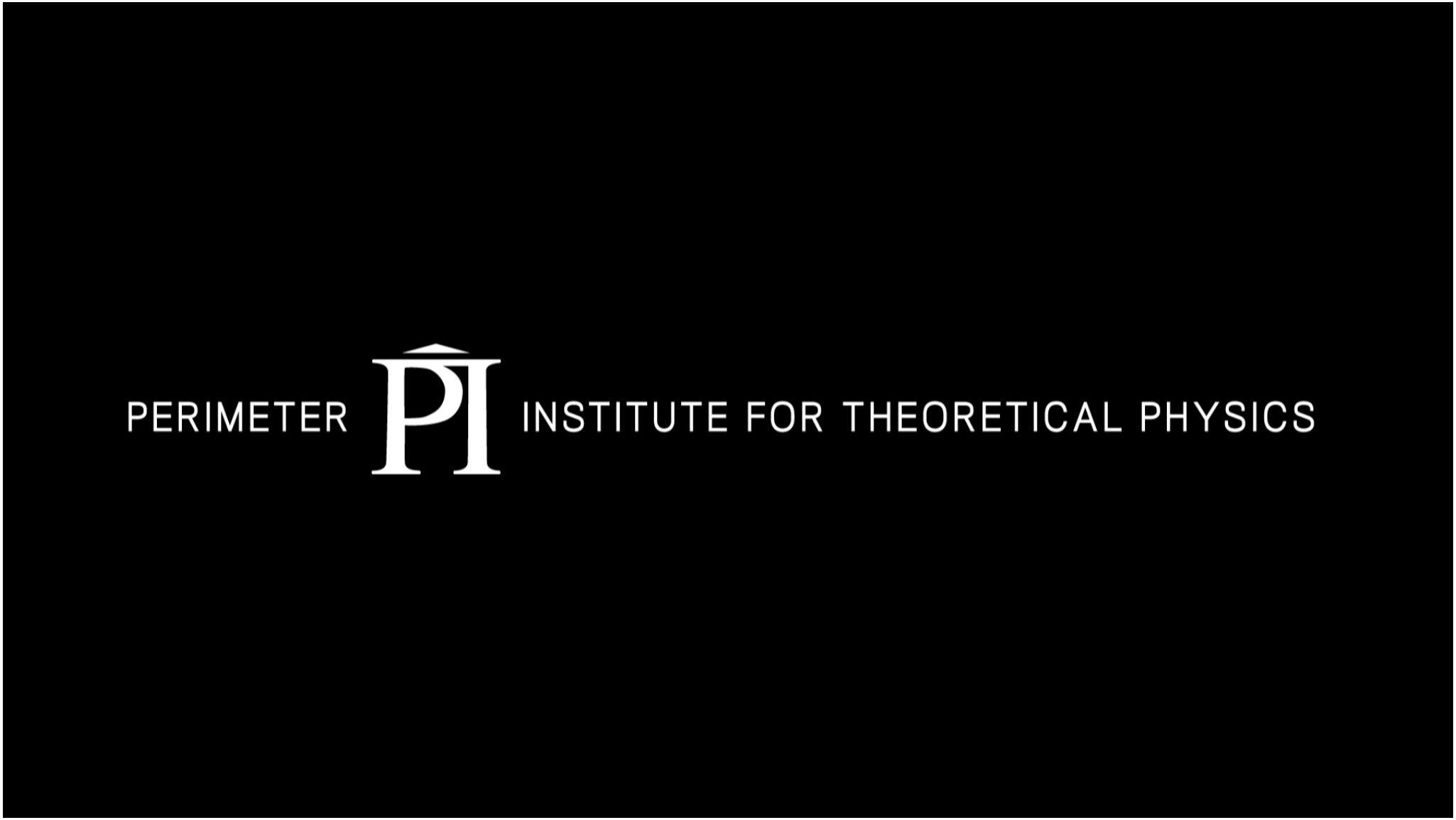
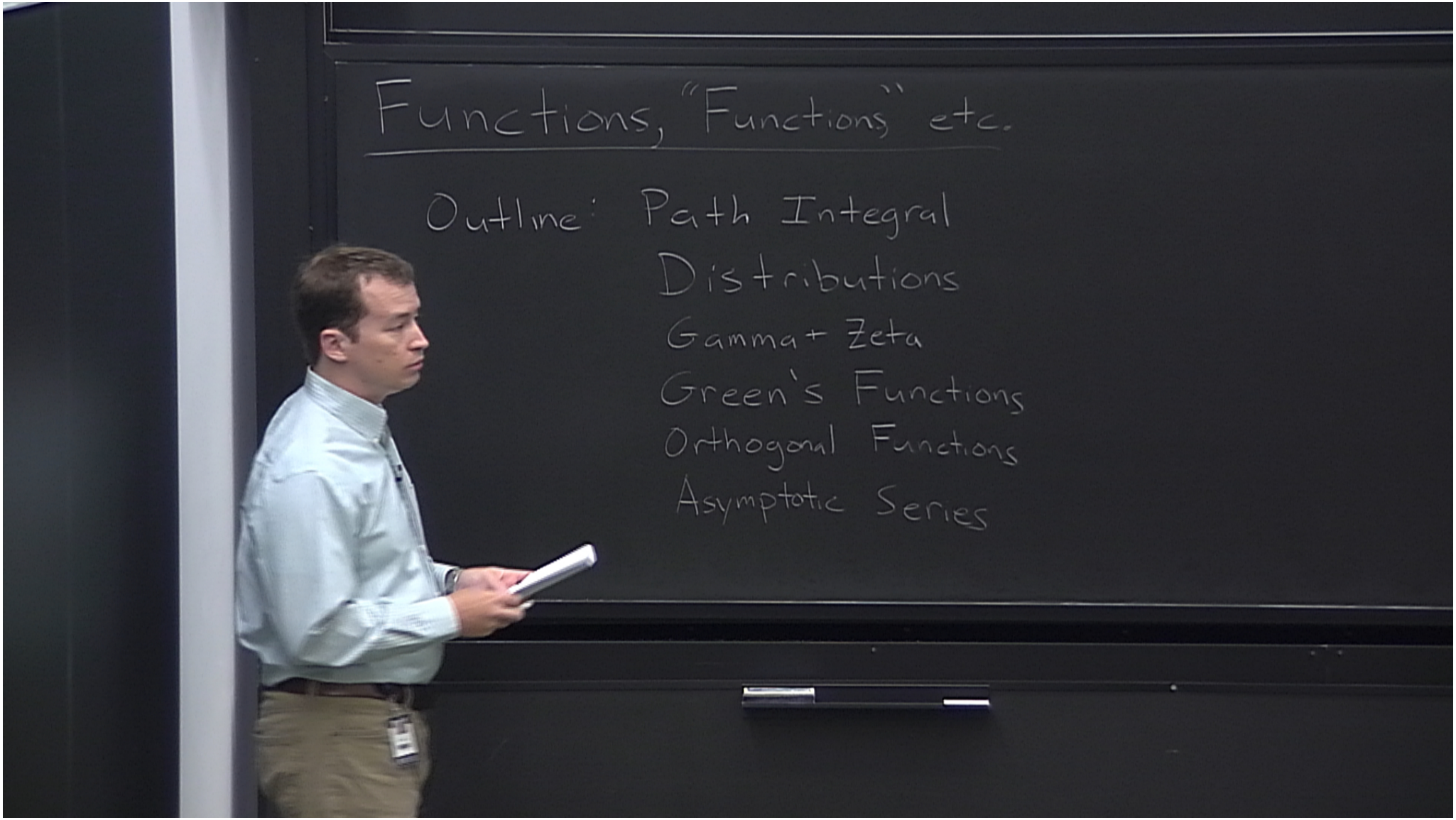
Title: Front End - Functions, "Functions", etc. - 1

Date: Aug 23, 2016 09:00 AM

URL: http://pirsa.org/16080075

Abstract:


PERIMETER
INSTITUTE FOR THEORETICAL PHYSICS


Functions, "Functions" etc.
Outline: Path Integral
Distributions
Gamma + Zeta
Green's Functions
Orthogonal Functions
Asymptotic Series

Path Integral

Motivation: equivalent to Heisenberg + Schrodinger

QFT

Particle in 1D:

$$H = \frac{p^2}{2m} + V(q)$$

$$A = \langle q' | e^{-iHT} | q \rangle \equiv K(q', T; q, 0)$$

amp to go from q at $t=0$ to q' at $t=T$

$\hbar = 1$

$$e^{-iHT} = e^{-iH(T-t_1)}\, e^{-iHt_1}$$

$$K = \langle q' | e^{-iH(T-t_1)}\, e^{-iHt_1} | q \rangle$$

$$\mathbb{1} = \int dq_1 \, |q_1\rangle\langle q_1|$$

$$= \int dq_1 \langle q' | e^{-iH(T-t_1)} |q_1\rangle\langle q_1| e^{-iHt_1} |q\rangle$$

$$= \int dq_1 \, K(q', T; q_1, t_1)\, K(q_1, t_1; q, 0)$$

$A = \langle q' | e^{-iHT} | q \rangle \equiv K(q', T; q, 0)$ $\quad \hbar = 1$

amp to go from q at $t=0$ to q' at $t=T$

Repeat using N

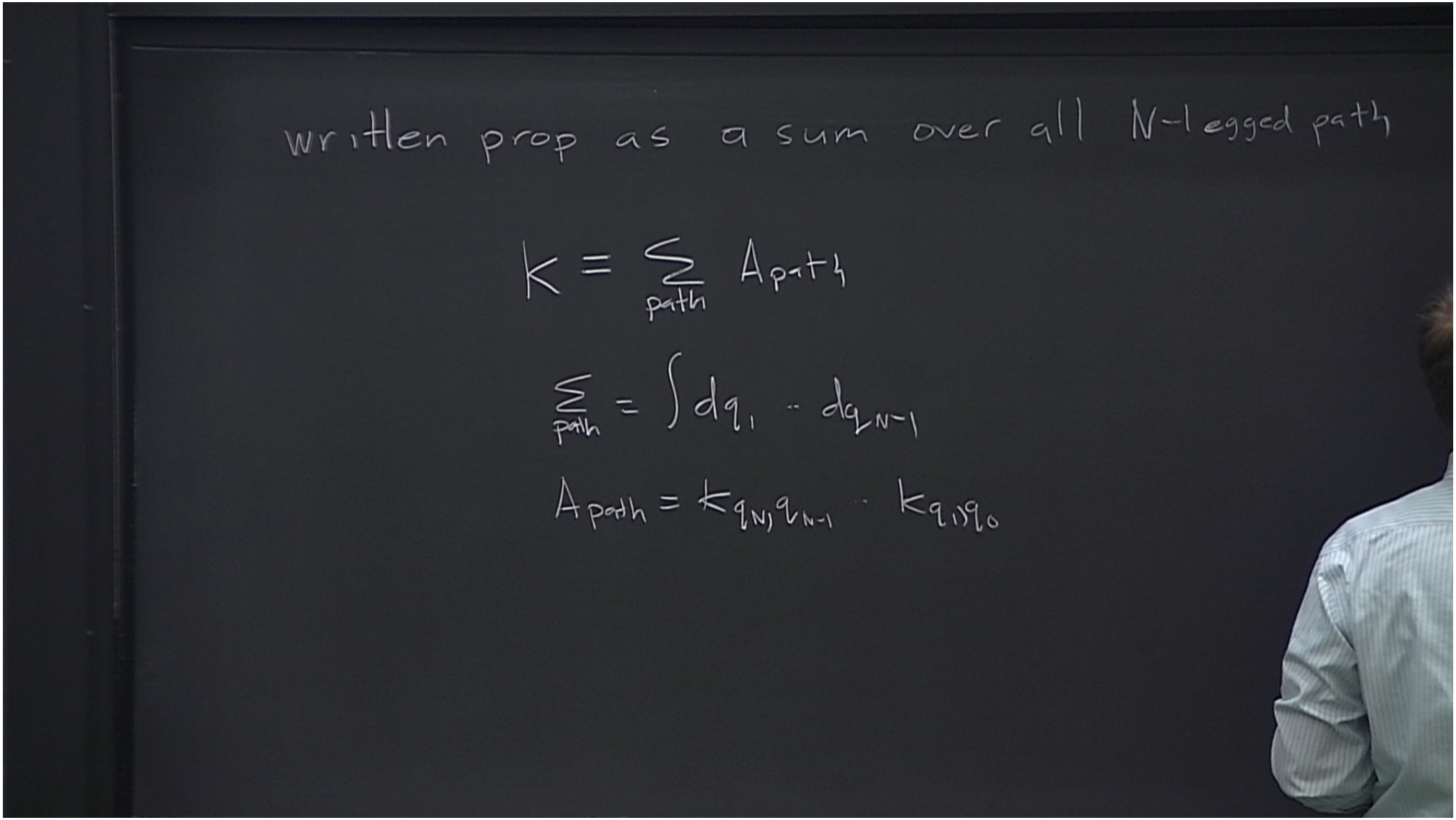
$$K = \langle q' | \underbrace{e^{-iH\Delta t} \cdots e^{-iH\Delta t}}_{N} | q \rangle$$

$$= \langle q' | e^{-iH\Delta t} \int dq_{N-1} \, | q_{N-1} \rangle \langle q_{N-1} | e^{-iH\Delta t} \cdots \int dq_1 \, | q_1 \rangle \langle q_1 | e^{-iH\Delta t} | q \rangle$$

$$= \int dq_1 \cdots dq_{N-1} \, K_{q_N, q_{N-1}} \cdots K_{q_1, q_0}$$

$$K_{q_{j+1}, q_j} \equiv \langle q_{j+1} | e^{-iH\Delta t} | q_j \rangle$$

$q \equiv q_0$
$q' = q_N$


written prop as a sum over all N-1 legged paths
$K = \sum_{\text{path}} A_{\text{path}}$
$\sum_{\text{path}} = \int dq_1 \cdots dq_{N-1}$
$A_{\text{path}} = K_{q_N q_{N-1}} \cdots K_{q_1 q_0}$

$\int dq_1 \, K(q', T; q_1, t_1) K(q_1, t_1; q, 0)$

$= \int dq_1 \cdots dq_{N-1} \, K_{q_N, q_{N-1}} \cdots K_{q_1, q_0}$

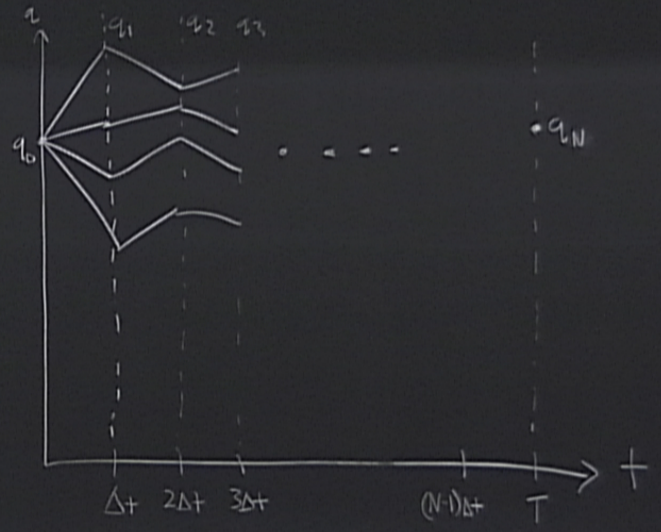
$K_{q_{j+1}, q_j} = \langle q_{j+1} | e^{-iH\Delta t} | q_j \rangle$ $\quad q \equiv q_0$, $q' = q_N$

prop as a sum over all N-1 legged paths

$$K = \sum_{\text{paths}} A_{\text{path}}$$

$$\sum_{\text{paths}} = \int dq_1 \cdots dq_{N-1}$$

$$A_{\text{path}} = K_{q_N, q_{N-1}} \cdots K_{q_1, q_0}$$



$dq_1\, K(q', T; q_1, t_1)\, K(q_1, t_1; q, 0)$

$= \int dq_1 \cdots dq_{N-1}\, K_{q_N, q_{N-1}} \cdots K_{q_1, q_0}$

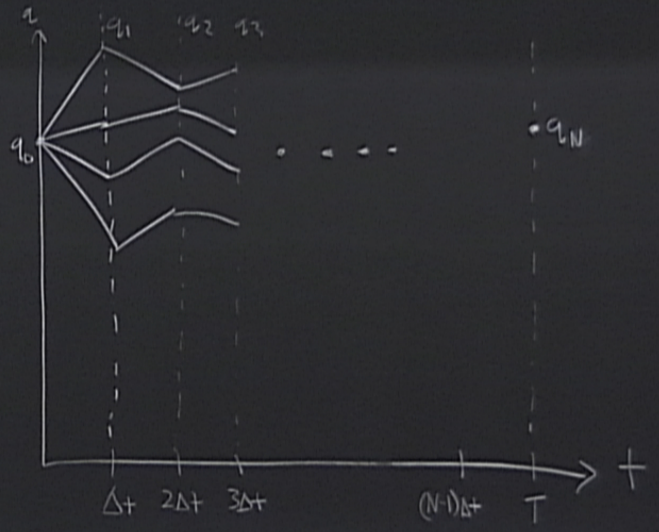
$K_{q_{j+1}, q_j} \equiv \langle q_{j+1} | e^{-iH\Delta t} | q_j \rangle$ $\quad q \equiv q_0$, $q' = q_N$

prop as a sum over all N-1 legged paths

$K = \sum_{\text{paths}} A_{\text{path}}$

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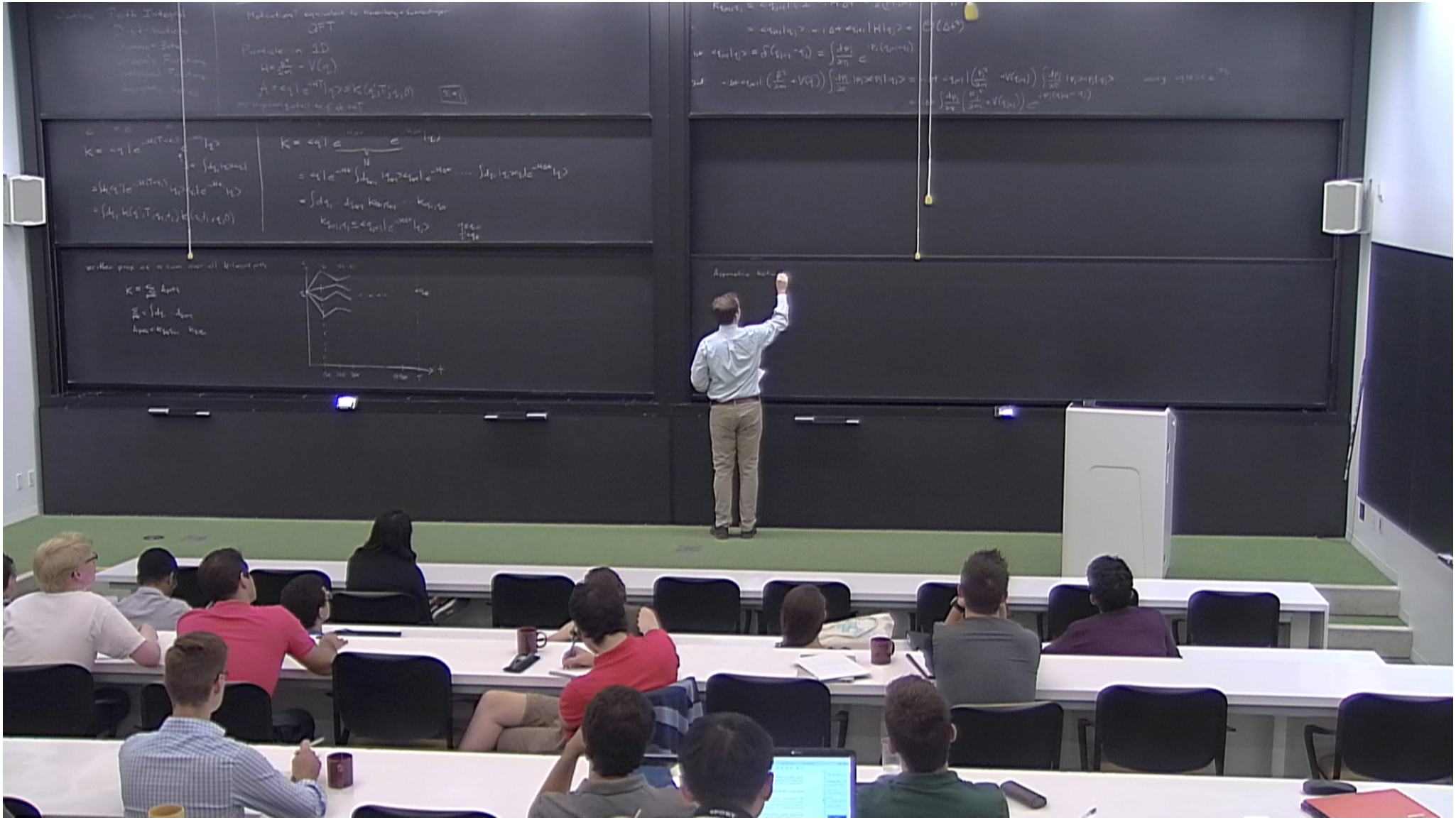


To get a path integral take $N \to \infty$, $\Delta t \to 0$

$$K_{q_{j+1},q_j} = \langle q_{j+1} | \left(1 - iH\Delta t + \frac{1}{2}(H\Delta t)^2 + \dots\right) | q_j \rangle$$

$$= \langle q_{j+1} | q_j \rangle - i\Delta t \langle q_{j+1} | H | q_j \rangle + \Theta(\Delta t^2)$$

$$\langle q_{j+1} | q_j \rangle = \delta(q_{j+1} - q_j) = \int \frac{dp_j}{2\pi\hbar} e^{ip_j(q_{j+1} - q_j)}$$


Particle in 1D
written prop as a sum over all histories

To get a path integral take $N \to \infty$, $\Delta t \to 0$

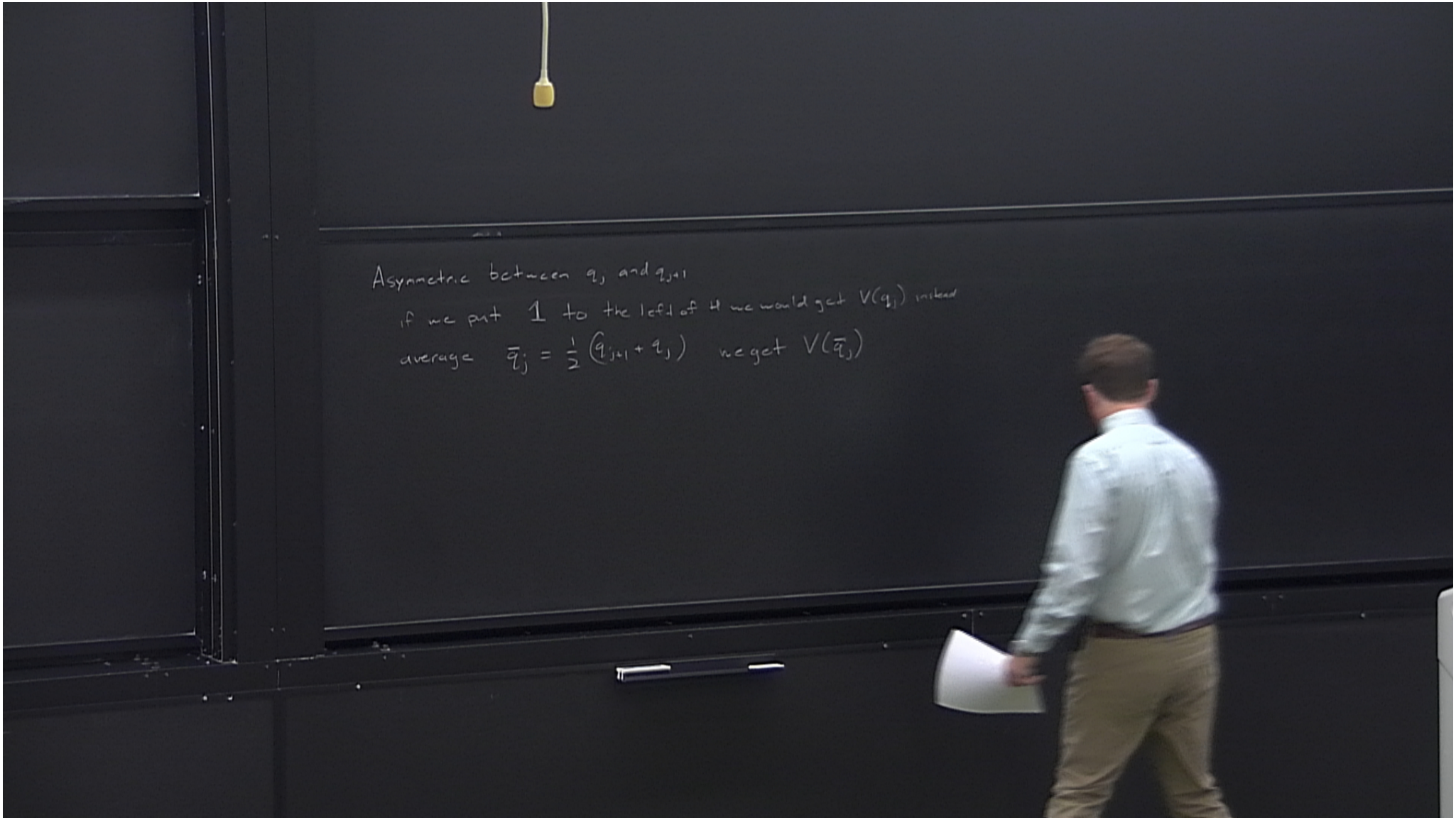
$$K_{q_{j+1} q_j} = \langle q_{j+1} | \left(1 - iH\Delta t + \tfrac{1}{2}(H\Delta t)^2 + \cdots\right) | q_j \rangle$$

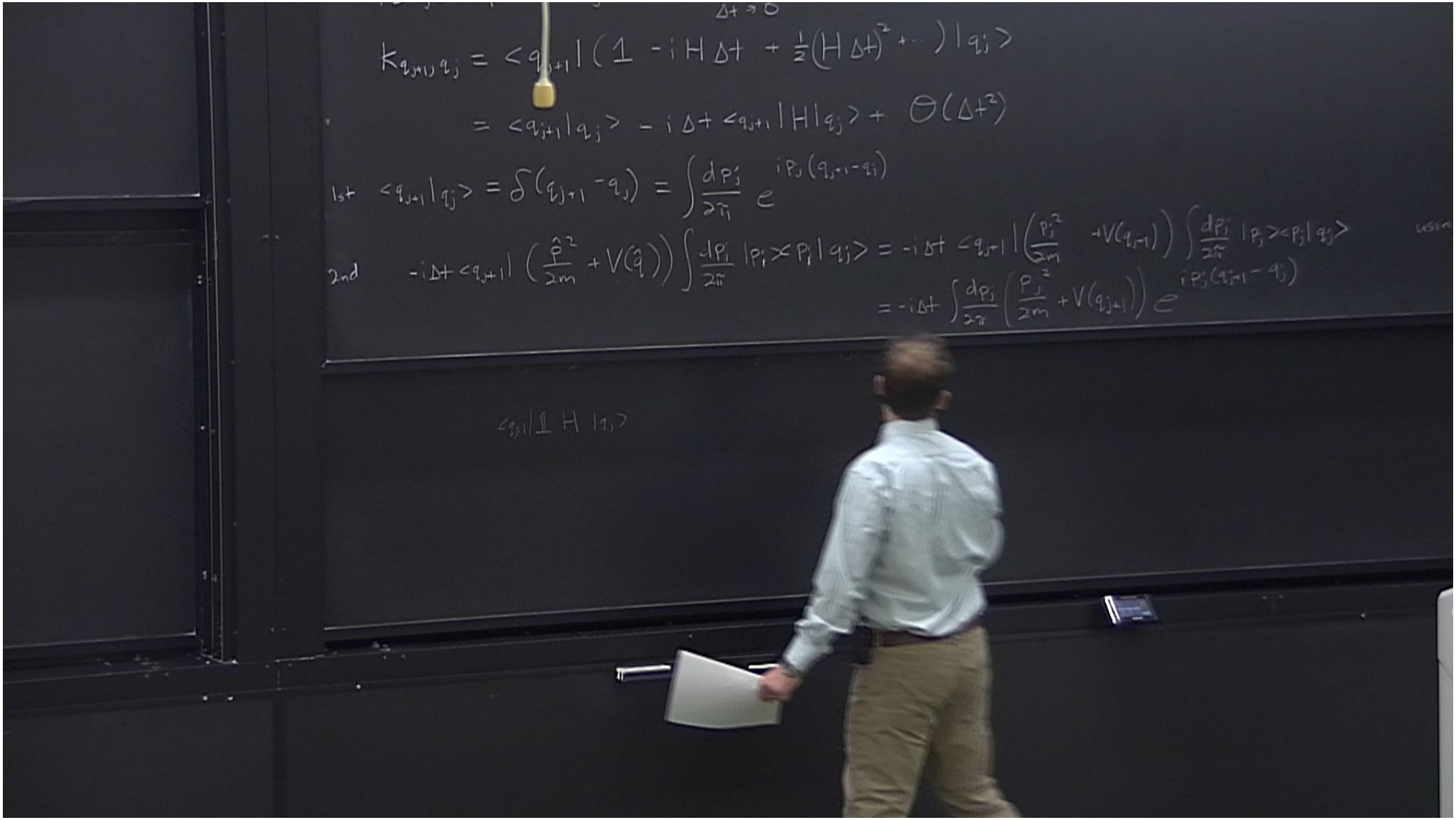
$$= \langle q_{j+1} | q_j \rangle - i\Delta t \langle q_{j+1} | H | q_j \rangle + \Theta(\Delta t^2)$$

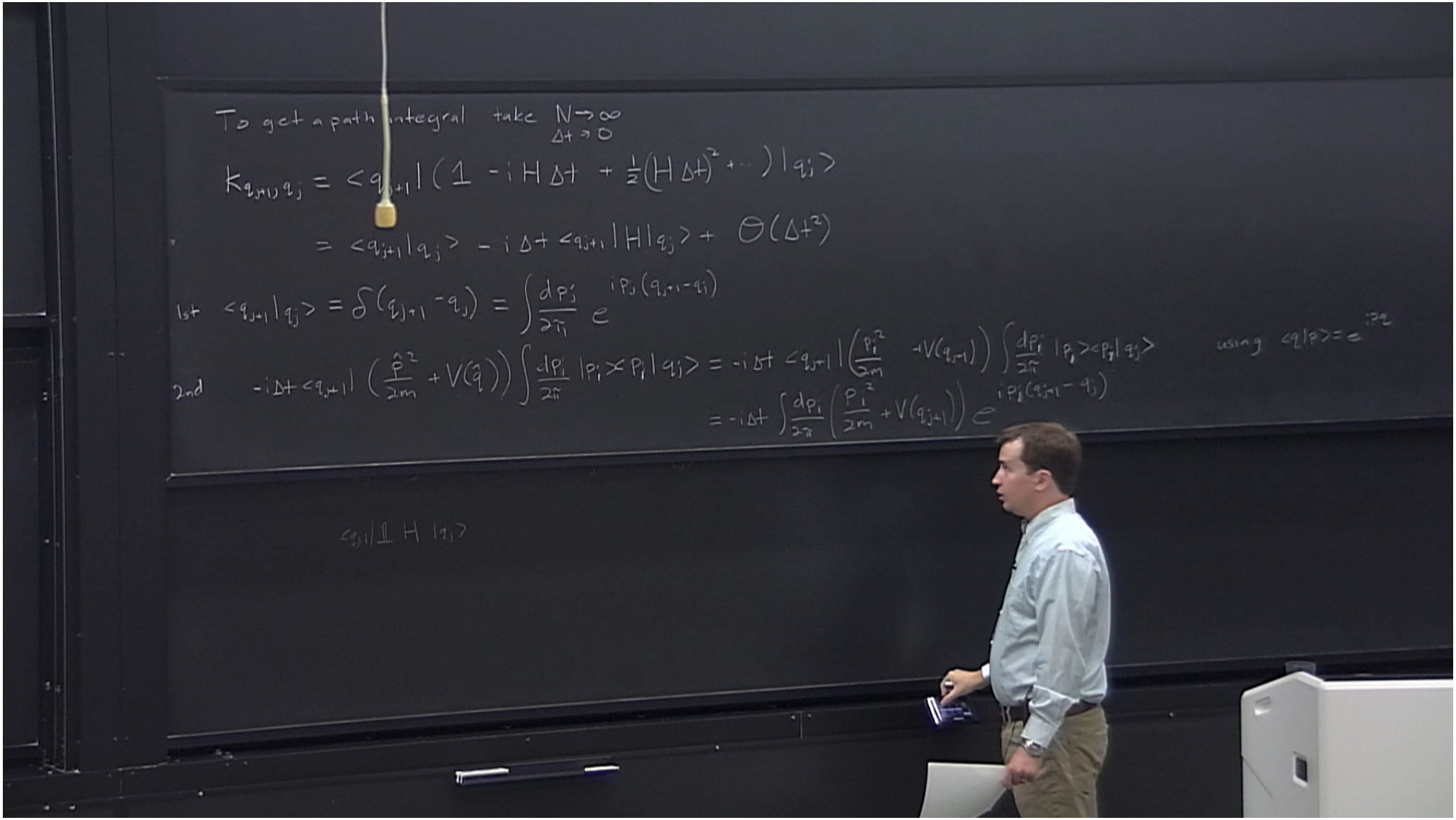
1st $\langle q_{j+1} | q_j \rangle = \delta(q_{j+1} - q_j) = \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1} - q_j)}$

2nd $-i\Delta t \langle q_{j+1} | \left(\frac{\hat{p}^2}{2m} + V(\hat{q})\right) \int \frac{dp_j}{2\pi} |p_j\rangle\langle p_j | q_j \rangle = -i\Delta t \langle q_{j+1} | \left(\frac{p_j^2}{2m} + V(q_{j+1})\right) \int \frac{dp_j}{2\pi} |p_j\rangle\langle p_j | q_j \rangle$ using $\langle q | p \rangle = e^{ipq}$

$$= -i\Delta t \int \frac{dp_j}{2\pi} \left(\frac{p_j^2}{2m} + V(q_{j+1})\right) e^{ip_j(q_{j+1} - q_j)}$$


Asymmetric between q_j and q_{j+1}
if we put 1 to the left of H we would get $V(q_j)$ instead
average $\bar{q}_j = \frac{1}{2}(q_{j+1} + q_j)$ we get $V(\bar{q}_j)$


$\Delta t \to 0$
$K_{q_{j+1},q_j} = \langle q_{j+1} | \left(1 - iH\Delta t + \frac{1}{2}(H\Delta t)^2 + \cdots\right) | q_j \rangle$
$= \langle q_{j+1} | q_j \rangle - i\Delta t \langle q_{j+1} | H | q_j \rangle + \Theta(\Delta t^2)$
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$\langle q_{j+1} | \mathbb{1} H | q_j \rangle$


To get a path integral take $N \to \infty$, $\Delta t \to 0$
$K_{q_{j+1},q_j} = \langle q_{j+1}| \left(1 - iH\Delta t + \frac{1}{2}(H\Delta t)^2 + \cdots\right) |q_j\rangle$
$= \langle q_{j+1}|q_j\rangle - i\Delta t \langle q_{j+1}|H|q_j\rangle + \Theta(\Delta t^2)$
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2nd $-i\Delta t \langle q_{j+1}| \left(\frac{\hat{p}^2}{2m} + V(q)\right) \int \frac{dp_i}{2\pi} |p_i\rangle\langle p_i|q_j\rangle = -i\Delta t \langle q_{j+1}| \left(\frac{p_i^2}{2m} + V(q_{j+1})\right) \int \frac{dp_i}{2\pi} |p_i\rangle\langle p_i|q_j\rangle$ using $\langle q|p\rangle = e^{ipq}$
$= -i\Delta t \int \frac{dp_i}{2\pi} \left(\frac{p_i^2}{2m} + V(q_{j+1})\right) e^{ip_i(q_{j+1} - q_j)}$
$\langle q_{j+1}| \mathbb{1} H |q_j\rangle$

$$= -i\Delta t \int \frac{dp_i}{2\pi} \left(\frac{p_i}{2m} + V(q_{j+1}) \right) e^{ip_i(q_{j+1}-q_j)}$$

Asymmetric between q_j and q_{j+1}

if we put $\mathbb{1}$ to the left of H we would get $V(q_j)$ instead

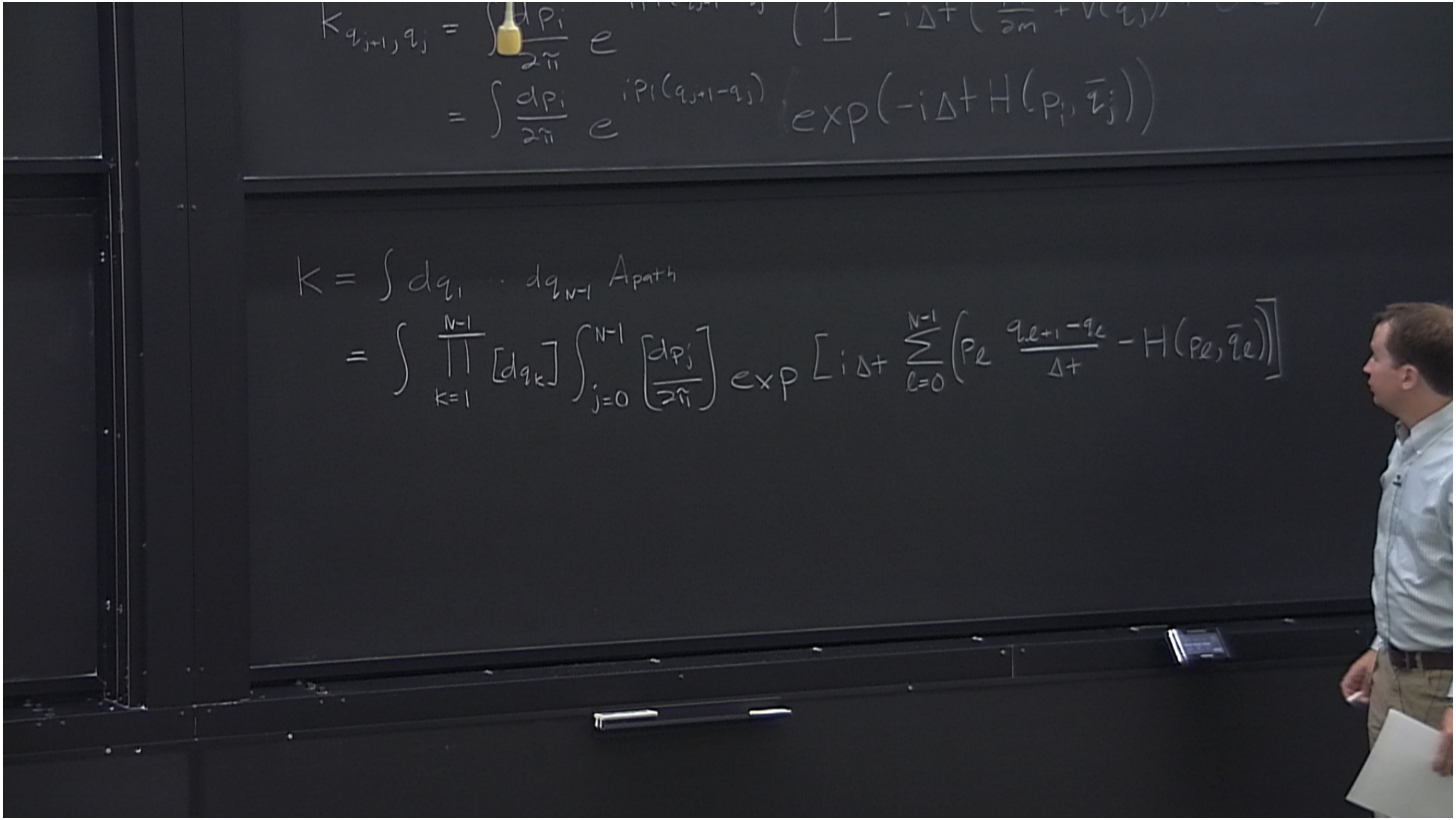
average $\bar{q}_j = \frac{1}{2}(q_{j+1} + q_j)$ we get $V(\bar{q}_j)$

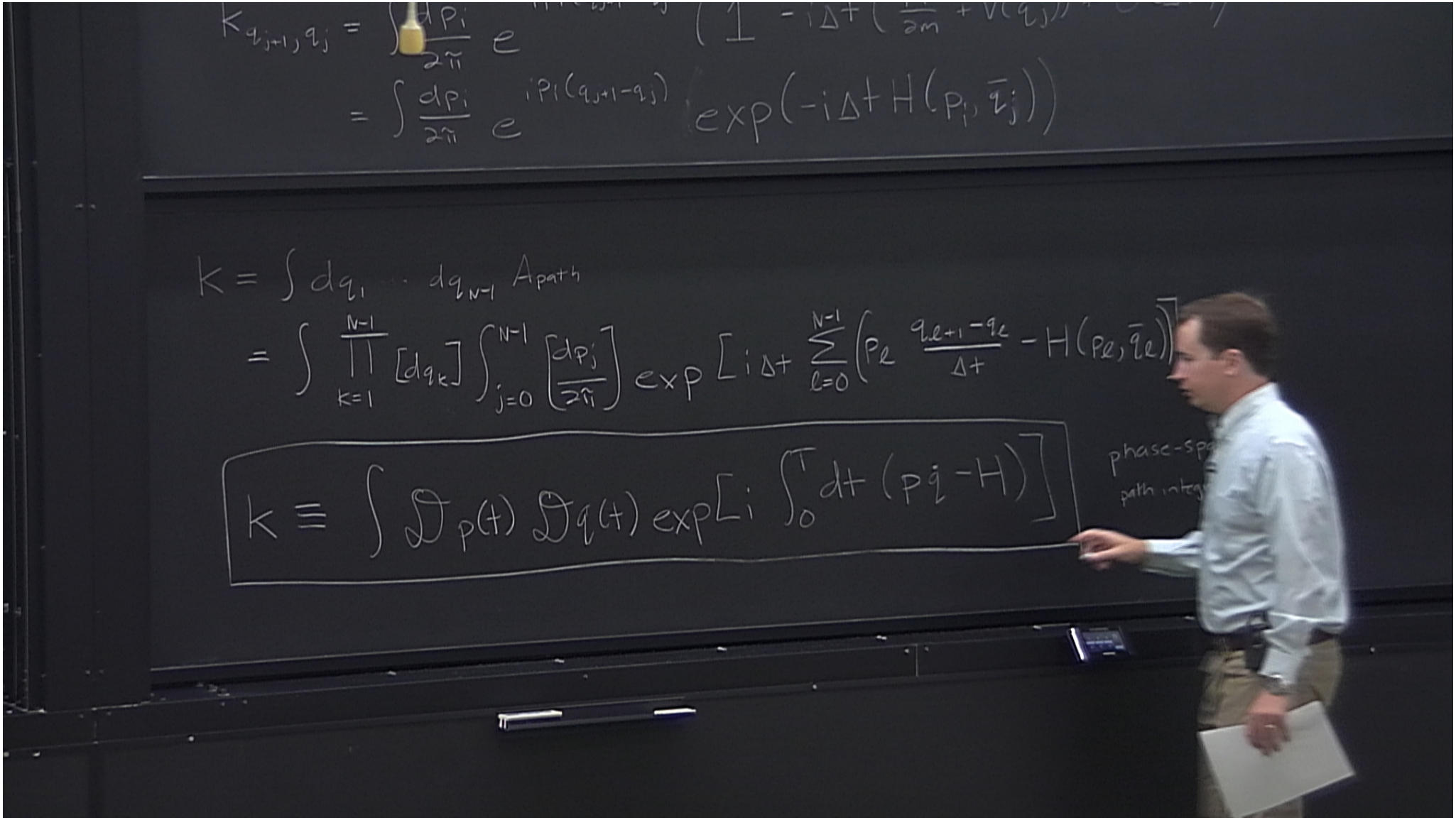
$$\langle q_{j+1} | \mathbb{1} H | q_j \rangle$$

$$K_{q_{j+1}, q_j} = \int \frac{dp_i}{2\pi} e^{ip_i(q_{j+1}-q_j)} \left(1 - i\Delta t \left(\frac{p_i^2}{2m} + V(\bar{q}_j) \right) + \mathcal{O}(\Delta t^2) \right)$$

$$= \int \frac{dp_i}{2\pi} e^{ip_i(q_{j+1}-q_j)} \exp\left(-i\Delta t H(p_i, \bar{q}_j) \right)$$

$$K = \int dq_1 \cdots dq_{N-1} A_{path}$$


$K_{q_{j+1},q_j} = \int \frac{dp_i}{2\pi} e^{ip_i(q_{j+1}-q_j)} \exp(-i\Delta t H(p_i, \bar{q}_j))$
$K = \int dq_1 \cdots dq_{N-1} A_{path}$
$= \int \prod_{k=1}^{N-1} [dq_k] \int_{j=0}^{N-1} \left[\frac{dp_j}{2\pi}\right] \exp\left[i\Delta t \sum_{\ell=0}^{N-1} \left(p_\ell \frac{q_{\ell+1}-q_\ell}{\Delta t} - H(p_\ell, \bar{q}_\ell)\right)\right]$


$K_{q_{j+1},q_j} = \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} \exp(-i\Delta t\, H(p_j, \bar{q}_j))$
$K = \int dq_1 \cdots dq_{N-1}\, A_{path}$
$= \int \prod_{k=1}^{N-1} [dq_k] \int_{j=0}^{N-1} \left[\frac{dp_j}{2\pi}\right] \exp\left[i\Delta t \sum_{\ell=0}^{N-1} \left(p_\ell \frac{q_{\ell+1}-q_\ell}{\Delta t} - H(p_\ell, \bar{q}_\ell)\right)\right]$
$K \equiv \int \mathcal{D}p(t)\, \mathcal{D}q(t) \exp\left[i \int_0^T dt\, (p\dot{q} - H)\right]$
phase-sp
path integ

$$K_{q_{j+1},q_j} = \int \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} \exp\left(-i\Delta t\, H(p_j, \bar{q}_j)\right)$$

$$K = \int dq_1 \cdots dq_{N-1}\, A_{path}$$

$$= \int \prod_{k=1}^{N-1} [dq_k] \int_{j=0}^{N-1} \left[\frac{dp_j}{2\pi}\right] \exp\left[i\Delta t \sum_{\ell=0}^{N-1} \left(p_\ell \frac{q_{\ell+1}-q_\ell}{\Delta t} - H(p_\ell, \bar{q}_\ell)\right)\right]$$

$$\boxed{K \equiv \int \mathcal{D}p(t)\, \mathcal{D}q(t) \exp\left[i \int_0^T dt\, (p\dot{q} - H)\right]}$$

phase-space path integral

$q(0) = q$ and $q(T) = q'$



$= \int dq_1 \cdots dq_{N-1} K_{q_N, q_{N-1}} \cdots K_{q_1, q_0}$

$(q_1, t_1; q, 0)$

$K_{q_{j+1}, q_j} \equiv \langle q_{j+1} | e^{-iH\Delta t} | q_j \rangle$ $\quad q \equiv q_0$, $q' = q_N$

etc.

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Interlude: Gaussian Integrals

Main tool for evaluating path or functional integrals

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$= \int dq_1 \cdots dq_{N-1} K_{q_N, q_{N-1}} \cdots K_{q_1, q_0}$

$K_{q_{j+1}, q_j} \equiv \langle q_{j+1} | e^{-iH\Delta t} | q_j \rangle$

$q \equiv q_0$

$q' = q_N$

Interlude: Gaussian Integrals

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$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$$\int dx \exp\left[-ax^2 + bx + c\right] = \sqrt{\frac{\pi}{a}}\left[\frac{b^2}{4a} + c\right]$$

$= \int dq_1 \cdots dq_{N-1} \, K_{q_N, q_{N-1}} \cdots K_{q_1, q_0}$

$q, 0)$

$K_{q_{j+1}, q_j} \equiv \langle q_{j+1} | e^{-iH\Delta t} | q_j \rangle$

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Orthogonal Functions
Asymptotic Series

If $H = \frac{p^2}{2m} + V(q)$ we can do p integrals

$$K = \int \prod_{j=1}^{N-1} [dq_j] \exp\left[-i\Delta t \sum_{k=0}^{N-1} V(\bar{q}_k)\right] \int \prod_{\ell=0}^{N-1} \left[\frac{dp_\ell}{2\pi}\right] \exp\left[i\Delta t \sum_{n=0}^{N-1} \left(p_n \dot{q}_n - \frac{p_n^2}{2m}\right)\right]$$

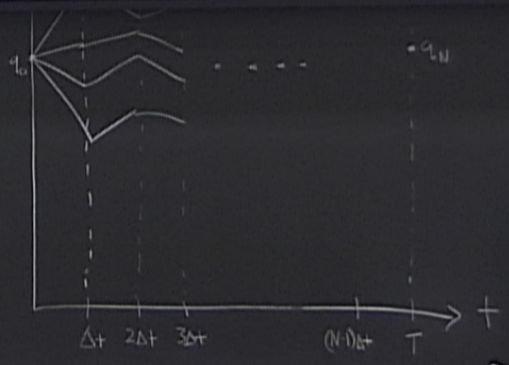
$$\int \frac{dp}{2\pi} e^{i\Delta t (p\dot{q} - \frac{p^2}{2m})} = \sqrt{\frac{m}{2\pi i \Delta t}}\, e^{i\Delta t\, m\dot{q}^2/2}$$

$$K = \int \prod_{j=1}^{N-1} [dq_j] \exp\left[-i\Delta t \sum_{k=0}^{N-1} V(\bar{q}_k)\right] \prod_{\ell=0}^{N-1} \left(\sqrt{\frac{m}{2\pi i \Delta t}}\, e^{i\Delta t\, m\dot{q}_\ell^2/2}\right) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} \int \prod_{j=1}^{N-1} [dq_j] \exp\left[i\Delta t \sum_{k=0}^{N-1} \left(\frac{m\dot{q}_k^2}{2} - V(\bar{q}_k)\right)\right]$$

$$K = \sum_{\text{paths}} A_{\text{path}}$$

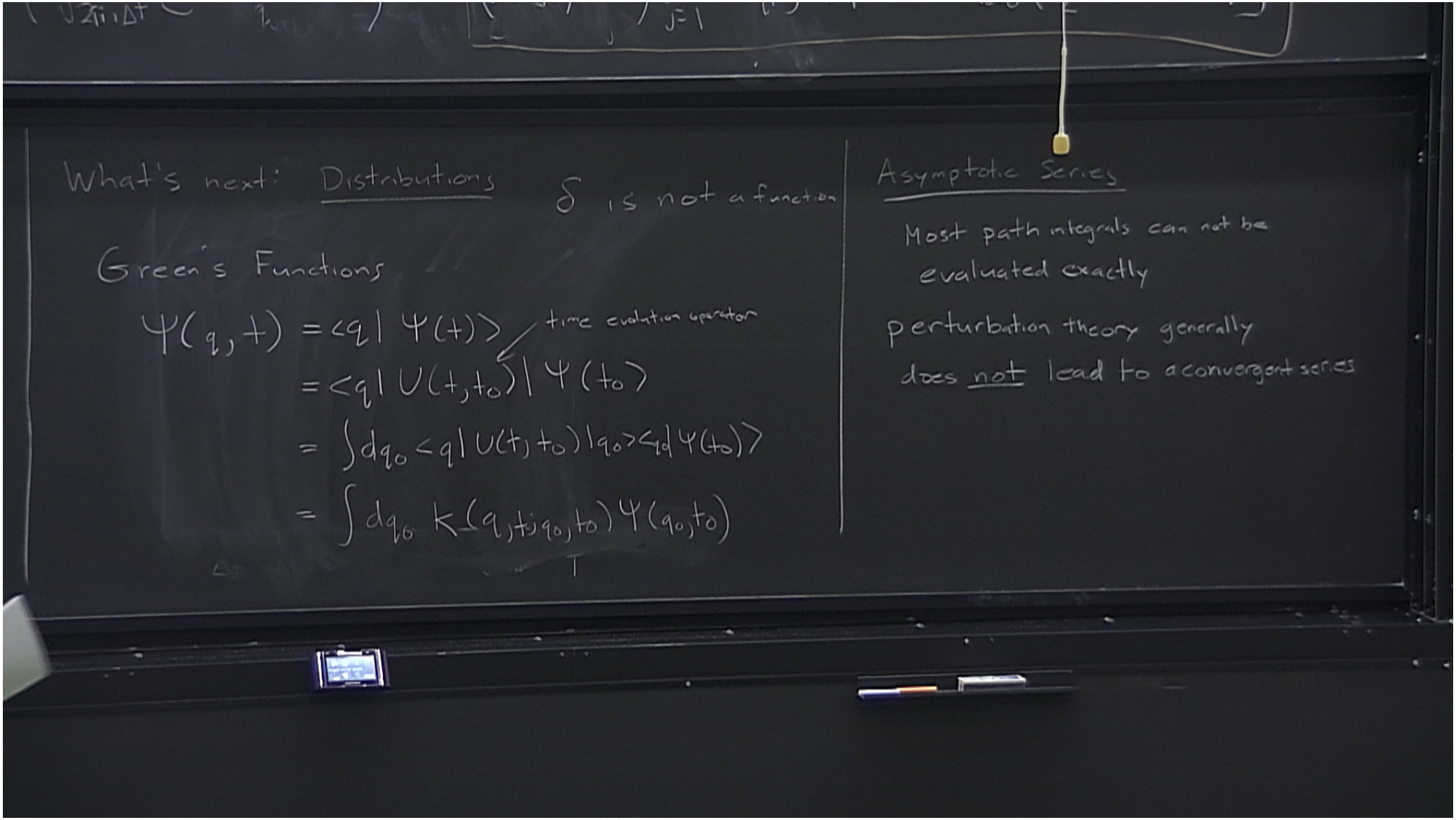
$$\sum_{\text{paths}} = \int dq_1 \cdots dq_{N-1}$$

$$A_{\text{path}} = K_{q_N q_{N-1}} \cdots K_{q_1 q_0}$$



$\Delta t \to 0$

$$K = \int \mathcal{D}q(t)\, e^{iS[q(t)]}$$


What's next: Distributions
δ is not a function
Green's Functions
$\Psi(q,t) = \langle q|\Psi(t)\rangle$
time evolution operator
$= \langle q|U(t,t_0)|\Psi(t_0)\rangle$
$= \int dq_0 \langle q|U(t,t_0)|q_0\rangle\langle q_0|\Psi(t_0)\rangle$
$= \int dq_0\, K(q,t;q_0,t_0)\Psi(q_0,t_0)$
Asymptotic Series
Most path integrals can not be evaluated exactly
perturbation theory generally does not lead to a convergent series

$$K_{q_{j+1},q_j} = \int \frac{dp_i}{2\pi} e^{ip_i(q_{j+1}-q_j)} \left(1 - i\Delta t\left(\frac{p_i^2}{2m} + V(\bar{q}_j)\right) + \mathcal{O}(\Delta t^2)\right)$$

$$= \int \frac{dp_i}{2\pi} e^{ip_i(q_{j+1}-q_j)} \exp\left(-i\Delta t H(p_i, \bar{q}_j)\right)$$

Distributions

Motivation:

- point charge $\rho(\vec{x}) = \begin{cases} 0 & \vec{x} \neq 0 \\ \infty & \vec{x} = 0 \end{cases}$

$$\int \rho(\vec{x}) d^n\vec{x} = q$$

- $\frac{d^n}{dx^n}|x|$ at $x = 0$?
- $\int_{\mathbb{R}} \frac{1}{x} dx$

$$= \int \frac{dp_i}{2\pi} e^{ip_i(q_{j+1}-q_j)} \exp\left(-i\Delta t H(p_i, \bar{q}_j)\right)$$

Distributions

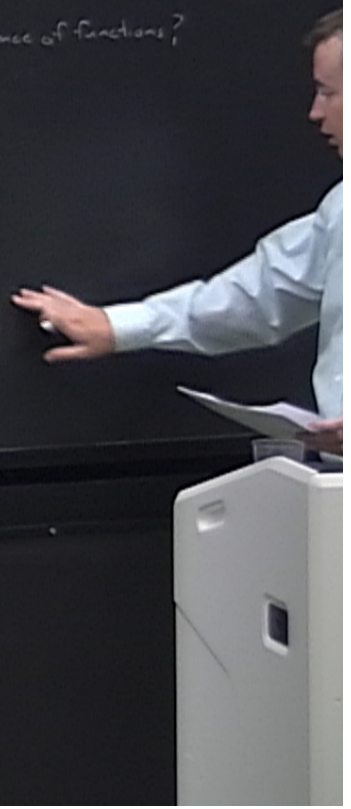
Motivation:

- point charge $\rho(\vec{x}) = \begin{cases} 0 & \vec{x} \neq 0 \\ \infty & \vec{x} = 0 \end{cases}$

$$\int \rho(\vec{x}) d^3\vec{x} = q$$

- $\frac{d^n}{dx^n}|x|$ at $x=0$?
- $\int_{\mathbb{R}} \frac{1}{x} dx$

- existence of limits of a sequence of functions?
- variational problems



Most important example is δ

$\delta(x)$ "=" $f(x) = \lim_{\sigma \to 0} f_\sigma(x)$

$$f_\sigma(x) = \begin{cases} \frac{1}{\sigma} & \text{for } |x| < \frac{\sigma}{2} \\ 0 & |x| \geq \frac{\sigma}{2} \end{cases}$$

$$\int_{-\infty}^{\infty} f_\sigma(x)\,dx = 1 \quad \text{for all } \sigma > 0$$

$f(x) = 0$ for $x \neq 0$

