Title: Front End - Lie Groups and Lie Algebras - 4

Date: Aug 19, 2016 09:00 AM

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Abstract:

Everything about SU(2)

Recap $[t_a, t_b] = i f_{abc} t_c$

f_{abc} real

anti-symmetric (if compact)

Jacobi Identity

Adjoint Representation $[T_a]_{bc} = -i f_{abc}$ $[T_a, T_b] = i f_{abc} T_c$

$O(B_1)$ is our only hope. Lie

$R^2 \times D_2$ X

$SU(2)$ is our only hope

$$[t_a, t_b] = i f_{abc} t_c$$

$$\varepsilon_{123} = 1$$

$$\varepsilon_{213} = -1$$

$$[J_i, J_j] = i \varepsilon_{ijk} J_k.$$

General Recipe. Representation Hunting.

1) Find the maximal set of generators that muturally commute

$\{H_1, H_2, \ldots\}$ $\quad [H_i, H_j] = 0$. Cartan subalgebra.

– rank.

For SU(2). – rank-1 $\quad J_3$

...ting

generators

commute

Cartan subalgebra

$\downarrow$

J_3

2) organize the rest of the generators.

$$[H_i, \cdot\,] = \text{const}\ \cdot$$

$$H_i |\cdot\rangle = \text{const}\, |\cdot\rangle$$

3) Highest weight (eigenvalue) method to build the irrep.

Representation $(T_a)_{bc} = -if_{abc}$ $\quad [T_a, T_b] = if_{abc} T_c$

1) J_3 is special

2) $J_1 + aJ_2$

$[J_3, J_1 + aJ_2]$

$= iJ_2 - iaJ_1$

$= \lambda(J_1 + aJ_2)$

$\lambda = -ia$

$i = \lambda a$

$\Downarrow$

$a^2 = -1$

$a = \pm i$

$J_\pm = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2)$

$[J_3, J_+] = J_+$

$[J_3, J_-] = J_-$

$[J_+, J_-] = \frac{1}{2}[J_1 + iJ_2, J_1 - iJ_2]$

$= \frac{1}{2}((-i)iJ_3 + i(-i)J_3)$

$= J_3$

3) $J_3|m\rangle = m|m\rangle$

$$J_3(J_\pm|m\rangle) = ([J_3, J_\pm] + J_\pm J_3)|m\rangle$$

$$= (\pm J_\pm + J_\pm m)|m\rangle$$

$$= (m\pm1)(J_\pm|m\rangle)$$

$$J_\pm|m\rangle \propto |m\pm1\rangle$$

$J_1 - iJ_2)$

$i) J_3)$

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "J_+"] & {|m-2\rangle} \arrow[l, bend left, "J_-"] \arrow[r, bend left, "J_+"] & {|m-1\rangle} \arrow[l, bend left, "J_-"] \arrow[r, bend left, "J_+"] & {|m\rangle} \arrow[l, bend left, "J_-"] \arrow[r, bend left, "J_+"] & {|m+1\rangle} \arrow[l, bend left, "J_-"] \arrow[r, bend left, "J_+"] & {|m+2\rangle} \arrow[l, bend left, "J_-"] & \cdots
\end{tikzcd}$$

In order to find a finite representation

$J_+|j\rangle = 0$ $\quad$ j: highest weight

$J_+|j\rangle = \underbrace{N(j+1)}_{0}|j+1\rangle$

J_3 is special

$J_1 + aJ_2$

$[J_3, J_1 + aJ_2]$

$= iJ_2 - iaJ_1$

$= \lambda(J_1 + aJ_2)$

$\lambda = -ia$

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$\Downarrow$

$a^2 = -1$

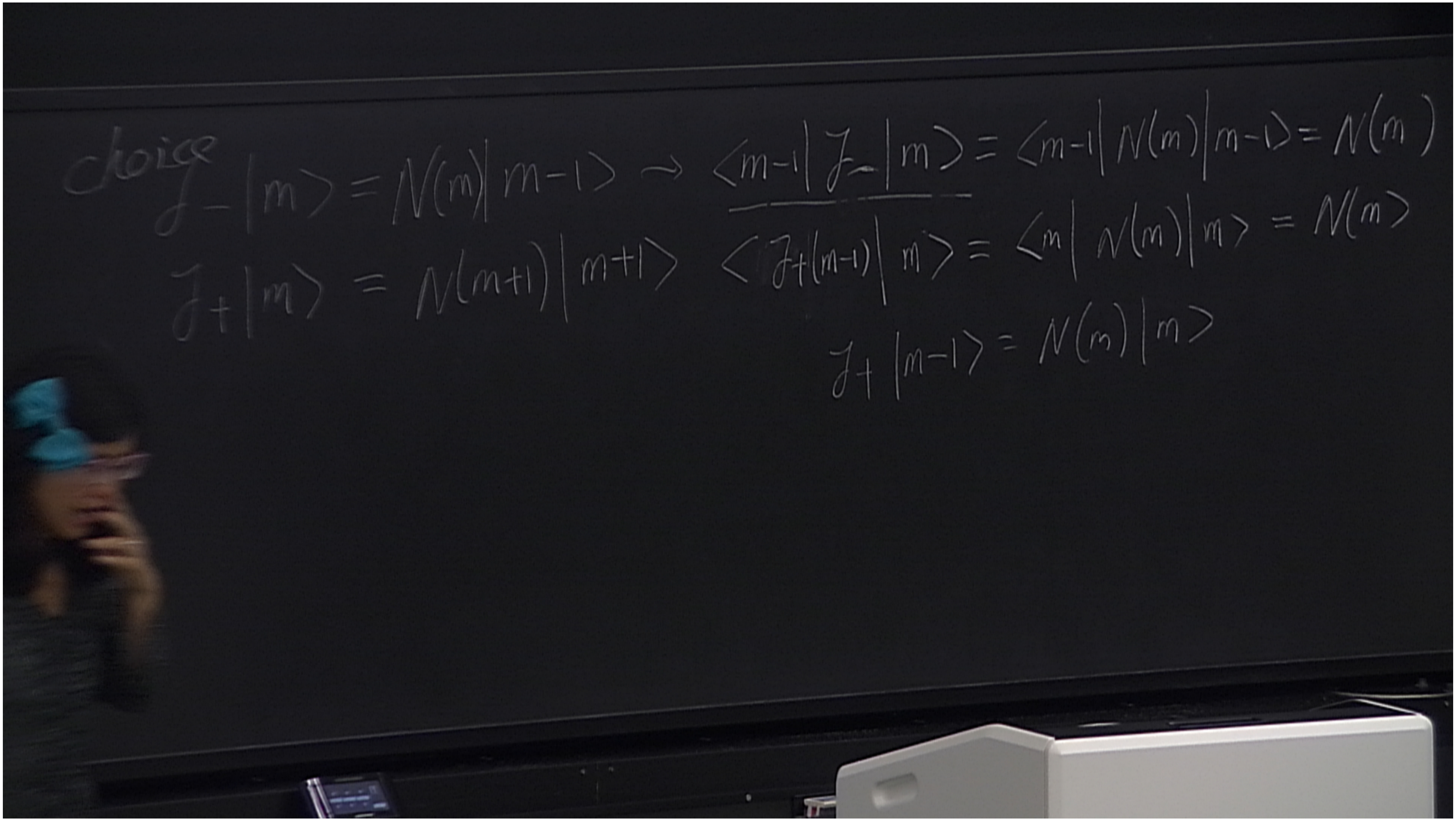
$a = \pm i$

$J_\pm = \frac{1}{\sqrt{2}}(J$

$[J_3, J$

$[J_3, J$

$[J_+, J_-] =$


choice
$J_-|m\rangle = N(m)|m-1\rangle \rightarrow \langle m-1|J_-|m\rangle = \langle m-1|N(m)|m-1\rangle = N(m)$
$\langle J_+(m-1)|m\rangle = \langle m|N(m)|m\rangle = N(m)$
$J_+|m\rangle = N(m+1)|m+1\rangle$
$J_+|m-1\rangle = N(m)|m\rangle$

$$J_+|m-1\rangle = N(m)|m\rangle$$

$$\langle m|J_3|m\rangle = m$$

$$= \langle m|[J_+, J_-]|m\rangle$$

$$= \langle m|J_+J_-|m\rangle - \langle m|J_-J_+|m\rangle$$

$$= |J_-|m\rangle|^2 - |J_+|m\rangle|^2 = N^2(m) - N^2(m+1)$$

$$+\begin{cases} j = N^2(j) - N^2(j+1) \\ j-1 = N^2(j-1) - N^2(j) \\ j-2 = N^2(j-2) - N^2(j-1) \\ \vdots \\ m = N^2(m) - N^2(m+1) \end{cases}$$

$N(j+1)=0$

$j+j$

$^2(j+1)$

$^2(j)$

$^2(j-1)$

$(m+1)$

$N(j+1)=0$

$$j + j-1 + \dots + m = N^2(m) - \underbrace{N^2(j+1)}_{\to 0}$$

$$\frac{j+m}{2}(j-m+1) = N^2(m)$$

$$N(m) = \sqrt{\frac{j+m}{2}(j-m+1)}$$

$m=j \quad N(-j)=0$

$\hbar^2(j+1) \to 0$

$$J_-|-j\rangle = N(-j)|-j-1\rangle$$
$$= 0.$$

$$|j\rangle \to |j-1\rangle \to |j-2\rangle \to \cdots |-j\rangle \to 0$$

$2j$ steps $\qquad 2j = \text{integer}$

$$j = \frac{\text{integer}}{2}$$

$j=0$ singlet

$j=\frac{1}{2}$ $\quad |\tfrac{1}{2}\rangle, |-\tfrac{1}{2}\rangle$

$\quad\quad\quad \Vert$

$\quad\quad |\tfrac{1}{2},\tfrac{1}{2}\rangle, |\tfrac{1}{2},-\tfrac{1}{2}\rangle \quad |j,m\rangle$

$M_{1,2}$

$$J_3 = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix} \qquad (J_3)_{\frac{1}{2},\frac{1}{2}} = \frac{1}{2}$$

$j=0$ singlet

$j=\frac{1}{2}$ $\left|\frac{1}{2}\right\rangle, \left|-\frac{1}{2}\right\rangle$ fundamental representation

$\Downarrow$

$\left|\frac{1}{2},\frac{1}{2}\right\rangle, \left|\frac{1}{2},-\frac{1}{2}\right\rangle$

$|j,m\rangle$

M_{12}

$\langle m'| J_{+}$

$$J_3 = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}$$

$$(J_3)_{\frac{1}{2},\frac{1}{2}} = \frac{1}{2}$$

ental
esentation

$,m\rangle$

M_{12}

$J_3\big|_{\frac{1}{2},\frac{1}{2}} = \frac{1}{2}$

$$\langle m' | J_+ | m \rangle = \langle m' | N(m+1) | m+1 \rangle$$

$$= N(m+1)\,\delta_{m',m+1}$$

$j=1$ $\quad |1\rangle, |0\rangle \; |-1\rangle$ $\quad$ 3 states

$$3 = 2^2 - 1$$

Ajoint Representation

$n+1\rangle$

$$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$$

$, m+1$

presentation