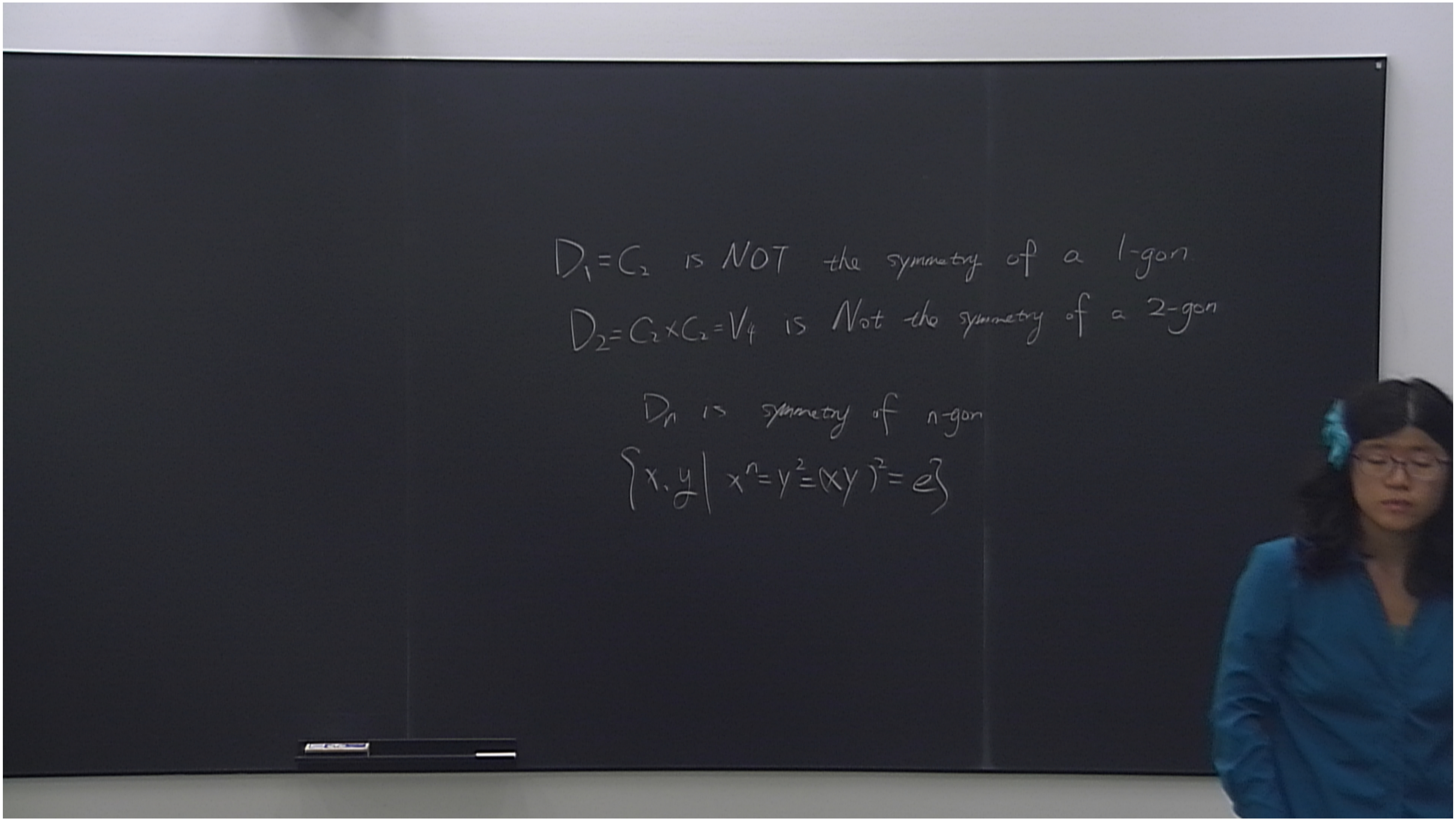
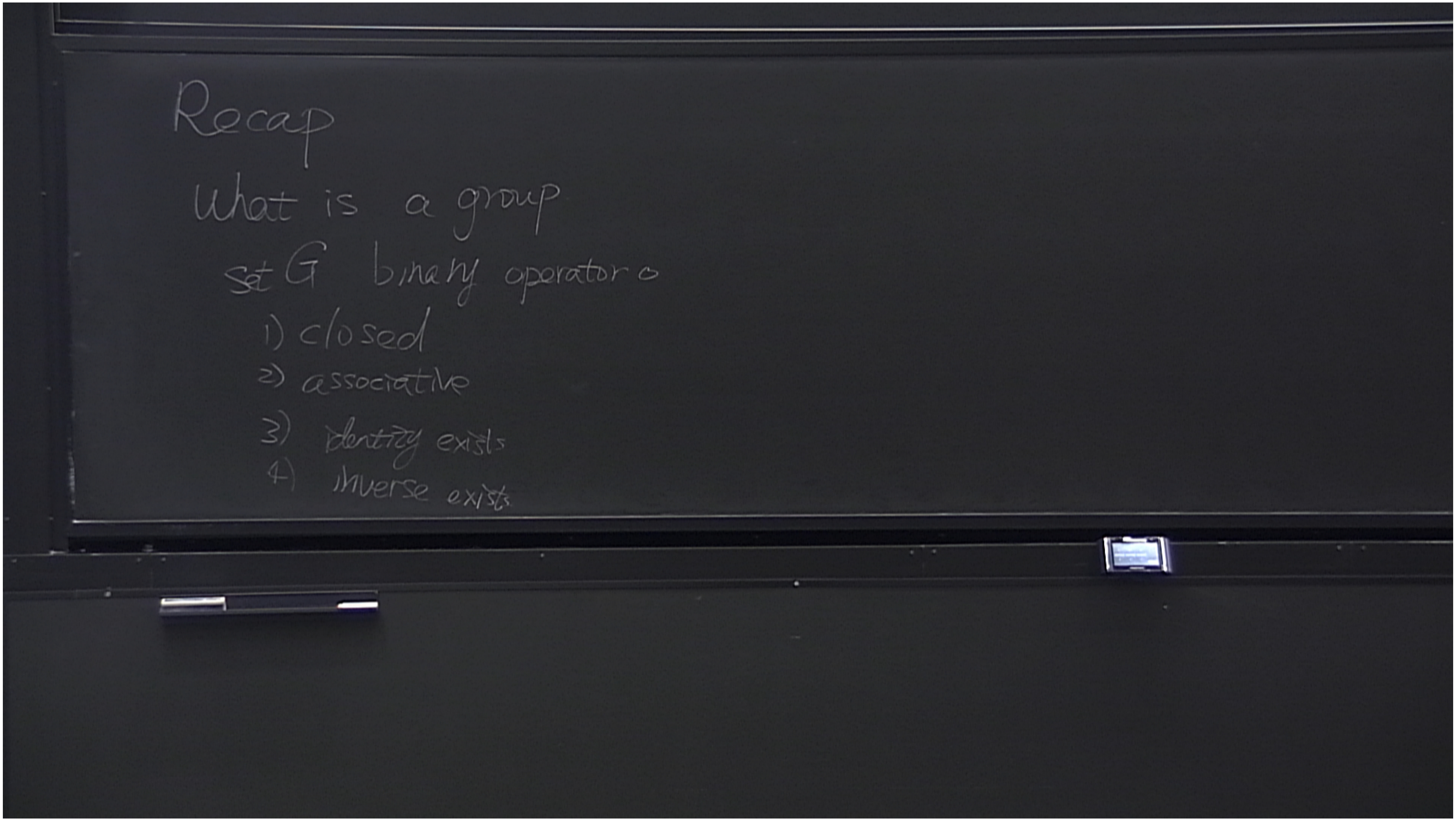
Title: Front End - Lie Groups and Lie Algebras - 2

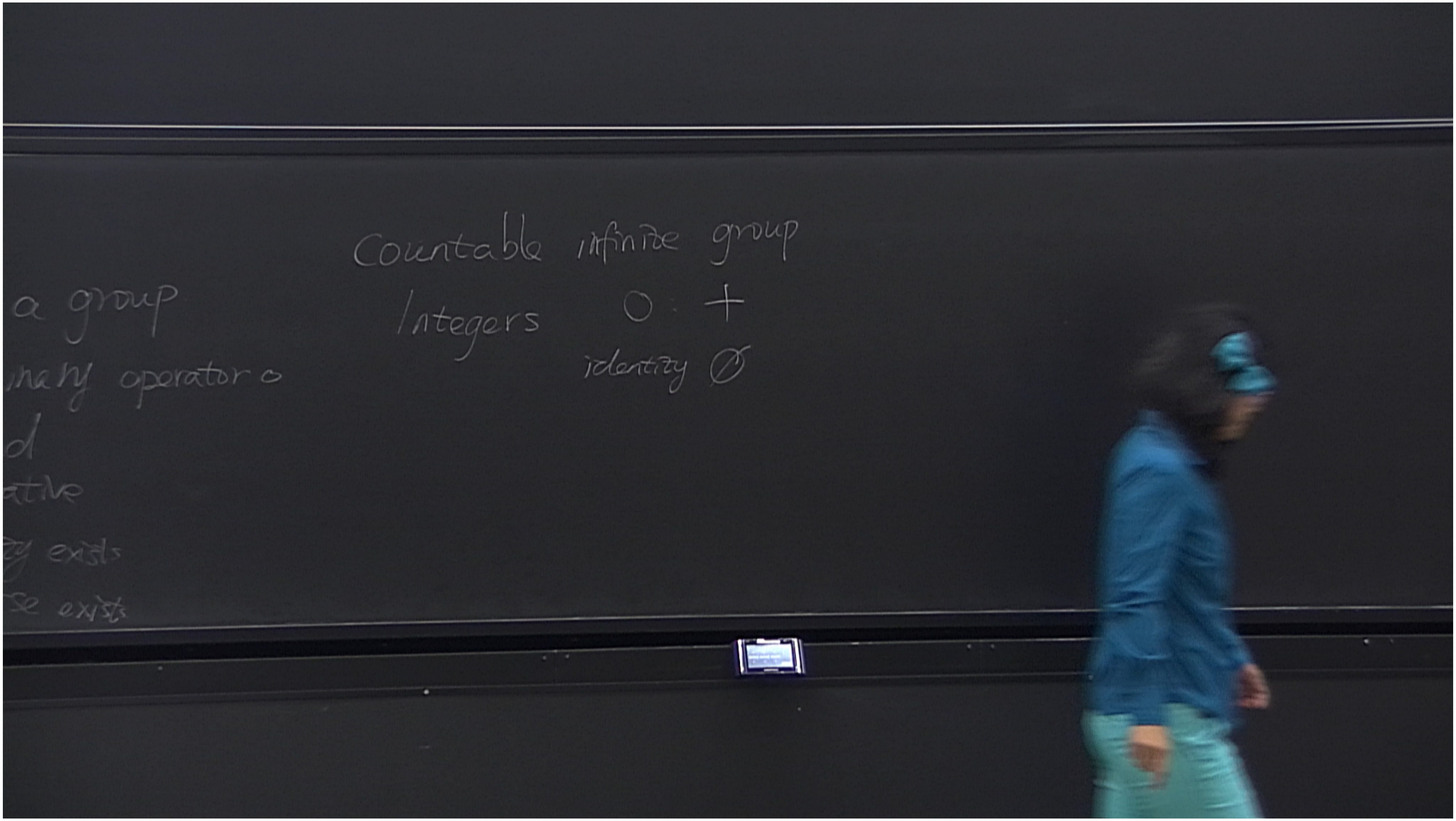
Date: Aug 17, 2016 09:00 AM

URL: http://pirsa.org/16080071

Abstract:


$D_1 = C_2$ is NOT the symmetry of a 1-gon
$D_2 = C_2 \times C_2 = V_4$ is Not the symmetry of a 2-gon
D_n is symmetry of n-gon
$\{x, y \mid x^n = y^2 = (xy)^2 = e\}$


Recap
What is a group
set G binary operator ∘
1) closed
2) associative
3) identity exists
4) inverse exists


Countable infinite group
Integers ○ : +
identity ∅
a group
nary operator ○
d
ative
ty exists
se exists

a group

inary operator ∘

d

ative

ty exists

se exists

Countable infinite group

Integers $\circ : +$

identity $\emptyset$

inverse $5^{-1} = -5$

a group
inary operator $\circ$
d
ative
y exists
se exists

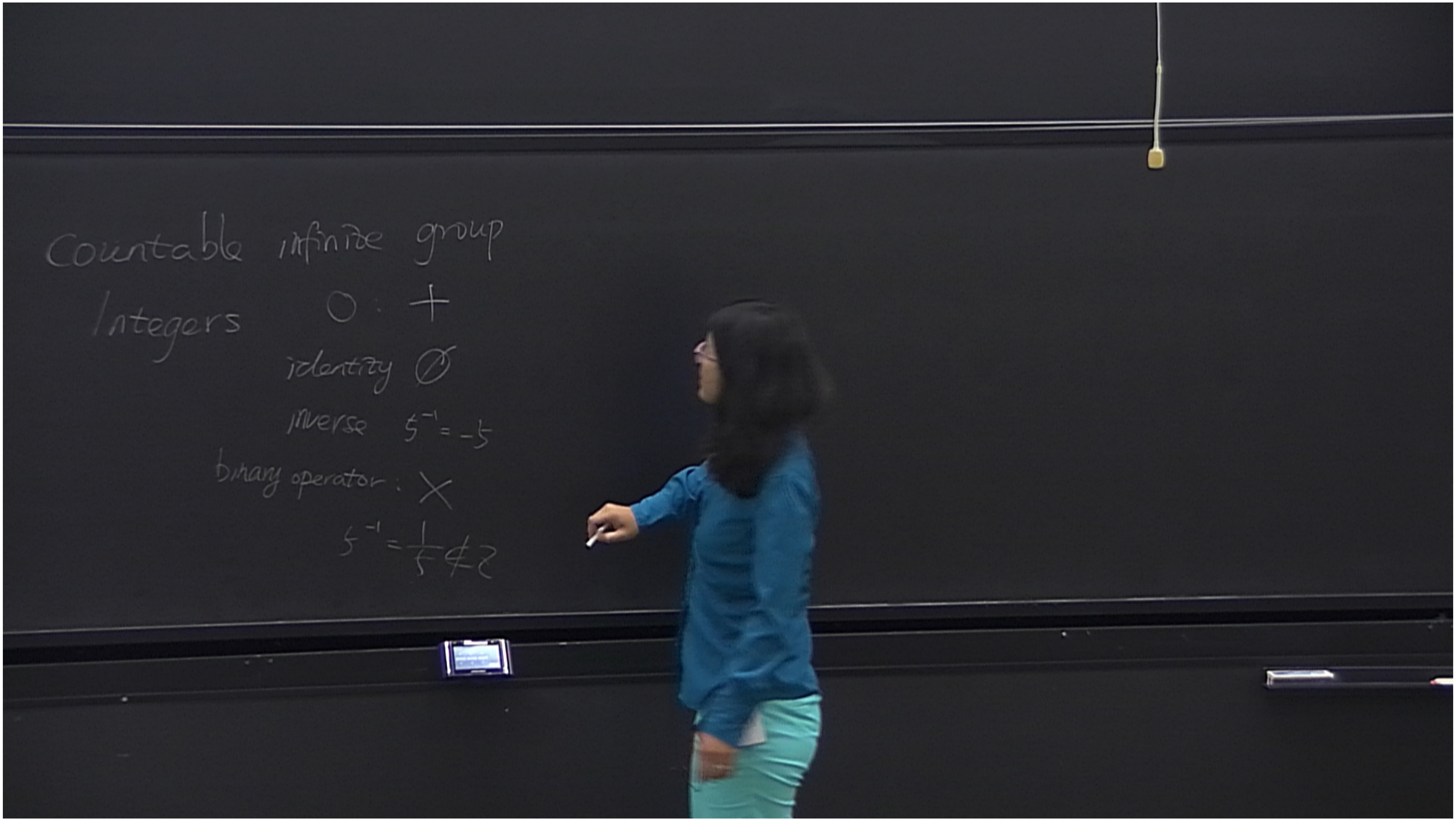
Countable infinite group

Integers $\circ$: $+$

identity $\emptyset$

inverse $5^{-1} = -5$

binary operator : $\times$


Countable infinite group
Integers 0 : +
identity ∅
inverse $5^{-1} = -5$
binary operator : ×
$5^{-1} = \frac{1}{5} \notin Z$

Countable infinite group

Integers O : $+$

identity $\emptyset$

inverse $5^{-1} = -5$

binary operator : $\times$

$5^{-1} = \frac{1}{5} \notin \mathbb{Z}$

Integers $(+, \times)$

abelian group

associative

distributive

$3 \times (2+5) = 3 \times 2 + 3 \times 5$

Countable infinite group

Integers $0 : +$

identity $\emptyset$

inverse $5^{-1} = -5$

binary operator : $\times$

$5^{-1} = \frac{1}{5} \notin Z$

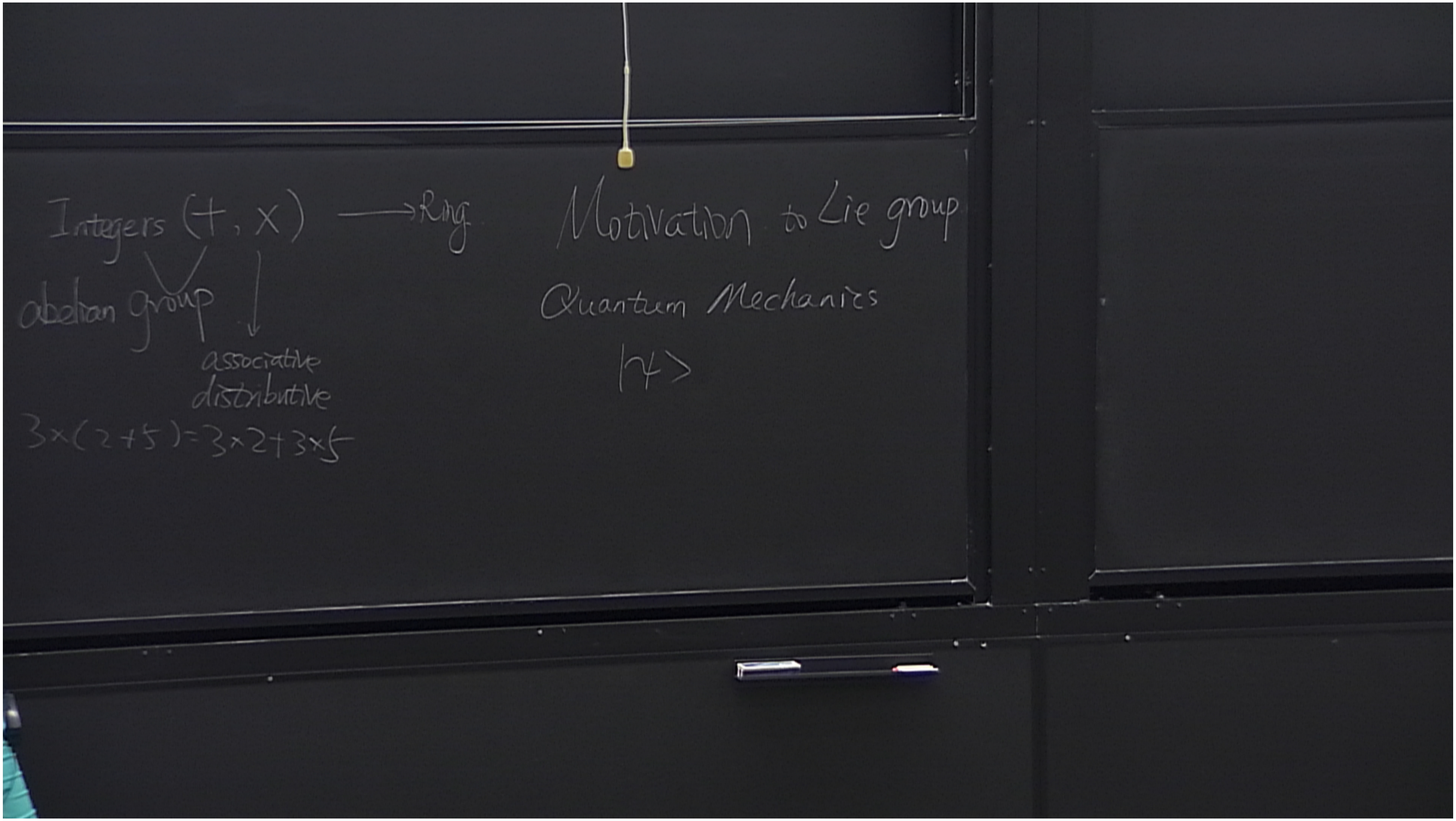
Integers $(+, \times) \longrightarrow$ Ring

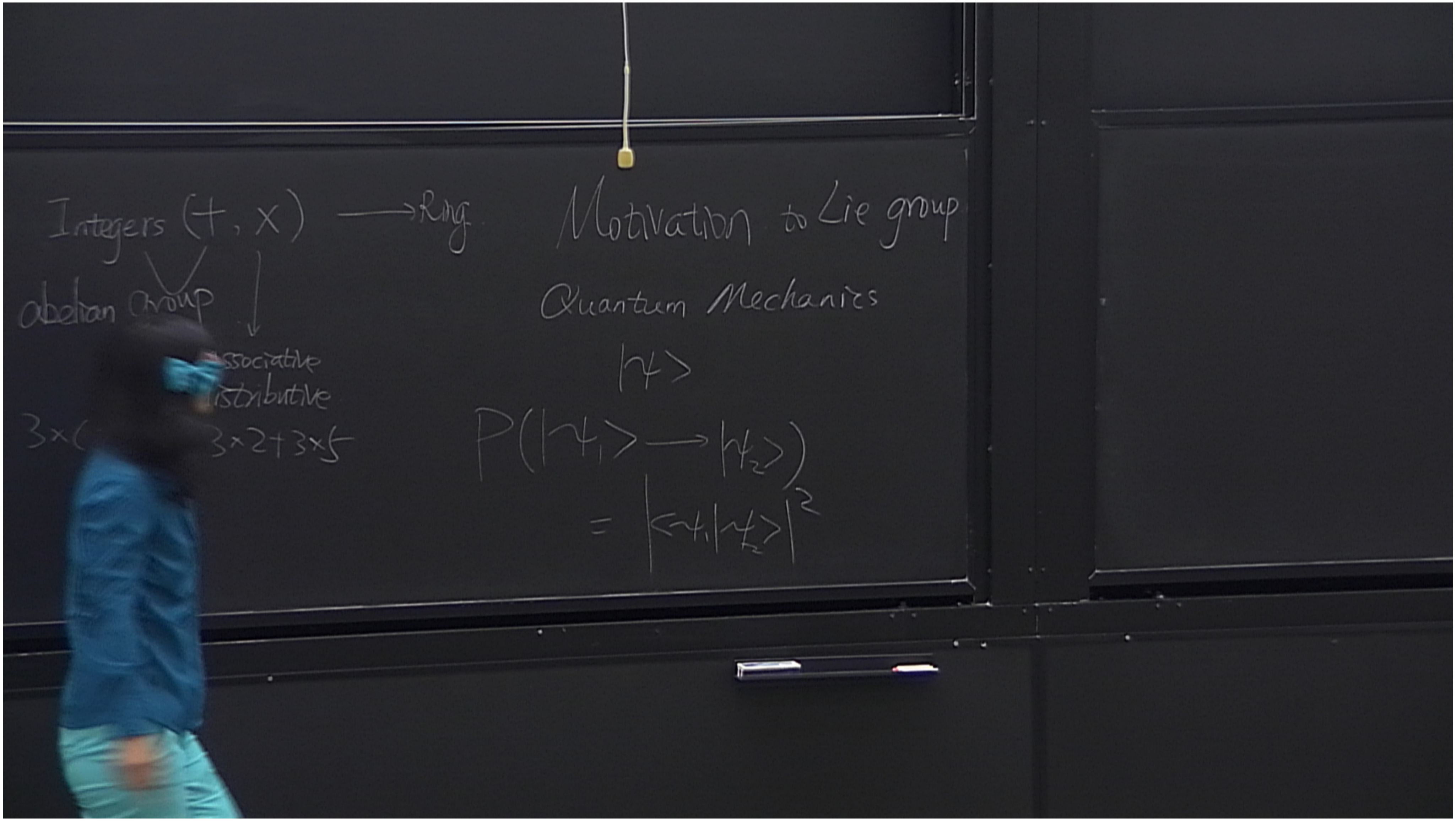
abelian group

associative

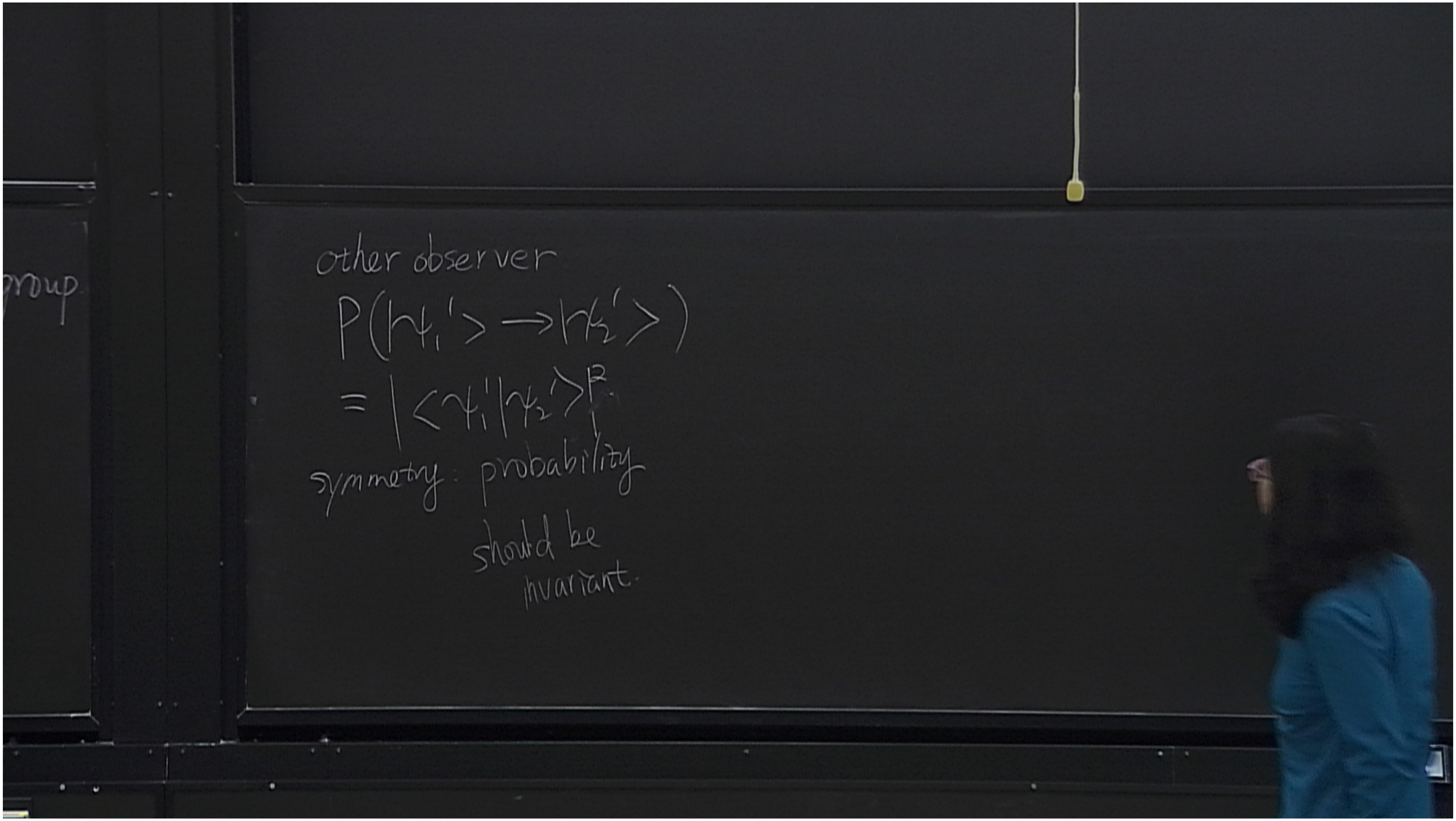
distributive

$3 \times (2+5) = 3 \times 2 + 3 \times 5$


Integers (+, x) → Ring
abelian group
associative
distributive
3×(2+5)=3×2+3×5
Motivation to Lie group
Quantum Mechanics
|ψ>



Integers (+, x) → Ring
abelian Group
$3\times2+3\times5$
Motivation to Lie group
Quantum Mechanics
$|\psi\rangle$
$P(|\psi_1\rangle \to |\psi_2\rangle)$
$= |\langle\psi_1|\psi_2\rangle|^2$


other observer
$P(|\psi_1'\rangle \rightarrow |\psi_2'\rangle)$
$= |\langle\psi_1'|\psi_2'\rangle|^2$
symmetry: probability
should be
invariant.

other observer

$$P(|\psi_1'\rangle \to |\psi_2'\rangle)$$

$$= |\langle\psi_1'|\psi_2'\rangle|^2$$

symmetry: probability should be invariant.

Wigner 30'

$$|\psi_1'\rangle = O|\psi_1\rangle$$

O — unitary and linear

— anti-unitary and anti-linear.

other observer

$$P(|\psi_1'\rangle \to |\psi_2'\rangle)$$

$$= |\langle \psi_1'|\psi_2'\rangle|^2$$

symmetry: probability should be invariant.

Wigner 30'

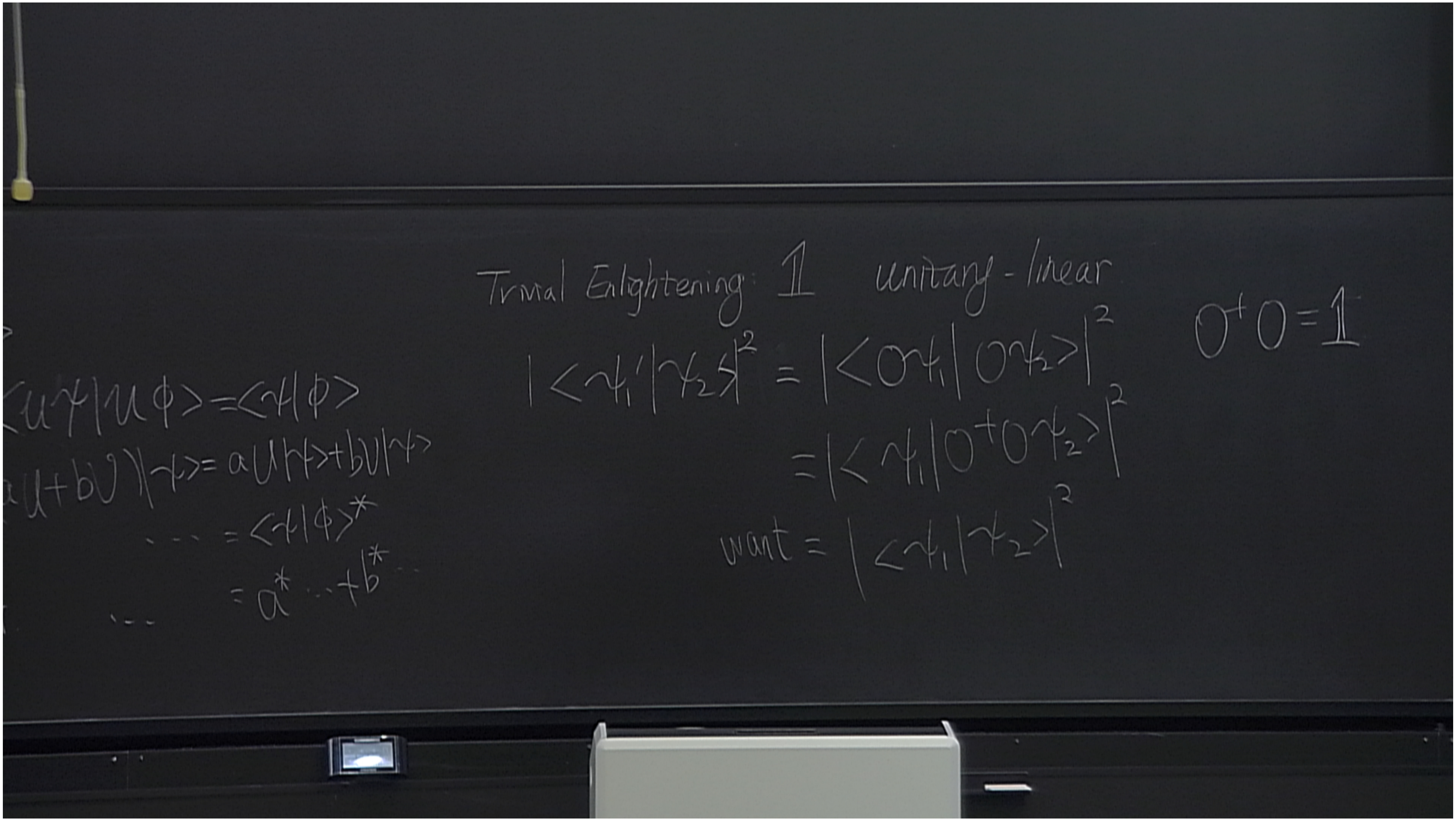
$$|\psi_1'\rangle = O|\psi_1\rangle$$

O – unitary $\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle$

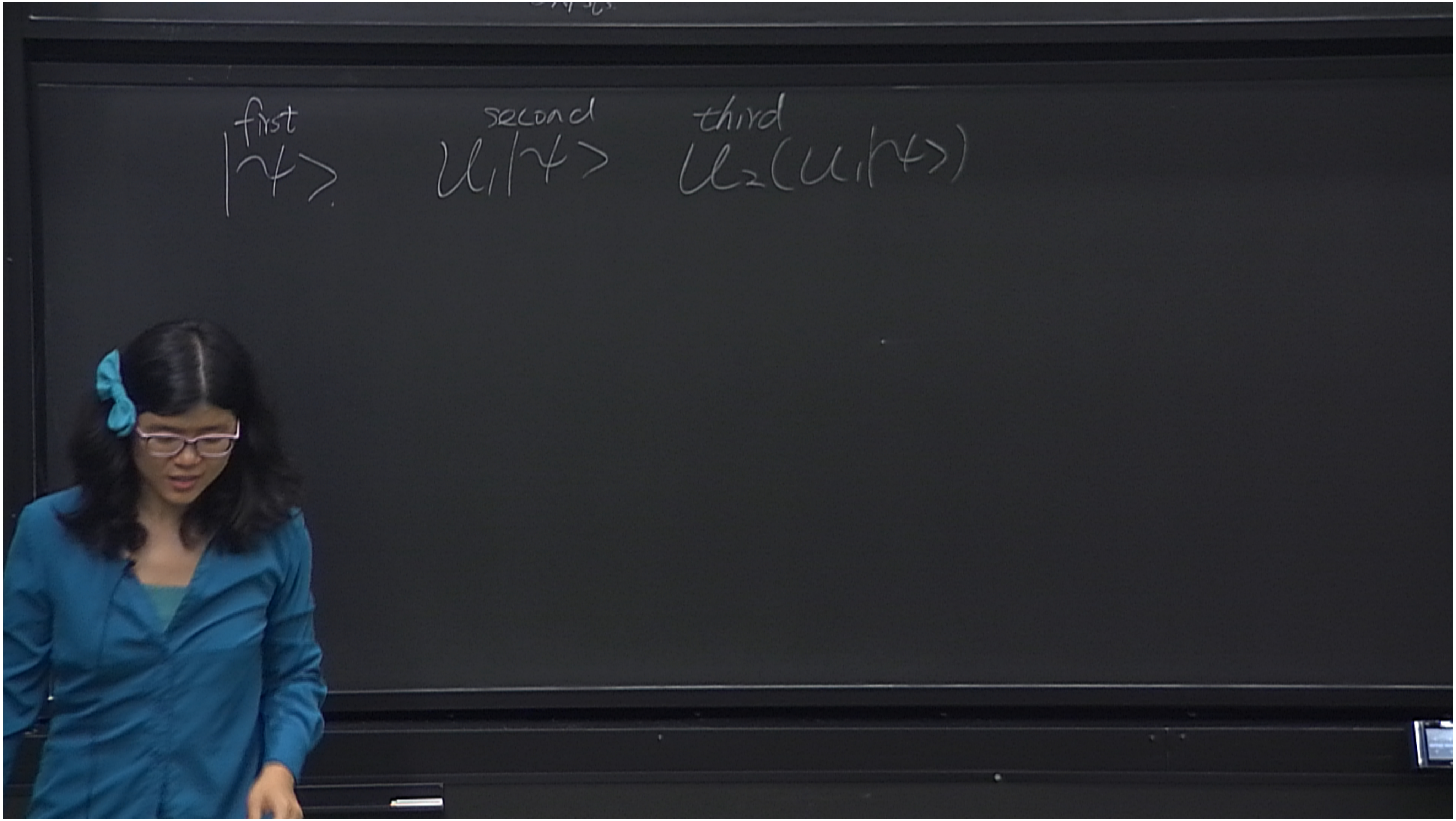
and linear $(aU+bU)|\psi\rangle = aU|\psi\rangle + bU|\psi\rangle$

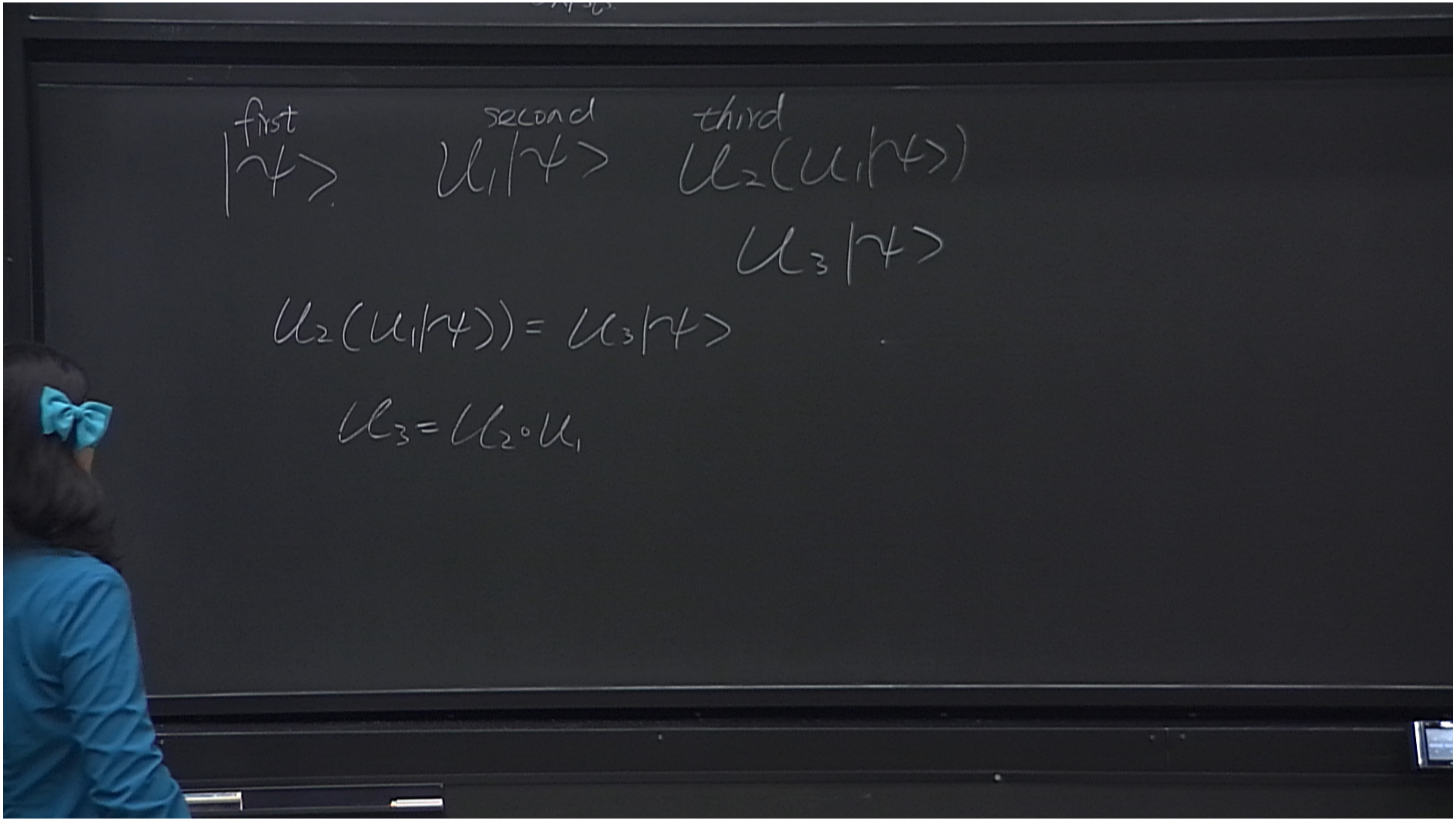
— anti-unitary $\dots = \langle\psi|\phi\rangle^*$

and anti-linear $\dots = a^* \dots + b^* \dots$



Trivial Enlightening: $\mathbb{1}$ unitary - linear
$|\langle \psi_1' | \psi_2' \rangle|^2 = |\langle O\psi_1 | O\psi_2 \rangle|^2$
$= |\langle \psi_1 | O^\dagger O \psi_2 \rangle|^2$
want $= |\langle \psi_1 | \psi_2 \rangle|^2$
$O^\dagger O = \mathbb{1}$
$\langle U\psi | U\phi \rangle = \langle \psi | \phi \rangle$
$(aU + bV)|\psi\rangle = aU|\psi\rangle + bV|\psi\rangle$
$\ldots = \langle \psi | \phi \rangle^*$
$= a^* \ldots + b^* \ldots$


first
$|\psi\rangle$
second
$U_1|\psi\rangle$
third
$U_2(U_1|\psi\rangle)$


first
$|\psi\rangle$
second
$U_1|\psi\rangle$
third
$U_2(U_1|\psi\rangle)$
$U_3|\psi\rangle$
$U_2(U_1|\psi\rangle) = U_3|\psi\rangle$
$U_3 = U_2 \circ U_1$

first $\quad$ second $\quad$ third

$|\psi\rangle \qquad \mathcal{U}_1|\psi\rangle \qquad \mathcal{U}_2(\mathcal{U}_1|\psi\rangle)$

Group. Representat

$\mathcal{U}_3|\psi\rangle$

$\mathcal{U}_2(\mathcal{U}_1|\psi\rangle) = \mathcal{U}_3|\psi\rangle$

$\mathcal{U}_1|\psi\rangle =$

$\mathcal{U}_3 = \mathcal{U}_2 \circ \mathcal{U}_1$

$|\psi\rangle = (\quad)\mathcal{U}_1|\psi\rangle$

Group Representations

$G = \{a, b, c \dots\}$

$\forall a, \exists\, D(a)_{m \times m}$

preserve the group structure

Group Representations

$G = \{a, b, c, \ldots\}$

$\forall a,\ \exists\ D(a)_{m\times m}$

preserve the group structure

$D(a) \circ D(b) = D(a \circ b)$

esentations

...c. ... }

$\exists\ D(a)_{m\times m}$

the group structure

$\circ D(b) = D(a\circ b)$

1) faithful representation distinctive

esentations

.. c. ... }

$\exists\ D(a)_{n\times m}$

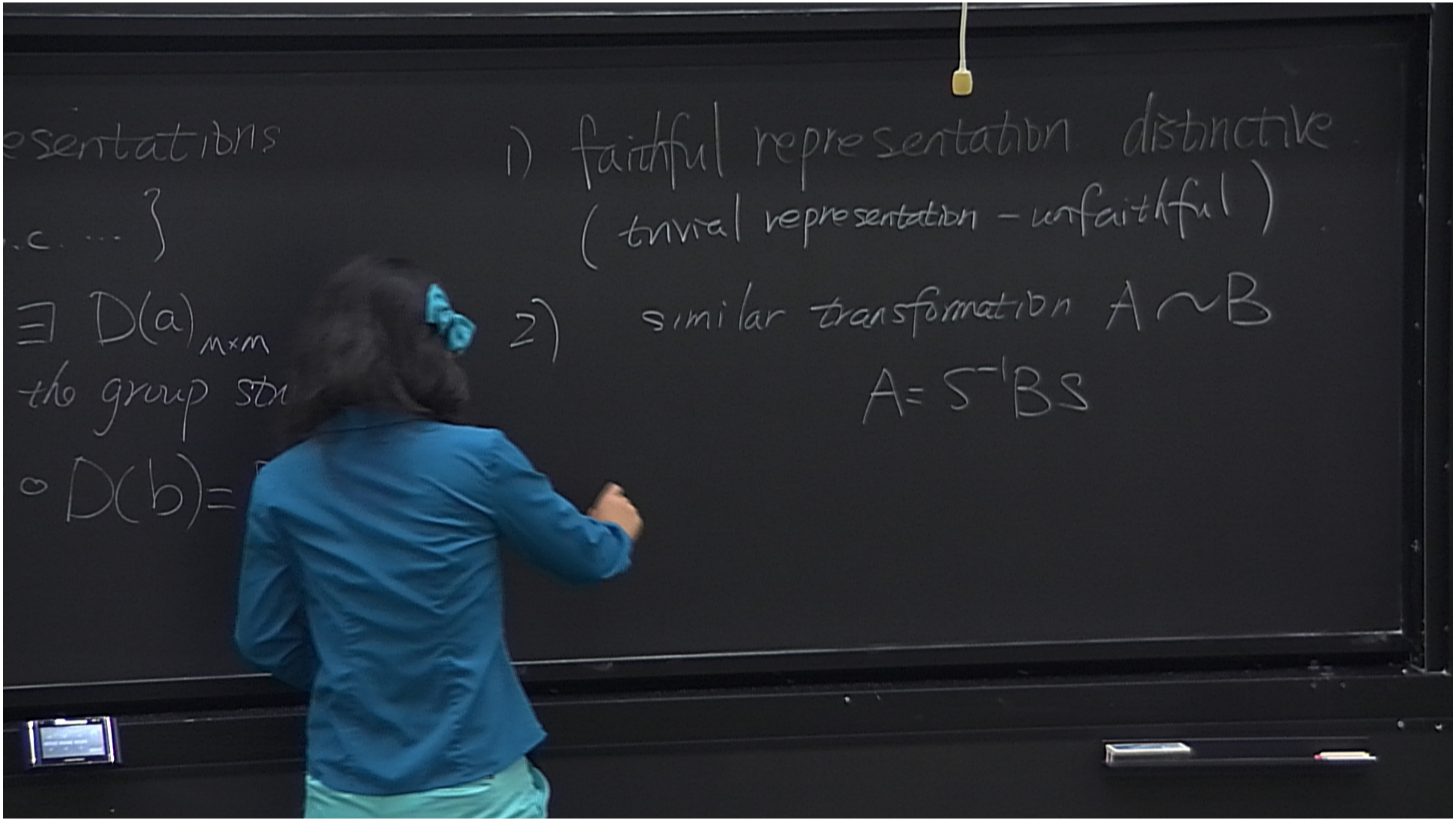
the group structure

$\circ\, D(b) = D(a\circ b)$

1) faithful representation distinctive

(trivial representation – unfaithful)

2)


esentations
..c....}
$\exists\ D(a)_{m\times m}$
the group str
$D(b)=$
1) faithful representation distinctive
(trivial representation - unfaithful)
2) similar transformation $A \sim B$
$A = S^{-1}BS$

esentations

..c. ... }

$\exists$ $D(a)_{m\times m}$

the group structure

$\circ D(b) = D(a\circ b)$

1) faithful representation distinctive

(trivial representation - unfaithful)

2) similar transformation $A \sim B$

$$A = S^{-1} B S$$

if representation $D(a)$ and $D'(a)$

$$D(a) = S^{-1} D'(a) S$$

$= |\langle \psi_1 | \psi_2 \rangle|^2$

s

n

structure

$D(a \circ b)$

1) faithful representation distinctive

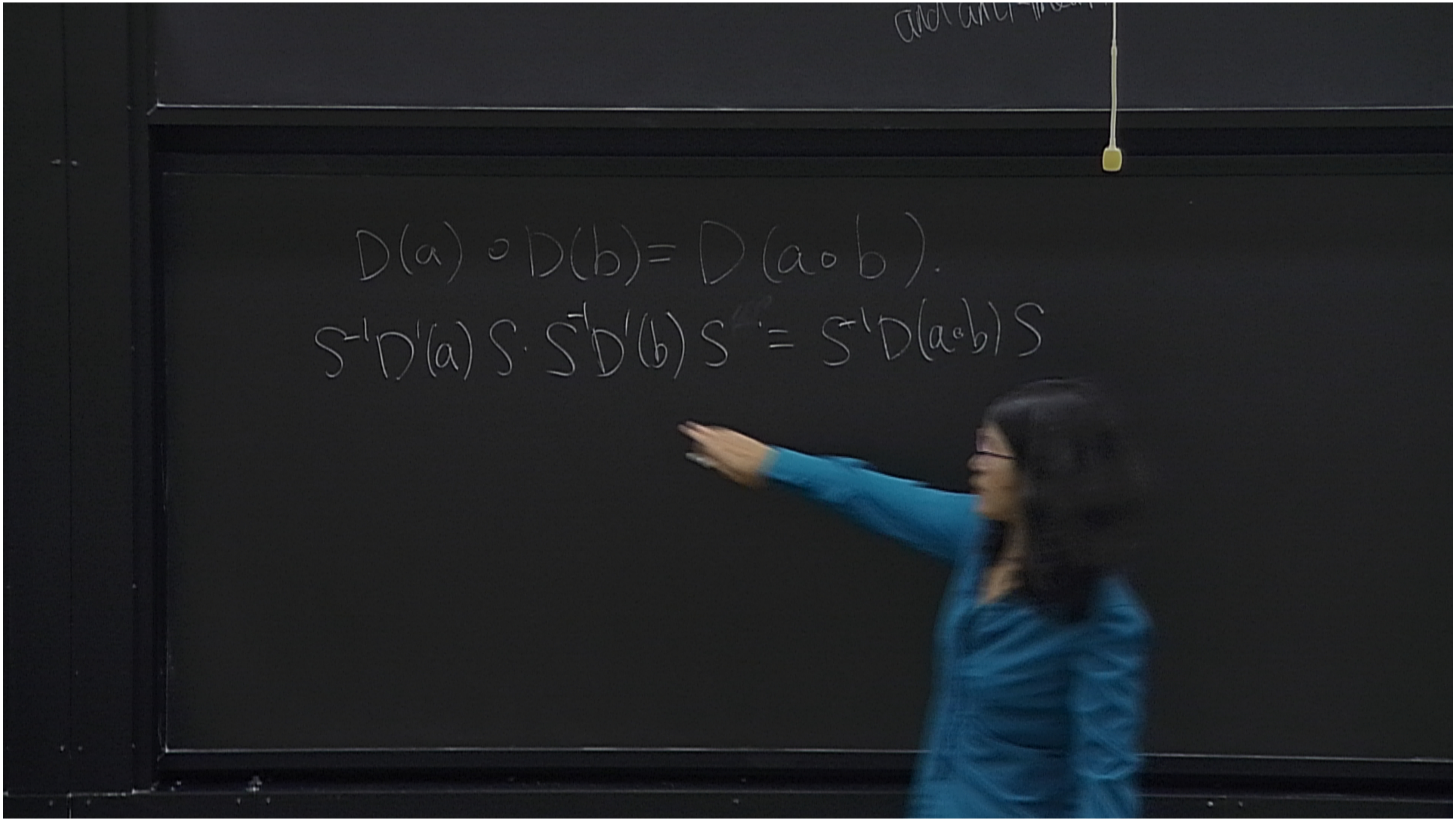
(trivial representation - unfaithful)

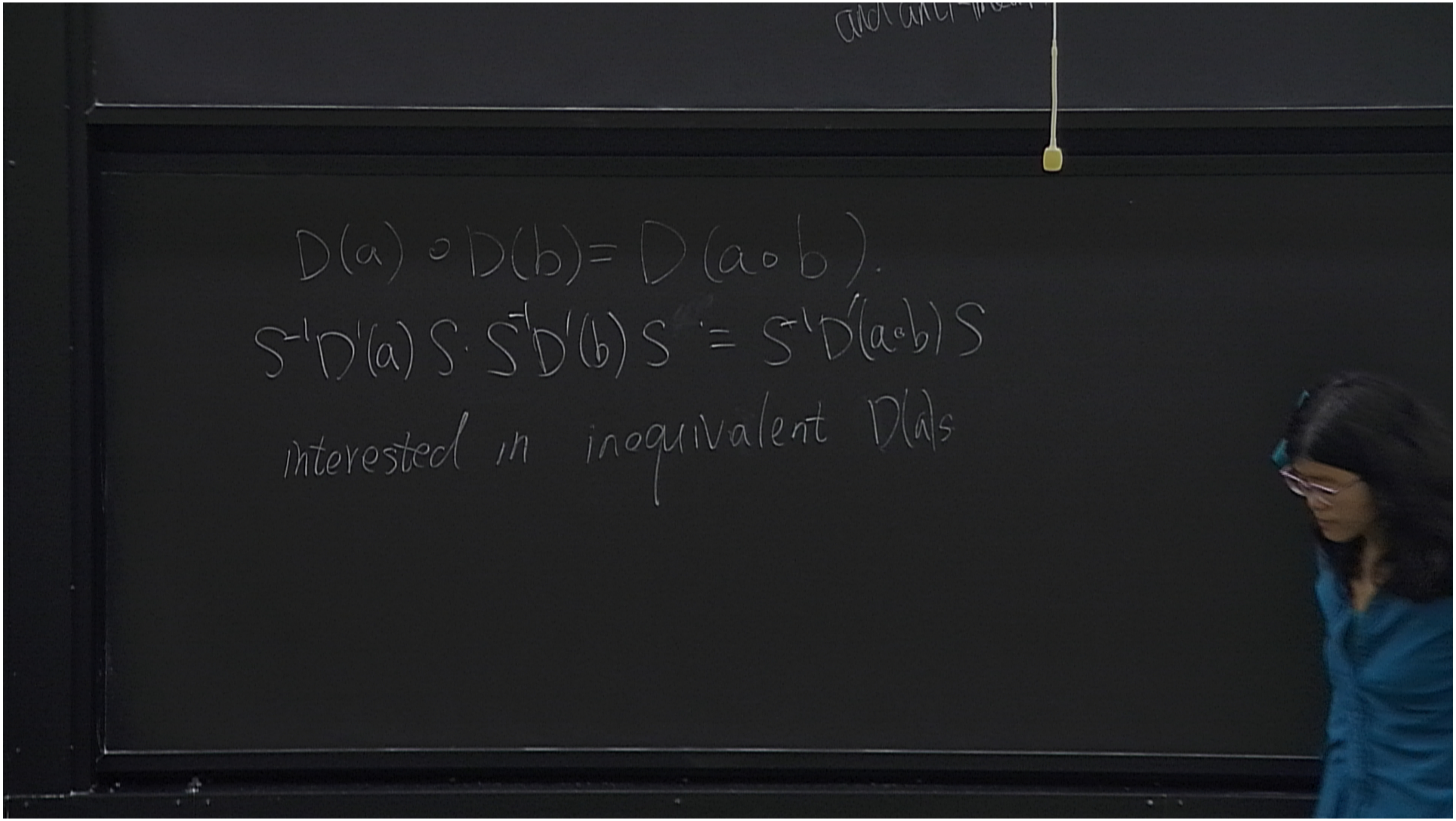
2) similar transformation $A \sim B$

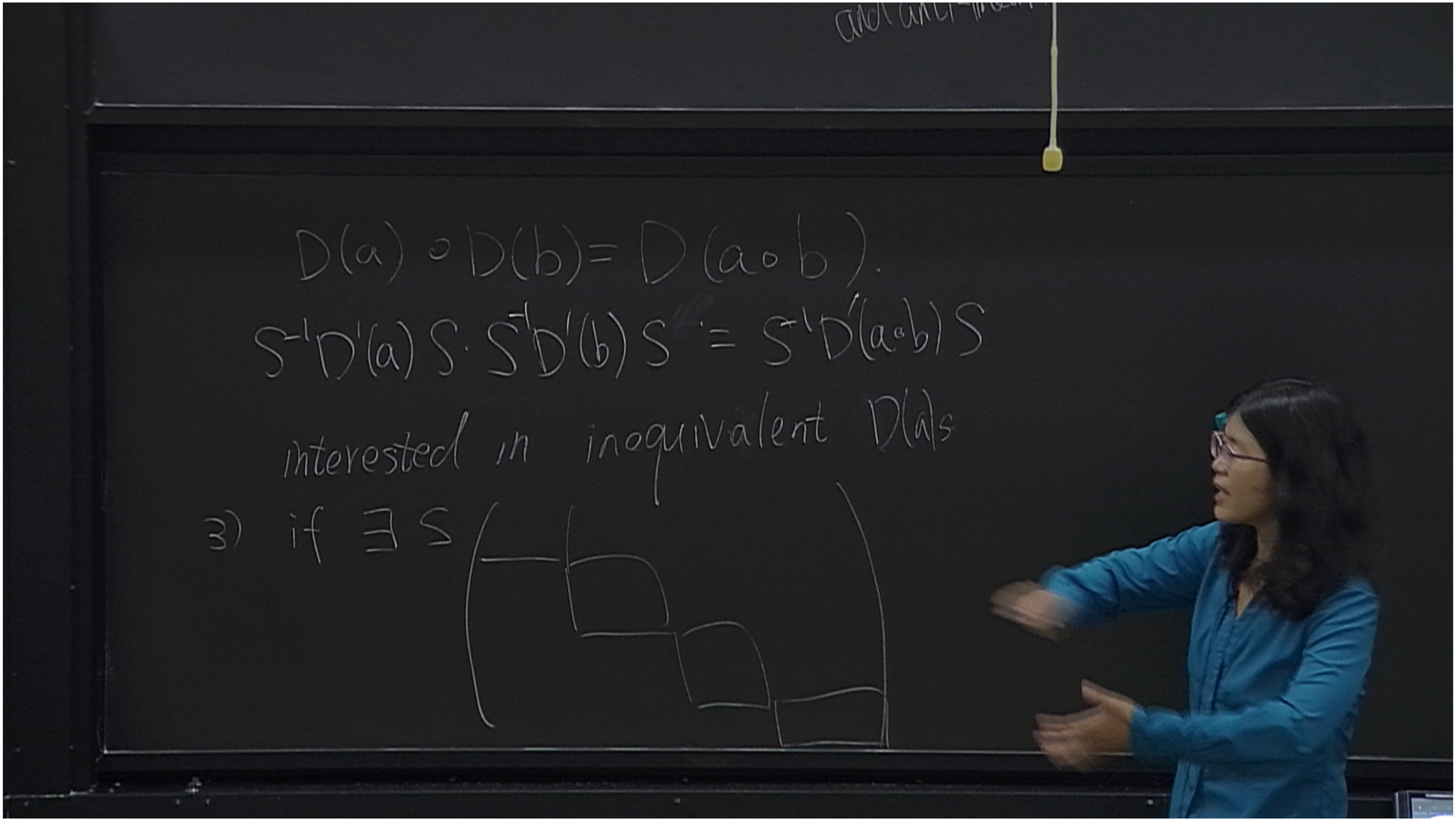
$$A = S^{-1} B S$$

if representation $D(a)$ and $D'(a)$ $\forall a$

$$D(a) = S^{-1} D'(a) S$$


$D(a) \circ D(b) = D(a \circ b).$
$S^{-1}D'(a)S \cdot S^{-1}D'(b)S = S^{-1}D(a \circ b)S$


$D(a) \circ D(b) = D(a \circ b)$.
$S^{-1} D'(a) S \cdot S^{-1} D'(b) S = S^{-1} D'(a \circ b) S$
interested in inequivalent $D(a)$s


$D(a) \circ D(b) = D(a \circ b)$.
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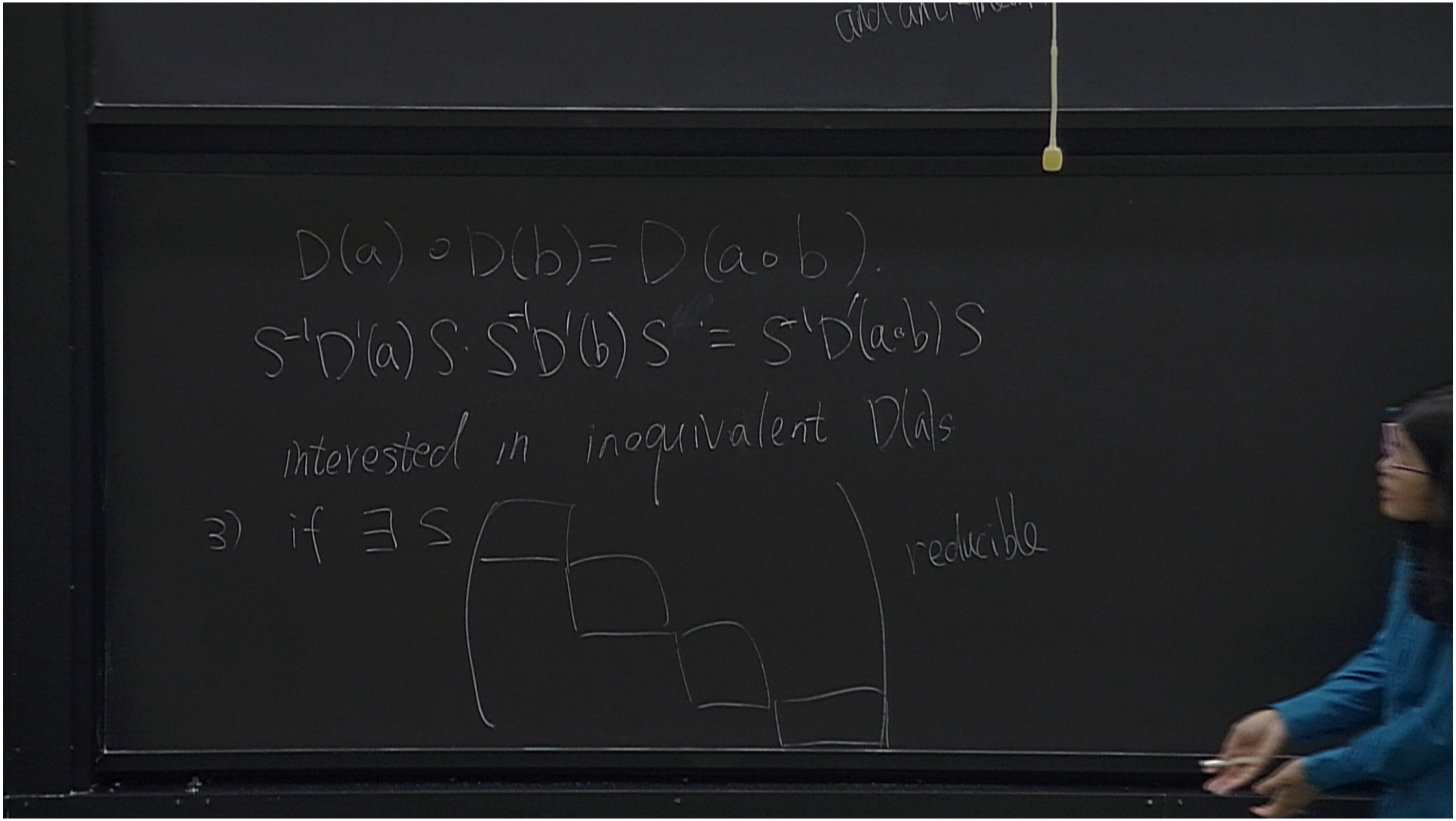
interested in inequivalent D(a)s
3) if $\exists S$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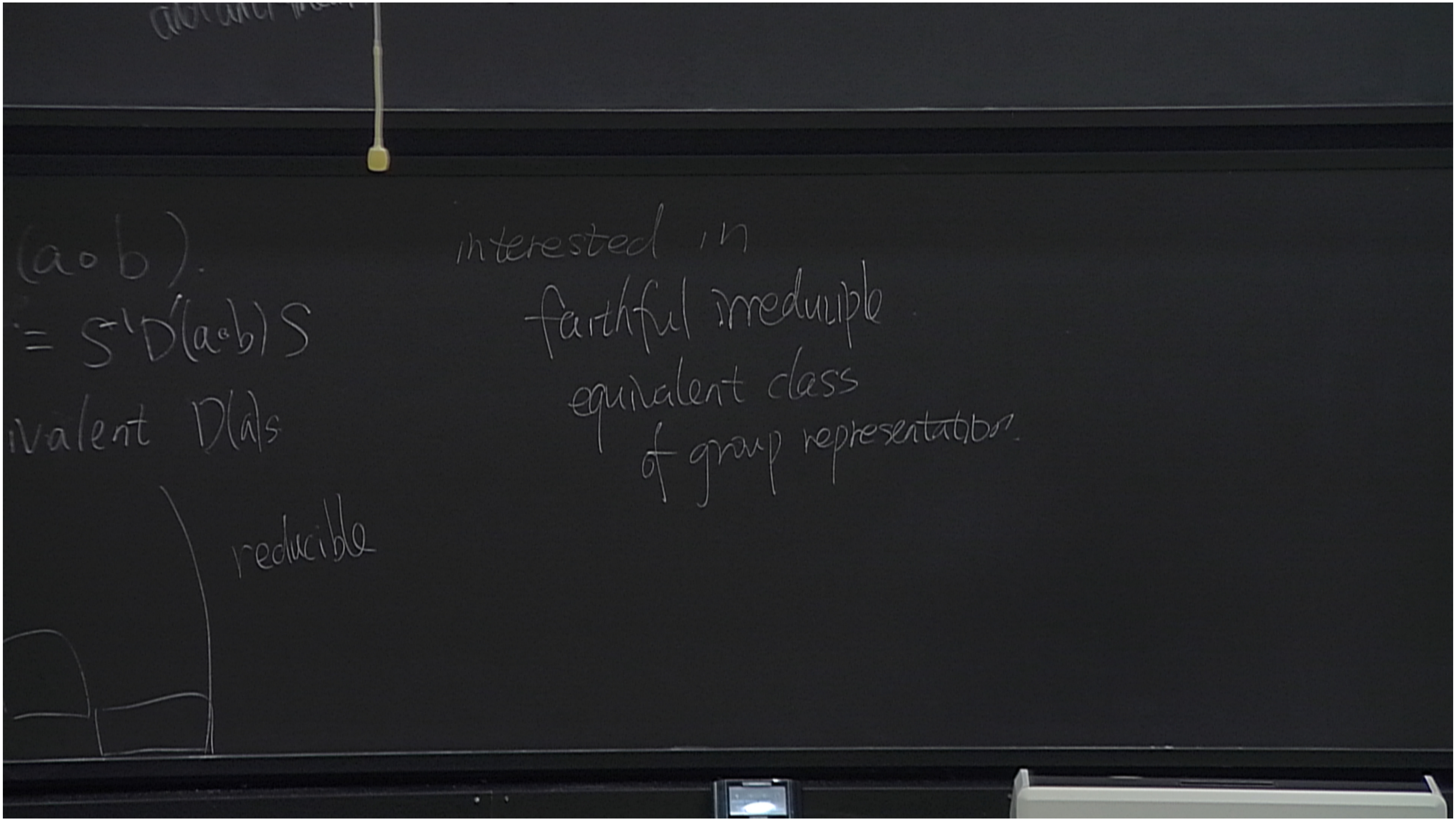
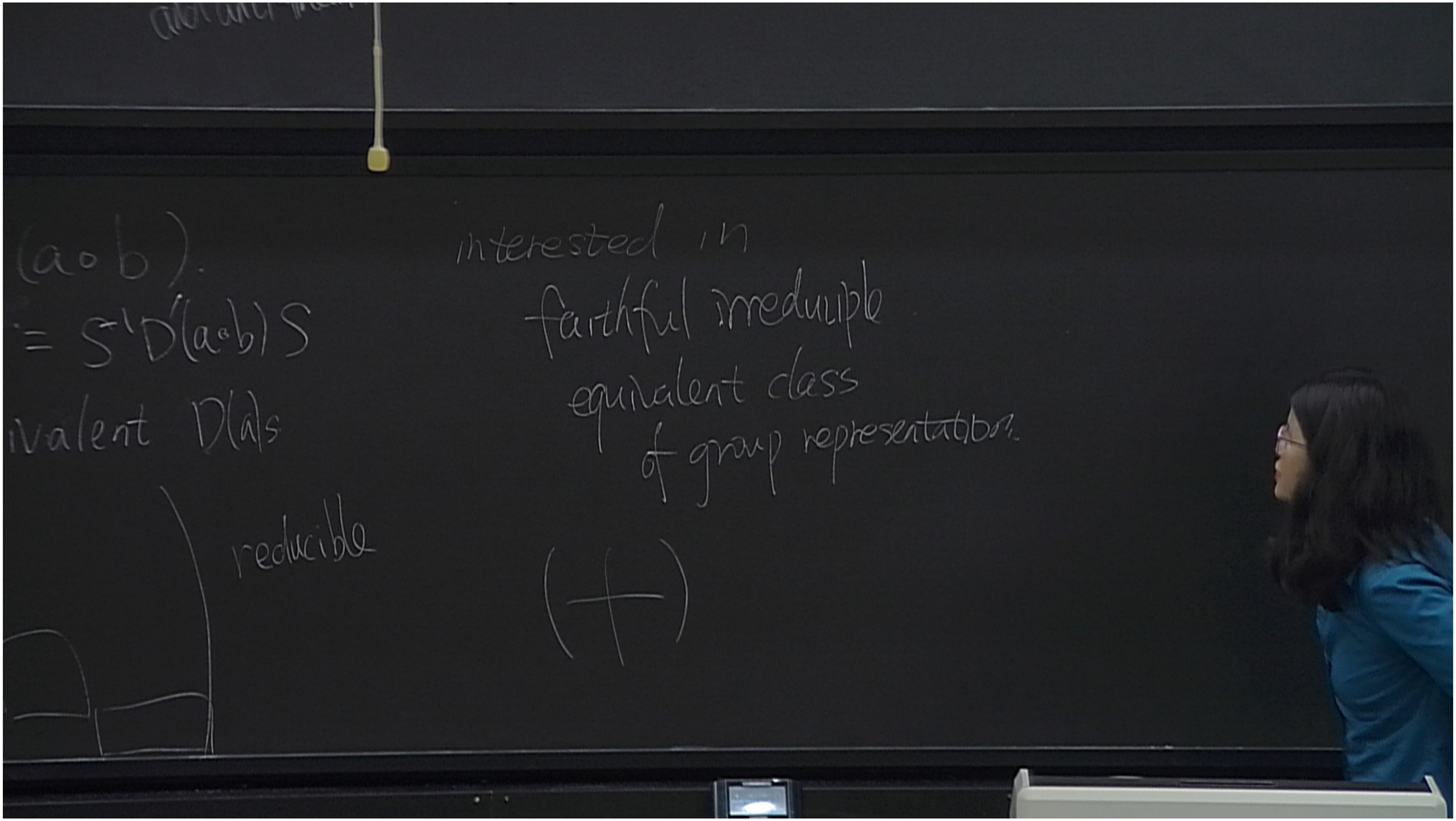
interested in inequivalent $D(a)$s
3) if $\exists\, S$
reducible


(a∘b).
= S⁻¹D′(a∘b) S
ivalent D(a)s
reducible
interested in
faithful irreducible
equivalent class
of group representation.



(a∘b).
= S⁻¹D'(a∘b)S
ivalent D(a)s
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of group representation.

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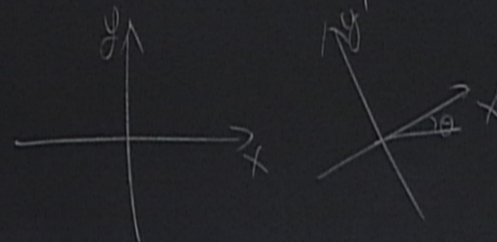
ul irreducible
alent class
f group representation

Which Lie Groups

what. group members are labeled by continuous parameter

$\vec{\theta} = (\theta_1, \theta_2, \cdots)$

start with one continuous parameter



$\{e^{i\theta}\}$

$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

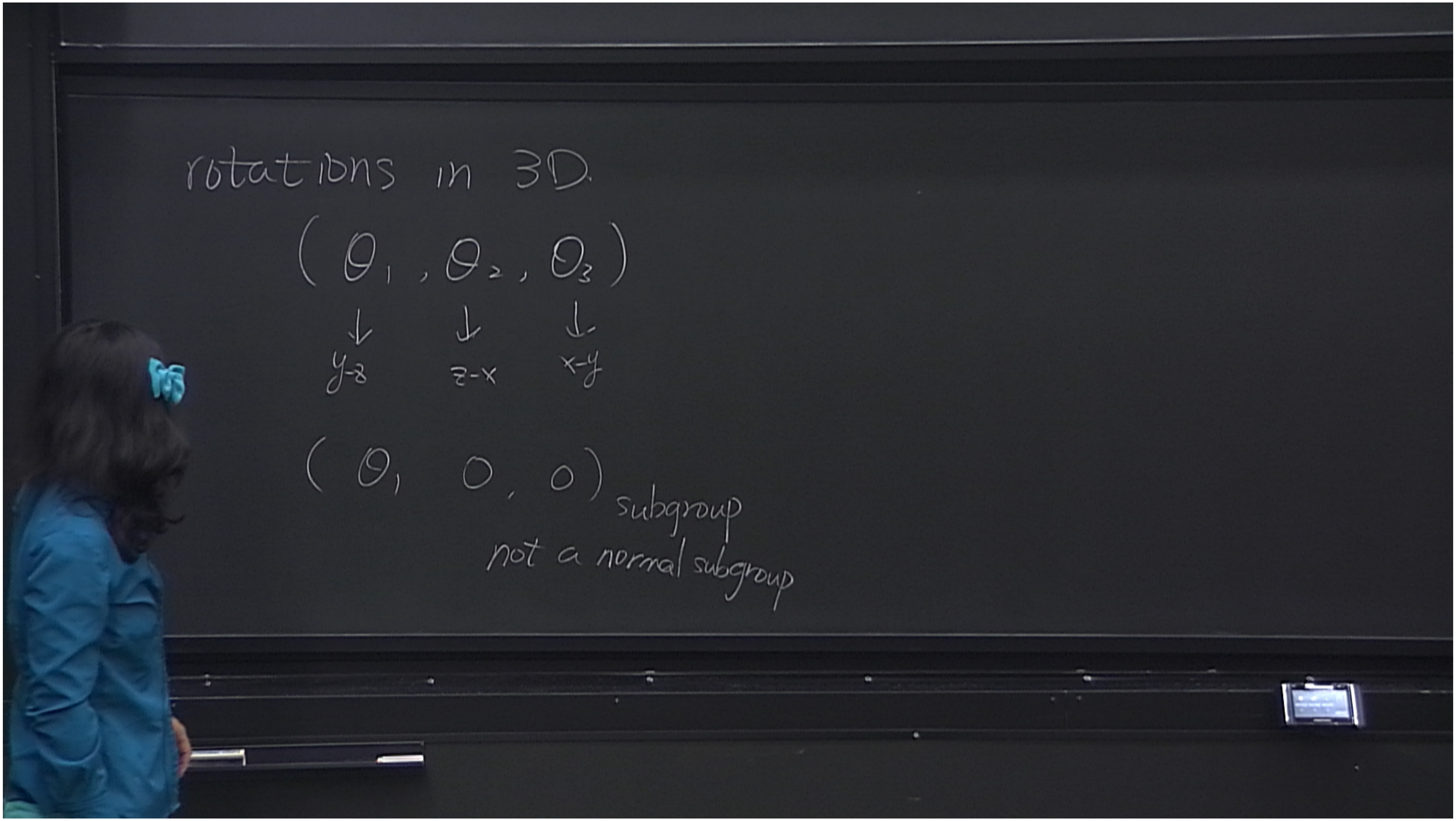
$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i\theta_2} \cdot e^{i\theta_1}$

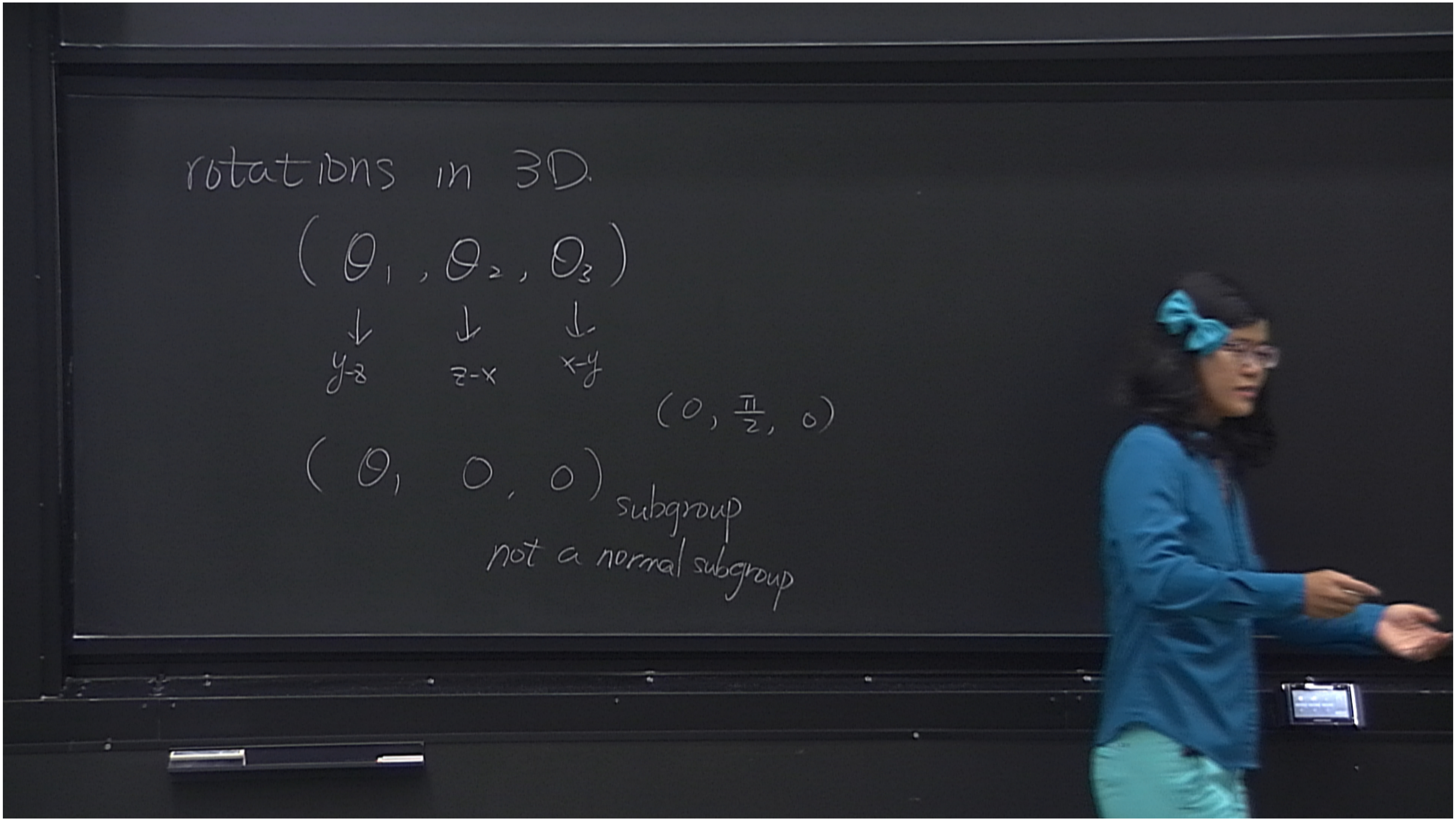
rotations in 3D.

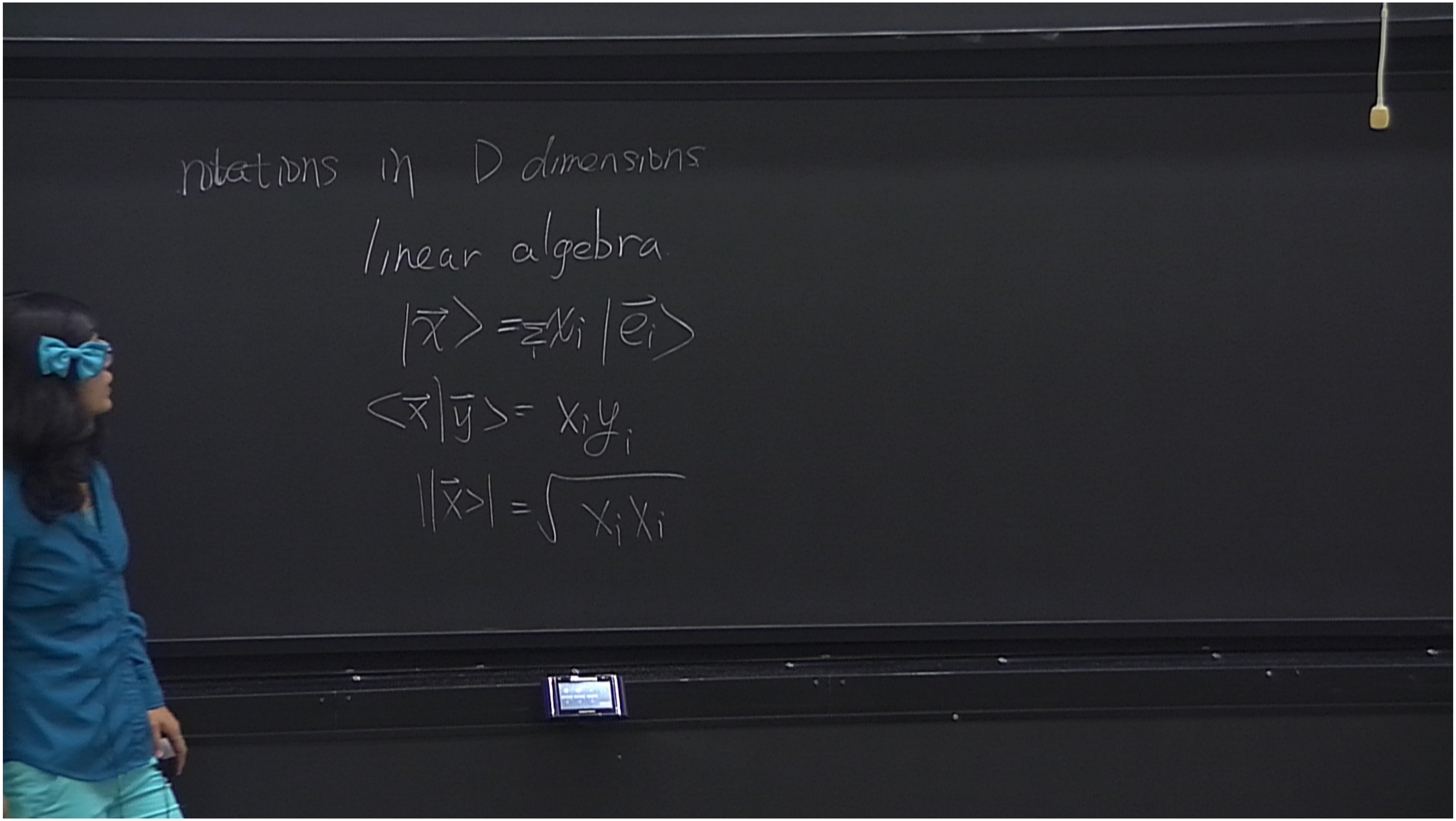
$$(\theta_1, \theta_2, \theta_3)$$

$$y\text{-}z \qquad z\text{-}x \qquad x\text{-}y$$

$$(\theta_1, 0, 0)_{\text{subgroup}}$$


rotations in 3D.
$(\theta_1, \theta_2, \theta_3)$
y-z
z-x
x-y
$(\theta_1\ \ 0, 0)$
subgroup
not a normal subgroup


rotations in 3D.
$(\theta_1, \theta_2, \theta_3)$
y-z z-x x-y
$(0, \frac{\pi}{2}, 0)$
$(\theta_1, 0, 0)$
subgroup
not a normal subgroup


rotations in D dimensions
linear algebra
$|\vec{x}\rangle = \sum_i x_i |\vec{e}_i\rangle$
$\langle\vec{x}|\vec{y}\rangle = x_i y_i$
$||\vec{x}\rangle| = \sqrt{x_i x_i}$

D dimensions

ear algebra

$\vec{x}\rangle = \sum_i x_i |\vec{e}_i\rangle$

$|\vec{y}\rangle = x_i y_i$

$|\vec{x}\rangle| = \sqrt{x_i x_i}$

rotation

$R|\vec{x}\rangle = |\vec{x}'\rangle$

$\langle \vec{x}' | \vec{y}' \rangle = \langle \vec{x} | \vec{y} \rangle$

D dimensions

ear algebra

$$\vec{x}\rangle = \sum_i x_i |\vec{e}_i\rangle$$

$$|\vec{y}\rangle = x_i y_i$$

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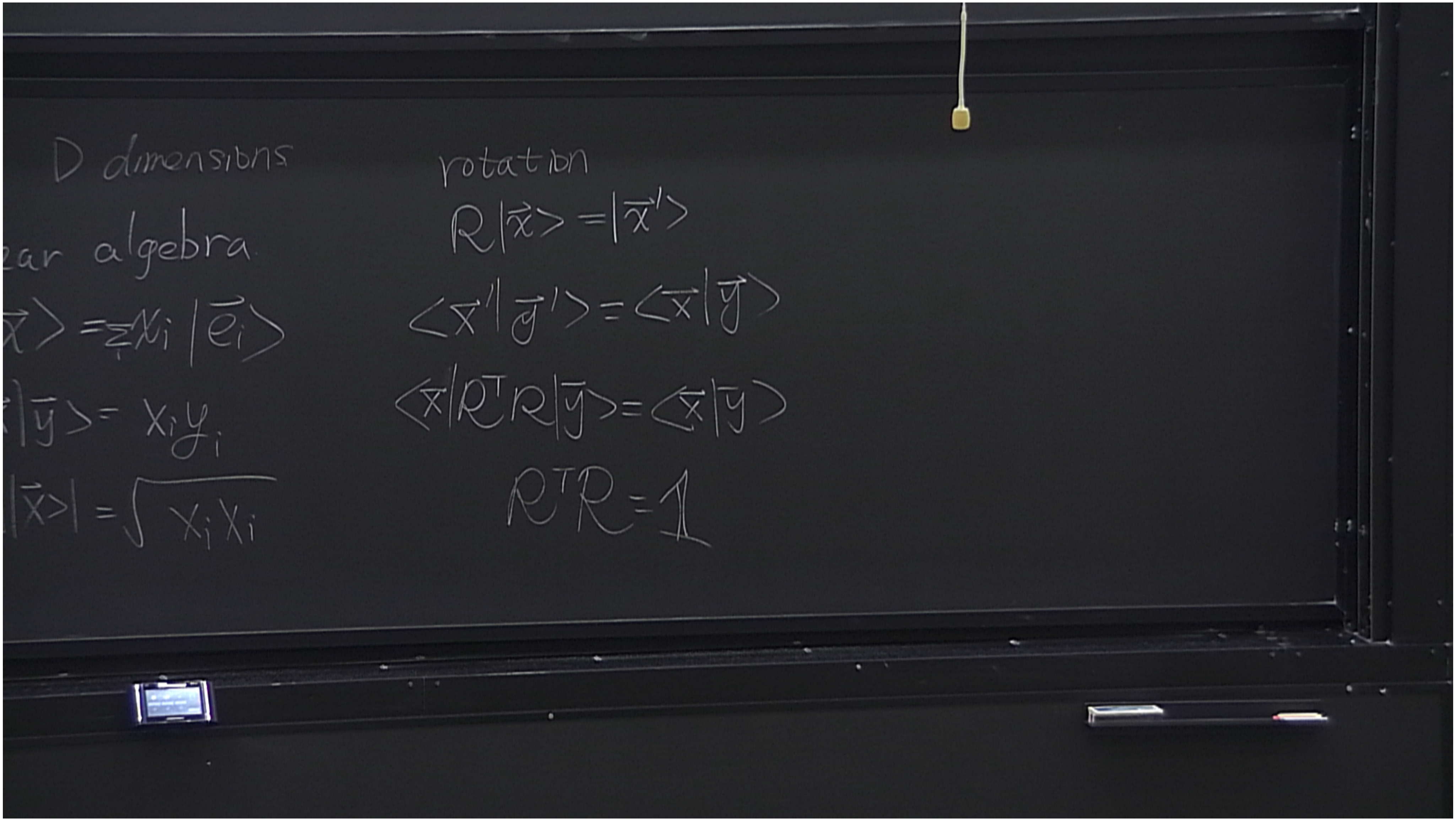
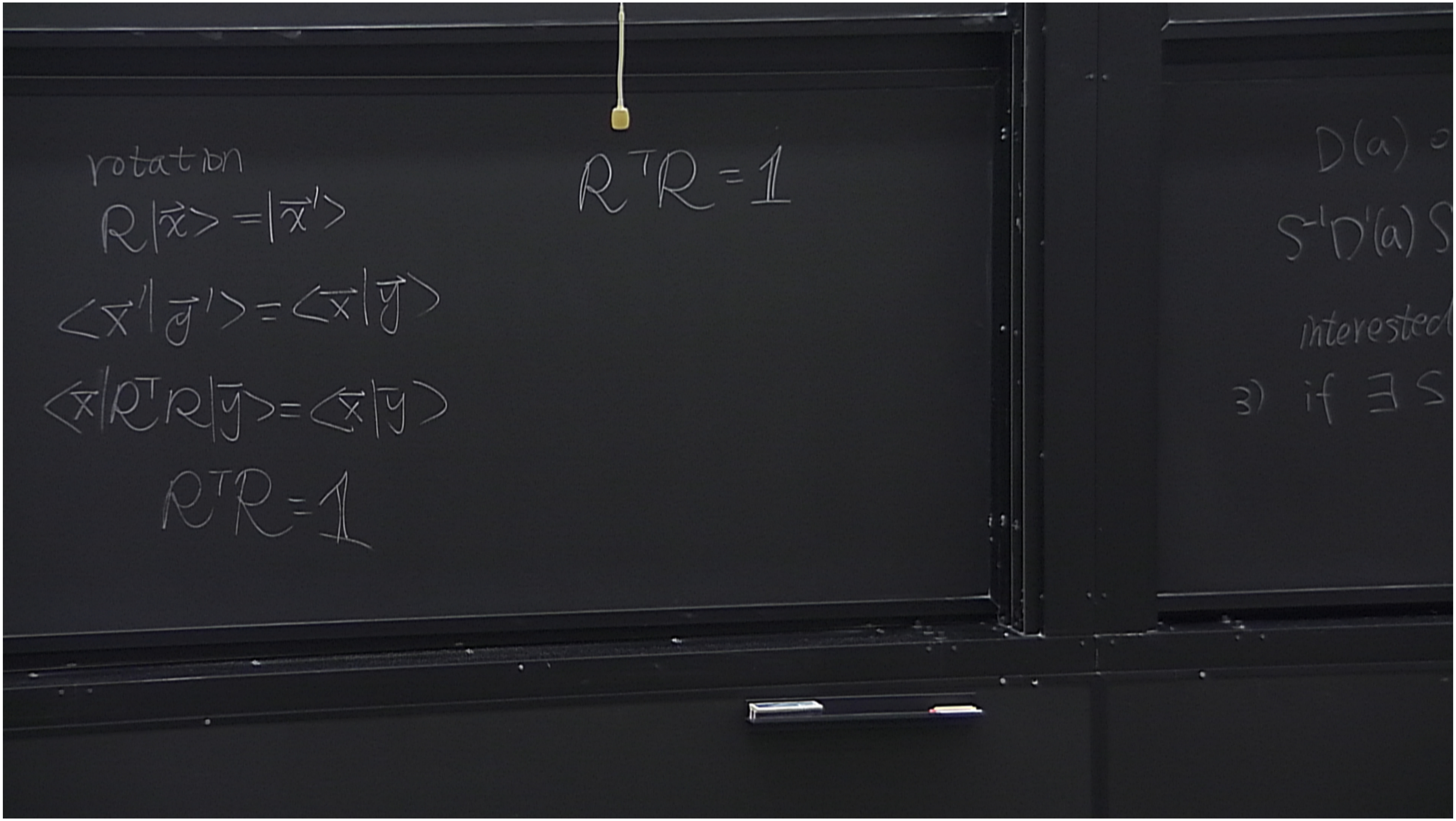
rotation

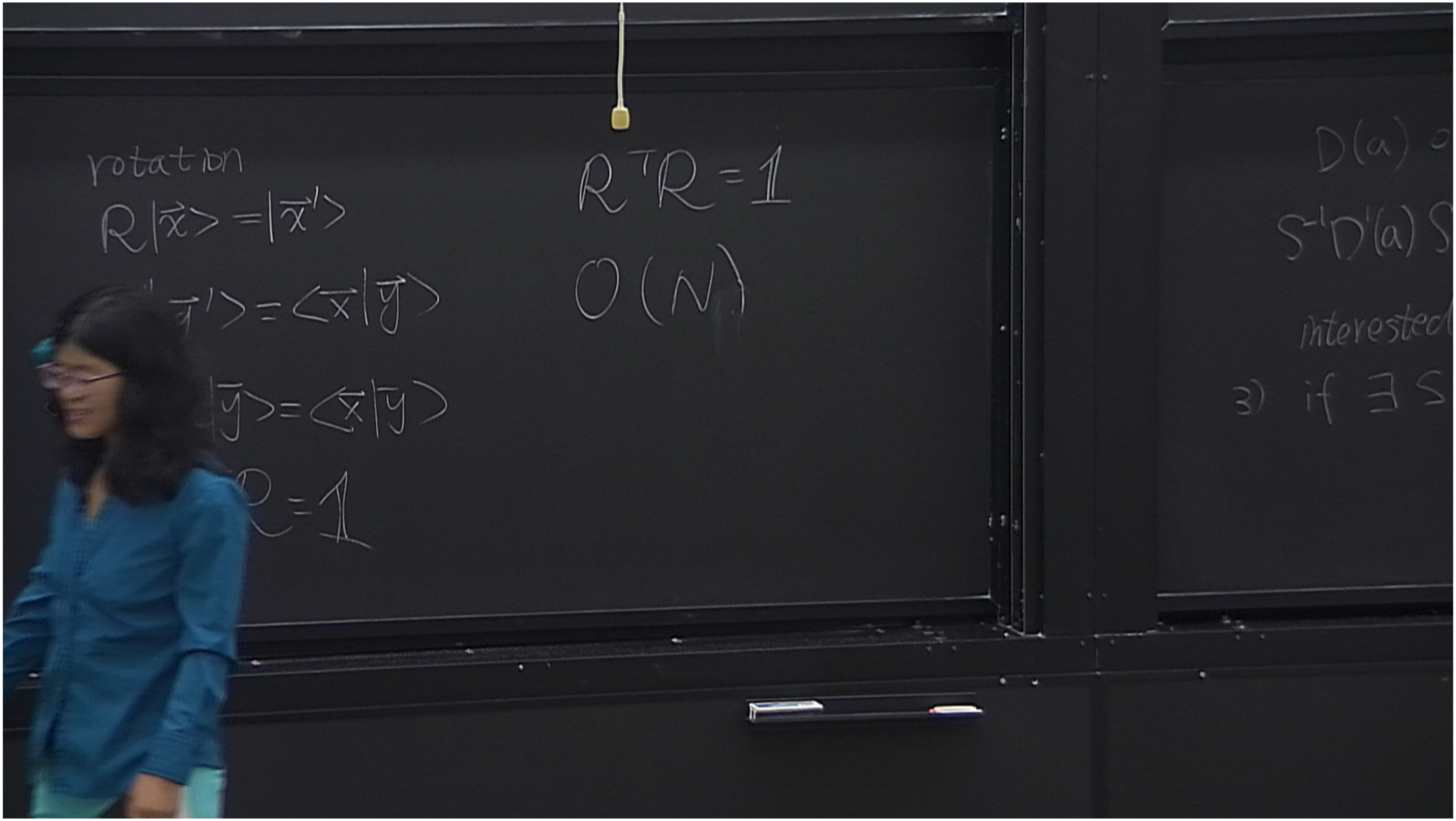
$$R|\vec{x}\rangle = |\vec{x}'\rangle$$

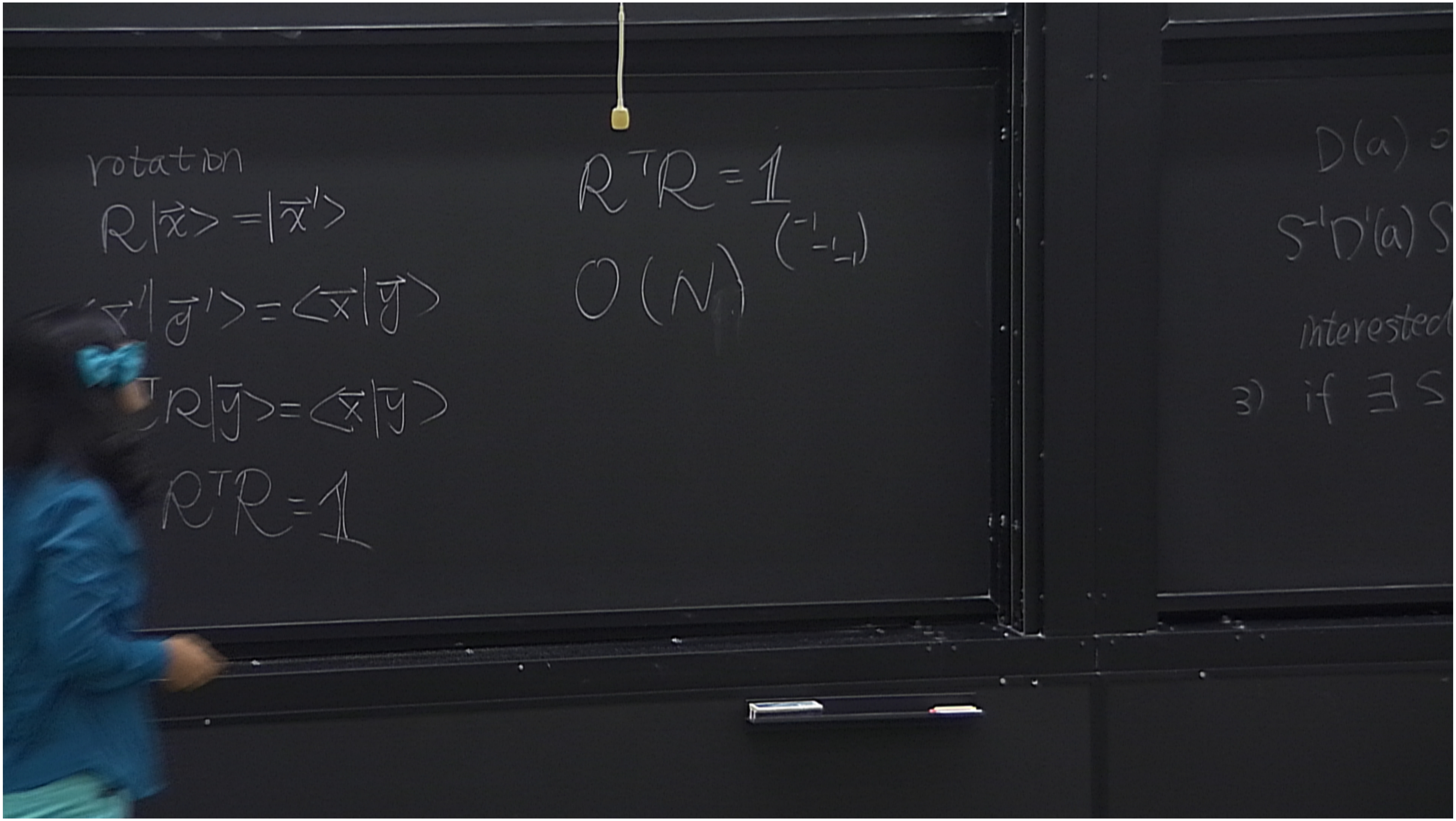
$$\langle \vec{x}'|\vec{y}'\rangle = \langle \vec{x}|\vec{y}\rangle$$

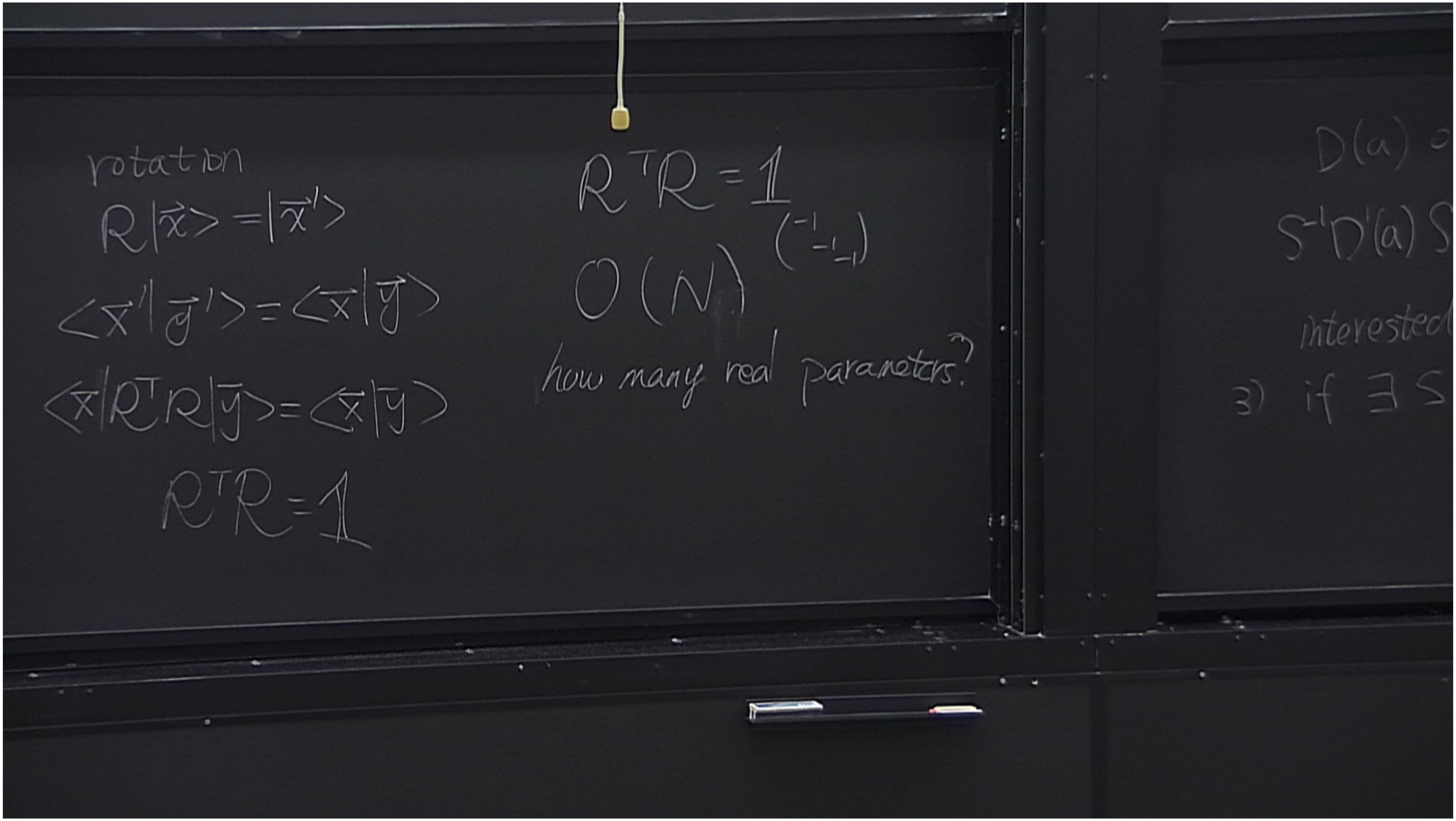
$$\langle \vec{x}|R^T R|\vec{y}\rangle = \langle \vec{x}|\vec{y}\rangle$$

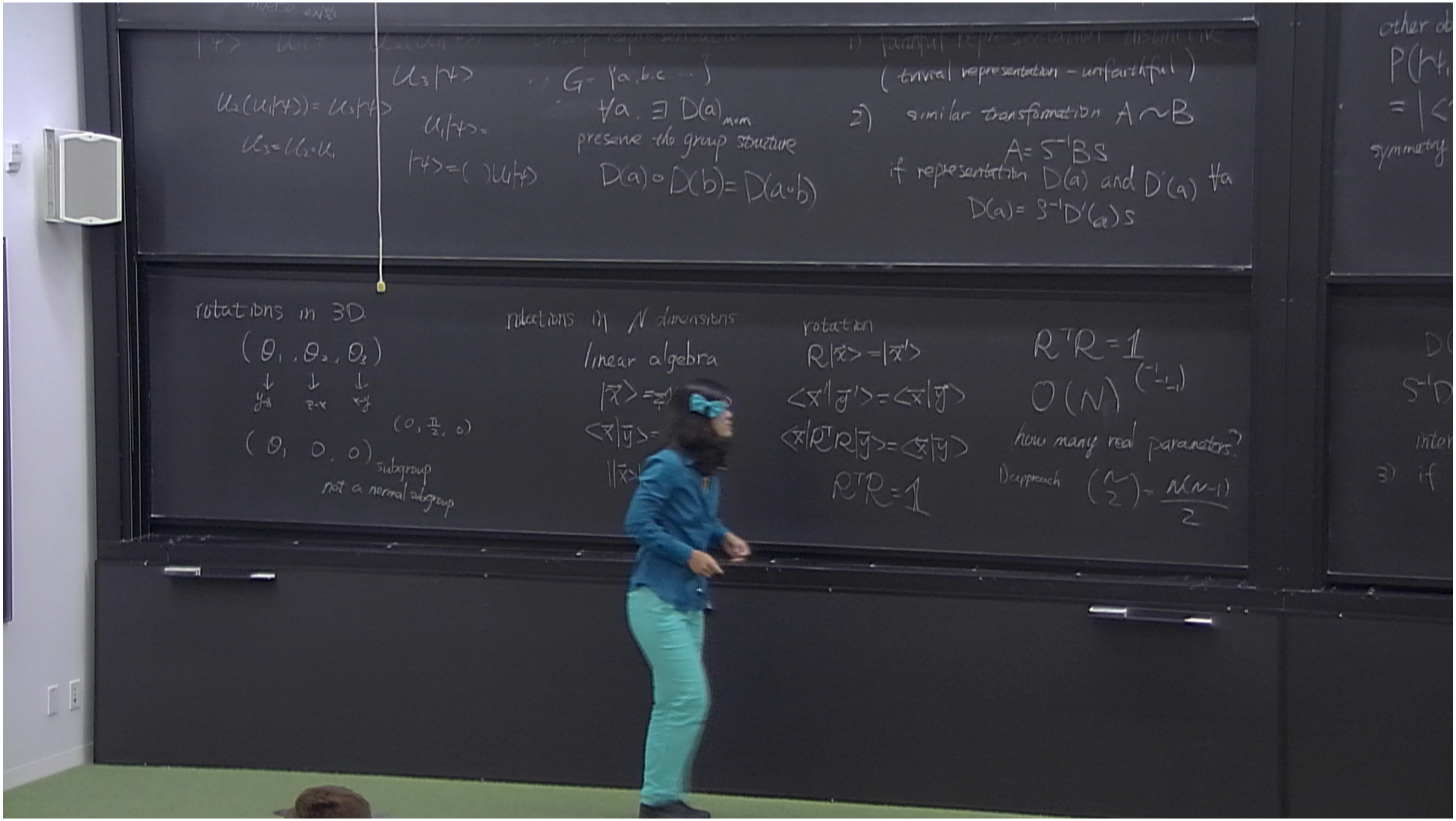

D dimensions
ear algebra
$|\vec{x}\rangle = \sum_i x_i |\vec{e}_i\rangle$
$|\vec{y}\rangle = x_i y_i$
$||\vec{x}\rangle| = \sqrt{x_i x_i}$
rotation
$R|\vec{x}\rangle = |\vec{x}'\rangle$
$\langle \vec{x}' | \vec{y}' \rangle = \langle \vec{x} | \vec{y} \rangle$
$\langle \vec{x} | R^T R | \vec{y} \rangle = \langle \vec{x} | \vec{y} \rangle$
$R^T R = 1$

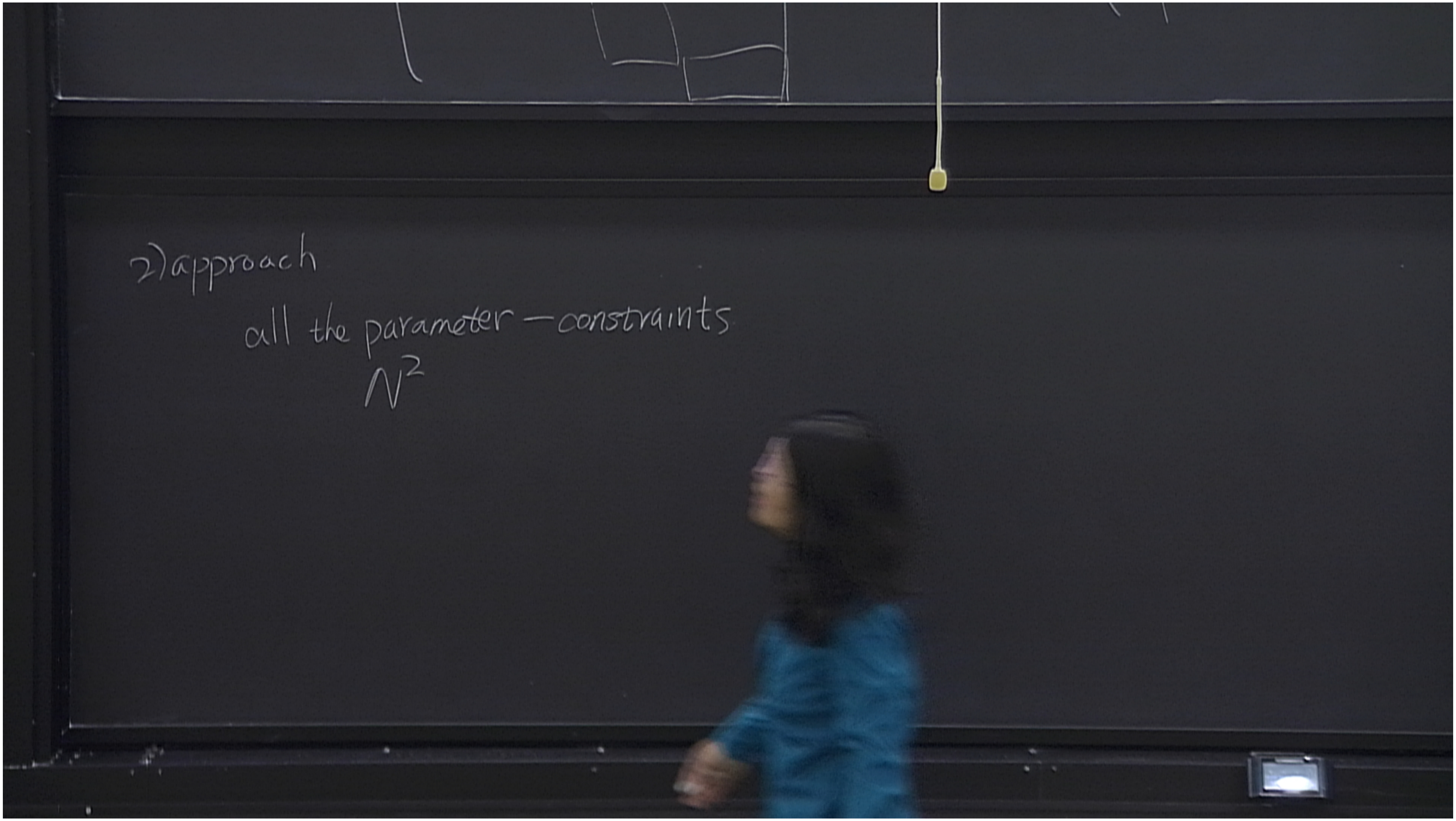

rotation
$R|\vec{x}\rangle = |\vec{x}'\rangle$
$\langle\vec{x}'|\vec{y}'\rangle = \langle\vec{x}|\vec{y}\rangle$
$\langle\vec{x}|R^T R|\vec{y}\rangle = \langle\vec{x}|\vec{y}\rangle$
$R^T R = \mathbb{1}$
$R^T R = \mathbb{1}$
$D(a)$
$S^{-1}D'(a)S$
interested
3) if $\exists S$

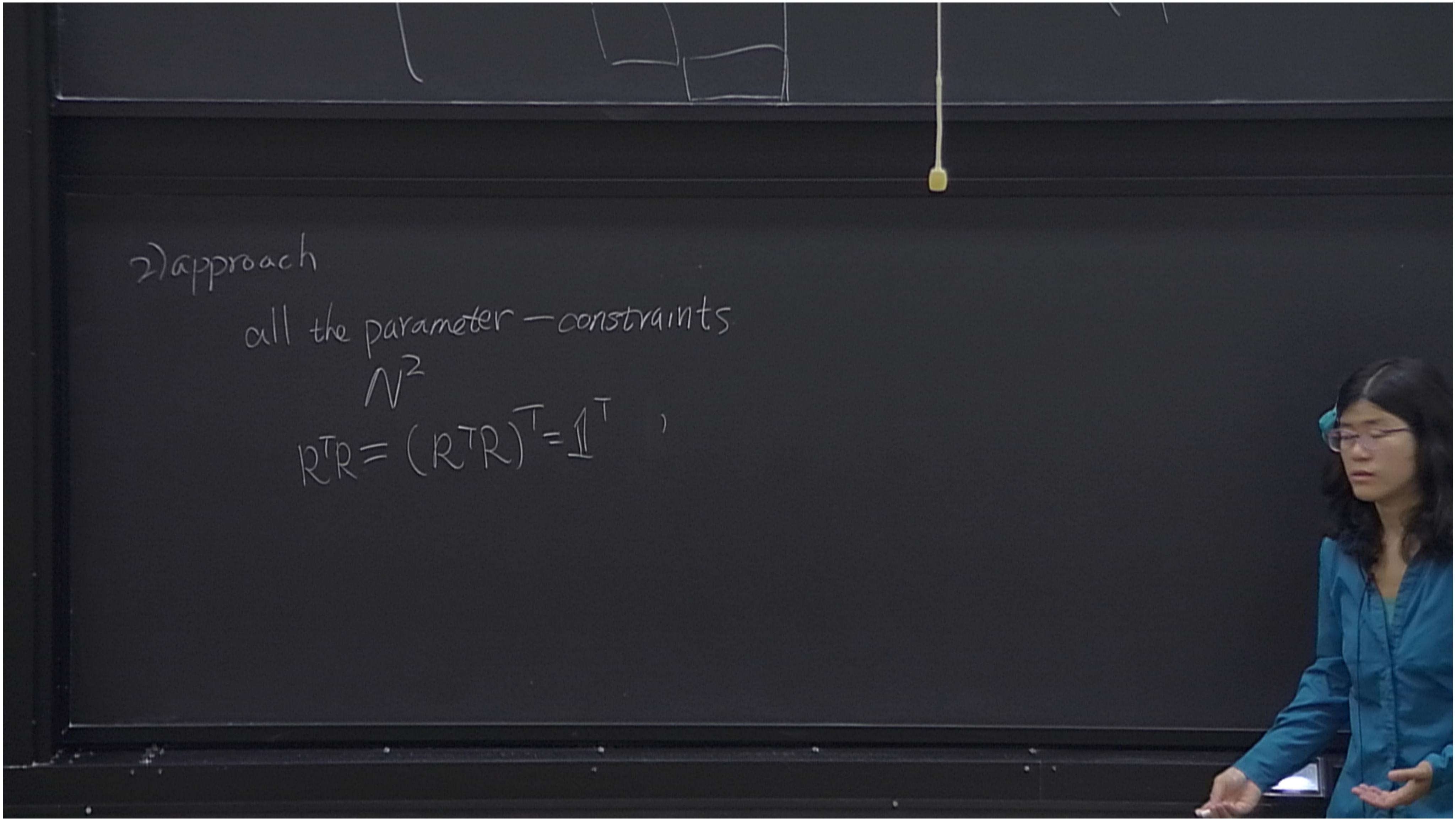

rotation
R|x⃗> = |x⃗'>
<x⃗|y⃗> = <x⃗|y⃗>
R = 1
R^T R = 1
O(N)
D(a)
S^-1 D'(a) S
interested
3) if ∃ S

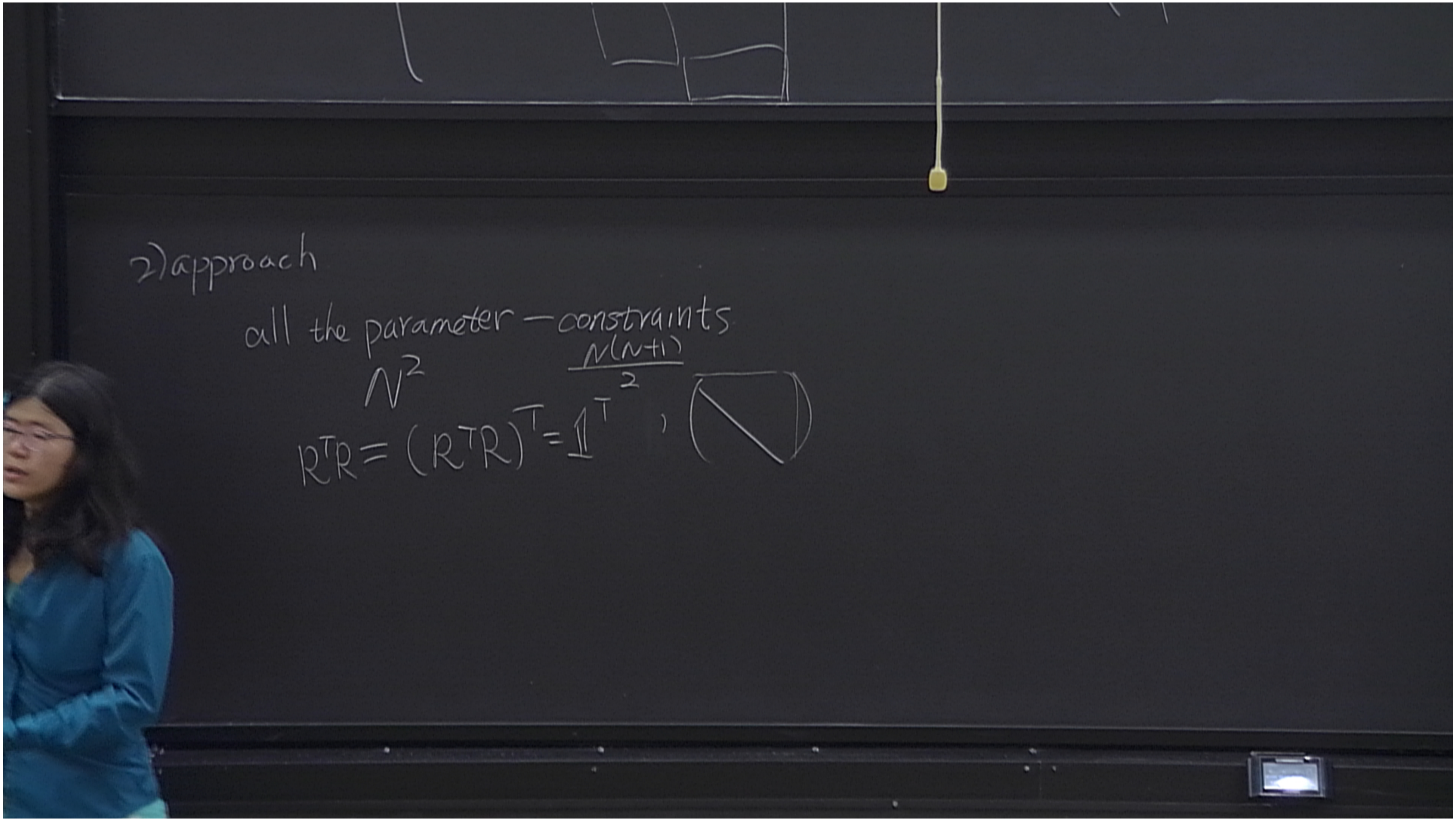

rotation
R|x> = |x'>
<x'|y'> = <x|y>
R|y> = <x|y>
R^T R = 1
R^T R = 1
O(N)
D(a)
S^-1 D'(a) S
3) if ∃ S


rotation
$R|\vec{x}\rangle = |\vec{x}'\rangle$
$\langle \vec{x}'|\vec{y}'\rangle = \langle \vec{x}|\vec{y}\rangle$
$\langle \vec{x}|R^T R|\vec{y}\rangle = \langle \vec{x}|\vec{y}\rangle$
$R^T R = 1$
$R^T R = 1$
$O(N)$
how many real parameters?
$D(a)$
$S^{-1}D'(a)S$
interested
3) if $\exists S$


trivial representation - unfaithful
similar transformation A ~ B
A = S⁻¹BS
if representation D(a) and D'(a) ∀a
D(a) = S⁻¹D'(a)S
preserve the group structure
D(a)∘D(b) = D(a∘b)
rotations in 3D
subgroup
not a normal subgroup
rotations in N dimensions
linear algebra
rotation
R|x⃗> = |x⃗'>
<x⃗'|y⃗'> = <x⃗|y⃗>
RᵀR = 1
O(N)
how many real parameters?
other d
symmetry


2) approach
all the parameter — constraints
N^2


2) approach
all the parameter — constraints
N^2
$R^TR = (R^TR)^T = \mathbb{1}^T$


2) approach
all the parameter − constraints
N^2
$\frac{N(N+1)}{2}$
$R^T R = (R^T R)^T = \mathbb{1}^T$

2) approach

all the parameter − constraints

$$N^2 - \frac{N(N+1)}{2} = \frac{N(N-1)}{2}$$

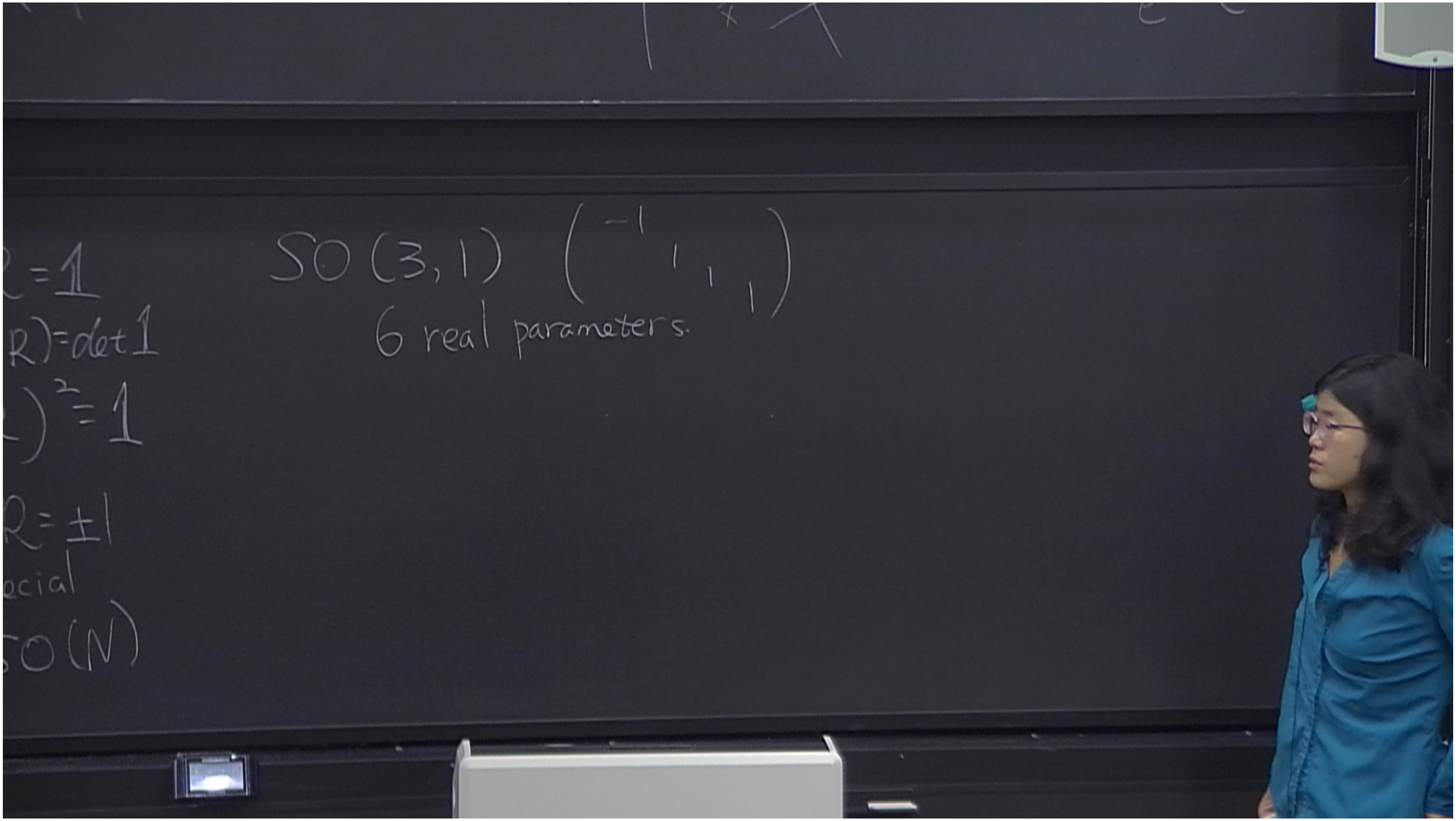
$$R^TR = (R^TR)^T = \mathbb{1}^T$$

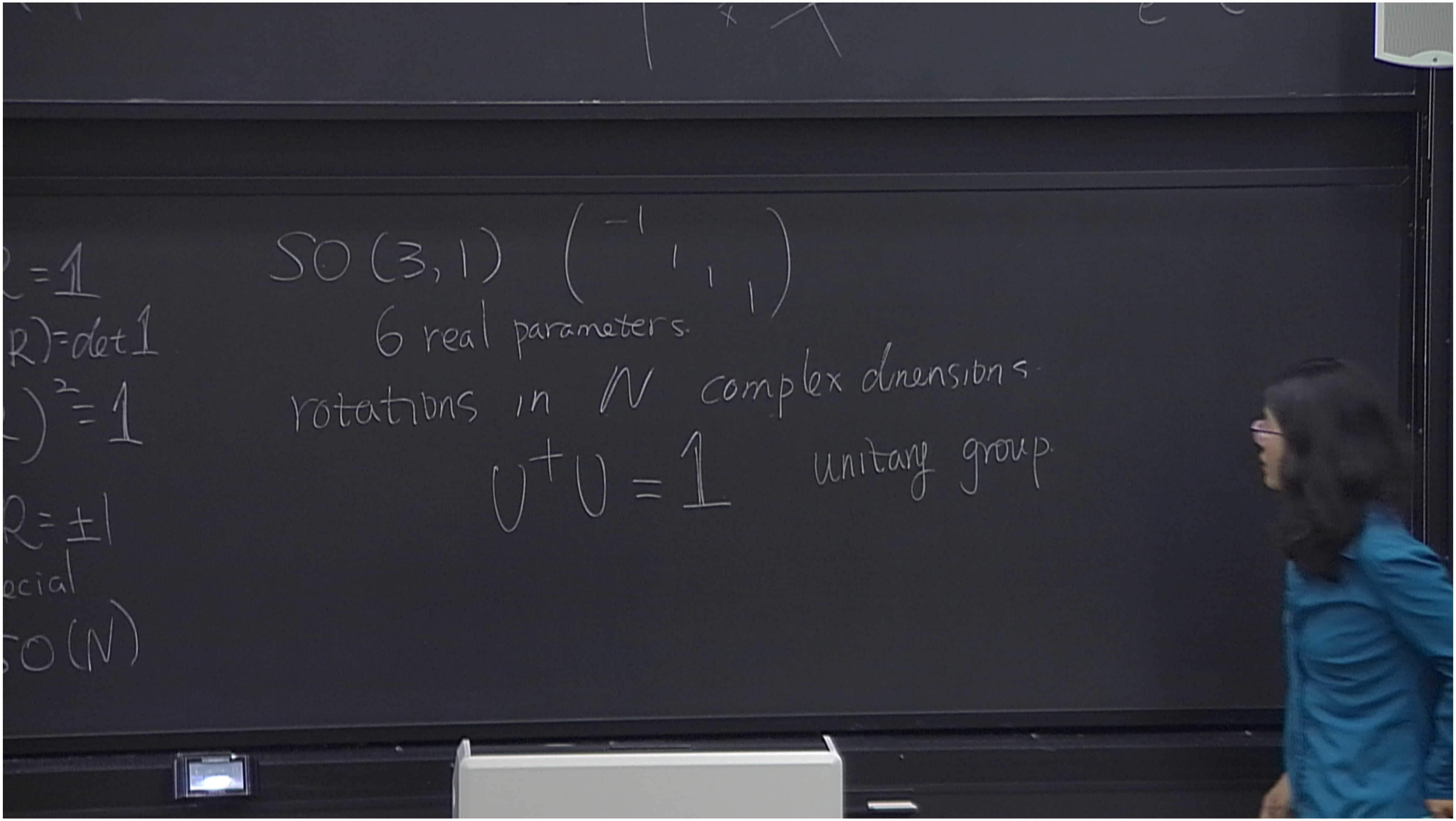
$$R^TR = \mathbb{1}$$

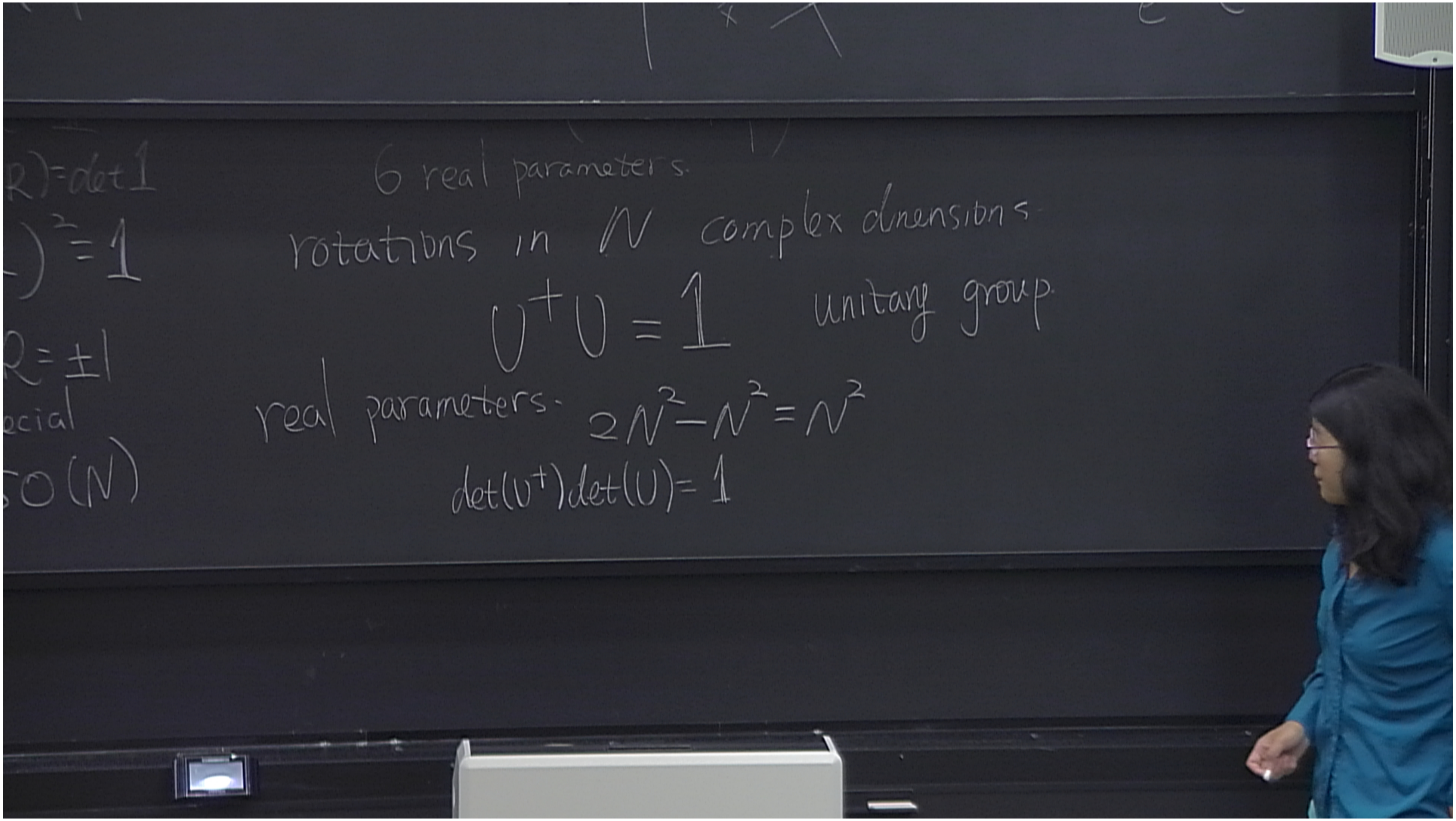
$$\det(R^TR) = \det \mathbb{1}$$

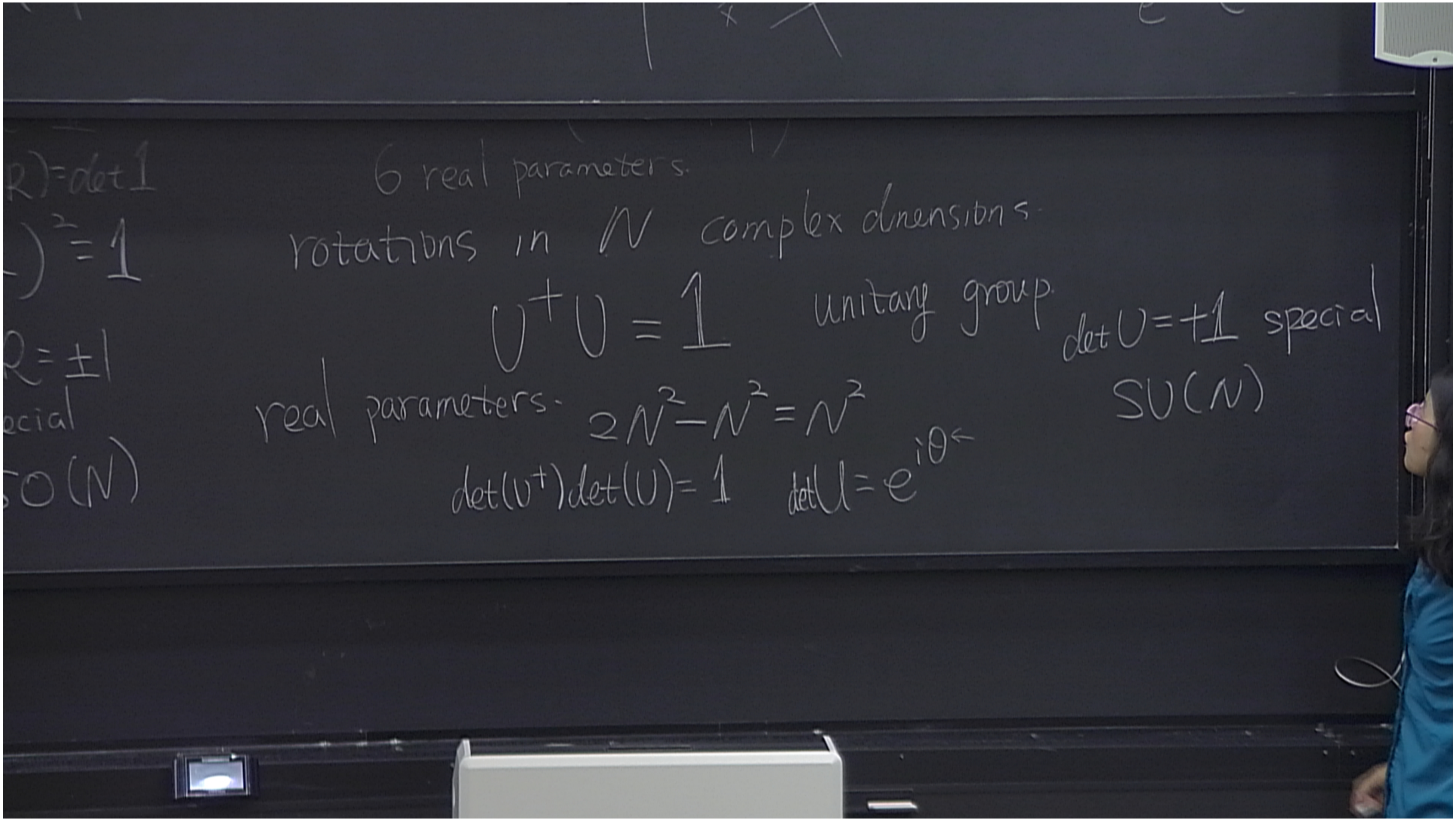
$$(\det R)^2 = 1$$

$$\det R = \pm 1$$


SO(3,1)
$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$
6 real parameters.
R)=det 1
R = ±1
SO(N)


SO(3,1)
6 real parameters.
rotations in N complex dimensions.
U†U = 1
unitary group.
SO(N)


6 real parameters.
rotations in N complex dimensions
$U^\dagger U = 1$ unitary group.
real parameters. $2N^2 - N^2 = N^2$
$\det(U^\dagger)\det(U) = 1$
$SO(N)$


R)=det 1
)² = 1
R = ±1
ecial
SO(N)
6 real parameters.
rotations in N complex dimensions.
U†U = 1 unitary group.
det U = +1 special
SU(N)
real parameters. 2N² − N² = N²
det(U†)det(U) = 1
det U = e^{iθ}

$R) = \det 1$

$)^2 = 1$

$R = \pm 1$

ecial

$SO(N)$

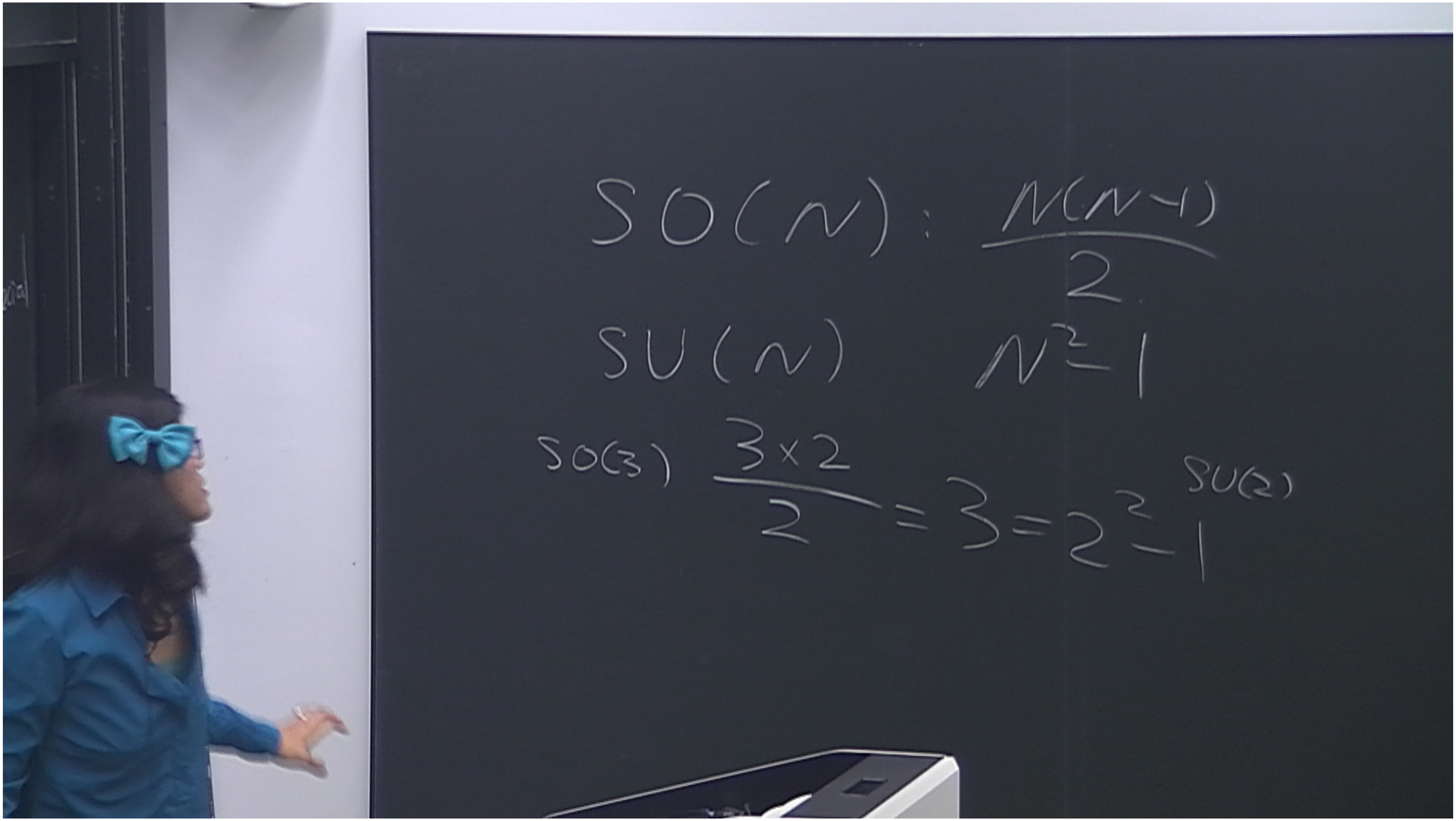
6 real parameters.

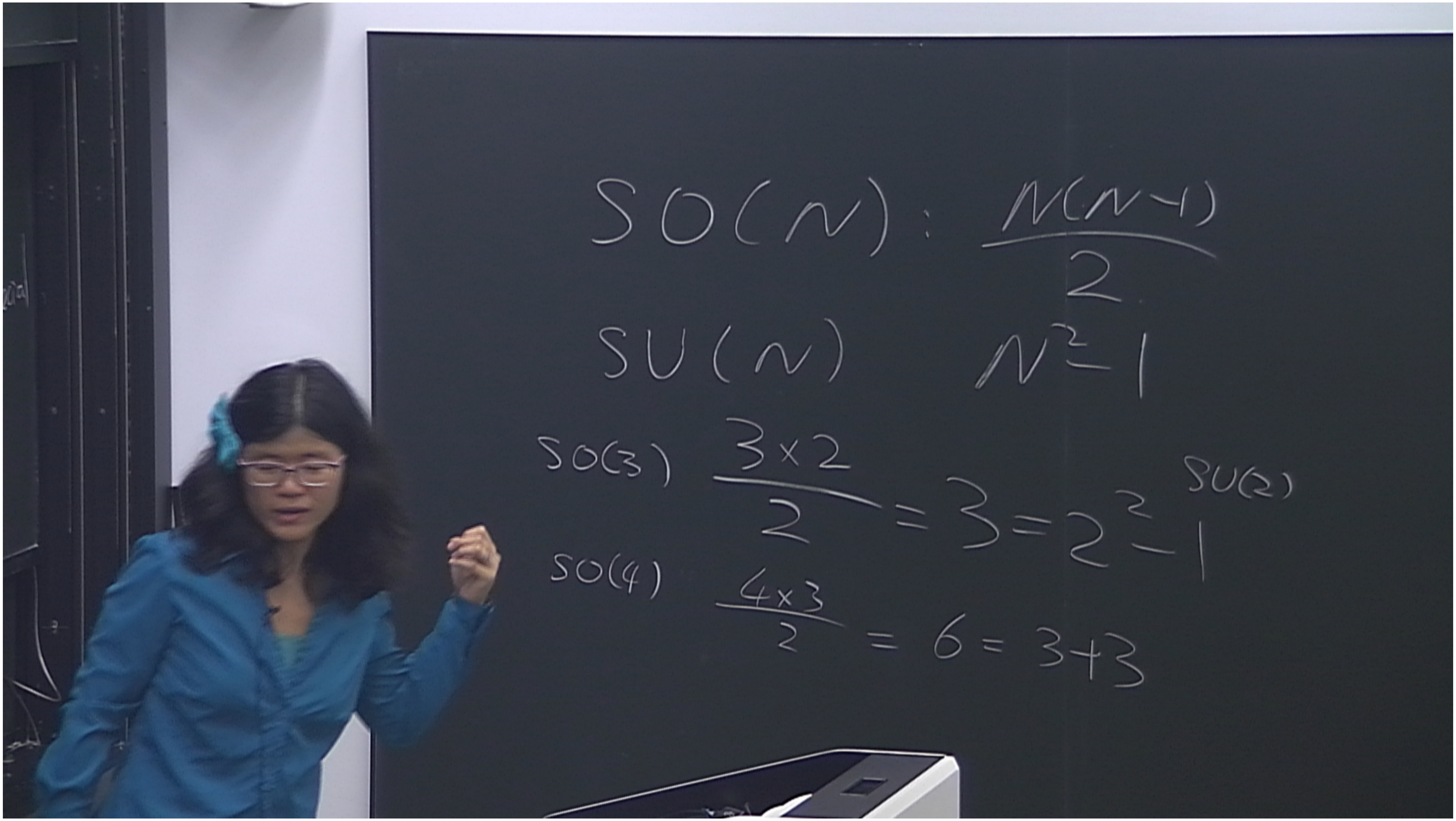
rotations in N complex dimensions.

$U^\dagger U = 1$ unitary group. $\det U = +1$ special

real parameters. $2N^2 - N^2 = N^2$ $SU(N)$

$\det(U^\dagger)\det(U) = 1$ $\det U = e^{i\theta}$ $N^2 - 1$


SO(N) : N(N-1)/2
SU(N) N²-1
SO(3) 3×2/2 = 3 = 2²-1
SU(2)


SO(N): N(N-1)/2
SU(N) N²-1
SO(3) 3×2/2 = 3 = 2²-1
SU(2)
SO(4) 4×3/2 = 6 = 3+3

$\frac{N(N-1)}{2}$

N^2-1

SU(2)

$= 2^2 - 1$

$3+3$

Poincaré Group 10 parameters

$SO(3,1)$ + translations in $\mathbb{R}^4$