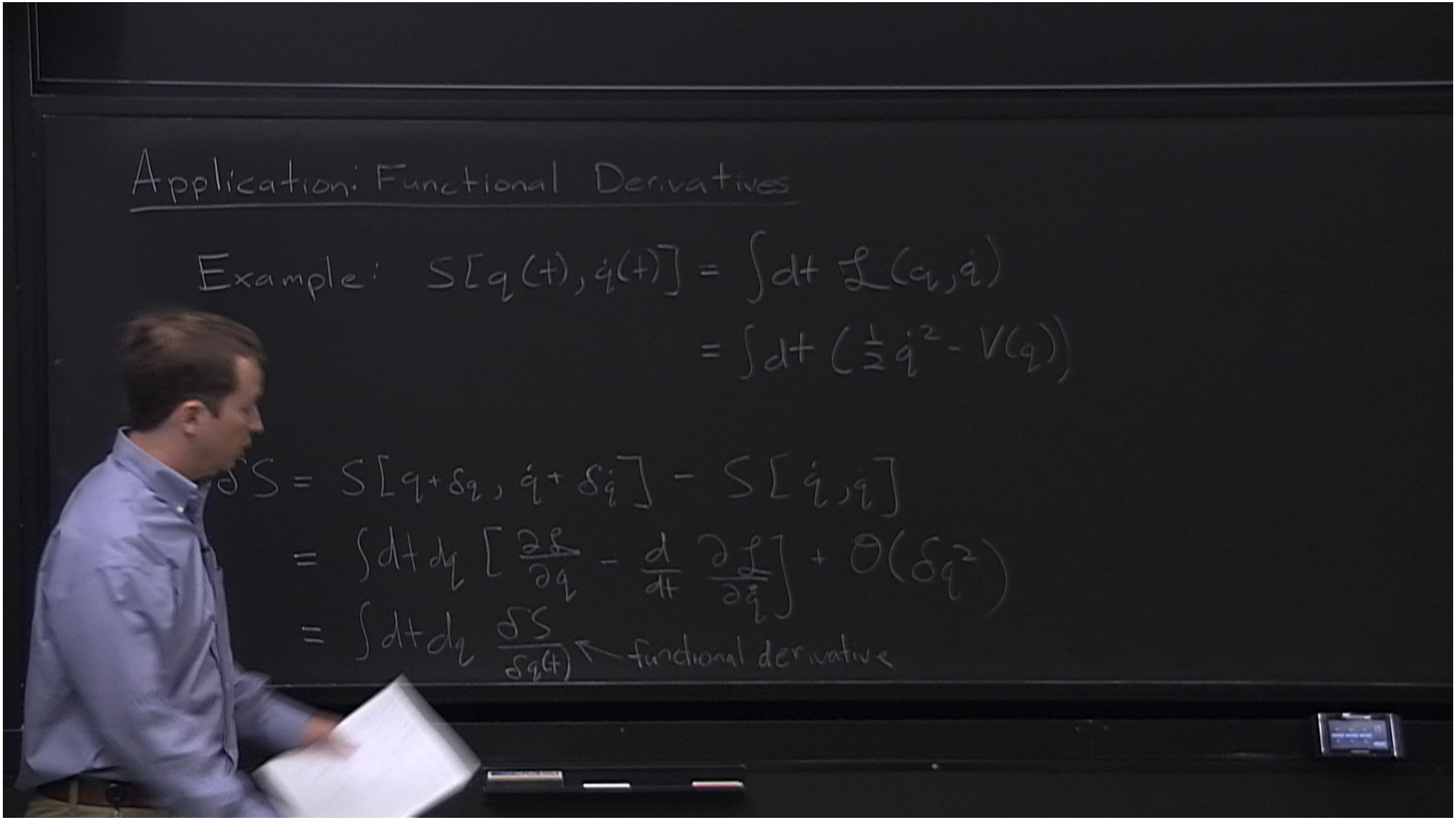
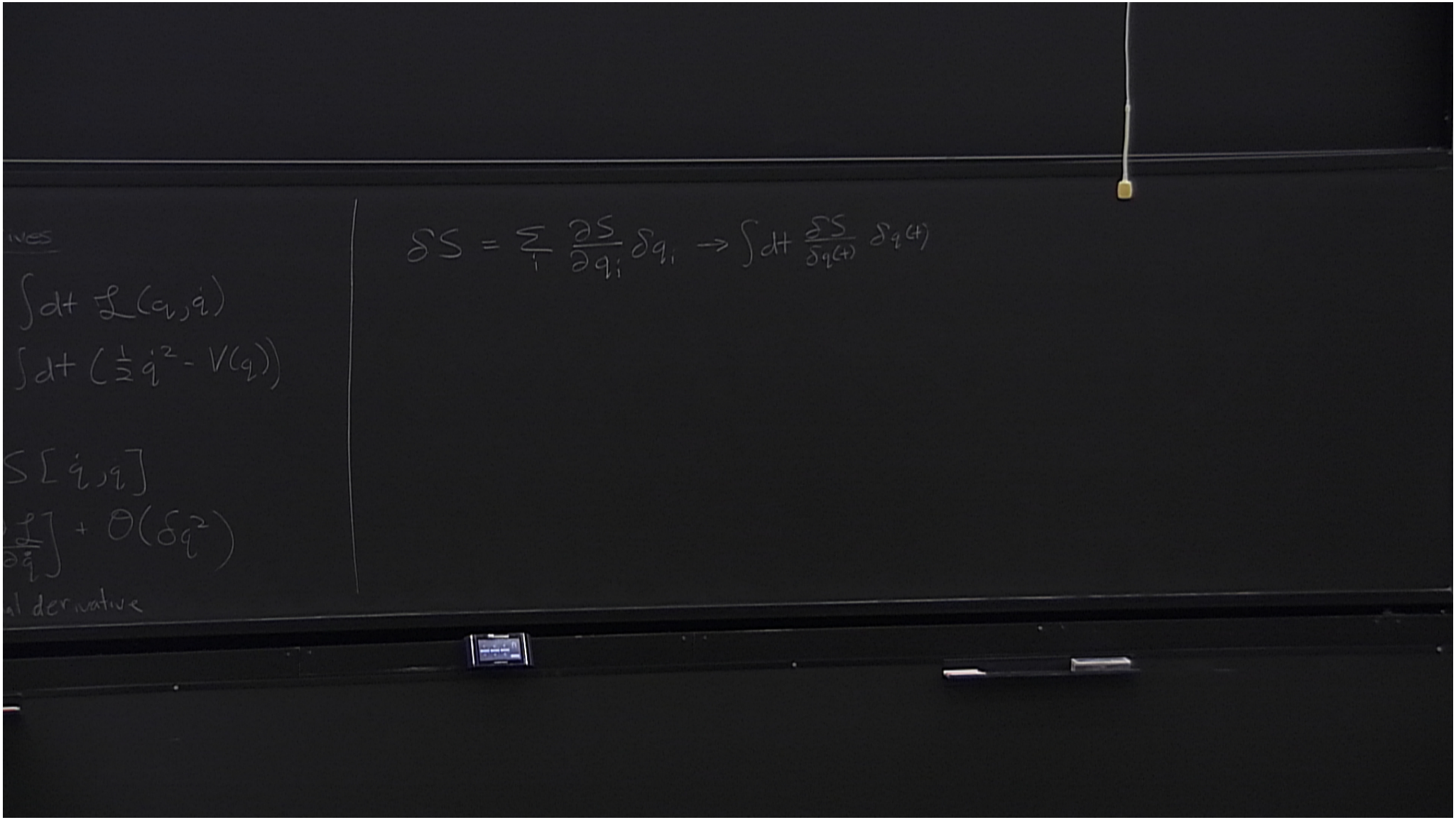
Title: Front End - Functions, "Functions", etc. - 3

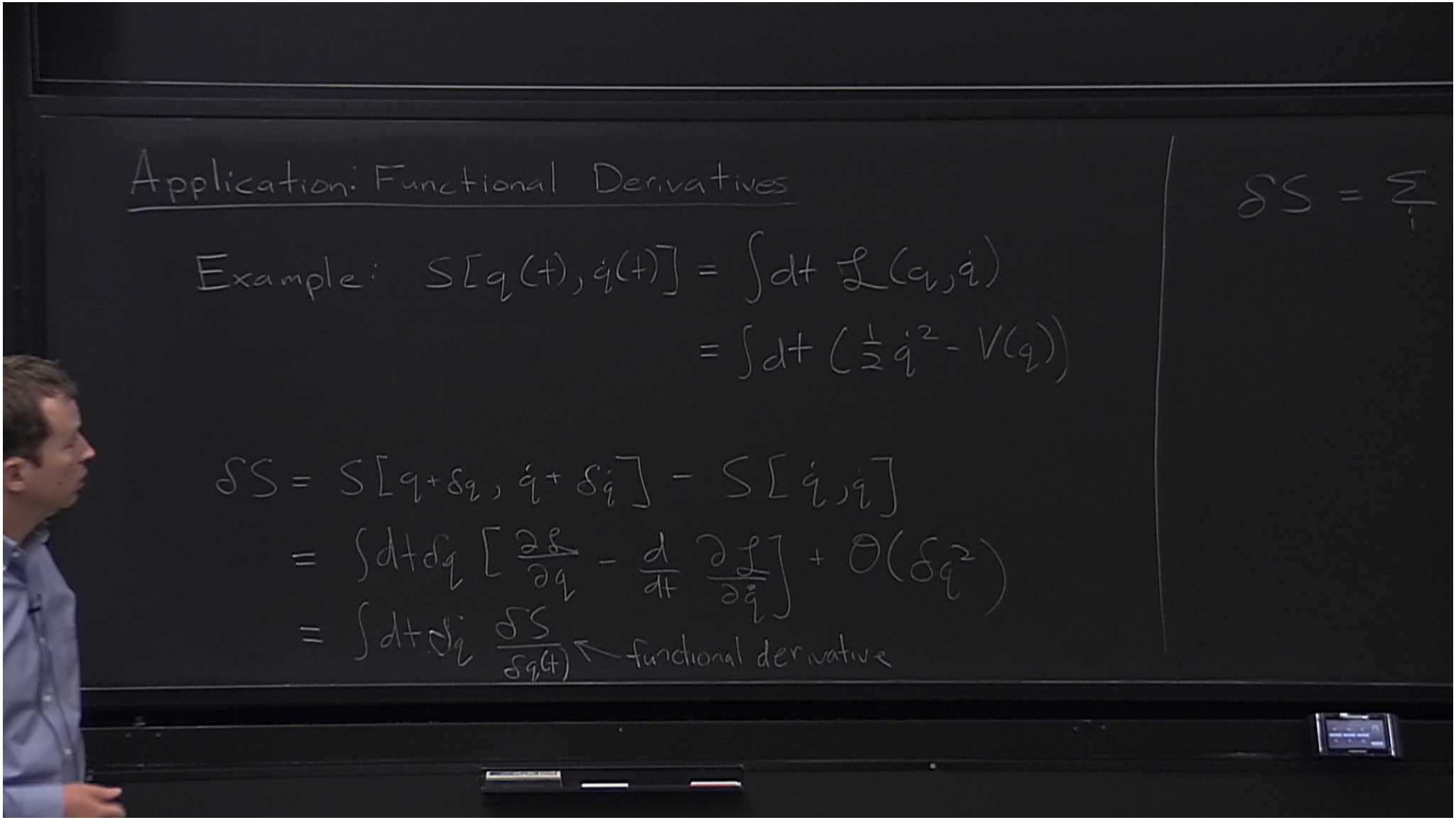
Date: Aug 25, 2016 09:00 AM

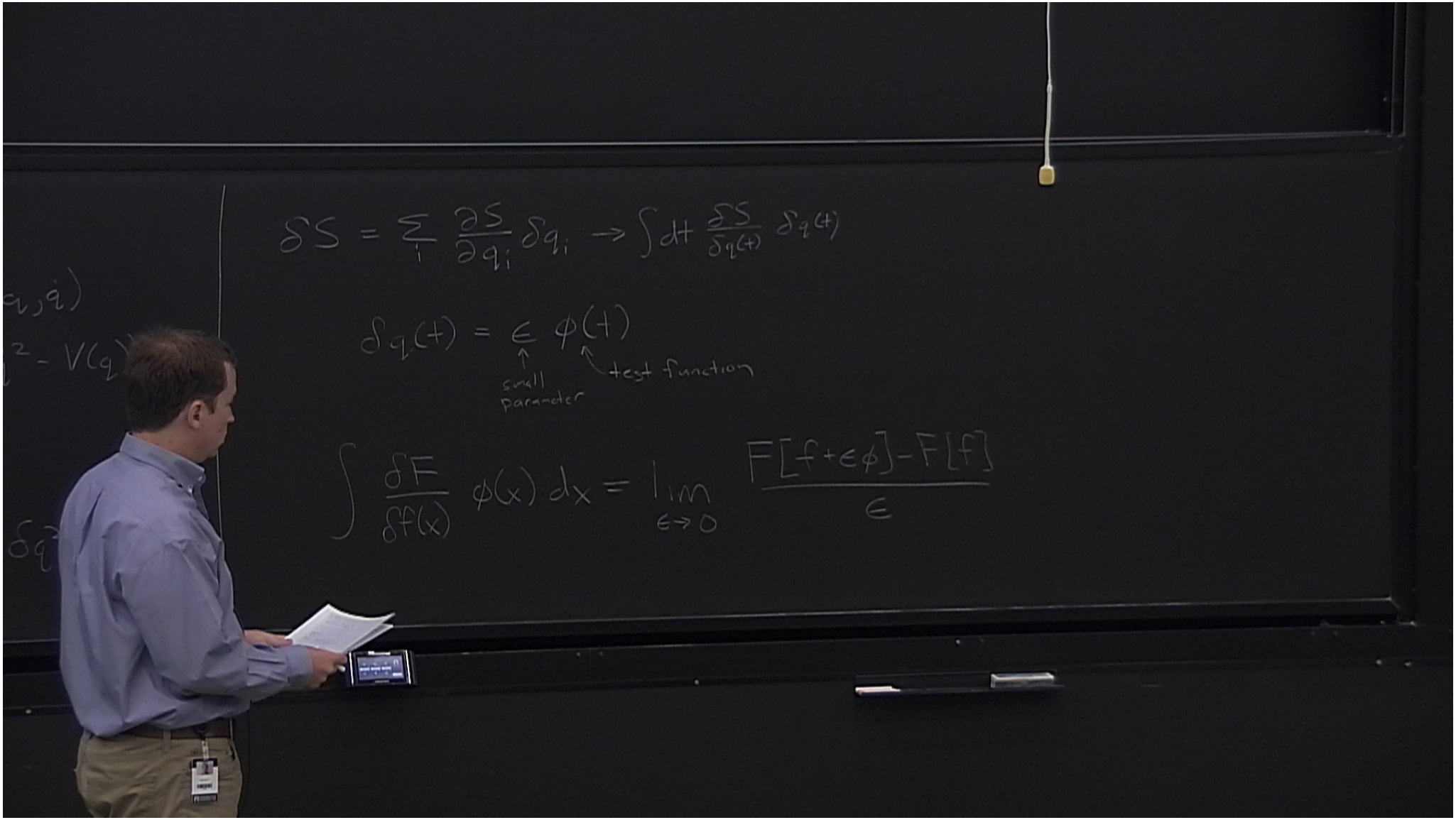
URL: http://pirsa.org/16080077

Abstract:


Application: Functional Derivatives
Example: $S[q(t), \dot{q}(t)] = \int dt\, \mathcal{L}(q, \dot{q})$
$= \int dt \left(\frac{1}{2}\dot{q}^2 - V(q)\right)$
$\delta S = S[q+\delta q, \dot{q}+\delta\dot{q}] - S[q, \dot{q}]$
$= \int dt\, dq \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] + O(\delta q^2)$
$= \int dt\, dq \frac{\delta S}{\delta q(t)}$
functional derivative


ives
$\int dt\, \mathcal{L}(q, \dot{q})$
$\int dt \left(\frac{1}{2}\dot{q}^2 - V(q)\right)$
$S[\dot{q}, q]$
$\frac{\partial}{\partial \dot{q}}] + \Theta(\delta q^2)$
al derivative
$\delta S = \sum_i \frac{\partial S}{\partial q_i} \delta q_i \to \int dt \frac{\delta S}{\delta q(t)} \delta q(t)$


Application: Functional Derivatives
Example: $S[q(t), \dot{q}(t)] = \int dt\, \mathcal{L}(q, \dot{q})$
$= \int dt \left(\frac{1}{2}\dot{q}^2 - V(q)\right)$
$\delta S = S[q+\delta q, \dot{q}+\delta\dot{q}] - S[q, \dot{q}]$
$= \int dt\, \delta q \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] + O(\delta q^2)$
$= \int dt\, \delta q \frac{\delta S}{\delta q(t)}$
functional derivative
$\delta S = \sum_i$


$\delta S = \sum_i \frac{\partial S}{\partial q_i} \delta q_i \to \int dt \frac{\delta S}{\delta q(t)} \delta q(t)$
$\delta q(t) = \epsilon \phi(t)$
small parameter
test function
$\int \frac{\delta F}{\delta f(x)} \phi(x) dx = \lim_{\epsilon \to 0} \frac{F[f+\epsilon\phi] - F[f]}{\epsilon}$

Example: $f^2(x_0) = F[f]$

$$\int \frac{\delta F}{\delta f(x)} \phi(x)\, dx = \lim_{\epsilon \to 0} \frac{f^2(x_0) + 2\epsilon f(x_0)\phi(x_0) + \epsilon^2 \phi^2(x_0) - f^2(x_0)}{\epsilon}$$

($\epsilon^2 \phi^2(x_0) \to 0$ as $\epsilon \to 0$)

$$= 2 f(x_0)\phi(x_0)$$

$$= \int 2 f(x_0)\, \delta(x - x_0)\, \phi(x)\, dx$$

$$\frac{\delta F}{\delta f(x)} = 2 f(x_0)\, \delta(x - x_0)$$

$\to 0$ as $\epsilon \to 0$

$\epsilon^2 \phi^2(x_0) - f^2(x_0)$

Properties:

Linearity: $\dfrac{\delta(\lambda F + \mu G)}{\delta f(x)} = \lambda \dfrac{\delta F}{\delta f(x)} + \mu \dfrac{\delta G}{\delta f(x)}$

Product Rule: $\dfrac{\delta(FG)}{\delta f(x)} = \dfrac{\delta F}{\delta f(x)} G + F \dfrac{\delta G}{\delta f(x)}$

Chain Rules: $\dfrac{\delta F[g(f)]}{\delta f(x)} = \dfrac{\delta F[g(f)]}{\delta g(f(x))} \dfrac{d g(f(x))}{d f(x)}$

$\dfrac{\delta g(F[f])}{\delta f(x)} = \dfrac{d g(F[f])}{\delta F[f]} \dfrac{\delta F[f]}{\delta f(x)}$

$\delta f(x)$

functional derivative

$\epsilon \to 0$ $\quad \epsilon$

$\phi^2(x_0) - f^2(x_0)$

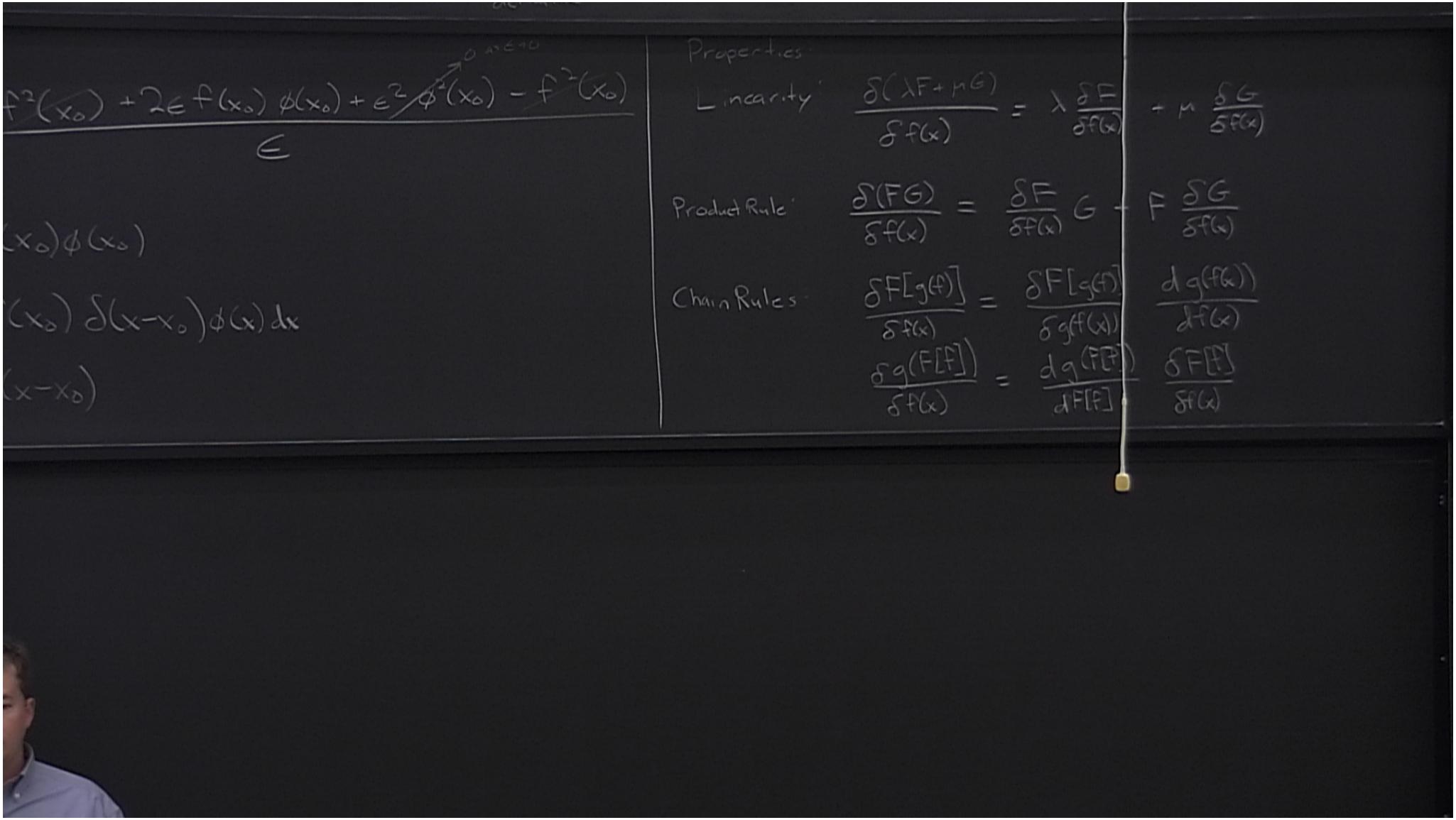
Properties:

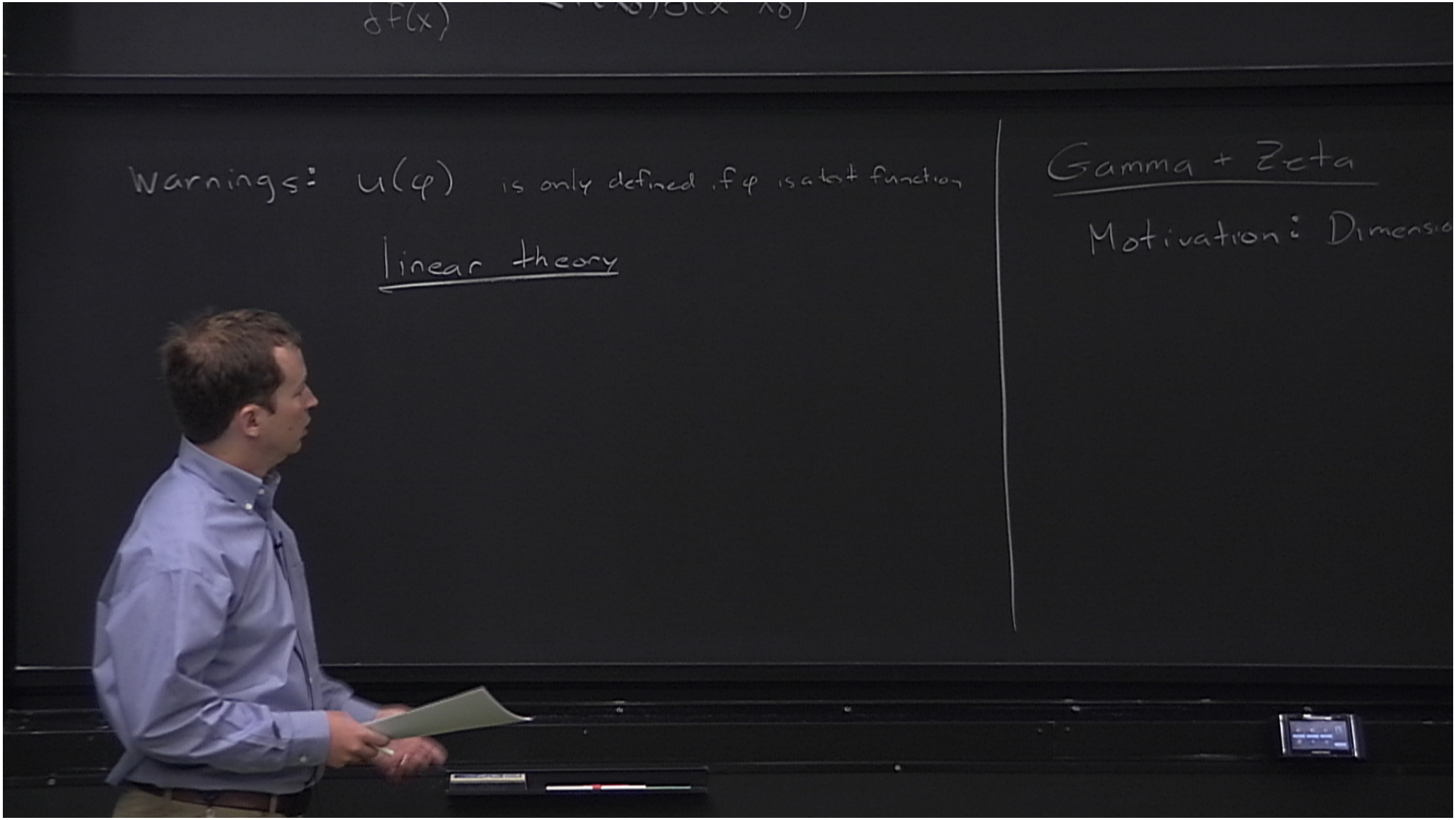
Linearity: $\dfrac{\delta(\lambda F + \mu G)}{\delta f(x)} = \lambda \dfrac{\delta F}{\delta f(x)} + \mu \dfrac{\delta G}{\delta f(x)}$

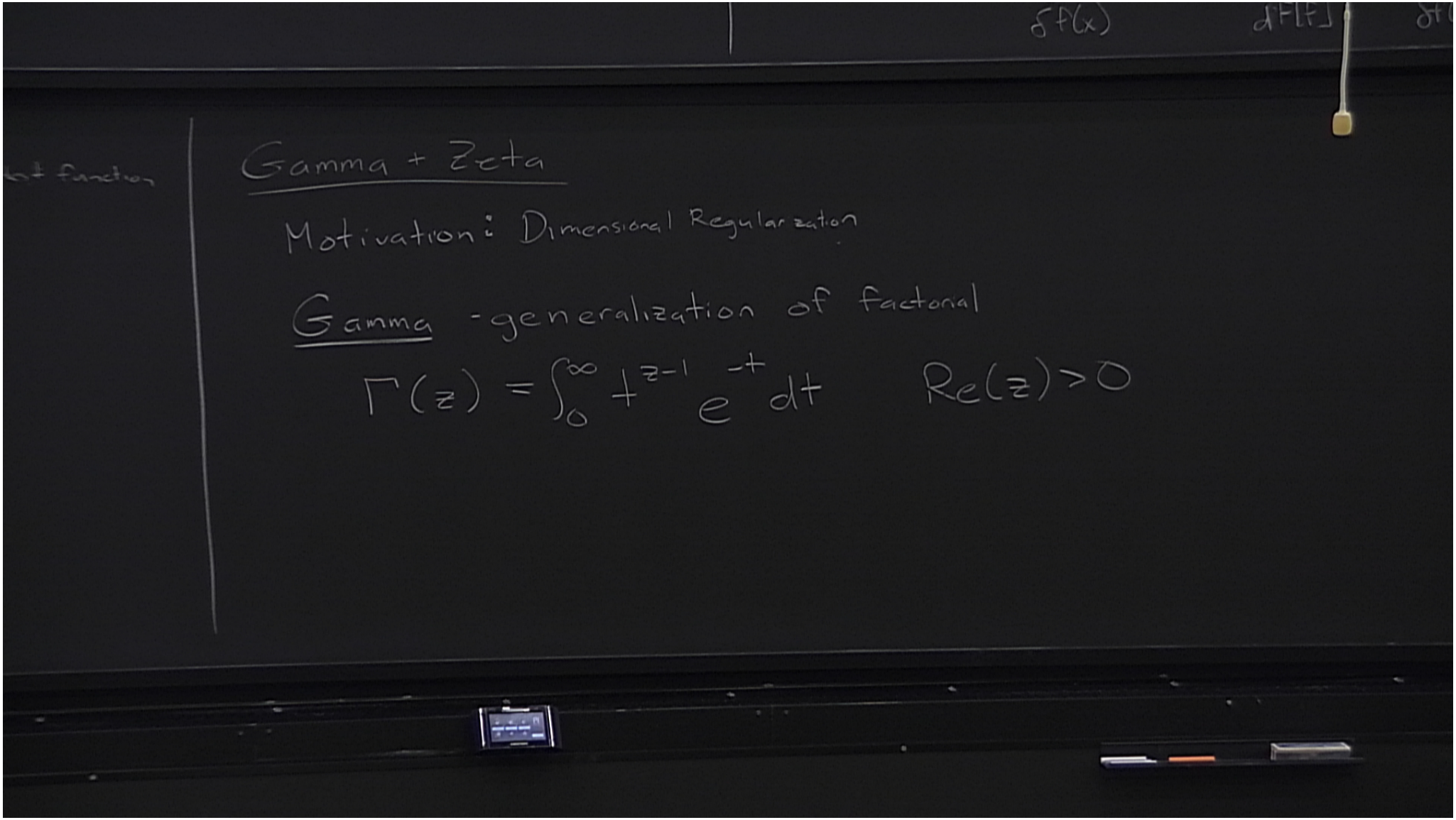
Product Rule: $\dfrac{\delta(FG)}{\delta f(x)} = \dfrac{\delta F}{\delta f(x)} G + F \dfrac{\delta G}{\delta f(x)}$

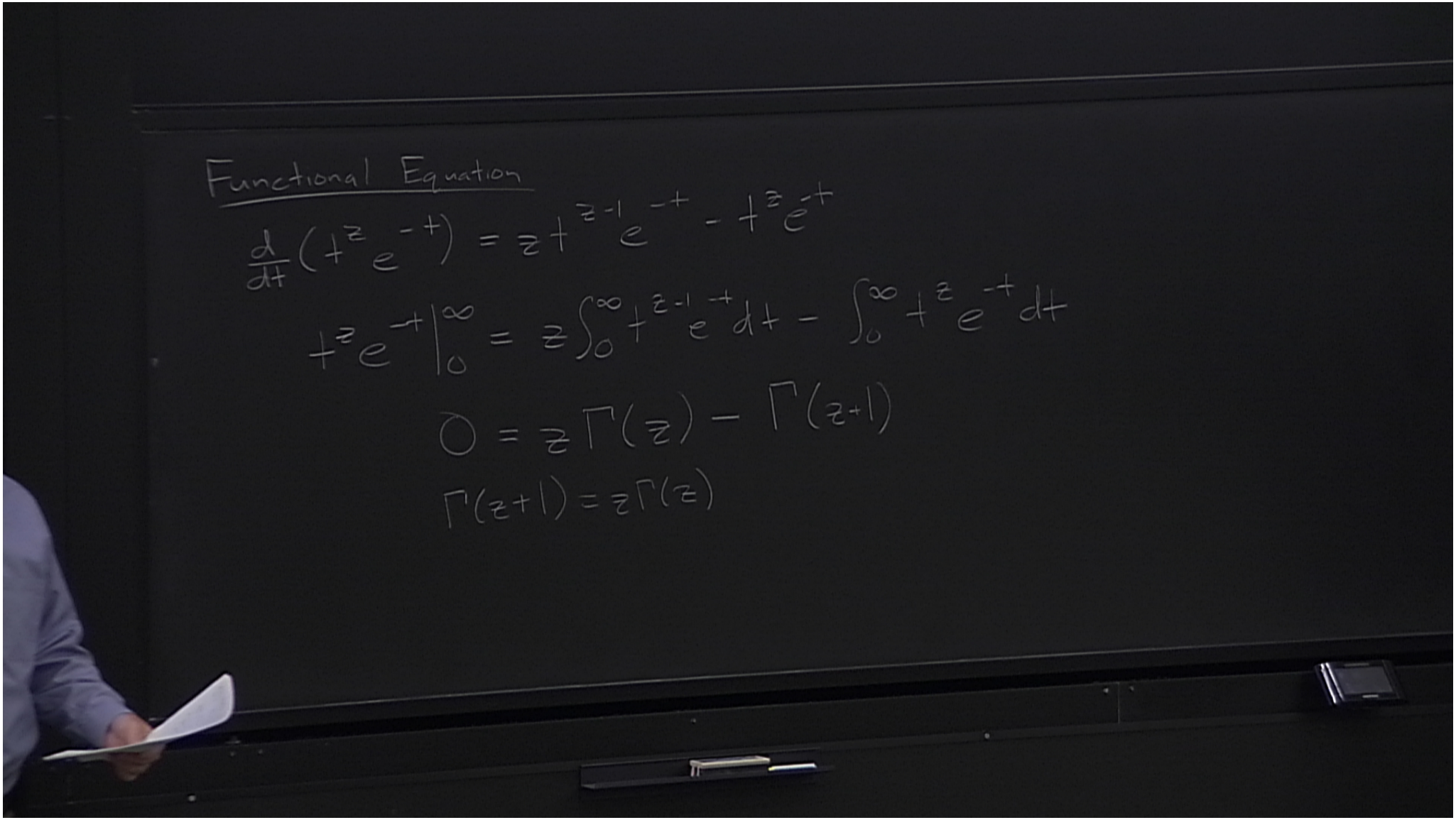
Chain Rules: $\dfrac{\delta F[g(f)]}{\delta f(x)} = \dfrac{\delta F[g(f)]}{\delta g(f(x))} \dfrac{d g(f(x))}{d f(x)}$

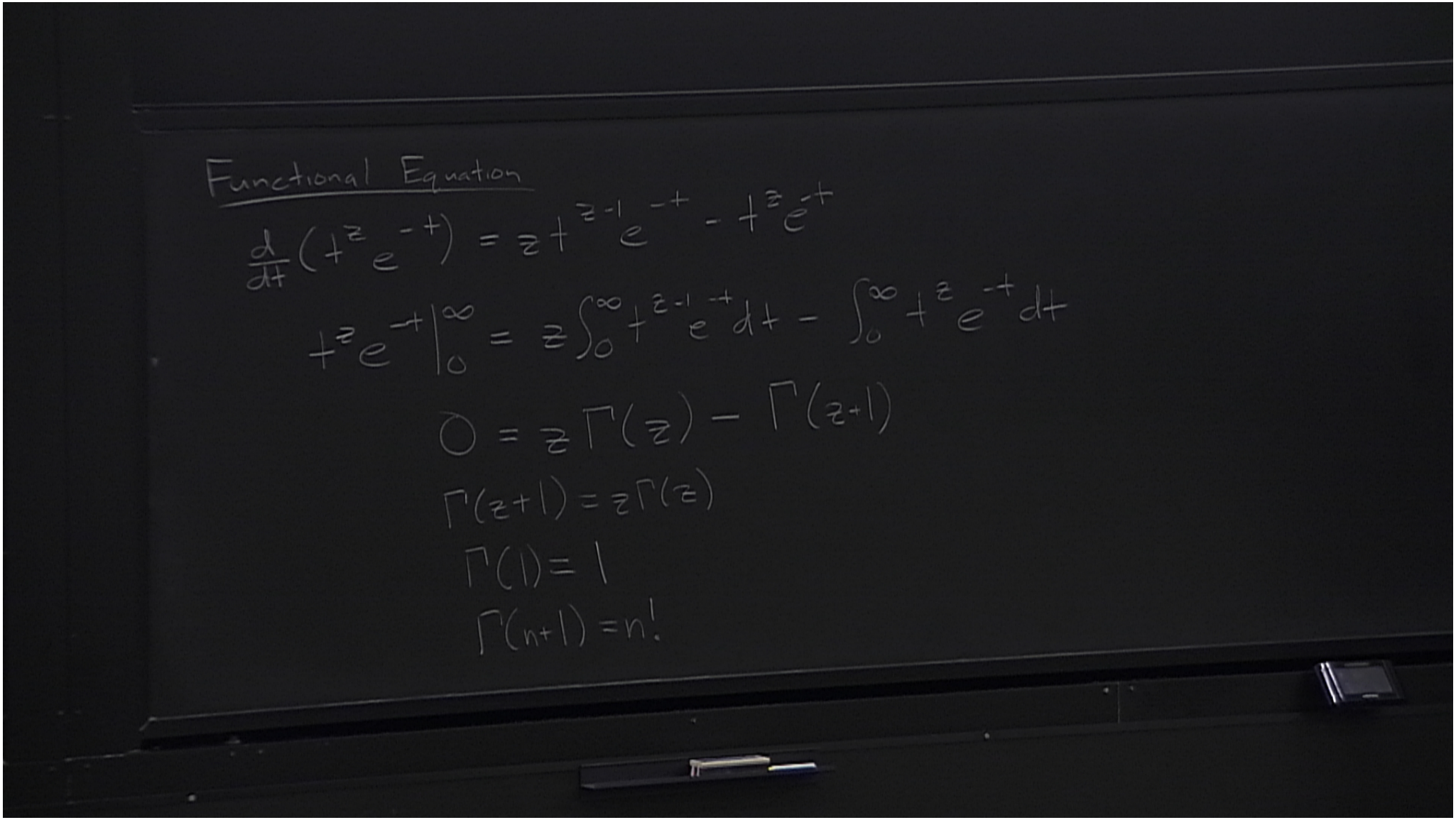
$\dfrac{\delta g(F[f])}{\delta f(x)} = \dfrac{d g(F[f])}{\delta F[f]} \dfrac{\delta F[f]}{\delta f(x)}$


$f^2(x_0) + 2\epsilon f(x_0)\phi(x_0) + \epsilon^2\phi^2(x_0) - f^2(x_0)$
ϵ
$\epsilon \to 0$
$(x_0)\phi(x_0)$
$(x_0)\,\delta(x-x_0)\phi(x)\,dx$
$(x-x_0)$
Properties:
Linearity: $\frac{\delta(\lambda F+\mu G)}{\delta f(x)} = \lambda\frac{\delta F}{\delta f(x)} + \mu\frac{\delta G}{\delta f(x)}$
Product Rule: $\frac{\delta(FG)}{\delta f(x)} = \frac{\delta F}{\delta f(x)}G + F\frac{\delta G}{\delta f(x)}$
Chain Rules: $\frac{\delta F[g(f)]}{\delta f(x)} = \frac{\delta F[g(f)]}{\delta g(f(x))}\frac{d\,g(f(x))}{d f(x)}$
$\frac{\delta g(F[f])}{\delta f(x)} = \frac{d\,g(F[f])}{d F[f]}\frac{\delta F[f]}{\delta f(x)}$


δf(x)
Warnings: u(φ) is only defined if φ is a test function
linear theory
Gamma + Zeta
Motivation: Dimensio


δf(x)
d[f[t]
Gamma + Zeta
Motivation: Dimensional Regularization
Gamma - generalization of factorial
$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad Re(z) > 0$


Functional Equation
$\frac{d}{dt}(t^z e^{-t}) = z t^{z-1} e^{-t} - t^z e^{-t}$
$t^z e^{-t}\Big|_0^\infty = z\int_0^\infty t^{z-1} e^{-t}\,dt - \int_0^\infty t^z e^{-t}\,dt$
$0 = z\Gamma(z) - \Gamma(z+1)$
$\Gamma(z+1) = z\Gamma(z)$


Functional Equation
$\frac{d}{dt}(t^z e^{-t}) = z t^{z-1} e^{-t} - t^z e^{-t}$
$t^z e^{-t}\Big|_0^\infty = z\int_0^\infty t^{z-1} e^{-t}\,dt - \int_0^\infty t^z e^{-t}\,dt$
$0 = z\Gamma(z) - \Gamma(z+1)$
$\Gamma(z+1) = z\Gamma(z)$
$\Gamma(1) = 1$
$\Gamma(n+1) = n!$

Extend to $Re(z) \leq 0$ with functional equation

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)} \qquad Re(z)+n > 0$$

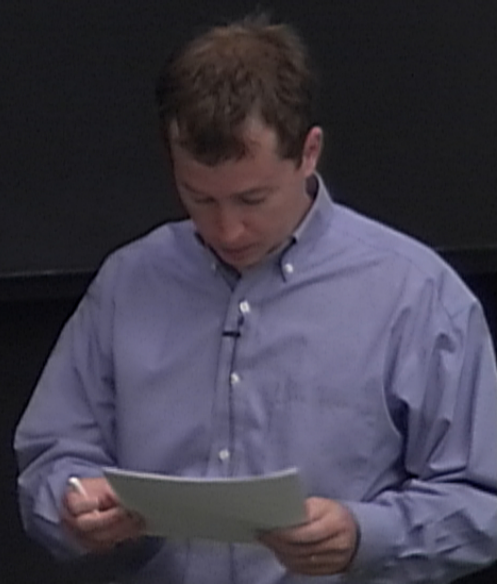
$$z = -n+1+\epsilon$$

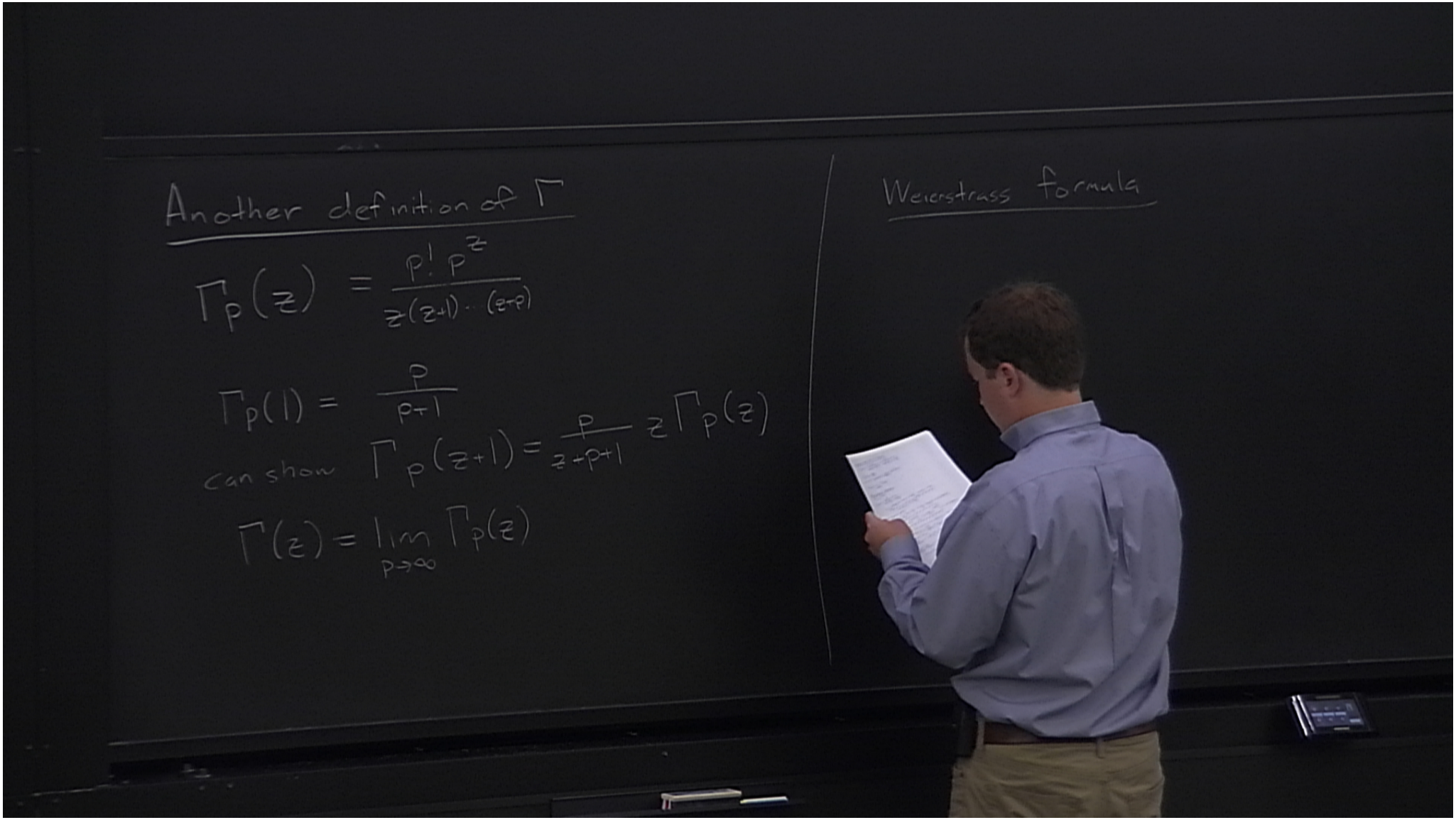
$$\Gamma(z) = \frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)\cdots(\epsilon-n+1)}$$

$$\underset{\epsilon \to 0}{\sim} \frac{(-1)^{n+1}}{(n-1)!\,\epsilon}$$

$\Gamma(z)$ has simple poles at $z = -n$

with residue $\frac{(-1)^n}{n!}$




Another definition of Γ
$\Gamma_p(z) = \frac{p!\, p^z}{z(z+1)\cdots(z+p)}$
$\Gamma_p(1) = \frac{p}{p+1}$
Can show $\Gamma_p(z+1) = \frac{p}{z+p+1} z\,\Gamma_p(z)$
$\Gamma(z) = \lim_{p\to\infty} \Gamma_p(z)$
Weierstrass formula

Weierstrass formula

$$\Gamma_p(z) = \frac{p^z}{z(1+\frac{z}{1})\cdots(1+\frac{z}{p})} \qquad p^z = e^{z\log p}$$

$$= \frac{e^{z(\log p - 1 - 1/2 \cdots - 1/p)}\; e^{z + z/2 + \cdots + z/p}}{z(1+z)\cdots\left(1+\frac{z}{p}\right)}$$

$$\gamma = \lim_{p\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{p}-\log p\right) \approx 0.577$$

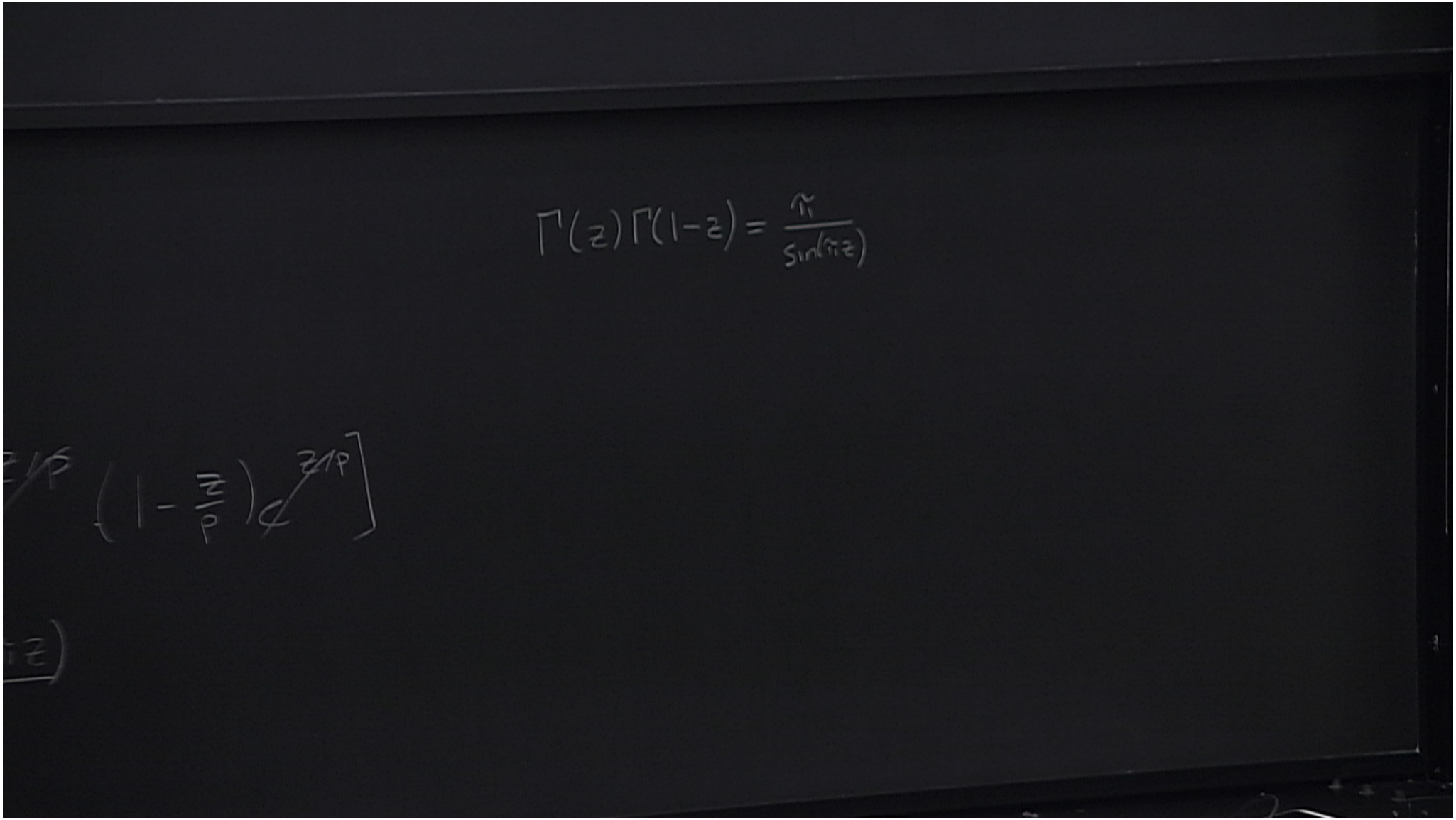
$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{p=1}^{\infty} \left(1 + \frac{z}{p}\right) e^{-z/p}$$

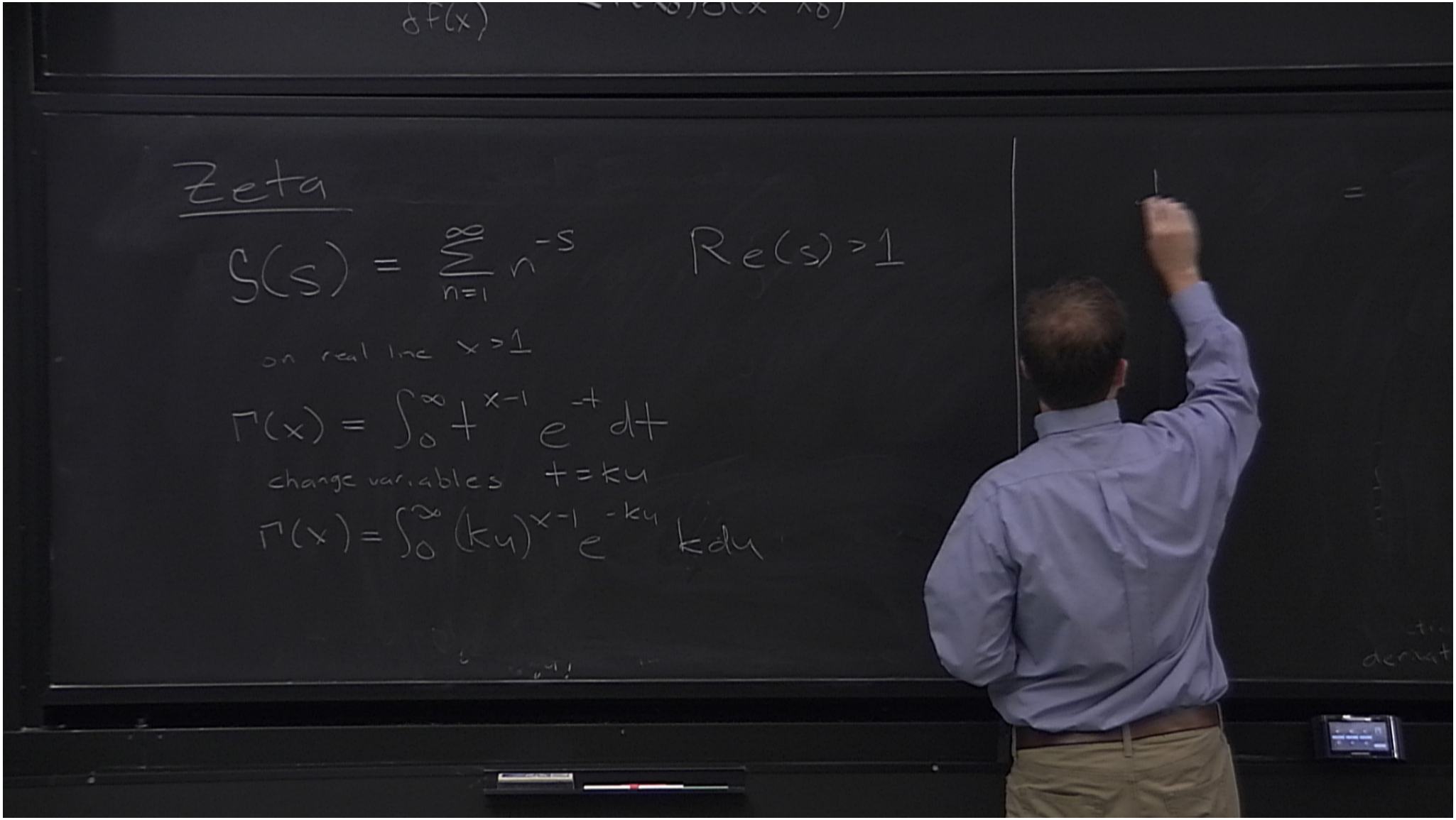
→ $\Gamma(z)$ has no zeros only poles at $z = -n$

Duplication formula: $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2z-1}}\Gamma(2z)$

complement formula: $\frac{1}{\Gamma(z)}\frac{1}{\Gamma(-z)} = -z^2 \cancel{e^{\gamma z}}\cancel{e^{-\gamma z}} \prod_{p=1}^{\infty}\left[\left(1+\frac{z}{p}\right)\cancel{e^{-z/p}}\left(1-\frac{z}{p}\right)\cancel{e^{z/p}}\right]$

$$\frac{1}{\Gamma(z)}\frac{-z}{\Gamma(1-z)} = -z^2 \prod_{p=1}^{\infty}\left(1-\frac{z^2}{p^2}\right) = -z\frac{\sin(\pi z)}{\pi}$$


$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$
$\left(1-\frac{z}{p}\right)e^{z/p}\Big]$
$\pi z)$


δf(x)
Zeta
$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ $\quad Re(s) > 1$
on real line $x > 1$
$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$
change variables $t = ku$
$\Gamma(x) = \int_0^\infty (ku)^{x-1} e^{-ku} k\,du$

$$\frac{\delta g(F[f])}{\delta f(x)} = \frac{dg(F[f])}{dF[f]} \frac{\delta F[f]}{\delta f(x)}$$

$$\frac{1}{k^x} = \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} e^{-ku} du$$

$$\sum_{k=1}^{\infty} \frac{1}{k^x} = \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} \sum_{k=1}^{\infty} e^{-ku} du$$

$$= \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} \frac{e^{-u}}{1-e^{u}} du$$

$$\zeta(x)\Gamma(x) = \int_0^\infty \frac{u^{x-1}}{e^u - 1} du$$

Functional Equation

$$\Lambda(s) = \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

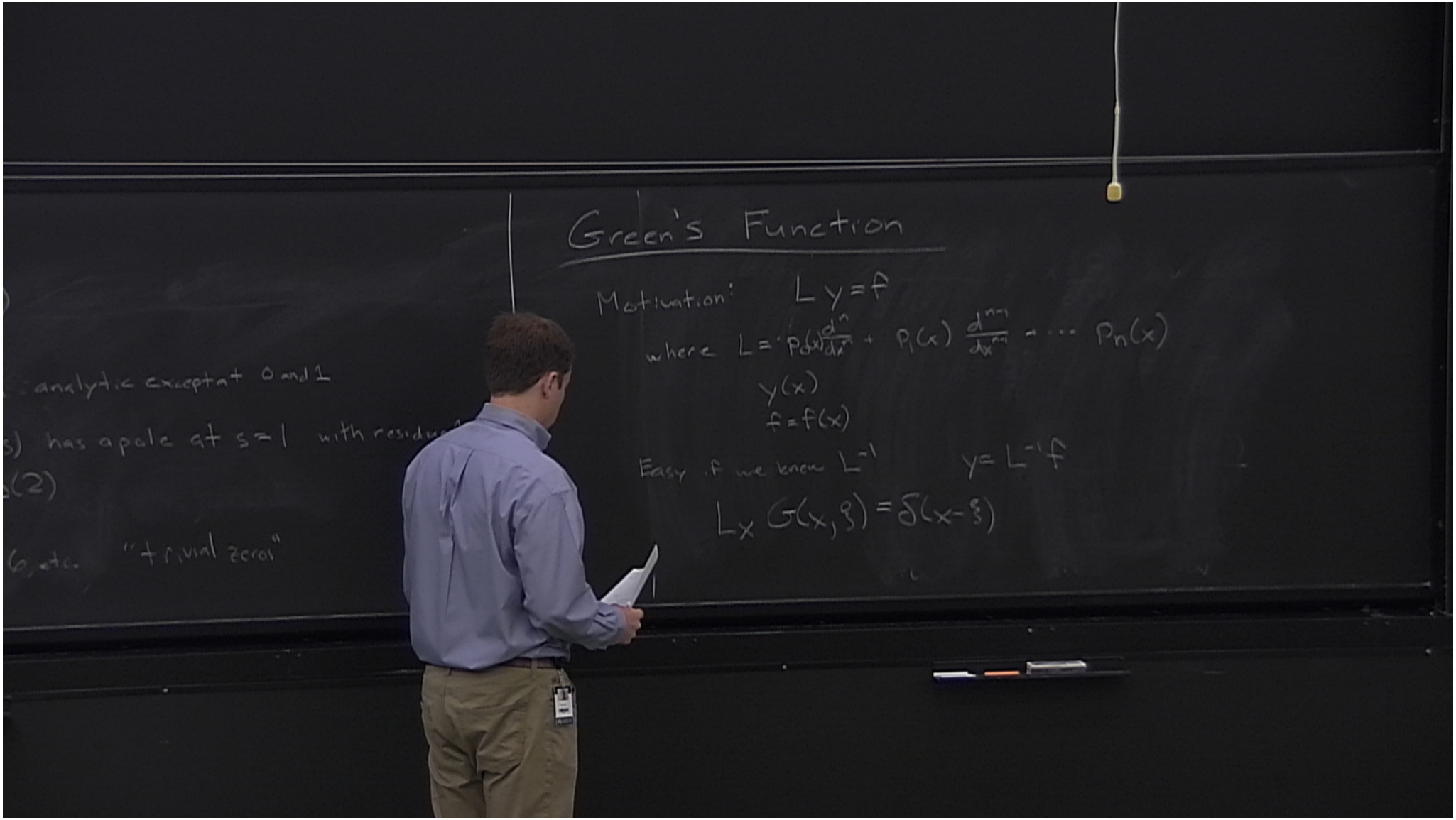
$$\Lambda(s) = \Lambda(1-s)$$

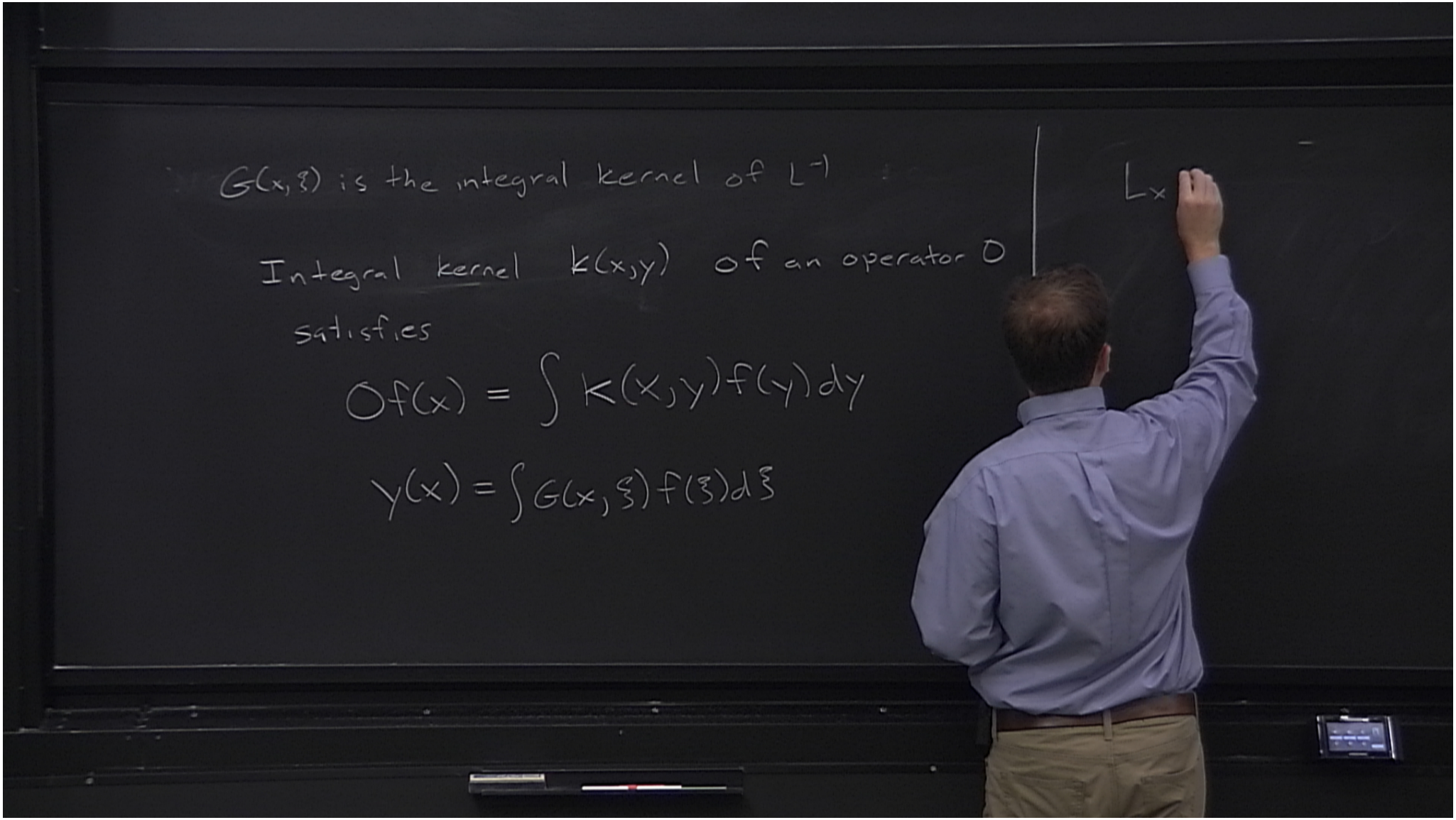
Λ: analytic except at 0 and 1

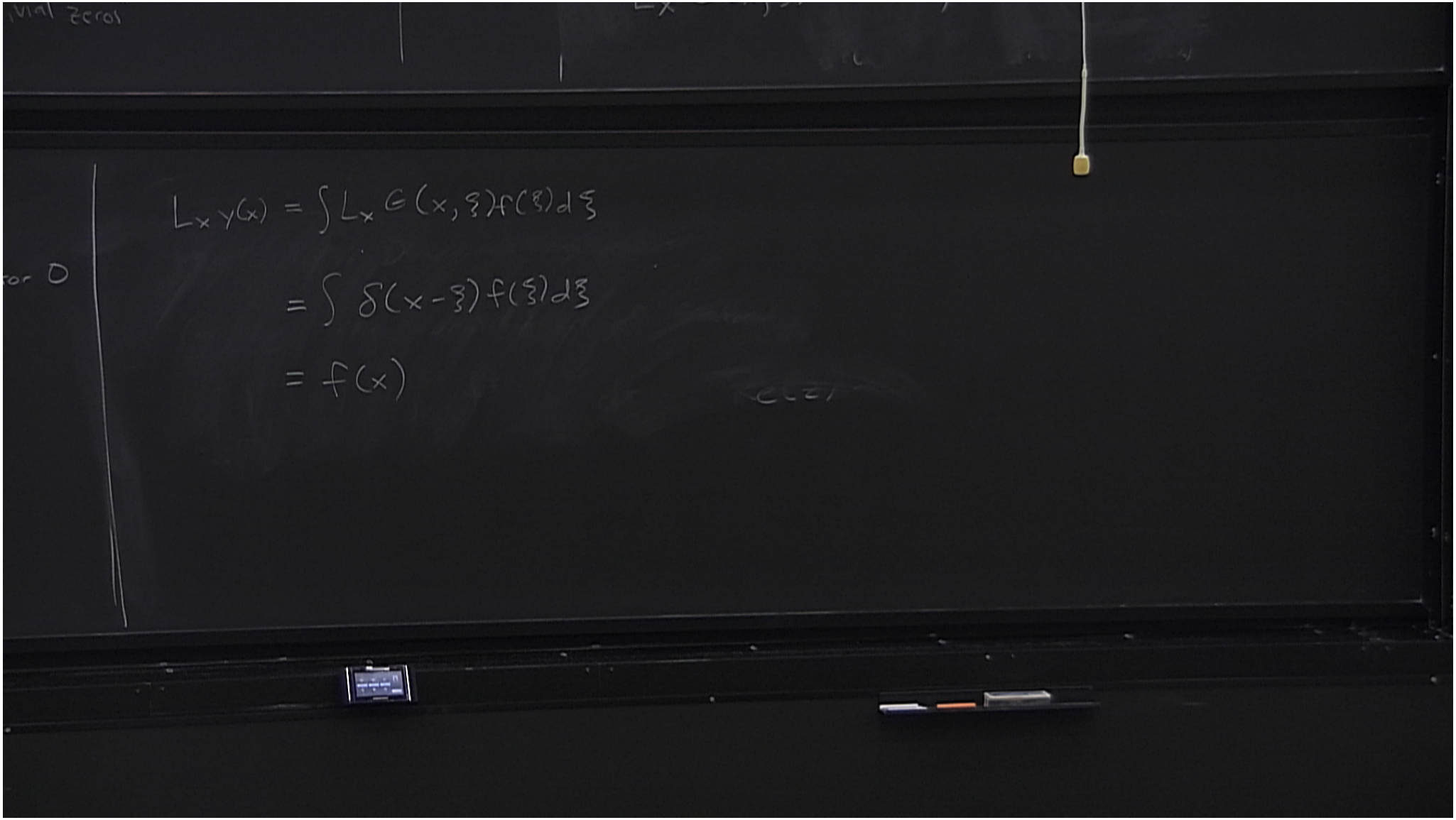
$\zeta(s)$ has a pole at $s=1$ with residue 1

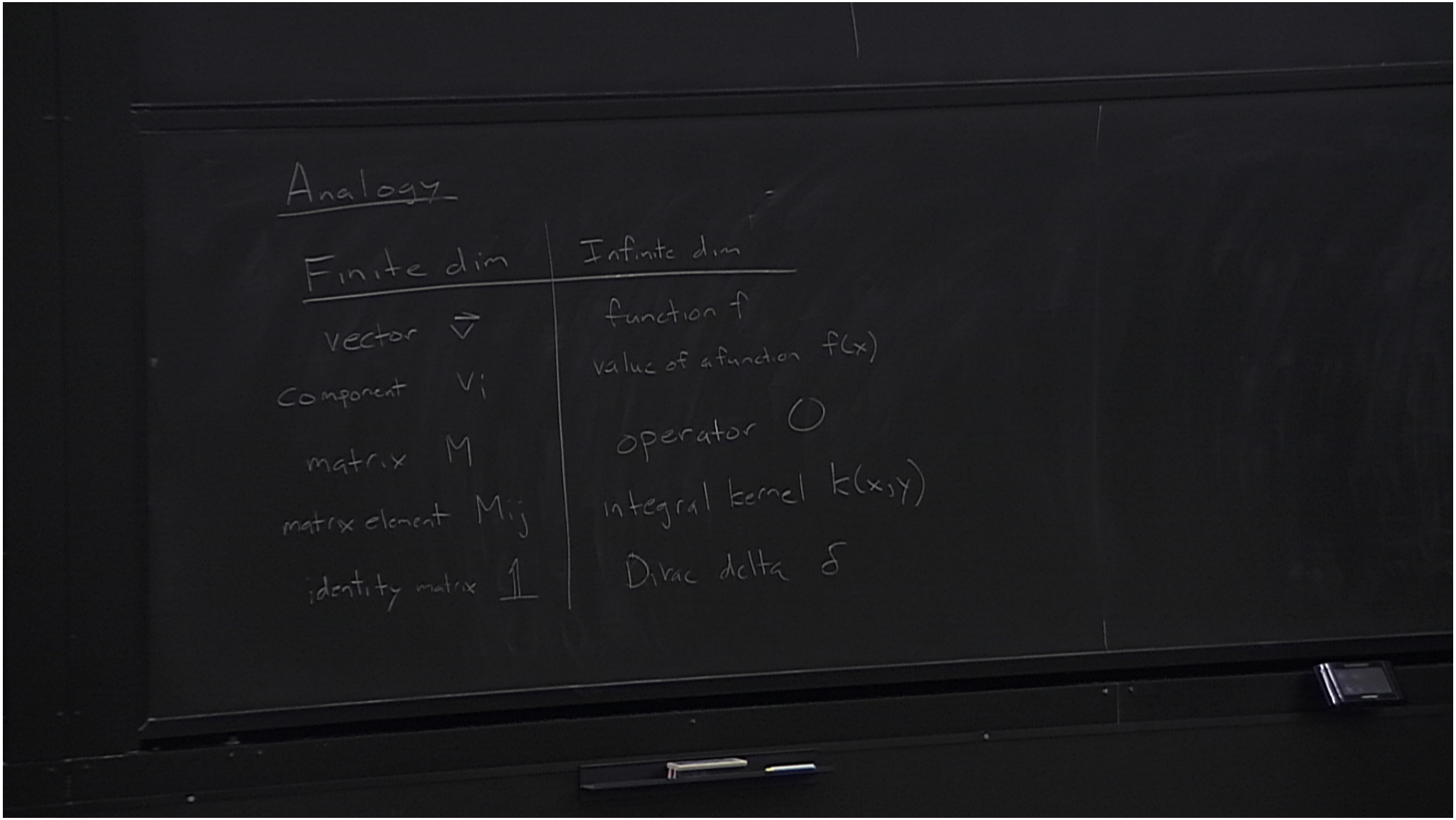
Func equat $\zeta(-1) = -\frac{1}{12}$ given $\zeta(2)$

$\zeta(s) = 0$ at $s = -2, -4, -6$, etc. "trivial zeros"


Green's Function
Motivation: $Ly = f$
where $L = P_0(x)\frac{d^n}{dx^n} + P_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots P_n(x)$
$y(x)$
$f = f(x)$
Easy if we know L^{-1} $y = L^{-1}f$
$L_x G(x,\xi) = \delta(x-\xi)$
analytic except at 0 and 1
has a pole at $s=1$ with residu
$\zeta(2)$
"trivial zeros"


$G(x,\xi)$ is the integral kernel of L^{-1}
Integral kernel $k(x,y)$ of an operator O
satisfies
$Of(x) = \int k(x,y) f(y)\, dy$
$y(x) = \int G(x,\xi) f(\xi)\, d\xi$
L_x


$L_x y(x) = \int L_x G(x,\xi) f(\xi) d\xi$
$= \int \delta(x-\xi) f(\xi) d\xi$
$= f(x)$


Analogy
Finite dim
Infinite dim
vector $\vec{v}$
function f
component v_i
value of a function f(x)
matrix M
operator O
matrix element M_{ij}
integral kernel k(x,y)
identity matrix $\mathbb{1}$
Dirac delta δ