

Title: Front End - Lie Groups and Lie Algebras - 3

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Abstract:

Structure of Lie Group

$$G : g(\tilde{\varphi}) \quad \tilde{\varphi} = (\varphi_1, \varphi_2, \dots, \varphi_n)$$

$$g^\alpha = g(\tilde{\varphi}^\alpha) \quad g^\beta = g(\tilde{\varphi}^\beta)$$

$$g^{\odot} \quad g^{\ominus} \quad g^{\oplus}$$

$$g^- \quad g^= \quad g^=$$

up

$$g^\alpha \circ g^\beta = g^\delta$$

near identity

$$\mathcal{U}(g(\tilde{\varphi}^\alpha)) \cdot \mathcal{U}(g(\tilde{\varphi}^\beta)) = \mathcal{U}(g(\tilde{\varphi}^\delta))$$

$$\mathcal{U}(\tilde{\varphi}^\alpha) \cdot \mathcal{U}(\tilde{\varphi}^\beta) = \mathcal{U}(\tilde{f}(\tilde{\varphi}^\alpha, \tilde{\varphi}^\beta))$$

$$g^\alpha \circ g^\beta = g^\delta$$

near identity

$$\mathcal{U}(g(\tilde{\varphi}^\alpha)) \cdot \mathcal{U}(g(\tilde{\varphi}^\beta)) = \mathcal{U}(g(\tilde{\varphi}^\delta))$$

$$\mathcal{U}(\tilde{\varphi}^\alpha) \mathcal{U}(\tilde{\varphi}^\beta) = \mathcal{U}(\mathcal{F}(\tilde{\varphi}^\alpha, \tilde{\varphi}^\beta))$$

trivial and enlightening

$$\mathcal{U}(0) = \mathbb{1}$$

$$\mathcal{F}(0, 0) =$$

trivial and enlightening

$$U(0) = 1$$

$$\tilde{F}(0, 0) = 0$$

$$\tilde{F}(\tilde{\varphi}^\alpha, 0) = \tilde{\varphi}^\alpha \quad g \circ e = g$$

$$\tilde{F}(0, \tilde{\varphi}^\beta) = \tilde{\varphi}^\beta$$

$\tilde{\epsilon}$ is a small parameter s.t. $\alpha=1, \dots, n$.

$$U(\tilde{\epsilon}) = \mathbb{1} + i\epsilon_a(t_a) \leftarrow \text{generators}$$

$$U^\dagger U = \mathbb{1}$$

$$(\mathbb{1} - i\epsilon_a(t_a)^\dagger)(\mathbb{1} + i\epsilon_a(t_a)) = \mathbb{1}$$

$$-i\epsilon_a(t_a)^\dagger + i\epsilon_a(t_a) = 0$$

$$t_a^\dagger = t_a$$

$$U(\tilde{\varepsilon}) = 1 + i\varepsilon_a t_a + \frac{1}{2}\varepsilon_a \varepsilon_b \cdot \text{lab}$$

$$f_a(\tilde{\varepsilon}^\alpha, \tilde{\varepsilon}^\beta) = A_a + B_{ba}\varepsilon_b^\alpha + C_{ba}\varepsilon_b^\beta + D_{bca}\varepsilon_b^\alpha \varepsilon_c^\beta + E_{bca}\varepsilon_b^\alpha \varepsilon_c^\alpha + F_{bca}\varepsilon_b^\beta \varepsilon_c^\beta$$

$$f_a(\tilde{\varepsilon}_{;0}^\alpha) = B_{ba}\varepsilon_b^\alpha + \cancel{E_{bca}\varepsilon_b^\alpha \varepsilon_c^\alpha} = \varepsilon_a^\alpha \quad B_{ba} = \delta_{ba}$$

$$C_{ba} = \delta_{ba}$$

$$U(\tilde{\epsilon}) = 1 + i\epsilon_a t_a + \frac{1}{2}\epsilon_a \epsilon_b \cdot \text{lab}$$

$$f_a(\tilde{\epsilon}^\alpha, \tilde{\epsilon}^\beta) = A_a + B_{ba} \epsilon_b^\alpha + C_{ba} \epsilon_b^\beta + D_{bca} \epsilon_b^\alpha \epsilon_c^\beta + \dots$$

$$f_a(\tilde{\epsilon}^\alpha, 0) = B_{ba} \epsilon_b^\alpha + \cancel{E_{bca} \epsilon_b^\alpha \epsilon_c^\alpha} = \epsilon_a^\alpha \quad B_{ba} = \delta_{ba} \\ C_{ba} = \delta_{ba}$$

$$f_a(\tilde{\epsilon}^\alpha, \tilde{\epsilon}^\beta) = \epsilon_a^\alpha + \epsilon_a^\beta + D_{bca} \epsilon_b^\alpha \epsilon_c^\beta$$

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$$

$$g^\alpha \circ g^\beta = g^\delta$$

near identity

$$\mathcal{U}(g(\tilde{\varphi}^\alpha)) \cdot \mathcal{U}(g(\tilde{\varphi}^\beta)) = \mathcal{U}(g(\tilde{\varphi}^\delta))$$

$$\boxed{\mathcal{U}(\tilde{\varphi}^\alpha) \cdot \mathcal{U}(\tilde{\varphi}^\beta) = \mathcal{U}(\tilde{f}(\tilde{\varphi}^\alpha, \tilde{\varphi}^\beta))}$$

trivial

if $\tilde{\varphi}^\alpha, \tilde{\varphi}^\beta$

Left

$$= \left(1 + i\varepsilon_a^\alpha t_a + \frac{1}{2}\varepsilon_a^\alpha \varepsilon_b^\alpha l_{ab} \right)$$

$$\left(1 + i\varepsilon_c^\beta t_c + \frac{1}{2}\varepsilon_c^\beta \varepsilon_d^\beta l_{cd} \right)$$

$$= \mathbb{1} + i\varepsilon_a^\alpha t_a + i\varepsilon_c^\beta t_c + \frac{1}{2}\varepsilon_a^\alpha \varepsilon_b^\alpha l_{ab} + \frac{1}{2}\varepsilon_c^\beta \varepsilon_d^\beta l_{cd} + (-1)\varepsilon_a^\alpha \varepsilon_b^\beta t_a t_b$$

Right

$$R_{\text{right}} = 1 + i(\epsilon_a^\alpha + \epsilon_a^\beta + D_{bca} \epsilon_b^\alpha \epsilon_c^\beta) t_a$$

$$+ \frac{1}{2} (\epsilon_a^\alpha + \epsilon_a^\beta + \cancel{D_{bca} \epsilon_b^\alpha \epsilon_c^\beta}) (\epsilon_d^\alpha + \epsilon_d^\beta + \cancel{D_{efd} \epsilon_e^\alpha \epsilon_f^\beta}) l_{ad}$$

$$\frac{1}{2} (\epsilon_a^\alpha \epsilon_b^\alpha l_{ab} + \epsilon_a^\beta \epsilon_b^\beta l_{ab} + \epsilon_a^\alpha \epsilon_b^\beta l_{ab} + \epsilon_a^\beta \epsilon_b^\alpha l_{ab})$$

Left

$$= (1 + i\varepsilon_a^\alpha t_a + \frac{1}{2}\varepsilon_a^\alpha \varepsilon_b^\alpha l_{ab})$$

$$\cdot (1 + i\varepsilon_c^\beta t_c + \frac{1}{2}\varepsilon_c^\beta \varepsilon_d^\beta l_{cd})$$

$$= \textcircled{1} + i\varepsilon_a^\alpha t_a + i\varepsilon_c^\beta t_c + \frac{1}{2}\varepsilon_a^\alpha \varepsilon_b^\alpha l_{ab} + \frac{1}{2}\varepsilon_c^\beta \varepsilon_d^\beta l_{cd} + (-1)\varepsilon_a^\alpha \varepsilon_c^\beta t_a t_c$$

Right

$$= \textcircled{1} + i(\varepsilon_a^\alpha + \varepsilon_c^\beta)$$

$$+ \frac{1}{2}(\varepsilon_a^\alpha + \varepsilon_c^\beta + \varepsilon_b^\alpha + \varepsilon_d^\beta)$$

$$\frac{1}{2}(\varepsilon_a^\alpha \varepsilon_b^\alpha l_{ab} + \varepsilon_c^\beta \varepsilon_d^\beta l_{cd})$$

$$+ \varepsilon_a^\alpha \varepsilon_c^\beta l_{ac}$$

$g^- \quad g^- \quad g^-$

Left

$$= (1 + i\varepsilon_a^\alpha t_a + \frac{1}{2}\varepsilon_a^\alpha \varepsilon_b^\alpha l_{ab})$$

$$\cdot (1 + i\varepsilon_c^\beta t_c + \frac{1}{2}\varepsilon_c^\beta \varepsilon_d^\beta l_{cd})$$

$$= \cancel{1} + i\varepsilon_a^\alpha t_a + i\varepsilon_a^\beta t_a + \frac{1}{2}\varepsilon_a^\alpha \varepsilon_b^\alpha l_{ab} + \frac{1}{2}\varepsilon_a^\beta \varepsilon_b^\beta l_{ab} + (-1)\varepsilon_a^\alpha \varepsilon_b^\beta t_a t_b$$

Right

$$= \cancel{1} + i(\varepsilon_a^\alpha + \varepsilon_a^\beta + D_{bca} \varepsilon_b^\alpha \varepsilon_c^\beta)$$

$$+ \frac{1}{2}(\varepsilon_a^\alpha + \varepsilon_a^\beta + D_{bca} \varepsilon_b^\alpha \varepsilon_c^\beta)$$

$$\frac{1}{2}(\varepsilon_a^\alpha \varepsilon_b^\alpha l_{ab} + \varepsilon_a^\beta \varepsilon_b^\beta l_{ab} + \varepsilon_a^\alpha \varepsilon_b^\beta l_{ab} + \varepsilon_a^\beta \varepsilon_b^\alpha l_{ab})$$

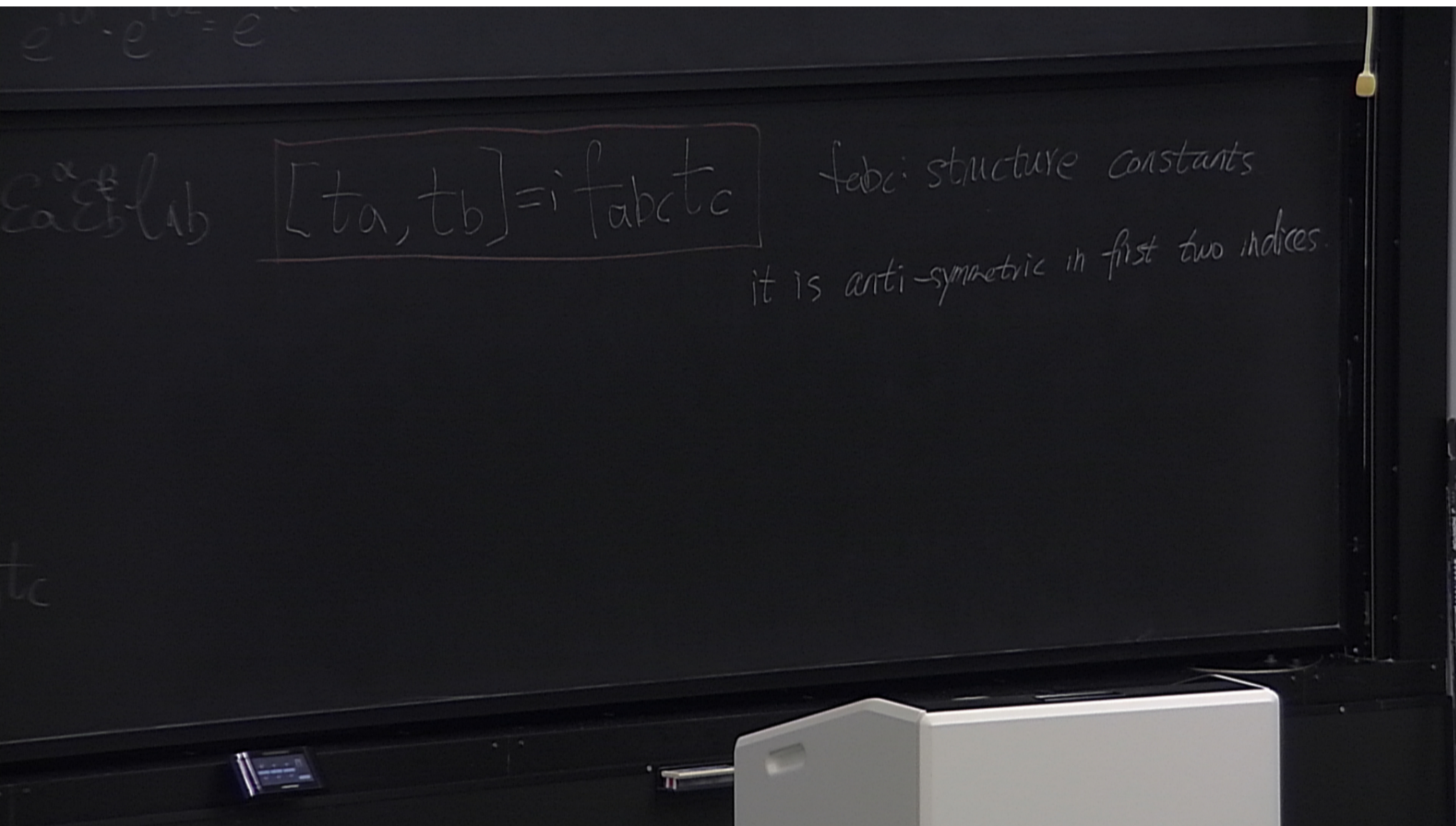
$$(-1) \epsilon_a^\alpha \epsilon_b^\beta t_a t_b = \underbrace{i D_{bca} \epsilon_b^\alpha \epsilon_c^\beta t_a + \epsilon_a^\alpha \epsilon_b^\beta l_{ab}}_{[t_a, t_b]}$$

$$i D_{abc} \epsilon_a^\alpha \epsilon_b^\beta t_c$$

$$-t_a t_b = i D_{abc} t_c + l_{ab} \quad ①$$

$$-t_b t_a = i D_{bac} t_c + l_{ba} \quad ②$$

$$② - ① \quad t_a t_b - t_b t_a = i \underbrace{(D_{bac} - D_{abc})}_{f_{abc}} t_c$$



$$e^x \cdot e^y = e^{x+y}$$

$$[a, b] = ab - ba$$

$$[t_a, t_b] = i f_{abc} t_c$$

f_{abc} : structure constants

$$(AB)^{\dagger} = B^{\dagger} A^{\dagger}$$

$$[t_a, t_b]^{\dagger} = -i f_{abc}^* t_c^{\dagger}$$

- ① it is anti-symmetric in first two
- ② f_{abc} are real

$$(t_a t_b - t_b t_a)^{\dagger} = t_b^{\dagger} t_a^{\dagger} - t_a^{\dagger} t_b^{\dagger} = t_b t_a - t_a t_b$$

$$f_{abc}^* = f_{abc}$$

$$[t_b, t_a] = i f_{bac} t_c = -i f_{abc} t_c$$

$$[[t_a, t_b], t_c] + [[t_b, t_c], t_a] + [[t_c, t_a], t_b] = 0$$

③
Jacobi identity $f_{ab}d f_{cd} + f_{bc}d f_{ad} + f_{ca}d f_{bd} = 0$

$$[t_c, t_a], t_b] = 0$$

$$e = 0$$

$$U(\tilde{\varphi}) \equiv \lim_{N \rightarrow \infty} \left(1 + i \frac{\varphi_a t_a}{N} \right)^N = e^{i \varphi_a t_a}$$

$$\lim_{N \rightarrow \infty} \left(1 + \frac{1}{N} \right)^N = e$$

Adjoint $[[t_a, t_b], t_c] + [[t_b, t_c], t_a] + [[t_c, t_a], t_b]$

Representator

$$n=M \quad \text{Jacobi identity} \quad (-i)f_{abd}f_{cde} + (-i)f_{bcd}f_{ade} + (-i)f_{cad}f_{bde} = 0$$

$$t_a \quad \boxed{(T_a)_{bc} = -if_{abc}} \quad (T_a)_{bd}(T_c)_{de} - (T_c)_{bd}(T_a)_{de} - (-i)f_{cad}(T_d)_{be} = 0$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ (T_a T_c)_{be} & - & (T_c T_a)_{be} \end{array}$$

$a, b, c = 1, \dots, n$

$$[t_c, t_a] + [[t_c, t_a], t_b] = 0$$

$$f_{ade} + (-i)f_{cad}f_{bde} = 0$$

$$(T_a)_{de} - (-i)f_{cad}(T_d)_{be} = 0$$

) be

$$U(\varphi) \equiv \lim_{N \rightarrow \infty} \left(1 + i \frac{\varphi_a t_a}{N} \right)^N = e^{i\varphi_a t_a}$$

$$\lim_{N \rightarrow \infty} \left(1 + \frac{1}{N} \right)^N = e$$

$$T_a T_c - T_c T_a = -i f_{cad} T_d$$

$$[T_a, T_c] = i f_{acd} T_d$$

Compact Lie Group

f_{abc} is completely
anti-symmetric.

$$\langle t_a | t_b \rangle = \text{Tr}[T_a T_b] \\ = \frac{1}{2} \delta_{ab}$$

$$\langle T_a | T_b \rangle = \delta_{ab}$$

$$T_a | t_b \rangle$$

$$T_a |t_b\rangle |\vec{e}_x\rangle \quad 1 = \sum_c |t_c\rangle \langle t_c|$$

$$= \sum_c |t_c\rangle \langle t_c| T_a |t_b\rangle$$

$$= \sum_c |t_c\rangle (T_a)_{cb}$$

$$= \sum_c |t_c\rangle (-i) f_{acb}$$

$$= \sum_c i f_{abc} |t_c\rangle = \sum_c i f_{abc} |t_c\rangle = i | [t_a, t_b] \rangle$$

$$|t_b\rangle |\vec{e}_x\rangle \quad 1 = \sum_c |t_c\rangle \langle t_c|$$

$$|T_a|t_b\rangle = |[t_a, t_b]\rangle$$

$$\langle t_c | T_a | t_b \rangle$$

$$(T_a)_{cb}$$

$$t_c \rangle (-i) f_{acb}$$

$$f_{abc} |t_c\rangle = \sum_c |f_{abc} t_c\rangle = |[t_a, t_b]\rangle$$