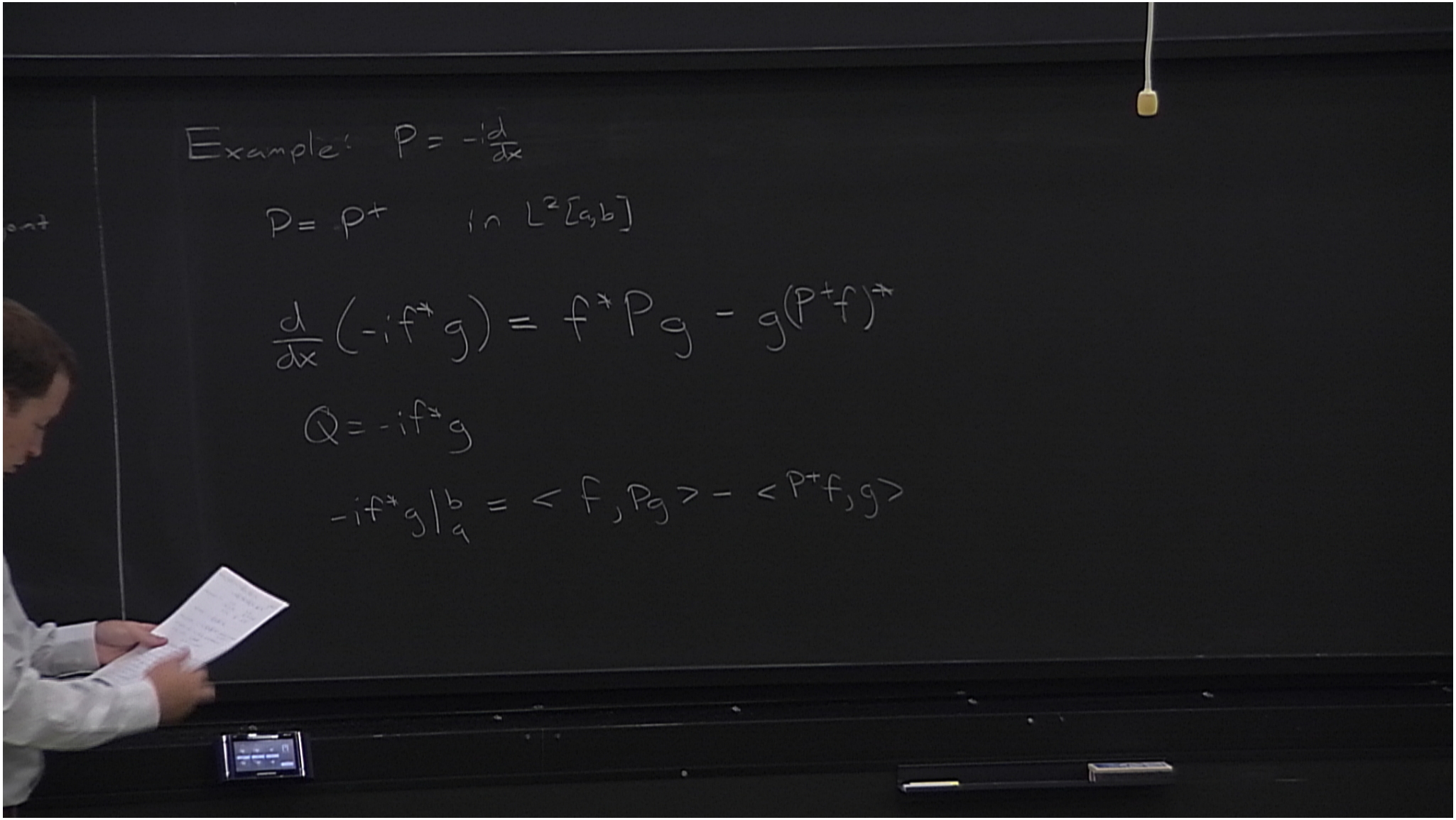
L is truly self-adjoint if it is formally self-adjoint

and we impose boundary cond. such that

- Q on RHS of Lagrange's identity vanishes $Q\Big|_a^b = 0$

- domain of L and L^+ are the same


Example: $P = -i\frac{d}{dx}$
$P = P^+$ in $L^2[a,b]$
$\frac{d}{dx}(-if^*g) = f^*Pg - g(Pf)^*$


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$Q = -if^*g$
$-if^*g\Big|_a^b = \langle f, Pg\rangle - \langle Pf, g\rangle$

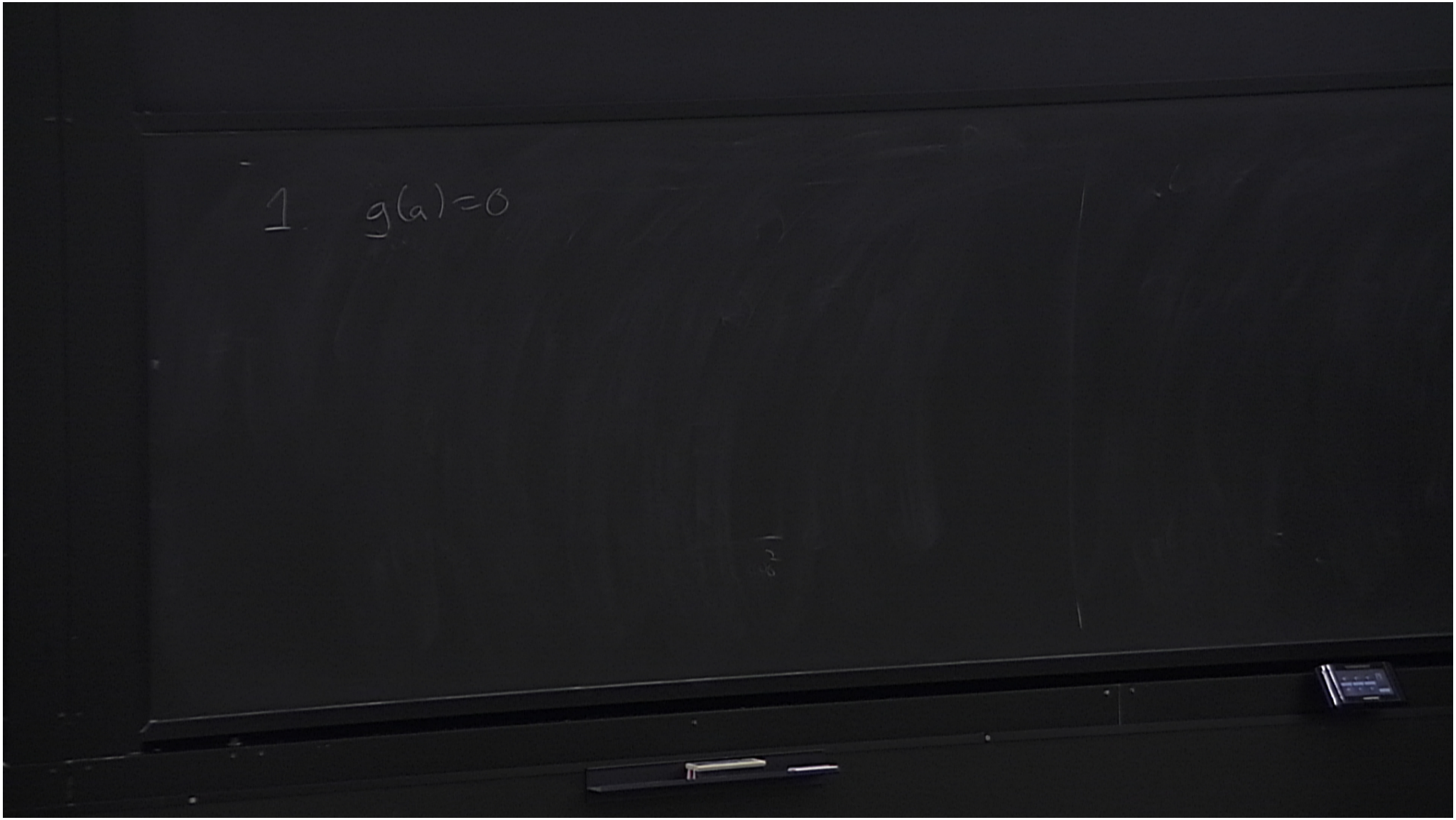
Example: $P = -i\frac{d}{dx}$

$P = P^+$ in $L^2[a,b]$

$$\frac{d}{dx}(-if^*g) = f^*Pg - g(P^+f)^*$$

$Q = -if^*g$

$$-if^*g\Big|_a^b = \langle f, Pg\rangle - \langle P^+f, g\rangle$$


1. g(a)=0

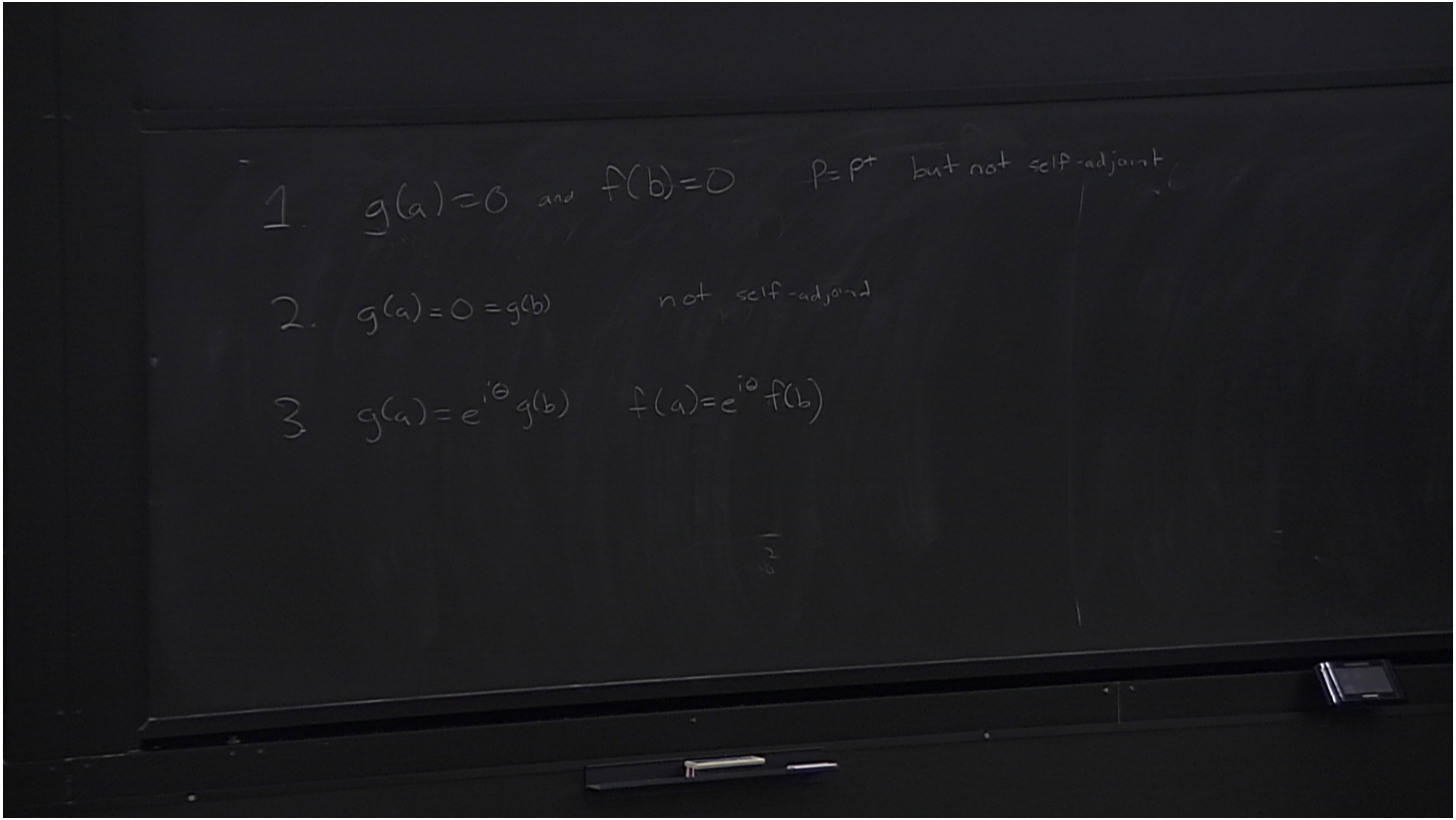
Example: $P = -i\frac{d}{dx}$

$P = P^+$ on $L^2[a,b]$

$$\frac{d}{dx}(-if^*g) = f^*Pg - g(P^+f)^*$$

$Q = -if^*g$

$$\underbrace{-if^*g\Big|_a^b}_{-if^*(b)g(b) + if^*(a)g(a)} = \langle f, Pg\rangle - \langle P^+f, g\rangle$$


1. $g(a)=0$ and $f(b)=0$ $P=P^\dagger$ but not self-adjoint
2. $g(a)=0=g(b)$ not self-adjoint
3 $g(a)=e^{i\theta}g(b)$ $f(a)=e^{i\theta}f(b)$

Example: $P = -i\frac{d}{dx}$

adjoint

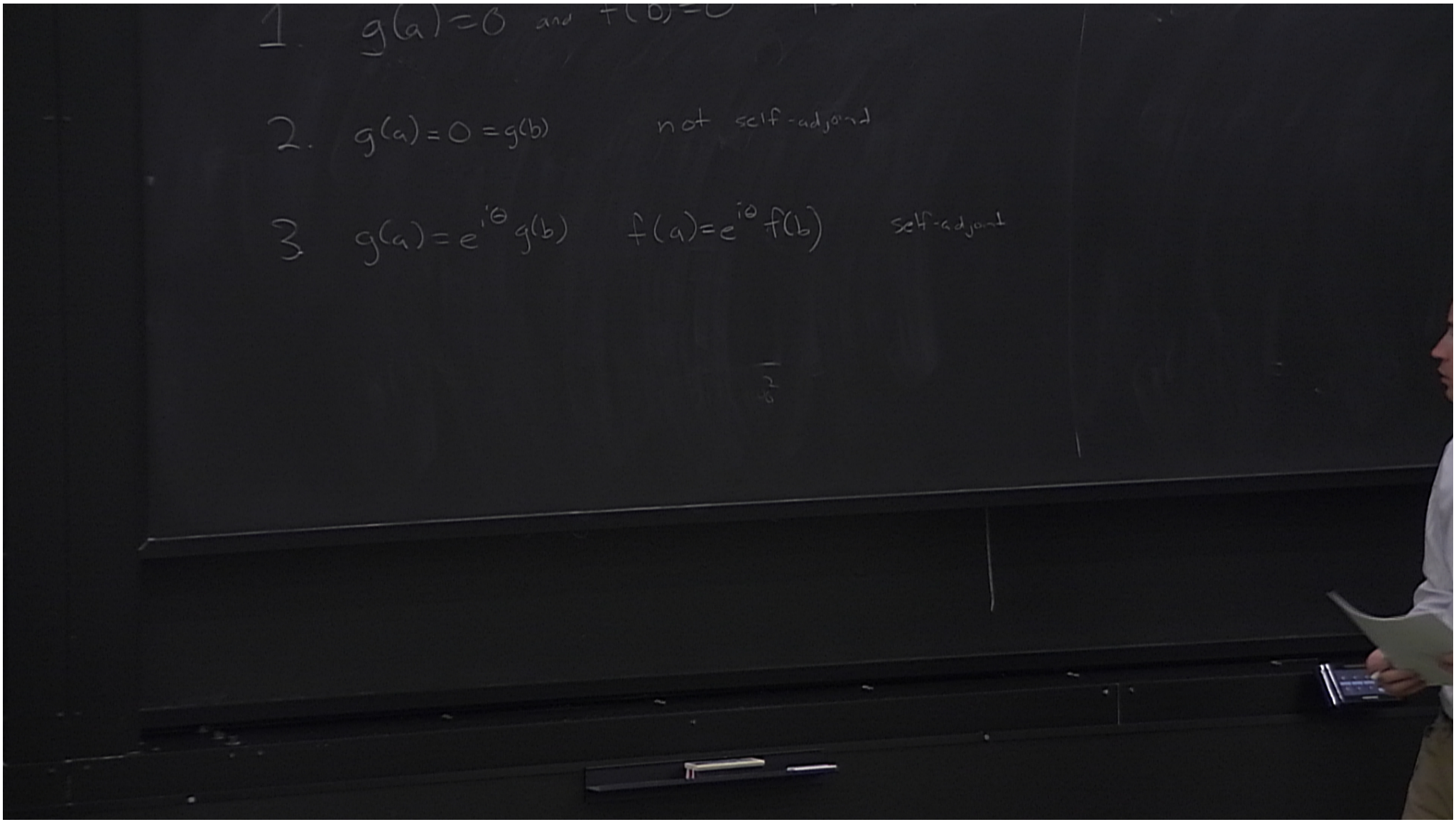
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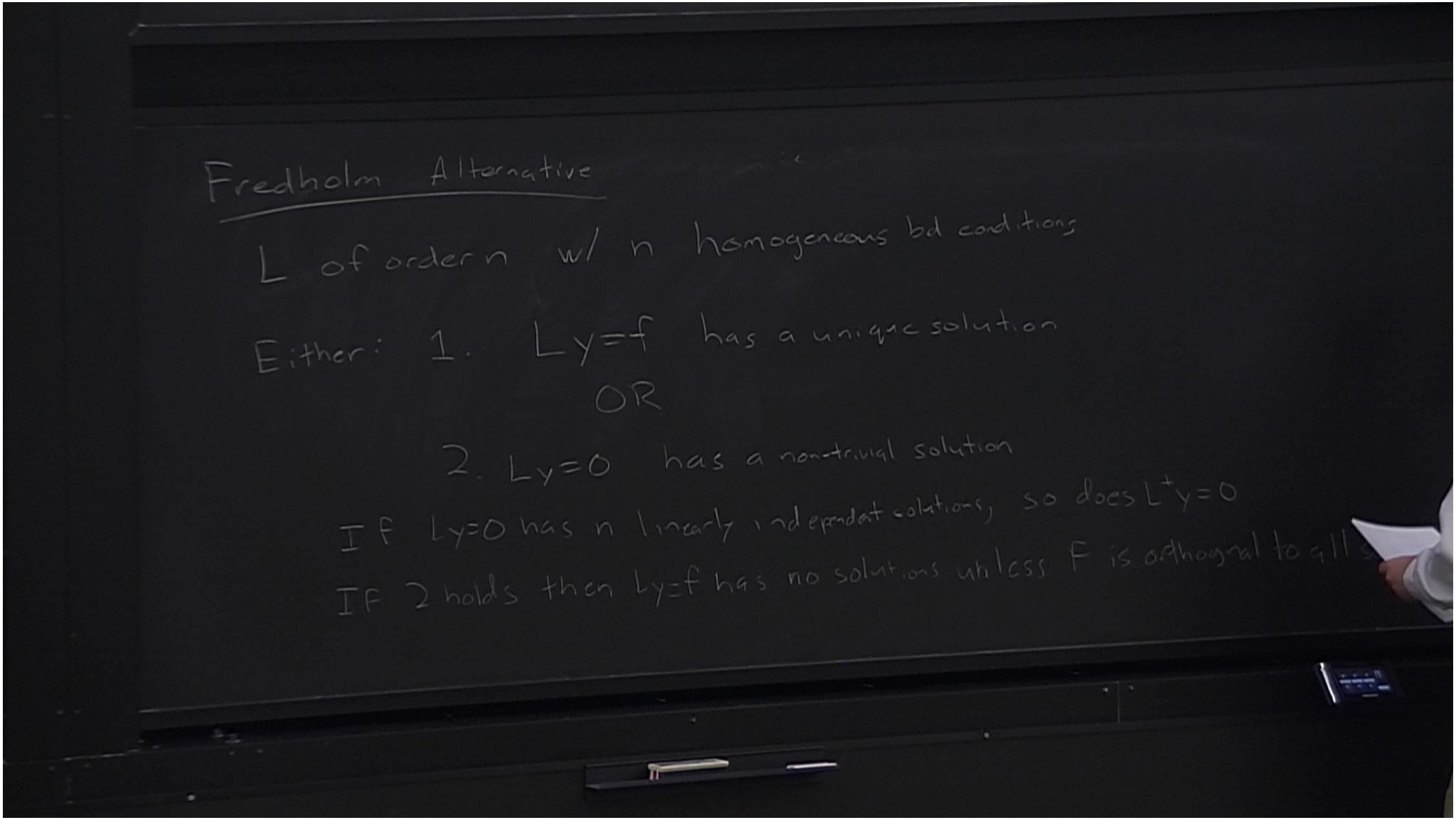
$$\frac{d}{dx}(-if^*g) = f^*Pg - g(P^+f)^*$$

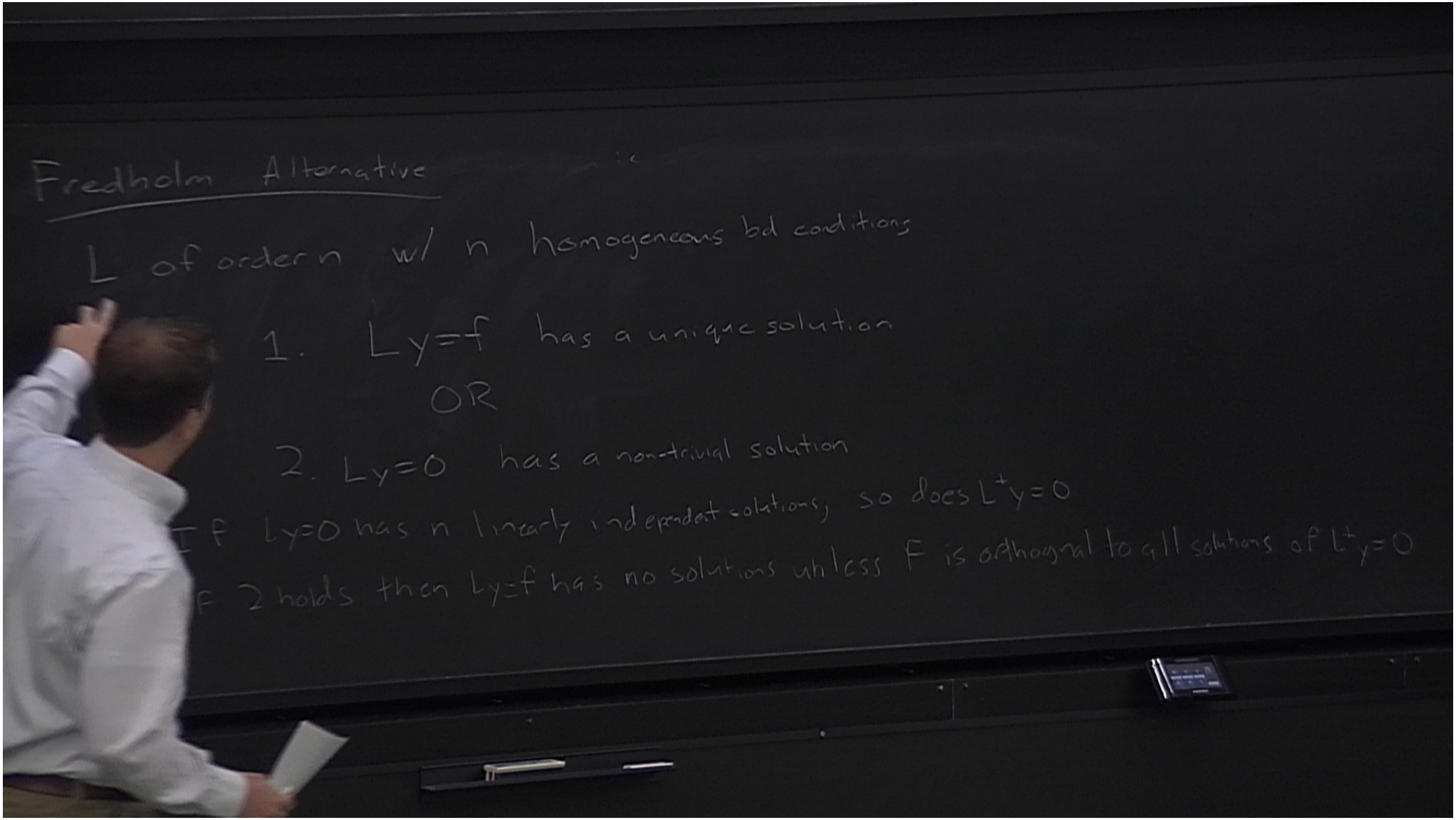
$Q = -if^*g$

$$\underbrace{-if^*g\Big|_a^b}_{} = \langle f, Pg\rangle - \langle P^+f, g\rangle$$

$-if^*(b)g(b) + if^*(a)g(a)$


1. g(a)=0 and
2. g(a)=0=g(b) not self-adjoint
3 g(a)=e^{iθ}g(b) f(a)=e^{iθ}f(b) self-adjoint


Fredholm Alternative
L of order n w/ n homogeneous bd conditions
Either: 1. Ly=f has a unique solution
OR
2. Ly=0 has a non-trivial solution
If Ly=0 has n linearly independent solutions, so does L†y=0
If 2 holds then Ly=f has no solutions unless F is orthogonal to all


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