Title: Front End - Functions, "Functions", etc. - 2

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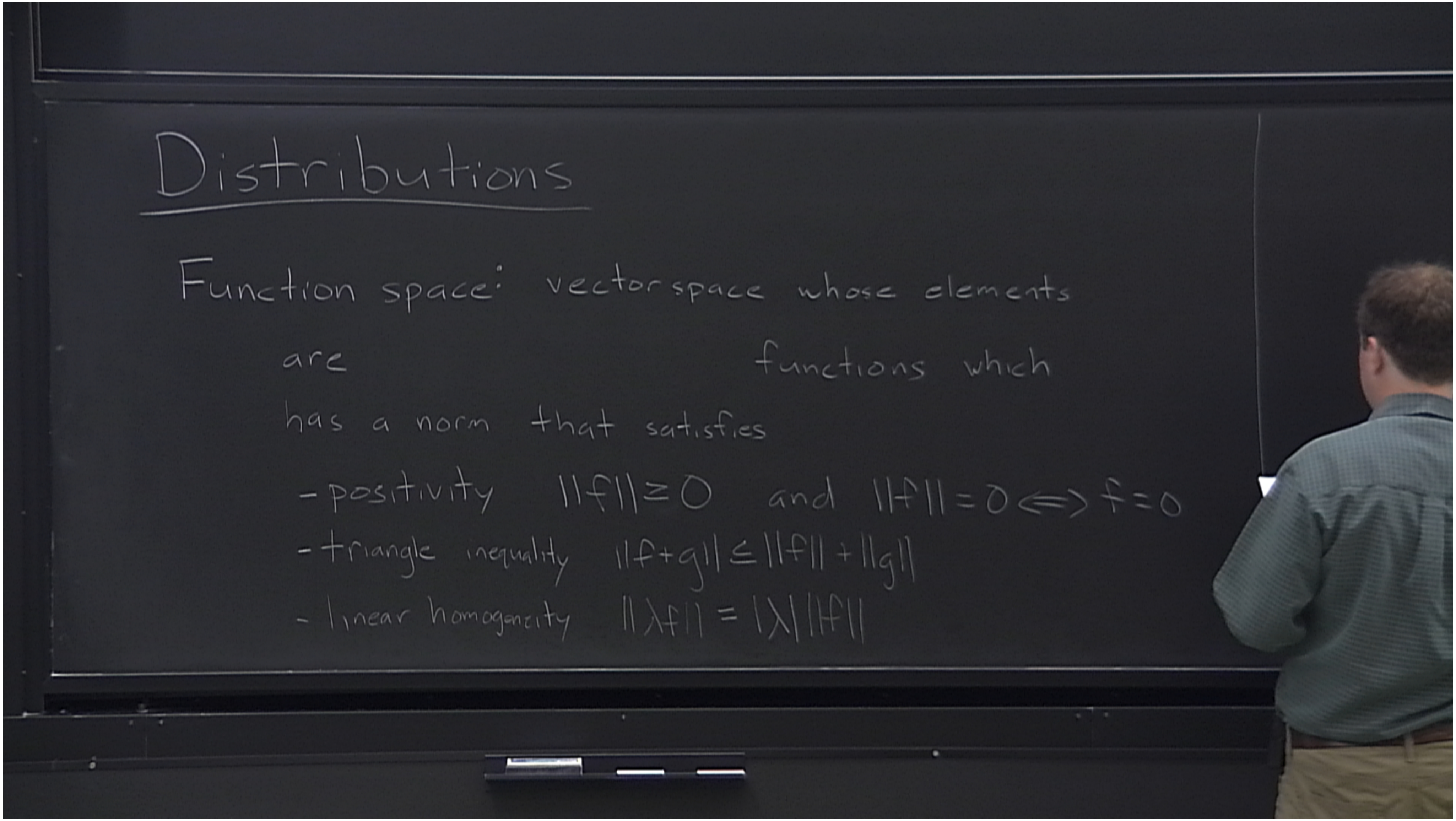
URL: http://pirsa.org/16080076

Abstract:


Distributions
Function space: vectorspace whose elements
are functions which
has a norm that satisfies
- positivity $||f|| \geq 0$ and $||f|| = 0 \Leftrightarrow f = 0$
- triangle inequality $||f+g|| \leq ||f|| + ||g||$
- linear homogeneity $||\lambda f|| = |\lambda| \, ||f||$

hose elements

ctions which

d $\|f\| = 0 \Longleftrightarrow f = 0$

$\| + \|g\|$

$\|f\|$

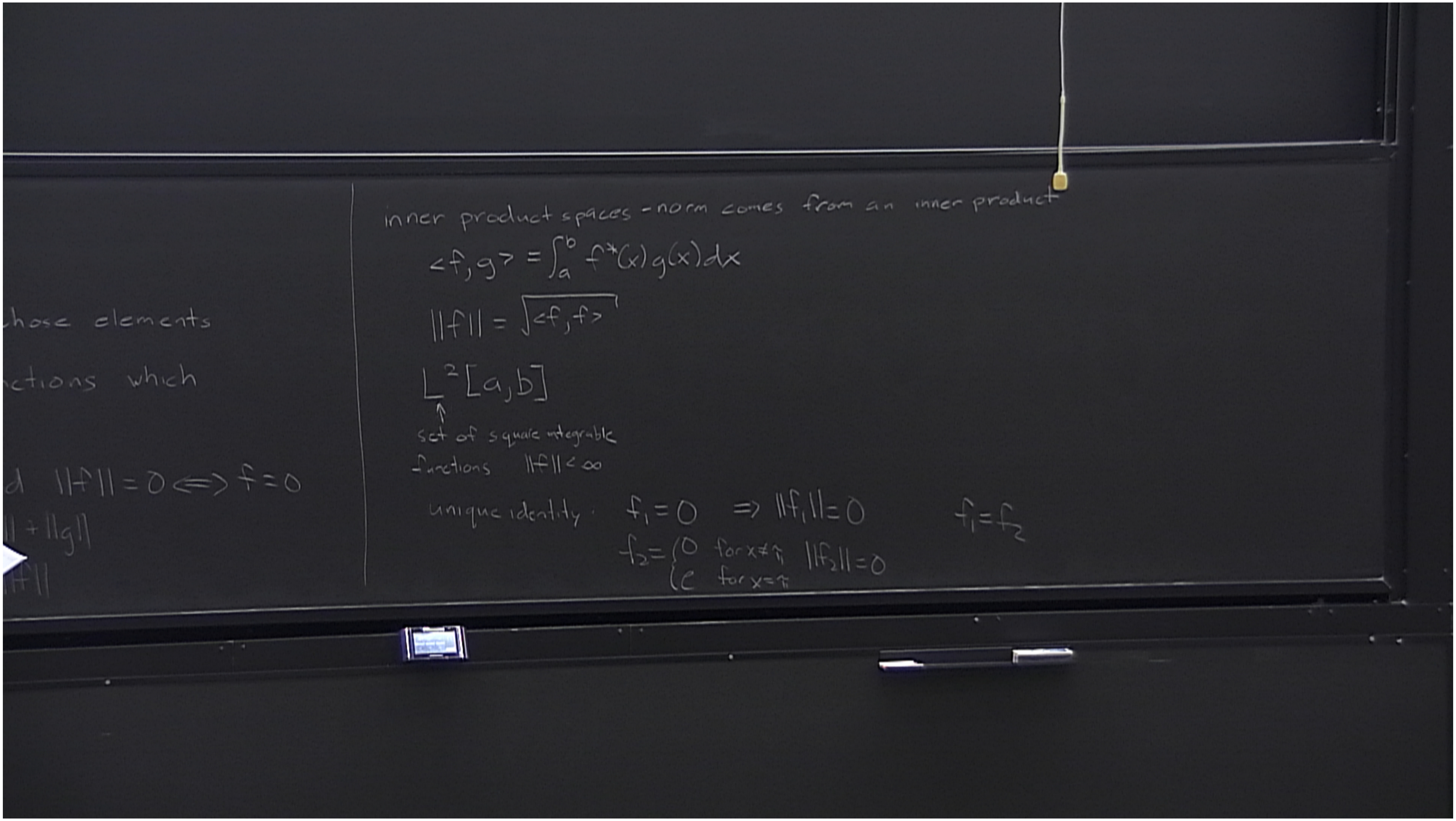
inner product spaces - norm comes from an inner product

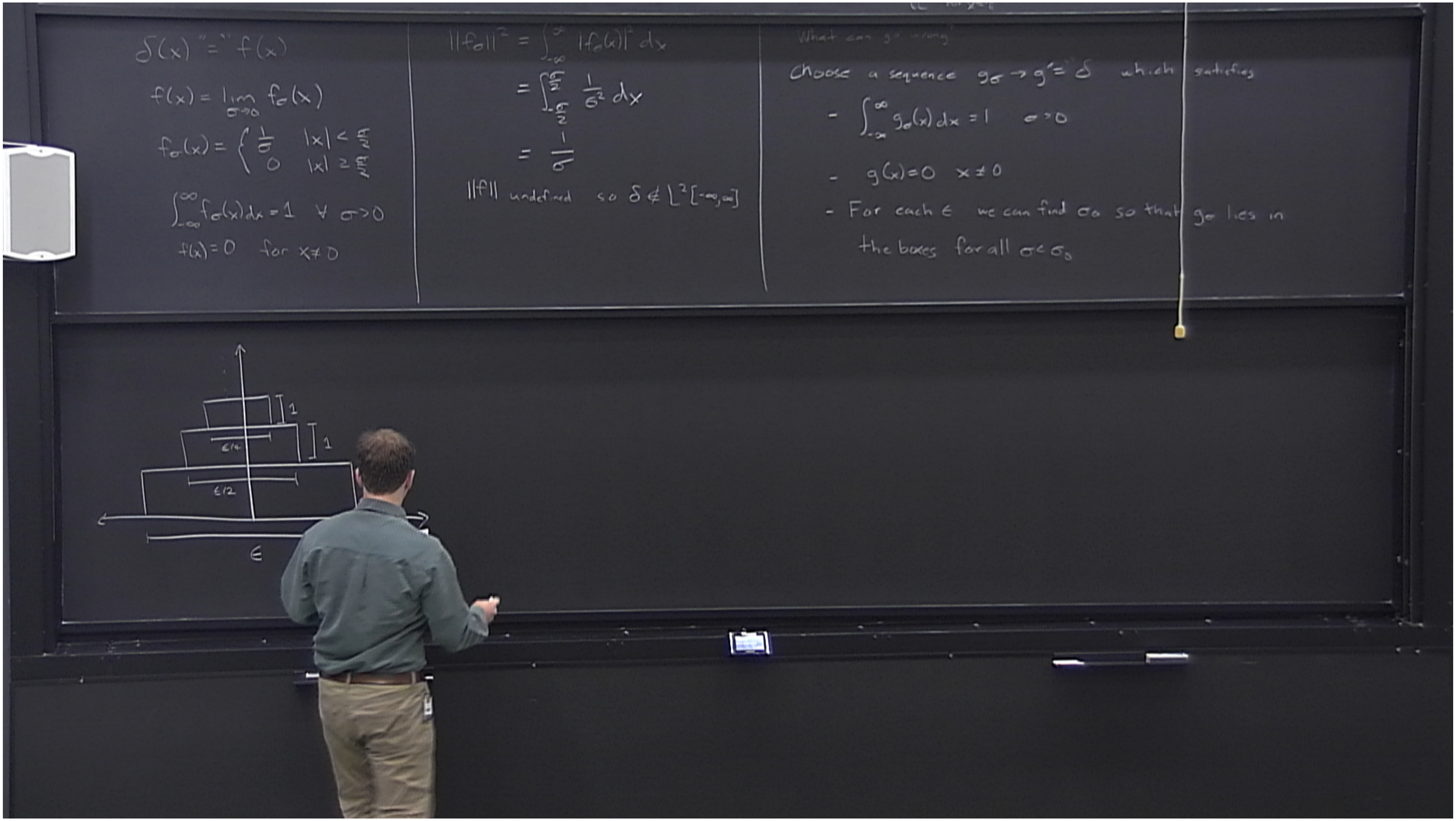
$$\langle f, g \rangle = \int_a^b f^*(x)\, g(x)\, dx$$

$$\|f\| = \sqrt{\langle f, f \rangle}$$

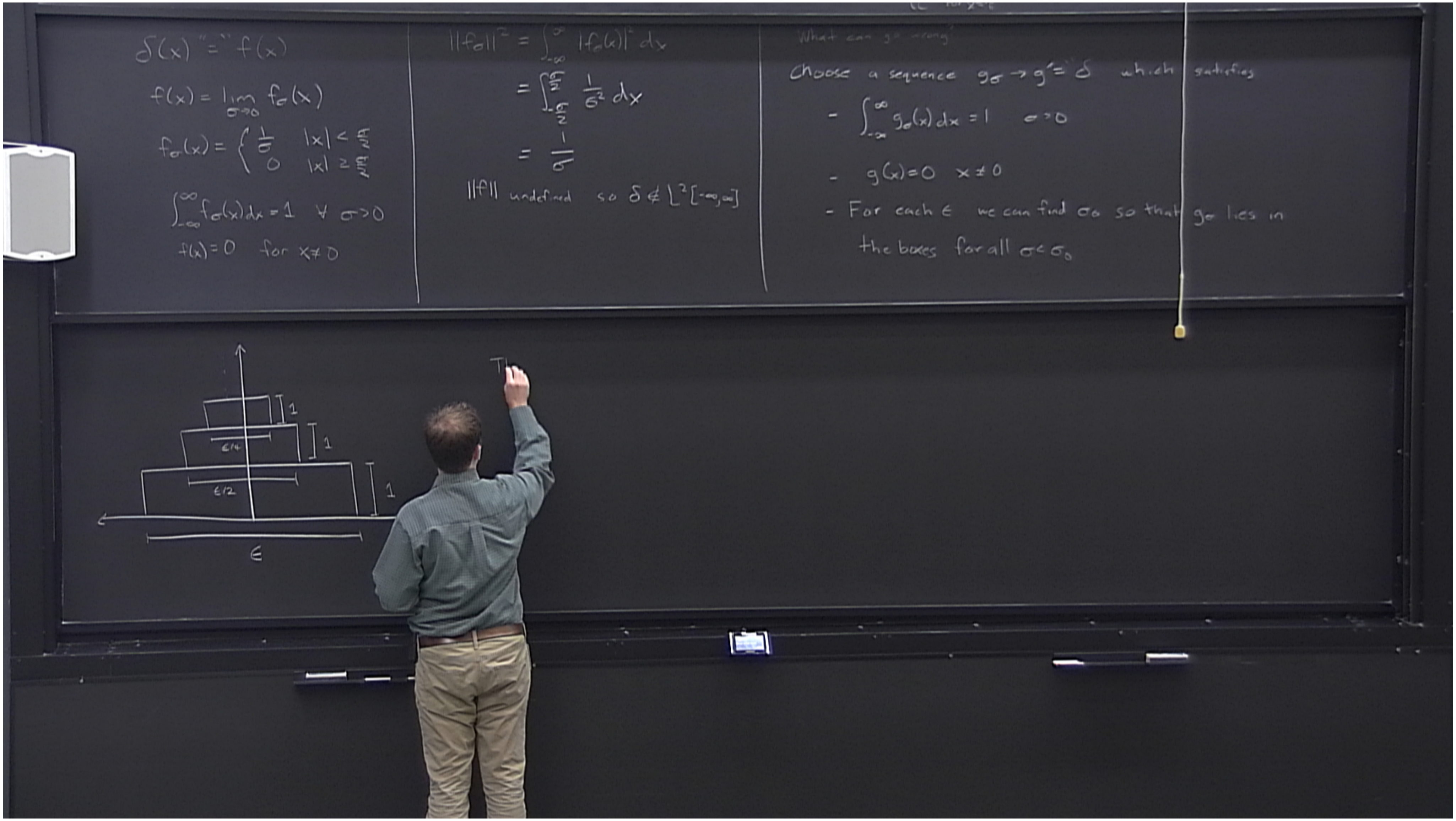
$$L^2[a,b]$$

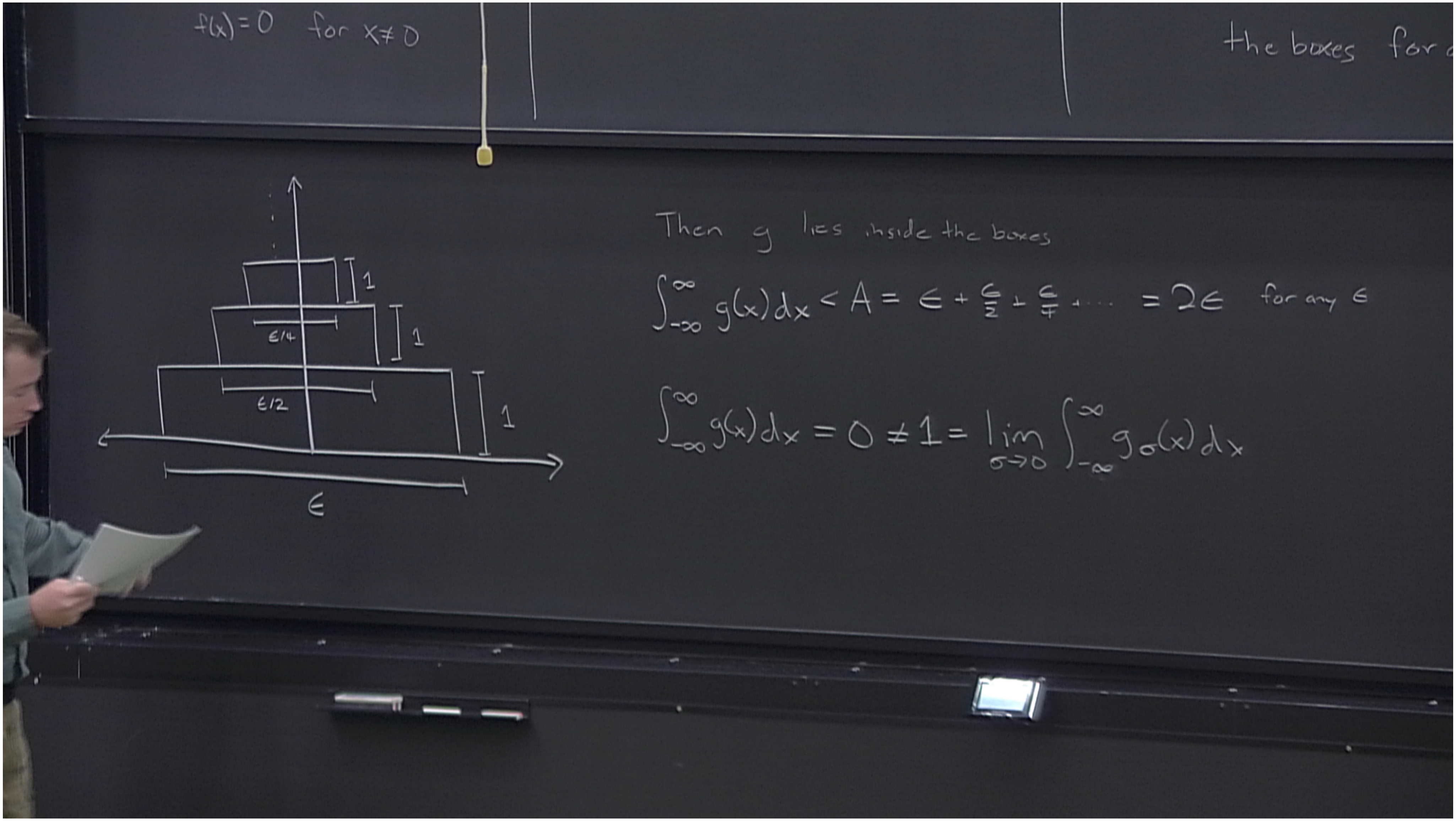
set of square integrable functions $\|f\| < \infty$


inner product spaces - norm comes from an inner product
<f, g> = ∫_a^b f*(x) g(x) dx
||f|| = √<f, f>
L²[a, b]
set of square integrable
functions ||f|| < ∞
unique identity: f₁ = 0 ⇒ ||f₁|| = 0
f₂ = { 0 for x ≠ x₁; c for x = x₁ } ||f₂|| = 0
f₁ = f₂
hose elements
ctions which
d ||f|| = 0 ⟺ f = 0
|| + ||g||



What can go wrong?
$\delta(x)$ "=" $f(x)$
$f(x) = \lim_{\sigma\to 0} f_\sigma(x)$
$f_\sigma(x) = \begin{cases} \frac{1}{\sigma} & |x| < \frac{\sigma}{2} \\ 0 & |x| \geq \frac{\sigma}{2} \end{cases}$
$\int_{-\infty}^{\infty} f_\sigma(x)\,dx = 1 \quad \forall\ \sigma > 0$
$f(x) = 0$ for $x \neq 0$
$\|f_\sigma\|^2 = \int_{-\infty}^{\infty} |f_\sigma(x)|^2\,dx$
$= \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{1}{\sigma^2}\,dx$
$= \frac{1}{\sigma}$
$\|f\|$ undefined so $\delta \notin L^2[-\infty,\infty]$
Choose a sequence $g_\sigma \to g$ "=" δ which satisfies
- $\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1 \quad \sigma > 0$
- $g(x) = 0 \quad x \neq 0$
- For each ϵ we can find σ_0 so that g_σ lies in the boxes for all $\sigma < \sigma_0$
1
1
$\epsilon/4$
$\epsilon/2$
ϵ


δ(x) "=" f(x)
f(x) = lim σ→0 fσ(x)
fσ(x) = { 1/σ |x| < σ/2 ; 0 |x| ≥ σ/2
∫-∞^∞ fσ(x)dx = 1 ∀ σ>0
f(x) = 0 for x≠0
||fσ||² = ∫-∞^∞ |fσ(x)|² dx
= ∫-σ/2^σ/2 1/σ² dx
= 1/σ
||f|| undefined so δ ∉ L²[-∞,∞]
Choose a sequence gσ → g "=" δ which satisfies
- ∫-∞^∞ gσ(x) dx = 1 σ>0
- g(x)=0 x≠0
- For each ε we can find σ0 so that gσ lies in
the boxes for all σ<σ0
1
1
1
ε/4
ε/2
ε



$f(x) = 0$ for $x \neq 0$
the boxes for
Then g lies inside the boxes
$\int_{-\infty}^{\infty} g(x)\,dx < A = \epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{4} + \dots = 2\epsilon$ for any ϵ
$\int_{-\infty}^{\infty} g(x)\,dx = 0 \neq 1 = \lim_{\sigma \to 0} \int_{-\infty}^{\infty} g_\sigma(x)\,dx$
1
1
1
$\epsilon/4$
$\epsilon/2$
ϵ

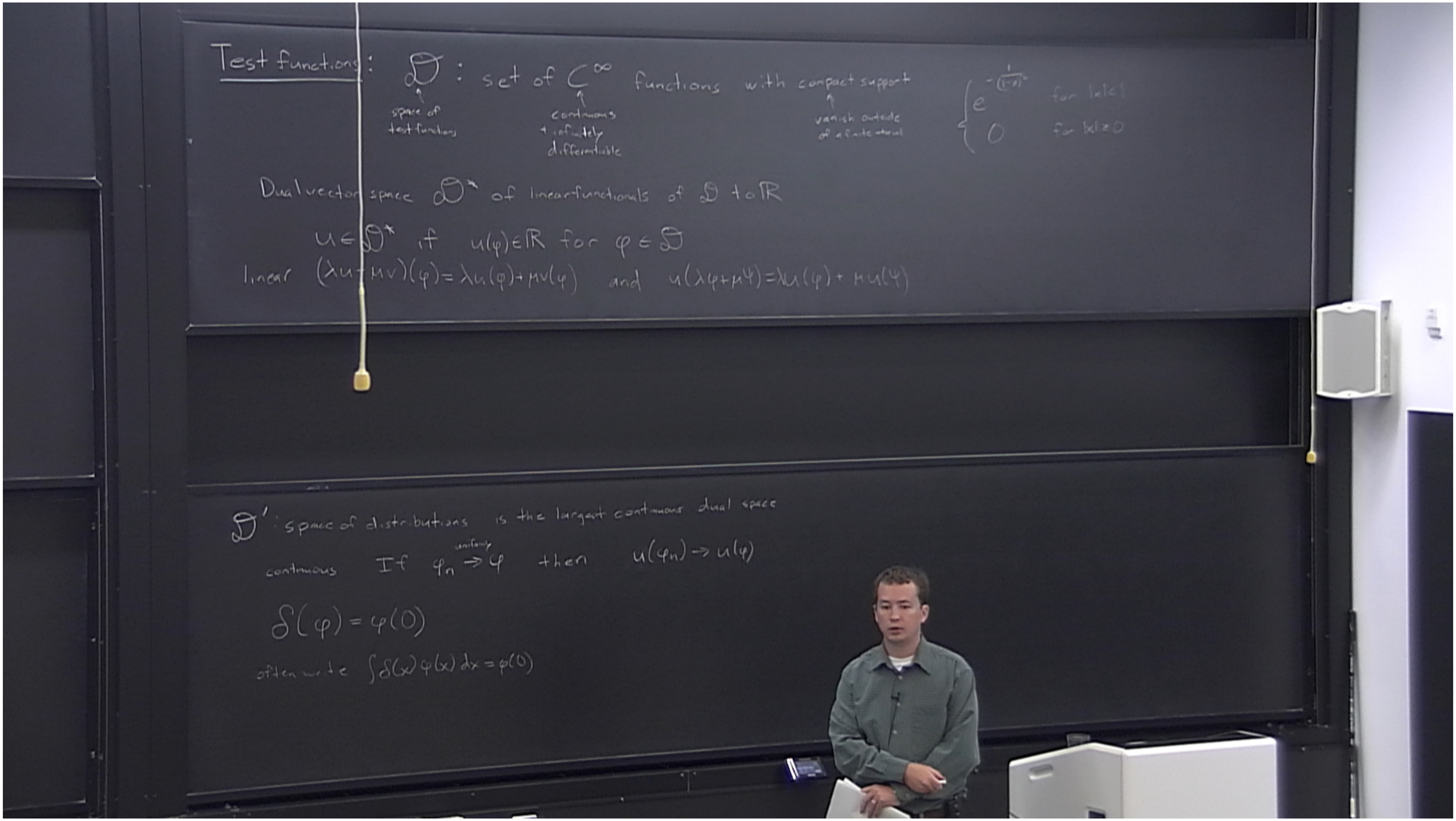
Test functions: $\mathcal{D}$: set of C^∞ functions with compact support

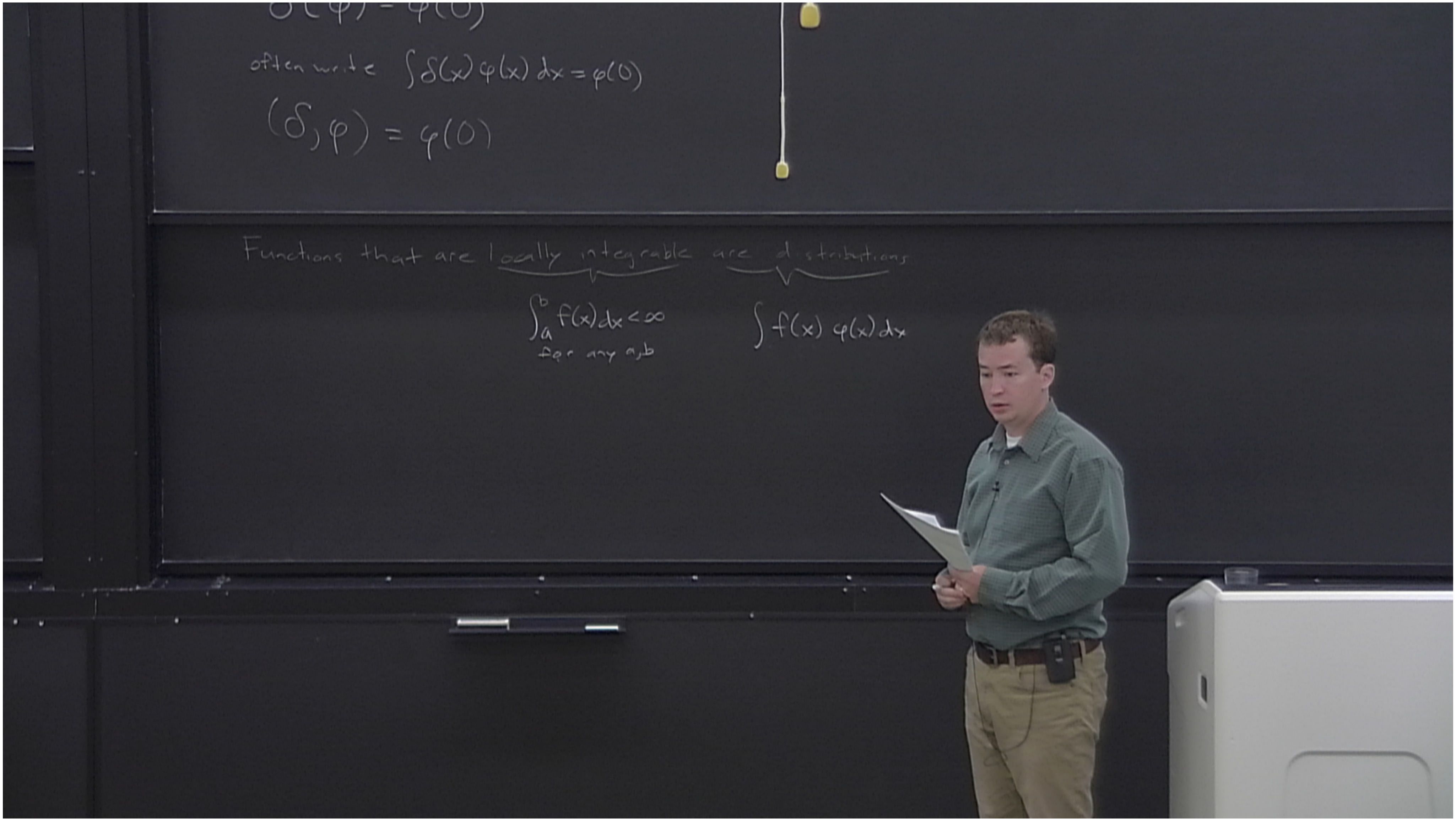
$\mathcal{D}$: space of test functions

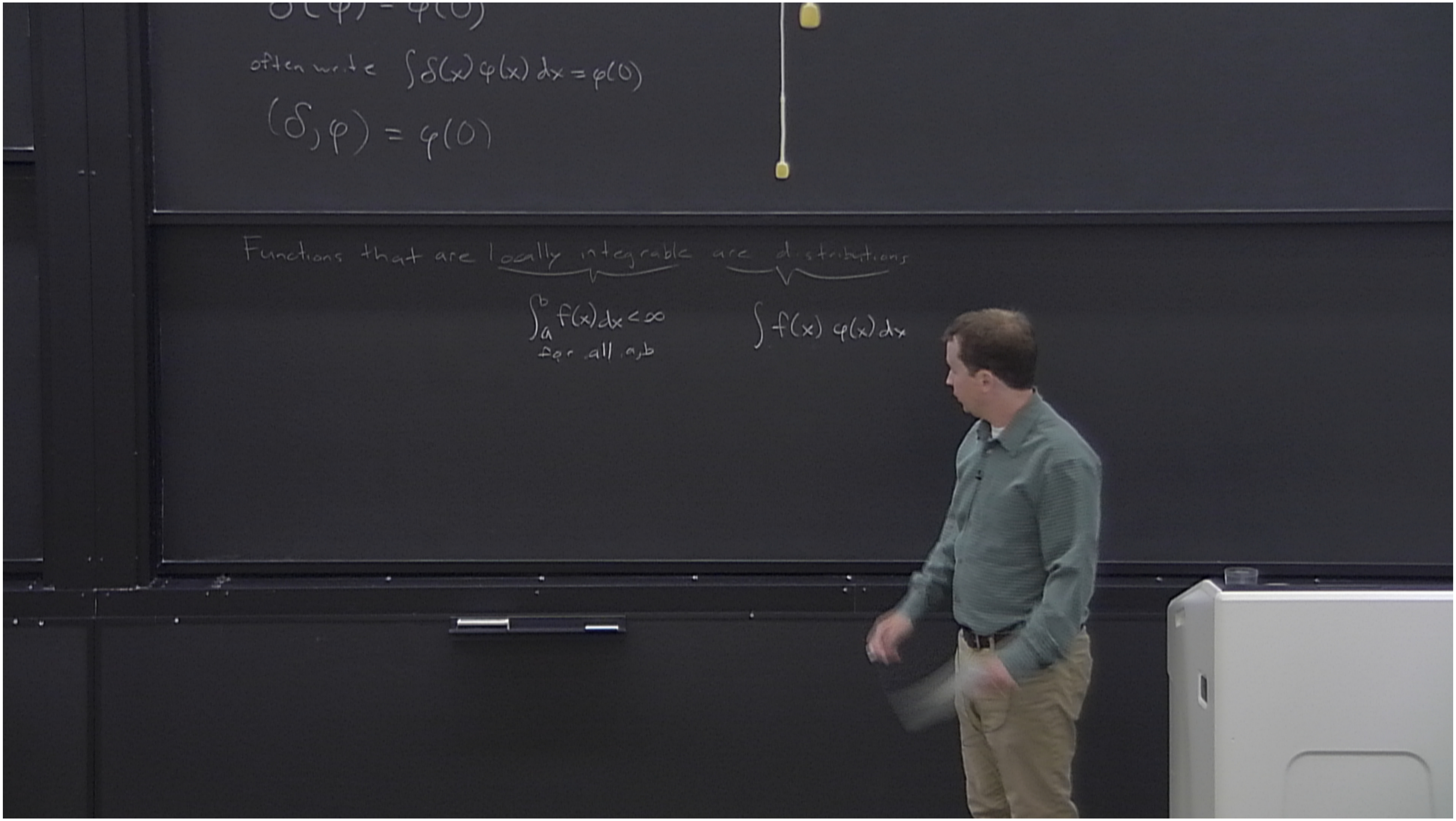
C: continuous + infinitely differentiable

compact support: vanish outside of a finite interval

$$\begin{cases} e^{-\frac{1}{(1-x)^2}} & \text{for } |x|<1 \\ 0 & \text{for } |x|\geq 0 \end{cases}$$


Test functions: $\mathfrak{D}$: set of C^∞ functions with compact support
space of test functions
continuous + infinitely differentiable
vanish outside of a finite interval
$\begin{cases} e^{-\frac{1}{(1-x)^2}} & \text{for } |x|<1 \\ 0 & \text{for } |x|\geq 0 \end{cases}$
Dual vector space $\mathfrak{D}^*$ of linear functionals of $\mathfrak{D}$ to $\mathbb{R}$
$u \in \mathfrak{D}^*$ if $u(\varphi)\in\mathbb{R}$ for $\varphi\in\mathfrak{D}$
linear $(\lambda u+\mu v)(\varphi)=\lambda u(\varphi)+\mu v(\varphi)$ and $u(\lambda\varphi+\mu\psi)=\lambda u(\varphi)+\mu u(\psi)$
$\mathfrak{D}'$: space of distributions is the largest continuous dual space
continuous If $\varphi_n \xrightarrow{\text{uniformly}} \varphi$ then $u(\varphi_n)\to u(\varphi)$
$\delta(\varphi)=\varphi(0)$
often write $\int\delta(x)\varphi(x)\,dx=\varphi(0)$


often write ∫δ(x) φ(x) dx = φ(0)
(δ, φ) = φ(0)
Functions that are locally integrable are distributions
∫_a^b f(x)dx < ∞
for any a,b
∫ f(x) φ(x) dx


often write $\int \delta(x)\varphi(x)\,dx = \varphi(0)$
$(\delta, \varphi) = \varphi(0)$
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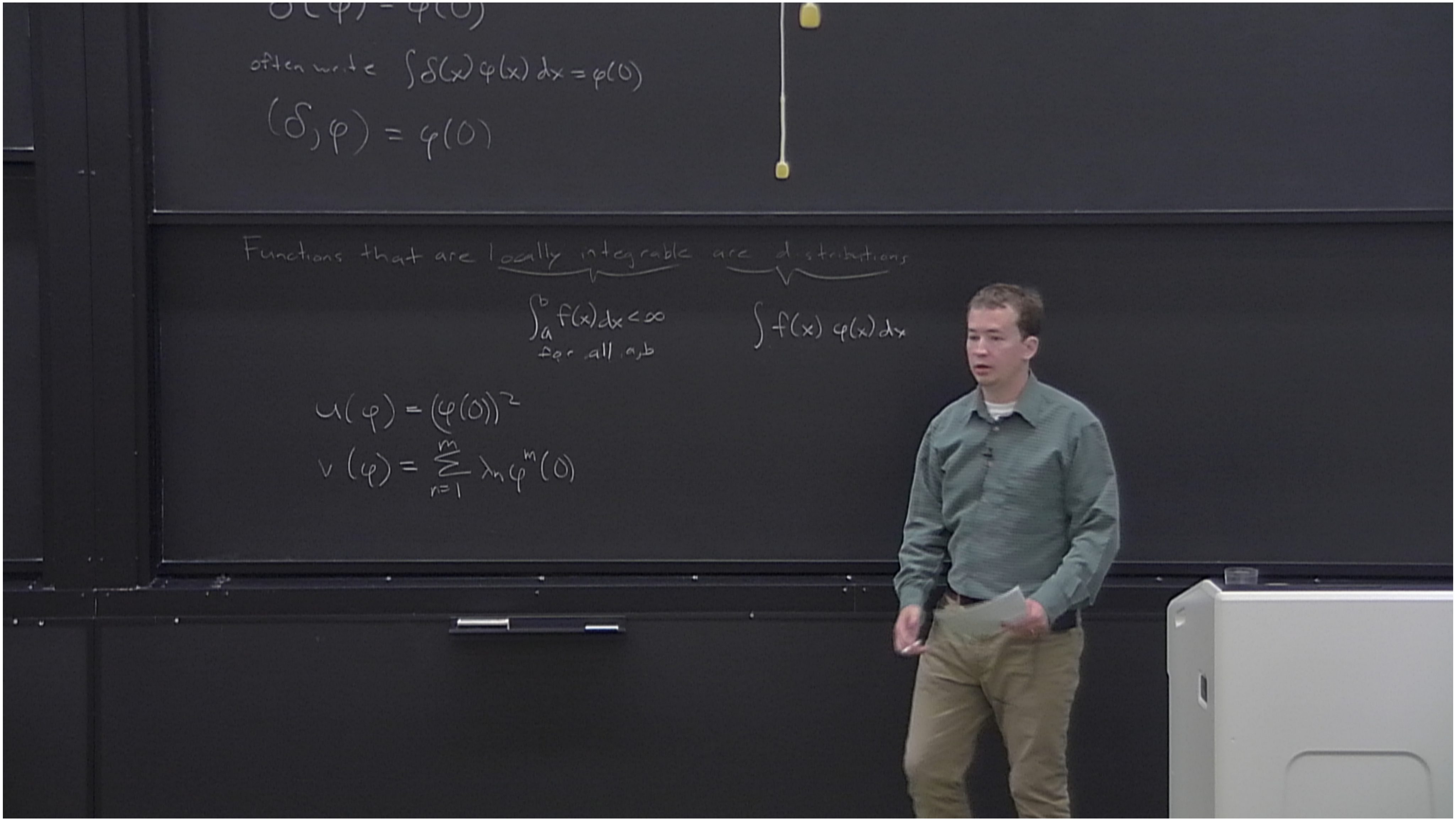
for all a,b
$\int f(x)\,\varphi(x)\,dx$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Functions that are locally integrable are distributions
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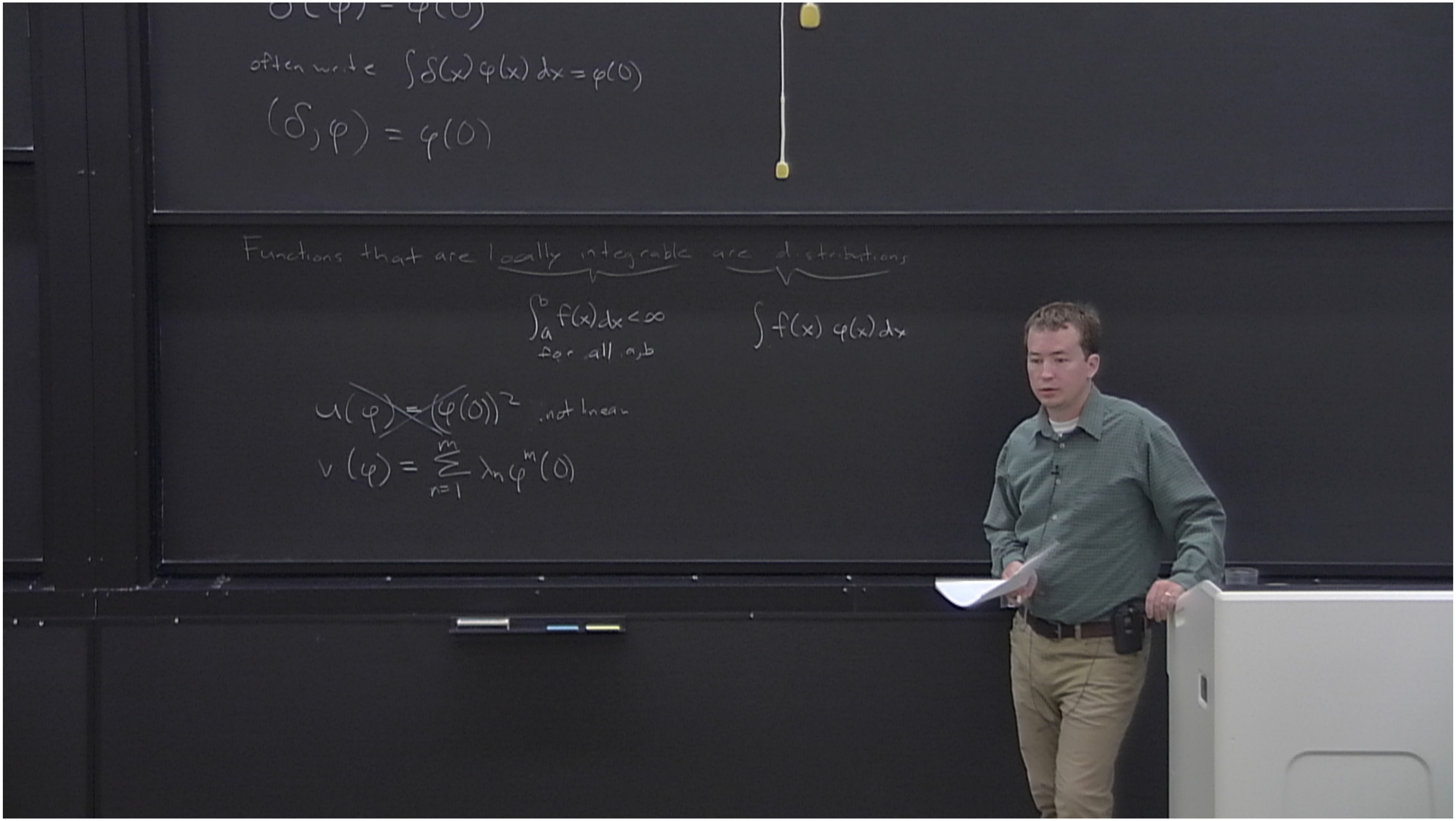
for all a,b
$\int f(x)\,\varphi(x)\,dx$
$u(\varphi) = (\varphi(0))^2$
$v(\varphi) = \sum_{n=1}^{m} \lambda_n \varphi^{n}(0)$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Functions that are locally integrable are distributions
$\int_a^b f(x)\,dx < \infty$
for all a,b
$\int f(x)\,\varphi(x)\,dx$
$\cancel{u(\varphi) = (\varphi(0))^2}$ not linear
$v(\varphi) = \sum_{n=1}^{m} \lambda_n \varphi^m(0)$

$\delta(\varphi) = \varphi(0)$

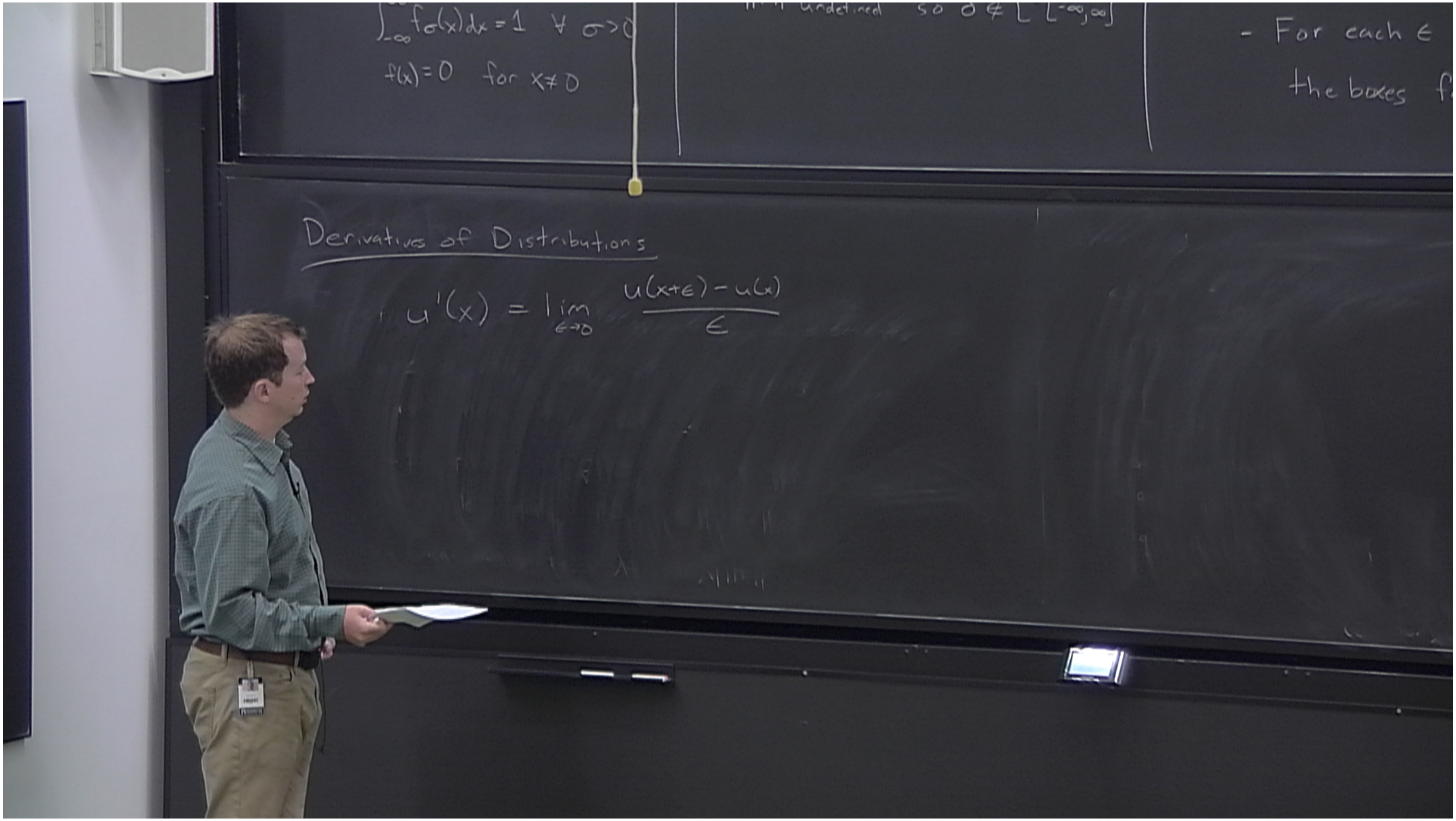
often write $\int \delta(x)\varphi(x)\,dx = \varphi(0)$

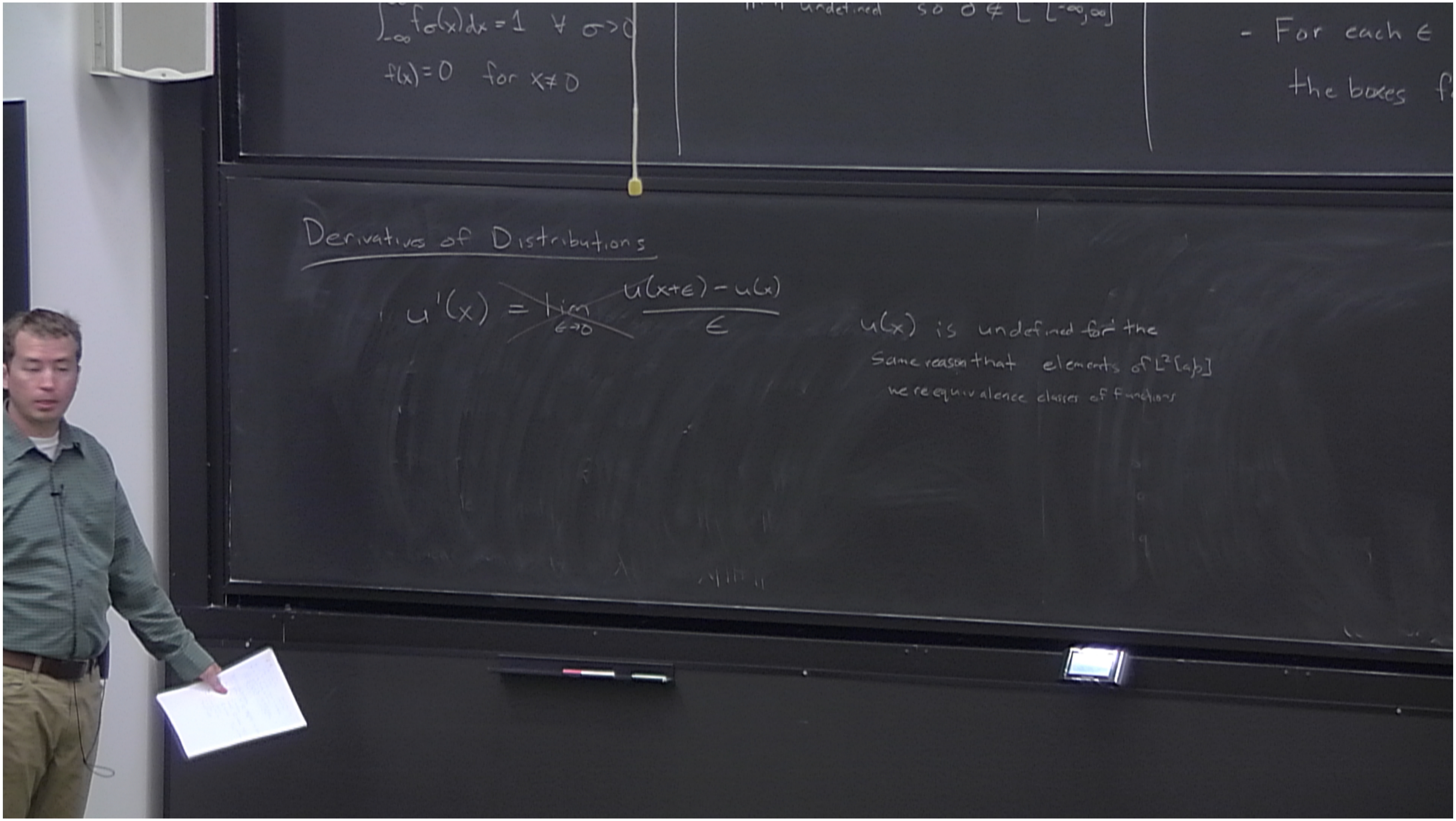
$(\delta, \varphi) = \varphi(0)$

Functions that are $\underbrace{\text{locally integrable}}_{\int_a^b f(x)dx < \infty \text{ for all } a,b}$ are $\underbrace{\text{distributions}}_{\int f(x)\,\varphi(x)\,dx}$

$\cancel{u(\varphi) = (\varphi(0))^2}$ not linear

$v(\varphi) = \sum_{n=1}^{m} \lambda_n \varphi^{(n)}(0)$ ✓ linear continuous finite


$\int_{-\infty} f_\sigma(x)dx = 1 \quad \forall\, \sigma > 0$
$f(x) = 0$ for $x \neq 0$
- For each ϵ
the boxes f
Derivatives of Distributions
$u'(x) = \lim_{\epsilon \to 0} \frac{u(x+\epsilon) - u(x)}{\epsilon}$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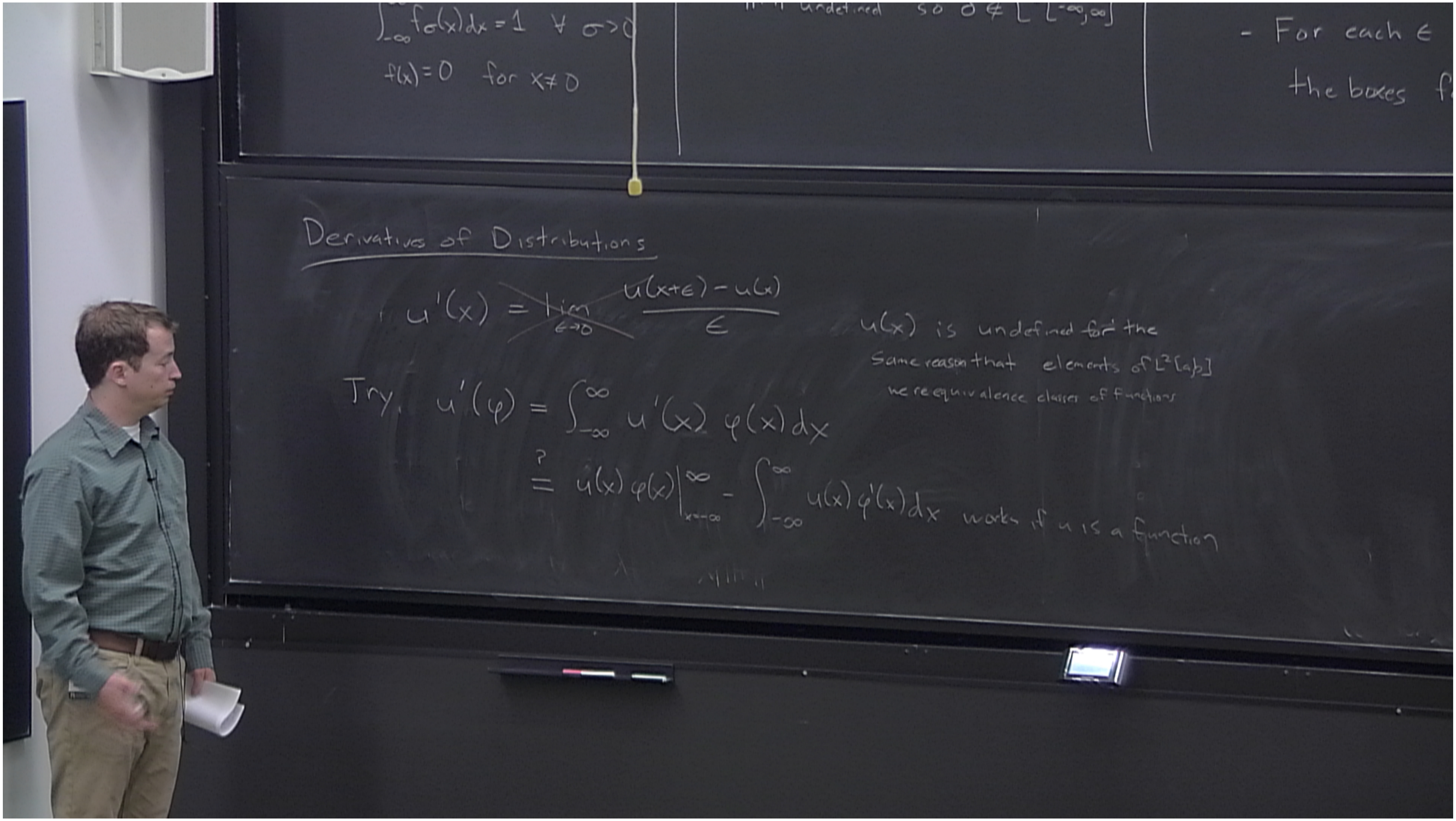
Derivatives of Distributions
$u'(x) = \cancel{\lim_{\epsilon \to 0}} \frac{u(x+\epsilon) - u(x)}{\epsilon}$
$u(x)$ is undefined for the
same reason that elements of $L^2[a,b]$
were equivalence classes of functions


∫_{-∞} f_σ(x)dx = 1 ∀ σ>0
f(x)=0 for x≠0
undefined so δ ∉ L²[-∞,∞]
- For each ε
Derivatives of Distributions
u'(x) = lim_{ε→0} (u(x+ε) - u(x))/ε
u(x) is undefined for the
Same reason that elements of L²[a,b]
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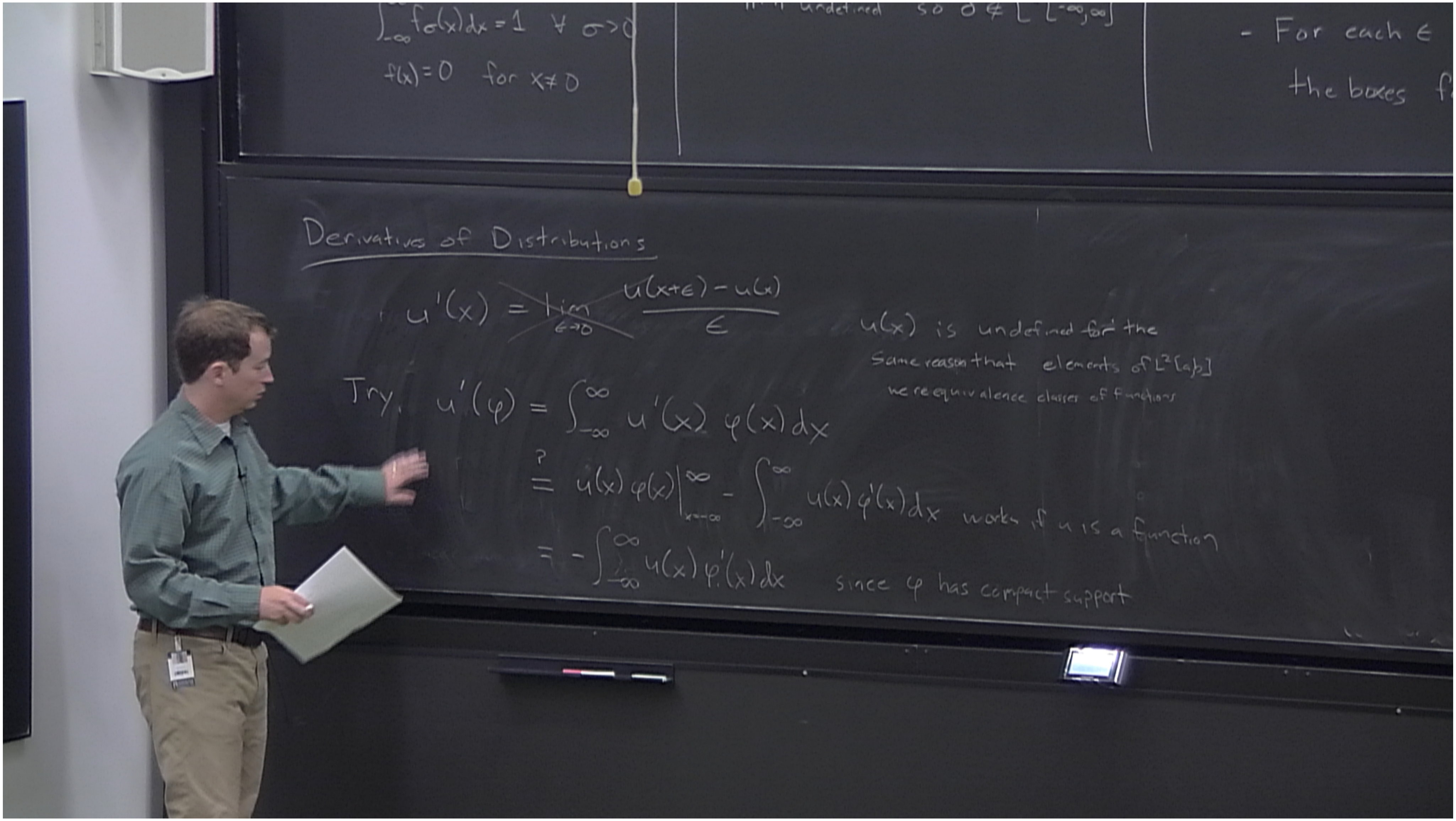
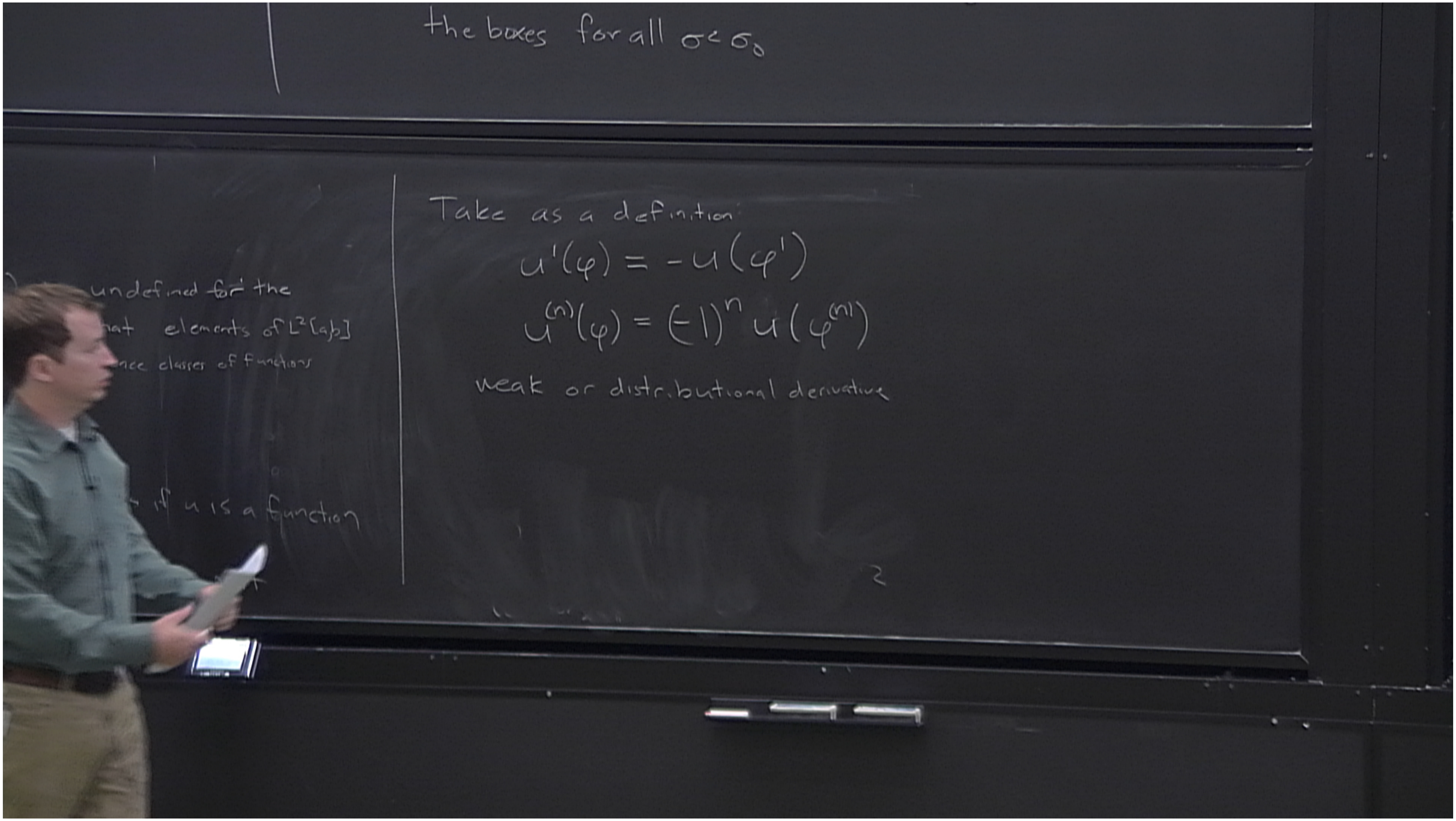
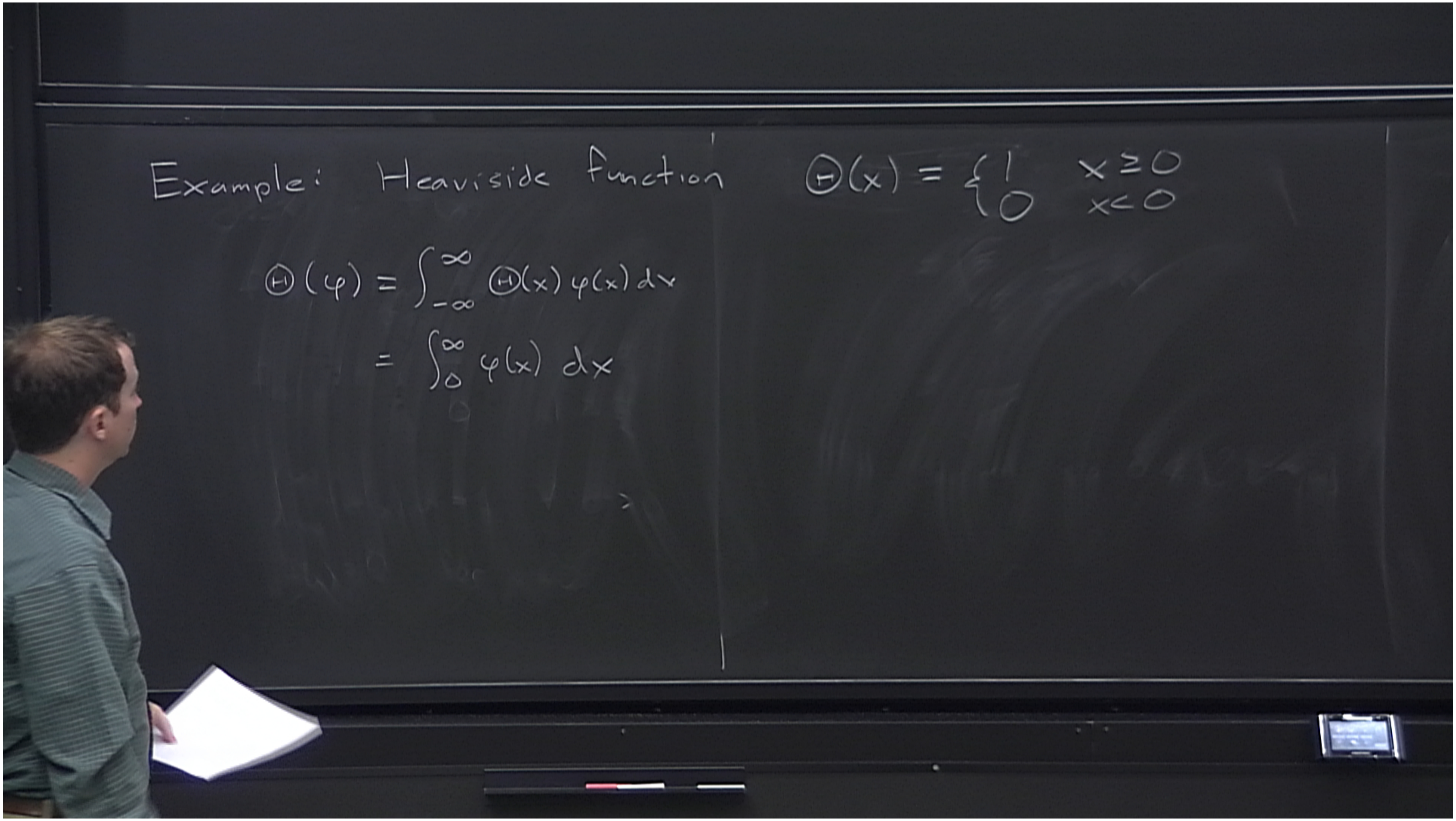
Try: u'(φ) = ∫_{-∞}^{∞} u'(x) φ(x) dx
?= u(x) φ(x)|_{x=-∞}^{∞} - ∫_{-∞}^{∞} u(x) φ'(x) dx works if u is a function



$\int_{-\infty}^{\infty} f_\sigma(x)\,dx = 1 \quad \forall\, \sigma>0$
$f(x)=0$ for $x\neq 0$
undefined so $\delta \notin L^2[-\infty,\infty]$
- For each ϵ
the boxes f
Derivatives of Distributions
$u'(x) = \cancel{\lim_{\epsilon\to 0}} \frac{u(x+\epsilon)-u(x)}{\epsilon}$
$u(x)$ is undefined for the same reason that elements of $L^2[a,b]$ were equivalence classes of functions
Try: $u'(\varphi) = \int_{-\infty}^{\infty} u'(x)\,\varphi(x)\,dx$
$\stackrel{?}{=} u(x)\varphi(x)\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)\varphi'(x)\,dx$ works if u is a function
$= -\int_{-\infty}^{\infty} u(x)\varphi'(x)\,dx$ since φ has compact support


the boxes for all $\sigma < \sigma_0$
undefined for the
that elements of $L^2[a,b]$
ence classes of functions
if u is a function
Take as a definition:
$u'(\varphi) = -u(\varphi')$
$u^{(n)}(\varphi) = (-1)^n u(\varphi^{(n)})$
weak or distributional derivative


Example: Heaviside function
$\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$
$\Theta(\varphi) = \int_{-\infty}^{\infty} \Theta(x)\, \varphi(x)\, dx$
$= \int_{0}^{\infty} \varphi(x)\, dx$

Example: $(\ln|x|)'$

$$(\ln|x|)(\varphi) = \int_{-\infty}^{\infty} \ln|x| \, \varphi(x)\, dx$$

$$= \lim_{\epsilon \to 0^+} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \ln|x| \, \varphi(x)\, dx$$

$$\lim_{\epsilon \to 0^+} \left| \int_{-\epsilon}^{\epsilon} \ln|x| \, \varphi(x)\, dx \right| \leq \lim_{\epsilon \to 0^+} \int_{-\epsilon}^{\epsilon} \big|\ln|x|\big| \, |\varphi(x)|\, dx$$

$$\leq \lim_{\epsilon \to 0^+} \int_{-\epsilon}^{\epsilon} \big|\ln|x|\big| \max|\varphi(x)|\, dx$$

$=$

$\Theta' = \delta$

$$\varphi(x)\,dx$$

$$\int_\epsilon^\infty \ln|x|\,\varphi(x)\,dx$$

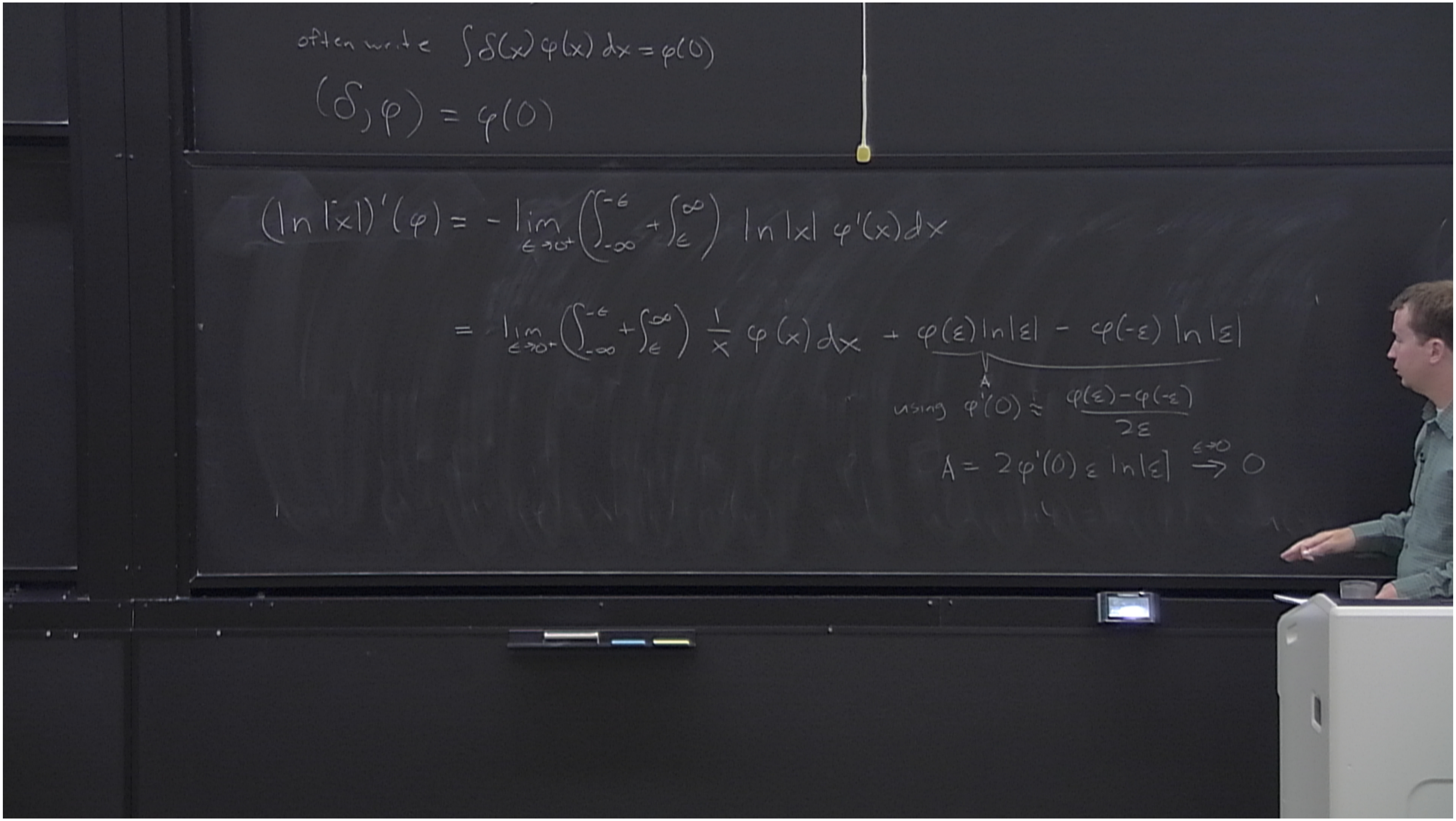
$$\leq \lim_{\epsilon\to 0^+} \int_{-\epsilon}^{\epsilon} \big|\ln|x|\big|\,|\varphi(x)|\,dx$$

$$\leq \lim_{\epsilon\to 0^+} \int_{-\epsilon}^{\epsilon} \big|\ln|x|\big| \max|\varphi(x)|\,dx$$

$$= \max|\varphi(x)| \lim_{\epsilon\to 0^+} 2\int_0^\epsilon \big|\ln|x|\big|\,dx$$

$$= \max|\varphi(x)| \lim_{\epsilon\to 0^+} \Big| -2x(\ln x - 1)\Big|_0^\epsilon$$

$$= 0$$


often write $\int \delta(x)\varphi(x)\,dx = \varphi(0)$
$(\delta, \varphi) = \varphi(0)$
$(\ln|x|)'(\varphi) = -\lim_{\epsilon\to0^+}\left(\int_{-\infty}^{-\epsilon}+\int_{\epsilon}^{\infty}\right)\ln|x|\,\varphi'(x)\,dx$
$= \lim_{\epsilon\to0^+}\left(\int_{-\infty}^{-\epsilon}+\int_{\epsilon}^{\infty}\right)\frac{1}{x}\varphi(x)\,dx + \underbrace{\varphi(\varepsilon)\ln|\varepsilon| - \varphi(-\varepsilon)\ln|\varepsilon|}_{A}$
using $\varphi'(0) \approx \frac{\varphi(\varepsilon)-\varphi(-\varepsilon)}{2\varepsilon}$
$A = 2\varphi'(0)\,\varepsilon\ln|\varepsilon| \xrightarrow{\epsilon\to0} 0$

often write $\int \delta(x)\varphi(x)\,dx = \varphi(0)$

$$(\delta, \varphi) = \varphi(0)$$

$$(\ln|x|)'(\varphi) = -\lim_{\epsilon\to 0^+}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) \ln|x|\, \varphi'(x)\,dx$$

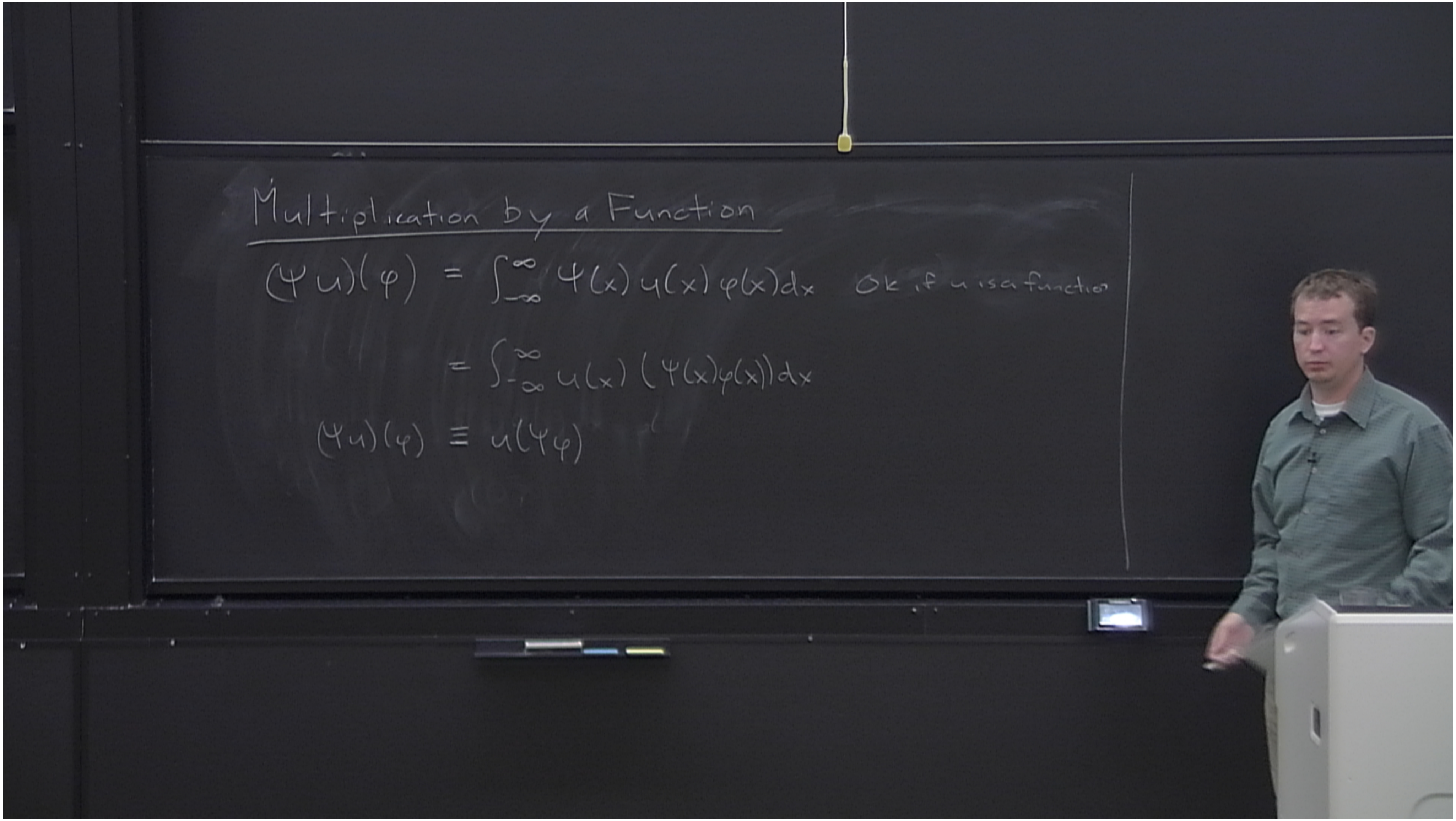
$$= \lim_{\epsilon\to 0^+}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) \frac{1}{x}\, \varphi(x)\,dx + \underbrace{\varphi(\varepsilon)\ln|\varepsilon| - \varphi(-\varepsilon)\ln|\varepsilon|}_{A}$$

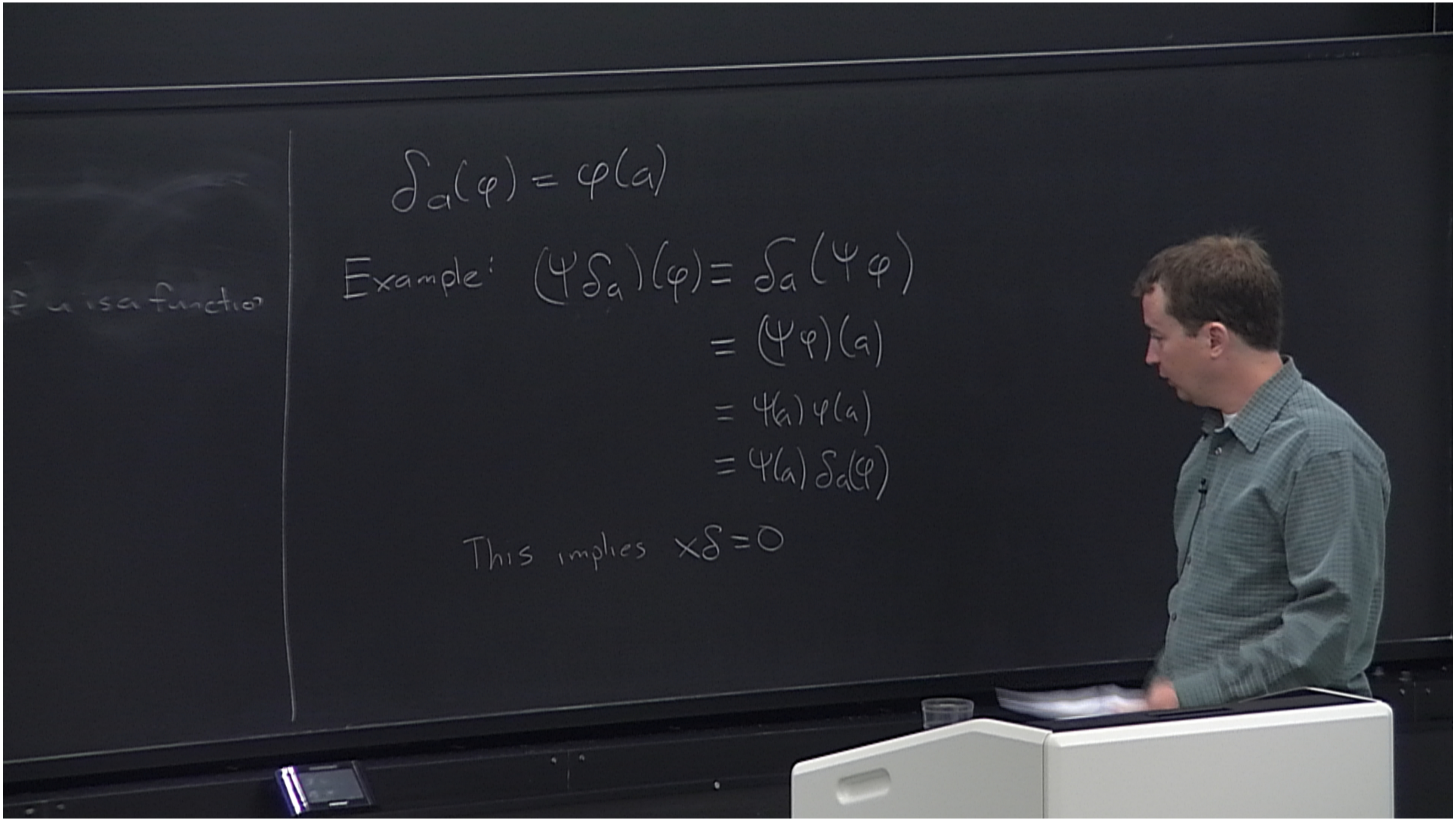
using $\varphi'(0) \approx \dfrac{\varphi(\varepsilon)-\varphi(-\varepsilon)}{2\varepsilon}$

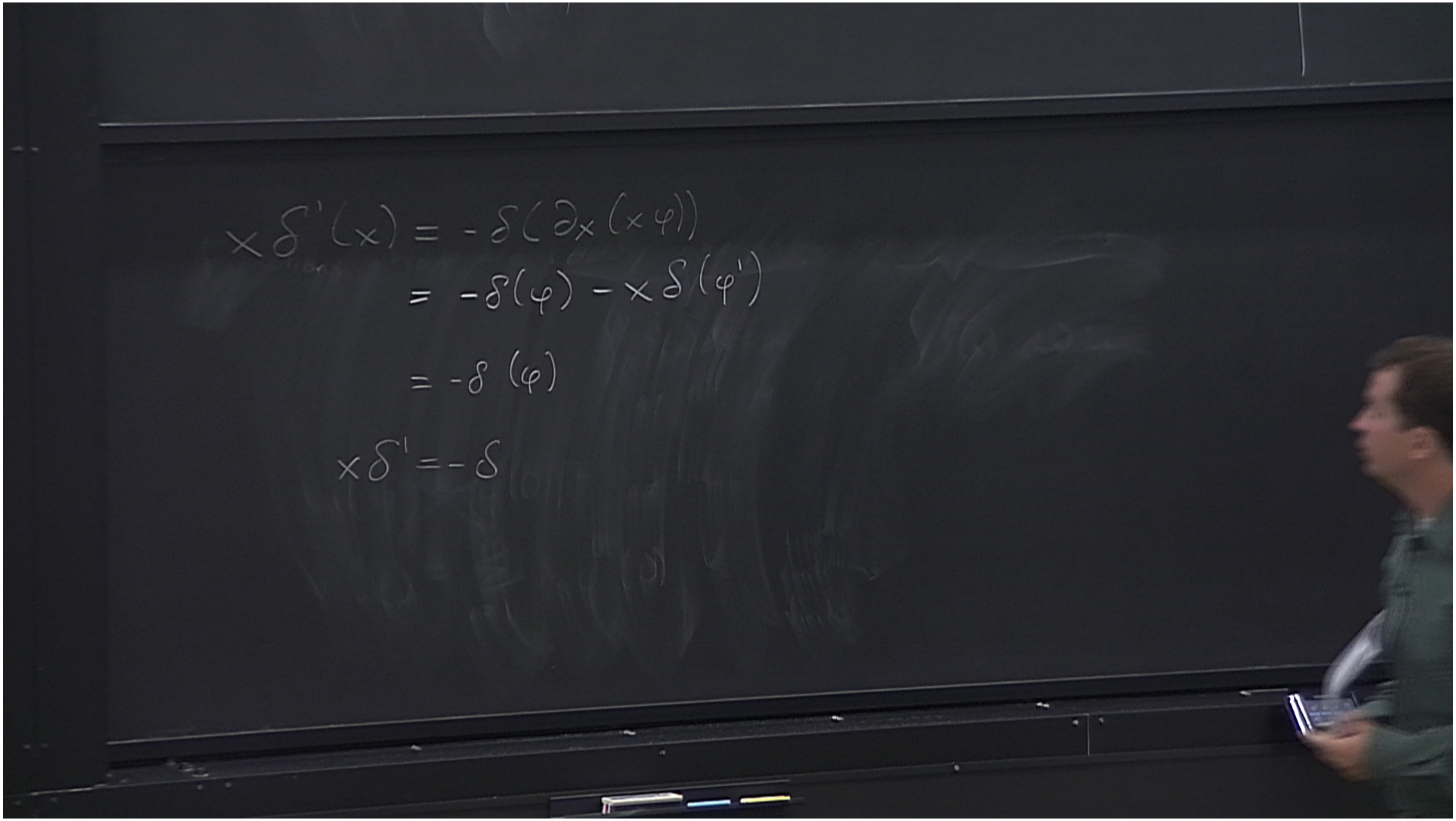
$$A = 2\varphi'(0)\,\varepsilon\ln|\varepsilon| \xrightarrow{\varepsilon\to 0} 0$$

$$(\ln|x|)'(\varphi) = \lim_{\epsilon\to 0^+}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) \frac{1}{x}\, \varphi(x)\,dx$$

$$= \left(PV\frac{1}{x}\right)(\varphi)$$


Multiplication by a Function
$(\Psi u)(\varphi) = \int_{-\infty}^{\infty} \Psi(x)\, u(x)\, \varphi(x)\, dx$
Ok if u is a function
$= \int_{-\infty}^{\infty} u(x)\, (\Psi(x)\varphi(x))\, dx$
$(\Psi u)(\varphi) \equiv u(\Psi\varphi)$


$\delta_a(\varphi) = \varphi(a)$
Example: $(\psi\delta_a)(\varphi) = \delta_a(\psi\varphi)$
$= (\psi\varphi)(a)$
$= \psi(a)\varphi(a)$
$= \psi(a)\delta_a(\varphi)$
This implies $x\delta = 0$


$x\delta'(x) = -\delta(\partial_x(x\varphi))$
$= -\delta(\varphi) - x\delta(\varphi')$
$= -\delta(\varphi)$
$x\delta' = -\delta$

Composition with a Function

$$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(f(x))\,\varphi(x)\,dx$$

Composition with a Function

$$(u\circ f)(\varphi) = \int_{-\infty}^{\infty} u(f(x))\,\varphi(x)\,dx$$

Natural to try $y=f(x)$

$(u\circ f)$ only guaranteed to exist if
- f is C^∞
- $y=f(x)$ has a unique solution x
- $g'(y)$ does not change sign

Composition with a Function

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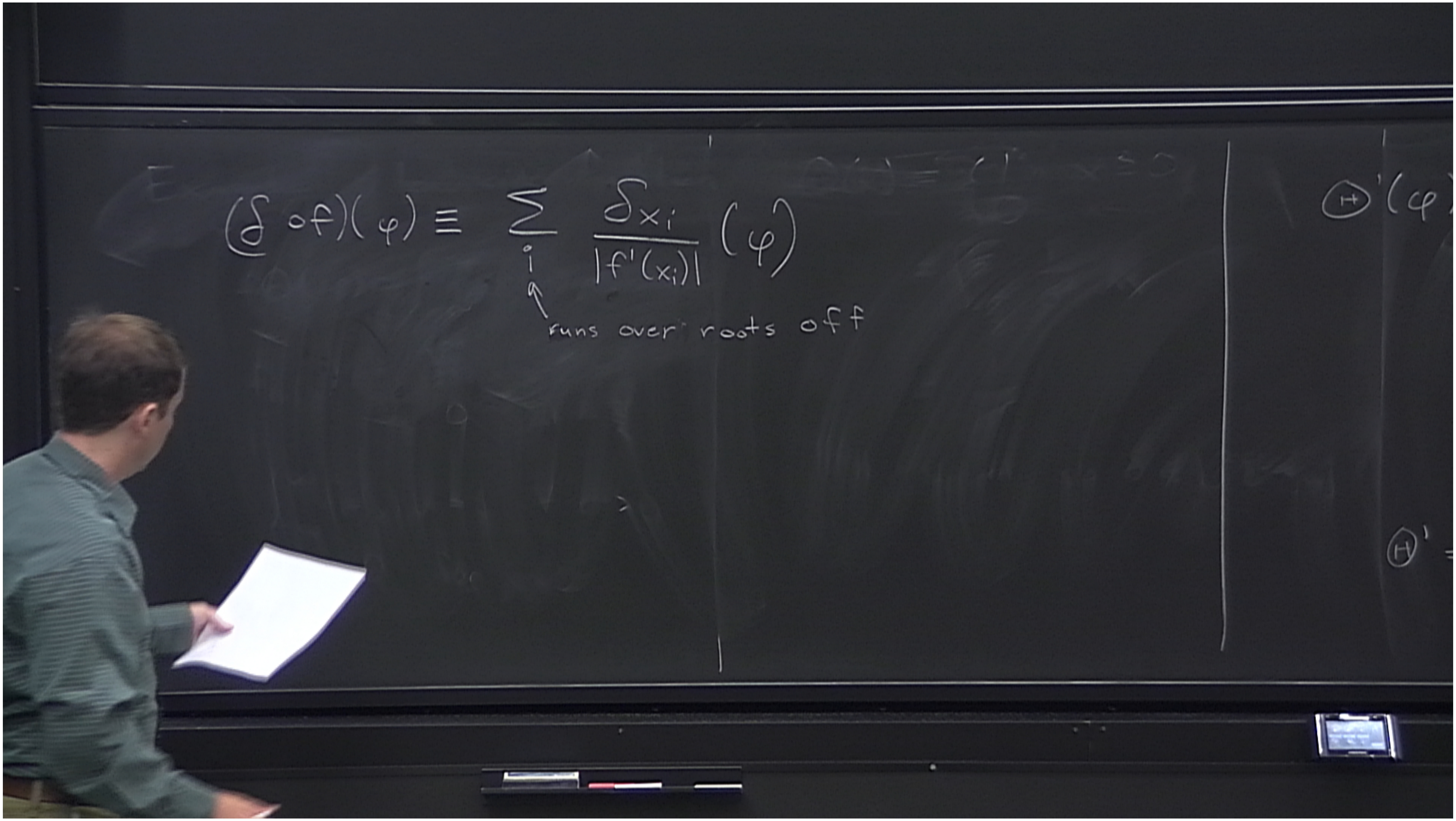
$(u \circ f)$ only guaranteed to exist if

- f is C^∞
- $y = f(x)$ has a unique solution, $x = g(y)$
- $g'(y)$ does not change sign


$(\delta \circ f)(\varphi) \equiv \sum_i \frac{\delta_{x_i}}{|f'(x_i)|}(\varphi)$
runs over roots of f

$$(\delta \circ f)(\varphi) \equiv \sum_i \frac{\delta_{x_i}}{|f'(x_i)|}(\varphi)$$

runs over roots of f

works if f has only simple roots
(no multiple roots)

$\Theta'(\varphi$

$\Theta' =$