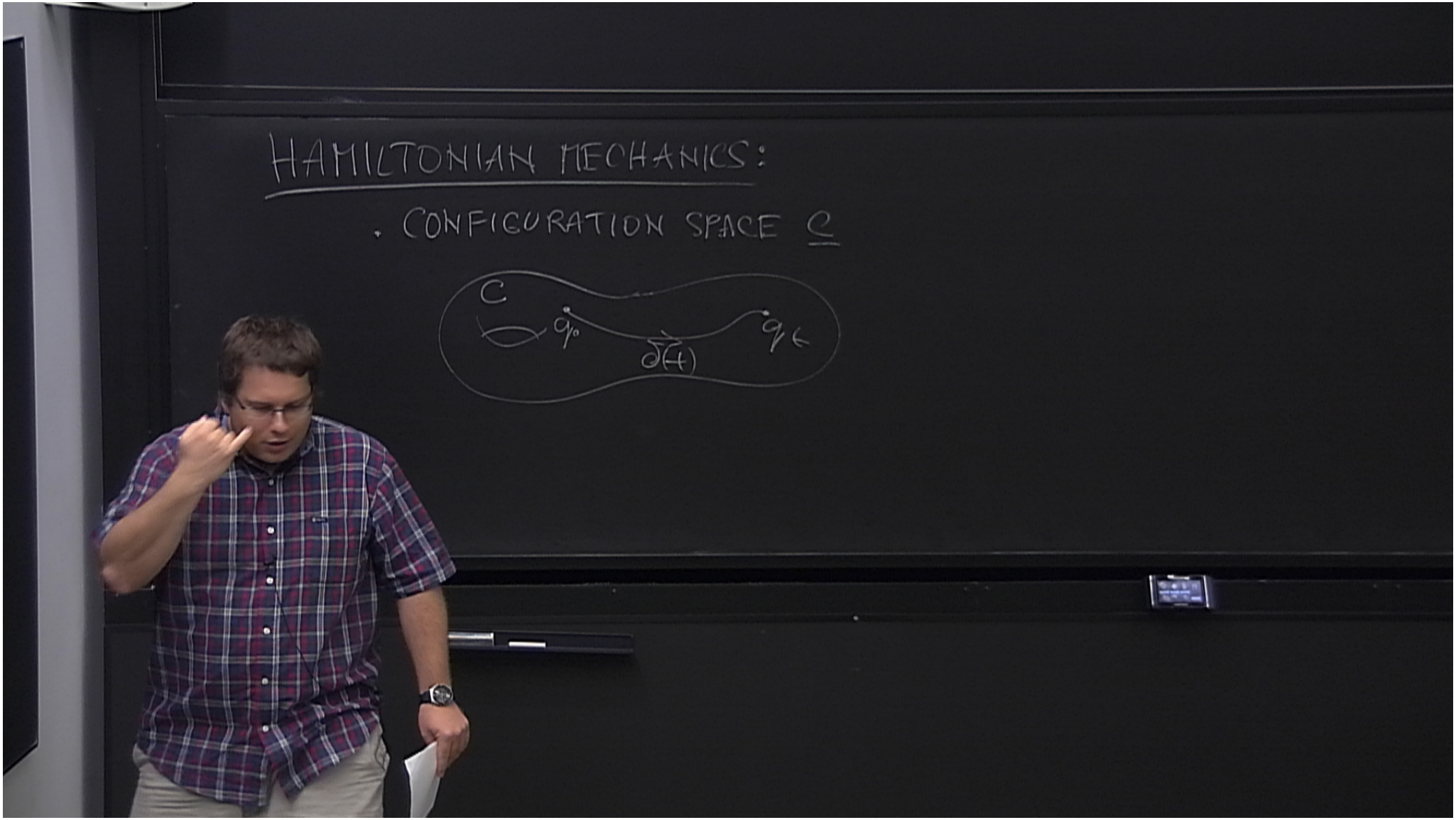
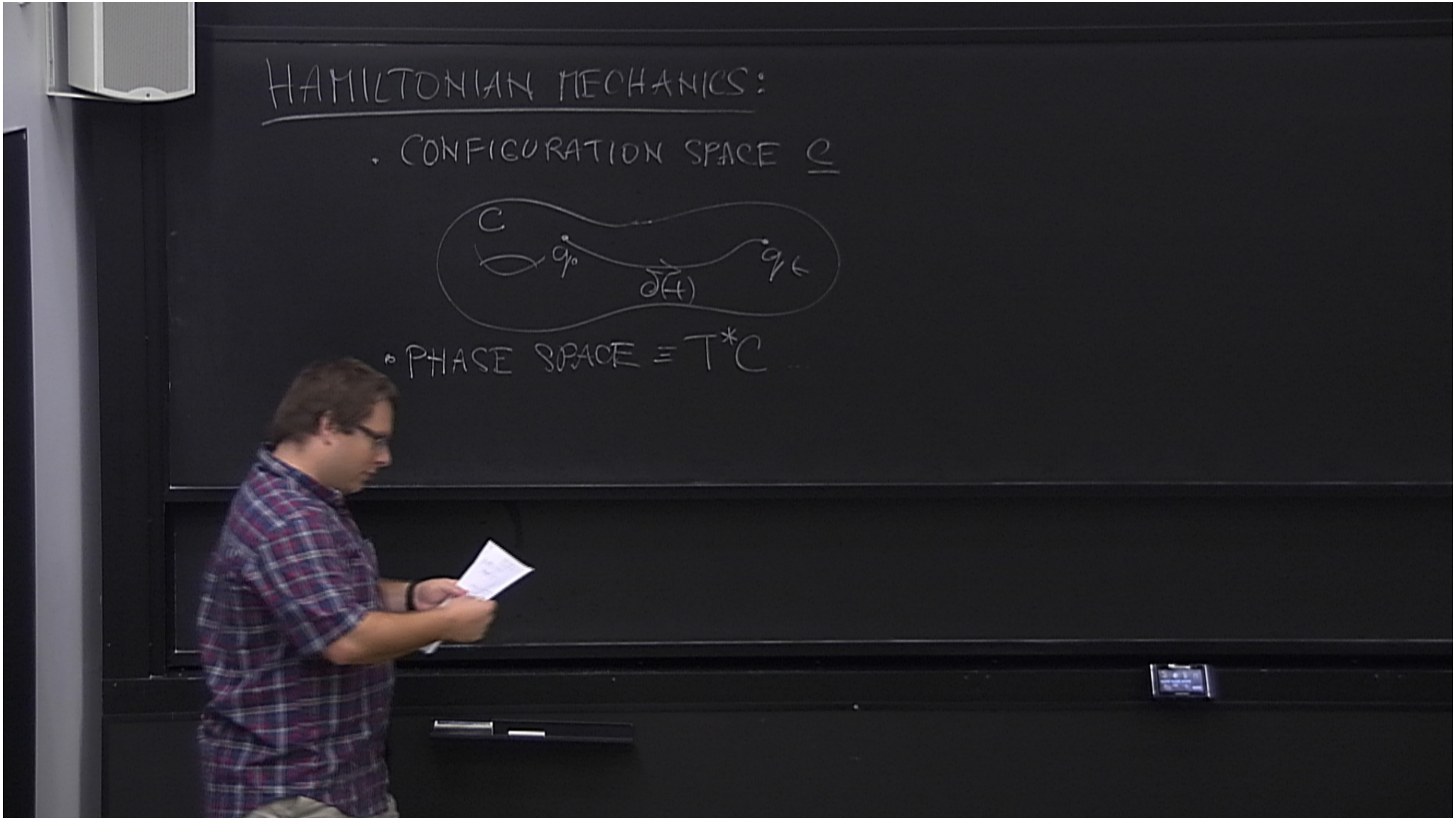
Title: Front End - Classical Mechanics 6

Date: Aug 18, 2016 10:30 AM

URL: http://pirsa.org/16080068

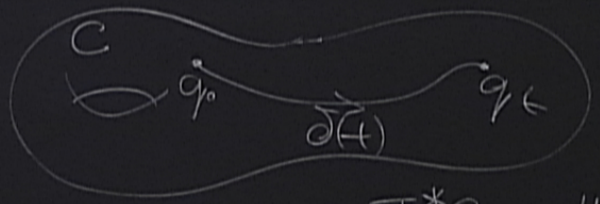
Abstract:


HAMILTONIAN MECHANICS:
. CONFIGURATION SPACE Q
C
q_0
$\gamma(t)$
q_t


HAMILTONIAN MECHANICS:
• CONFIGURATION SPACE C
C
q_0
$\gamma(t)$
q_t
• PHASE SPACE $= T^*C$

ILTONIAN MECHANICS:

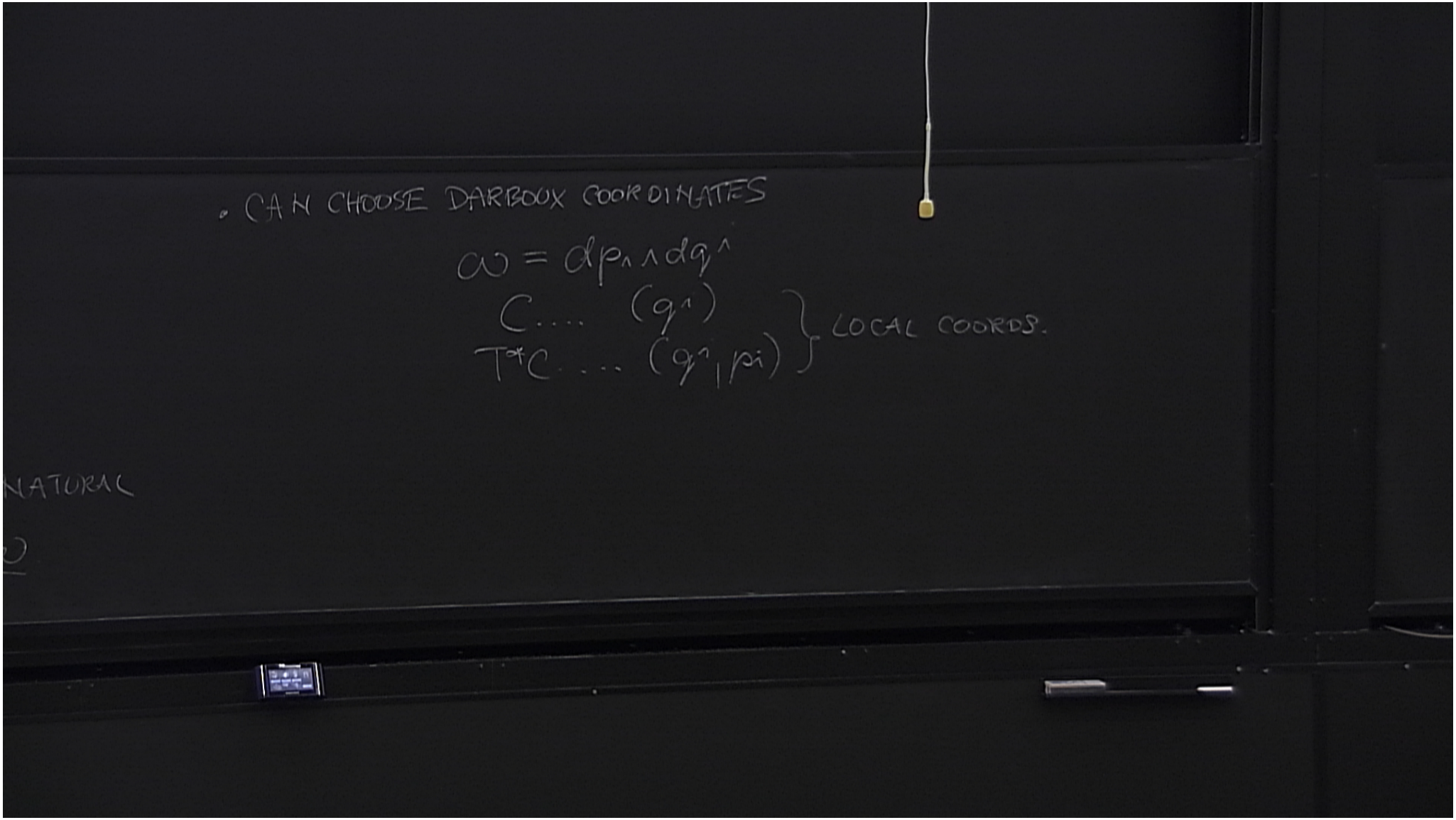
- CONFIGURATION SPACE C

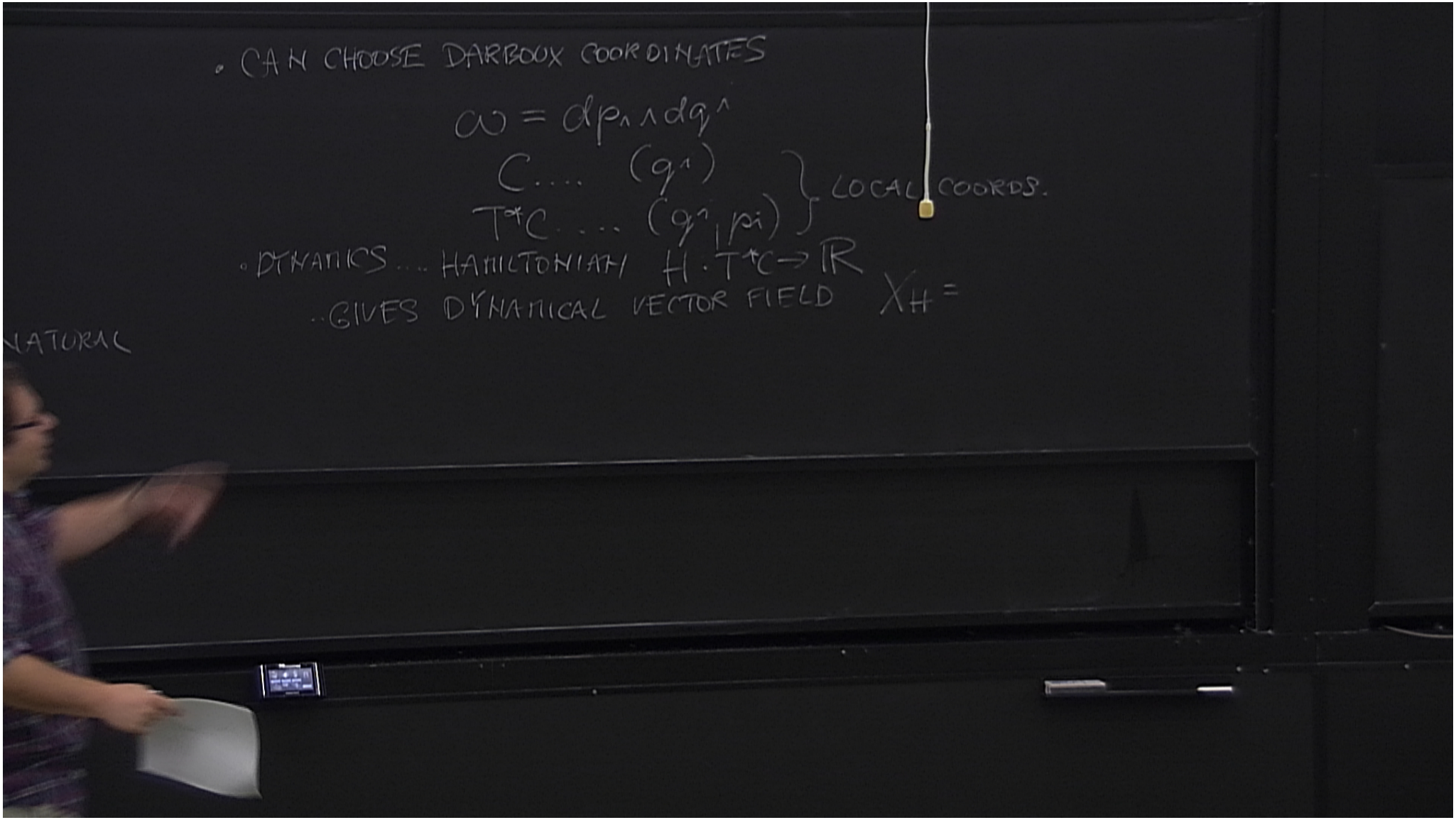


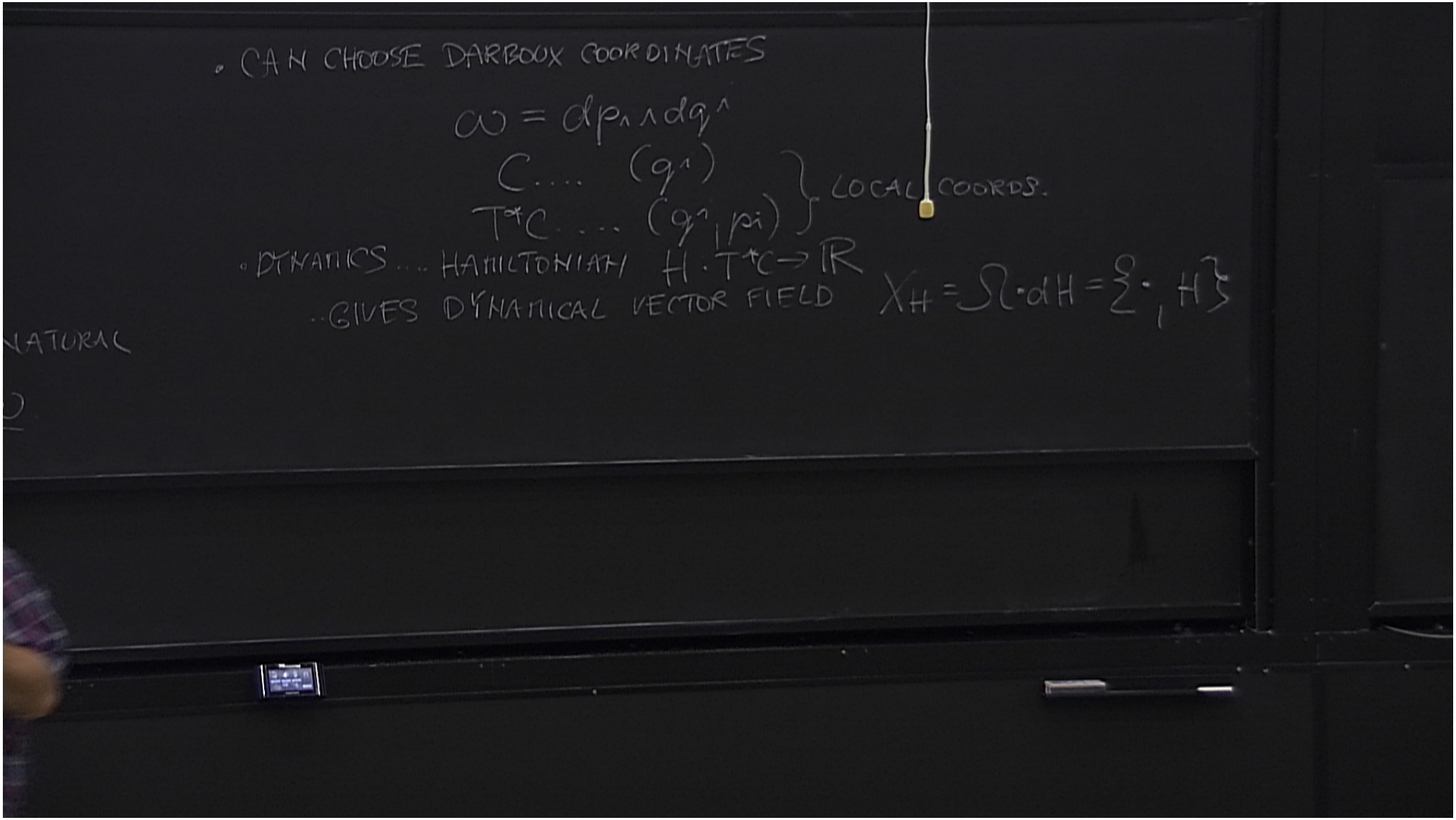
- PHASE SPACE $= T^*C$ HAS NATURAL SYMPLECTIC STRUCTURE ω

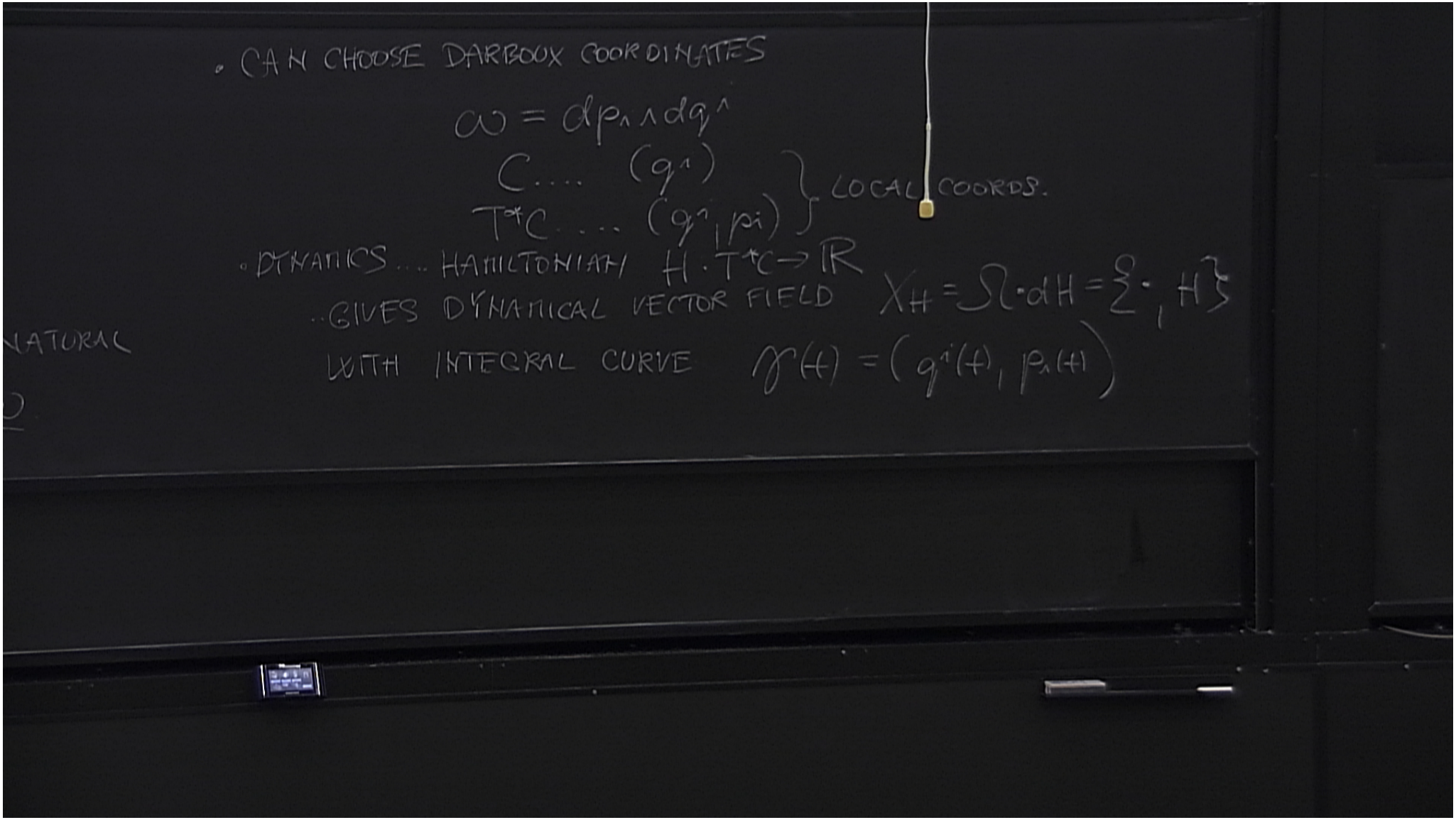
- CAN CHOOSE DARBOUX COORDINATES

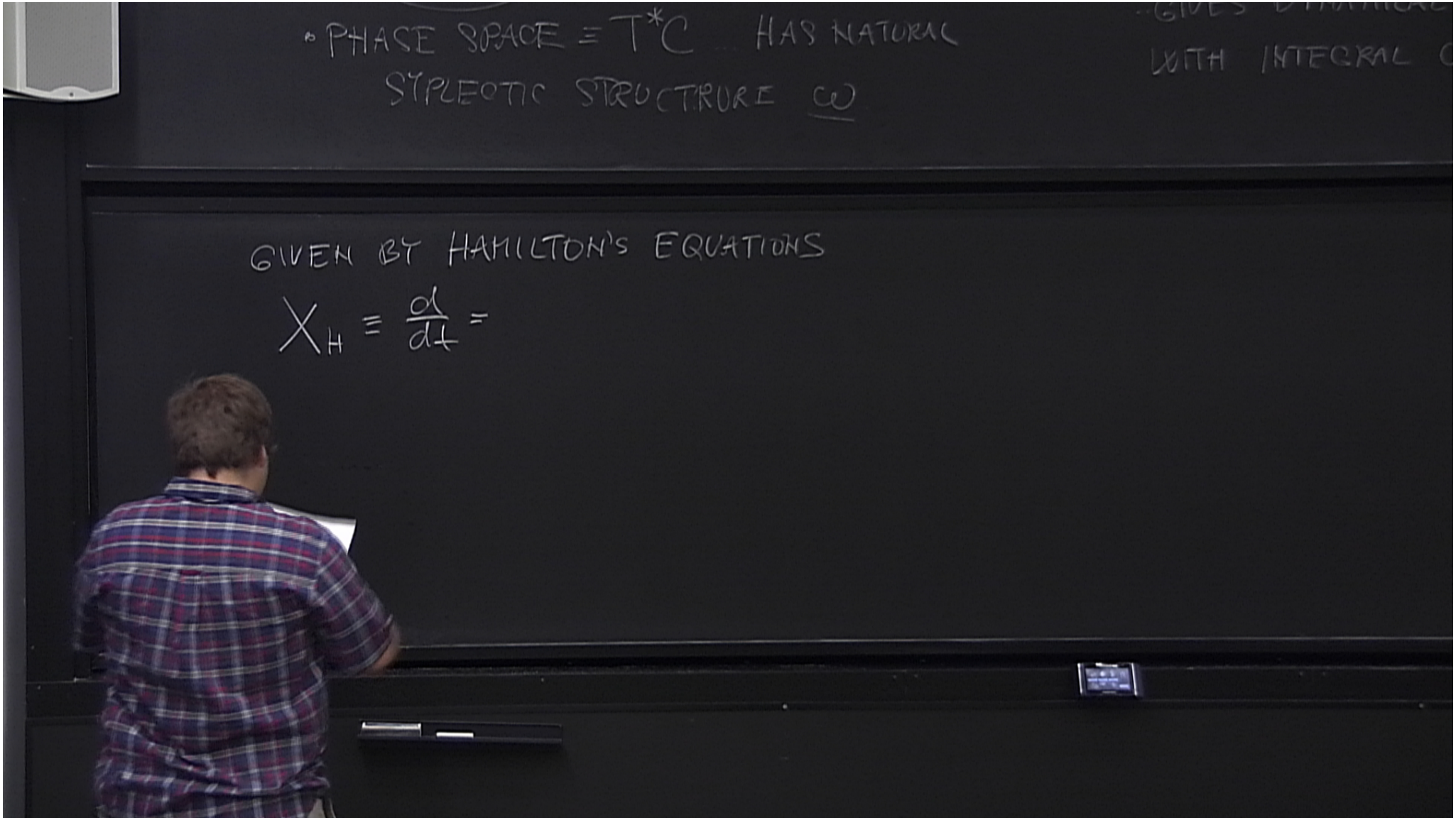
$$\omega = dp_i \wedge dq^i$$

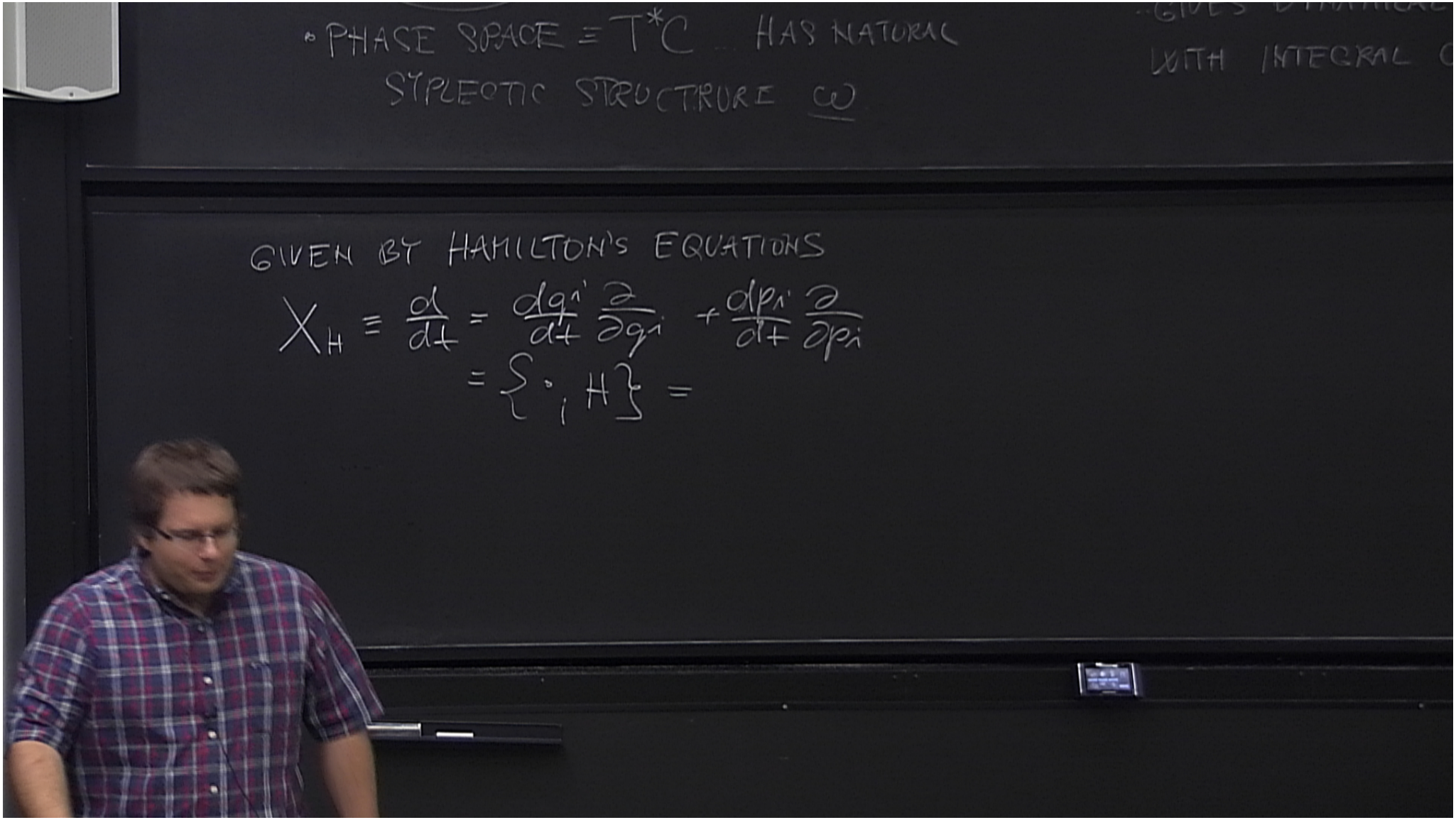

• CAN CHOOSE DARBOUX COORDINATES
$\omega = dp_i \wedge dq^i$
$C \ldots (q^i)$
$T^*C \ldots (q^i, p_i)$
} LOCAL COORDS.
NATURAL

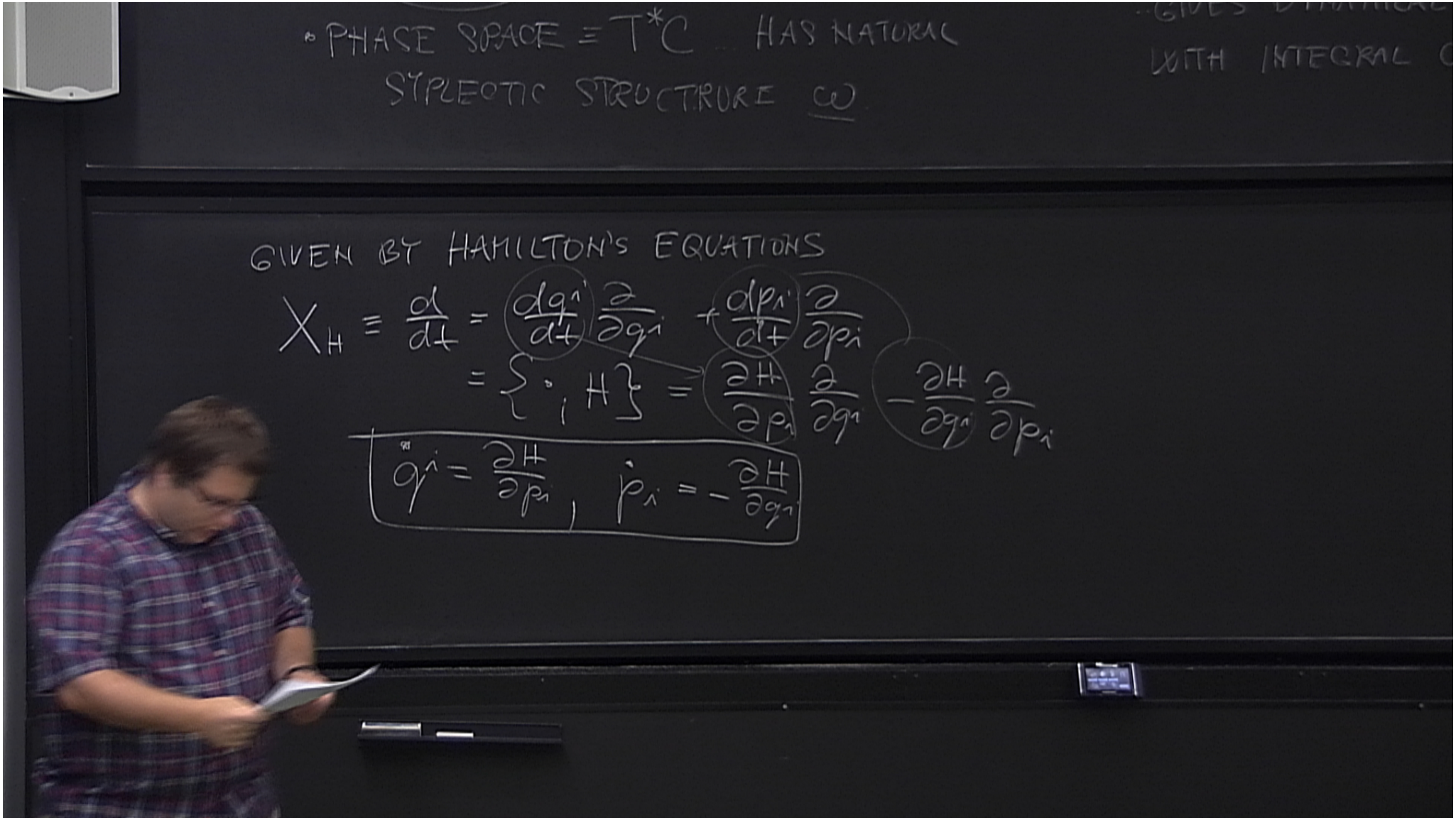

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LOCAL COORDS.
• DYNAMICS ... HAMILTONIAN $H : T^*C \to \mathbb{R}$
.. GIVES DYNAMICAL VECTOR FIELD $X_H =$
NATURAL

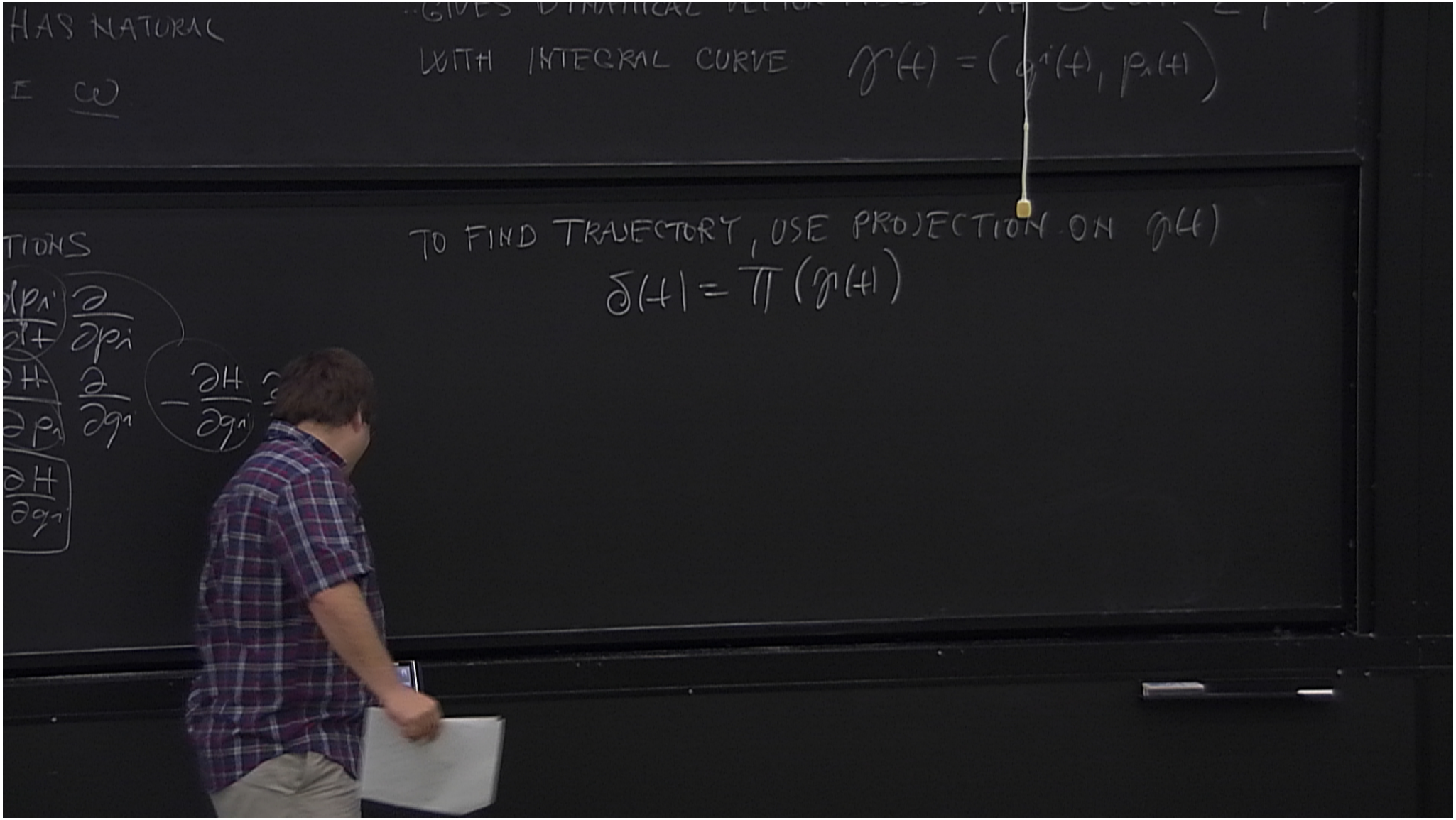

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..GIVES DYNAMICAL VECTOR FIELD $X_H = \Omega \cdot dH = \{\cdot, H\}$
WITH INTEGRAL CURVE $\gamma(t) = (q^i(t), p_i(t))$
NATURAL


PHASE SPACE = T*C HAS NATURAL
SYMPLECTIC STRUCTURE ω
WITH INTEGRAL
GIVEN BY HAMILTON'S EQUATIONS
$X_H \equiv \frac{d}{dt} =$


• PHASE SPACE = T^*C HAS NATURAL
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$X_H \equiv \frac{d}{dt} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i}$
$= \{\cdot, H\} =$


PHASE SPACE = T^*C HAS NATURAL
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GIVEN BY HAMILTON'S EQUATIONS
$X_H \equiv \frac{d}{dt} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i}$
$= \{\cdot, H\} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$
$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}$


HAS NATURAL
WITH INTEGRAL CURVE
TO FIND TRAJECTORY, USE PROJECTION ON

TIONS

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

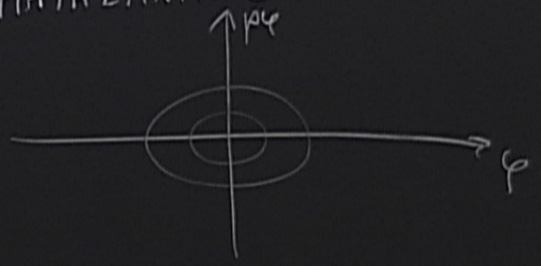
$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}$$

$$\frac{\partial H}{\partial q^i}$$

TO FIND TRAJECTORY, USE PROJECTION ON $\mathcal{D}(H)$

$$\delta(H) = \Pi(\mathcal{D}(H))$$

EX: MATHEMATICAL PENDULUM



TIONS

$\dot{p}_i = -\frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$

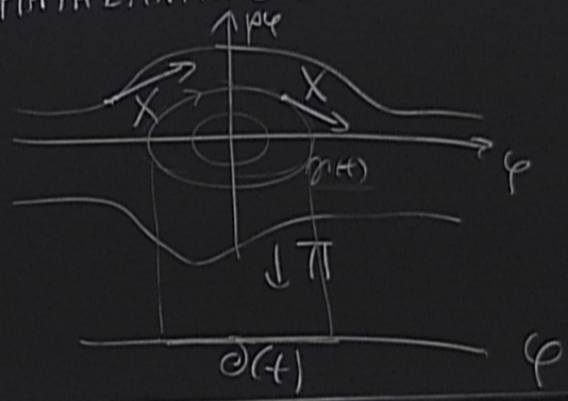
$\dot{q}^i = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}$

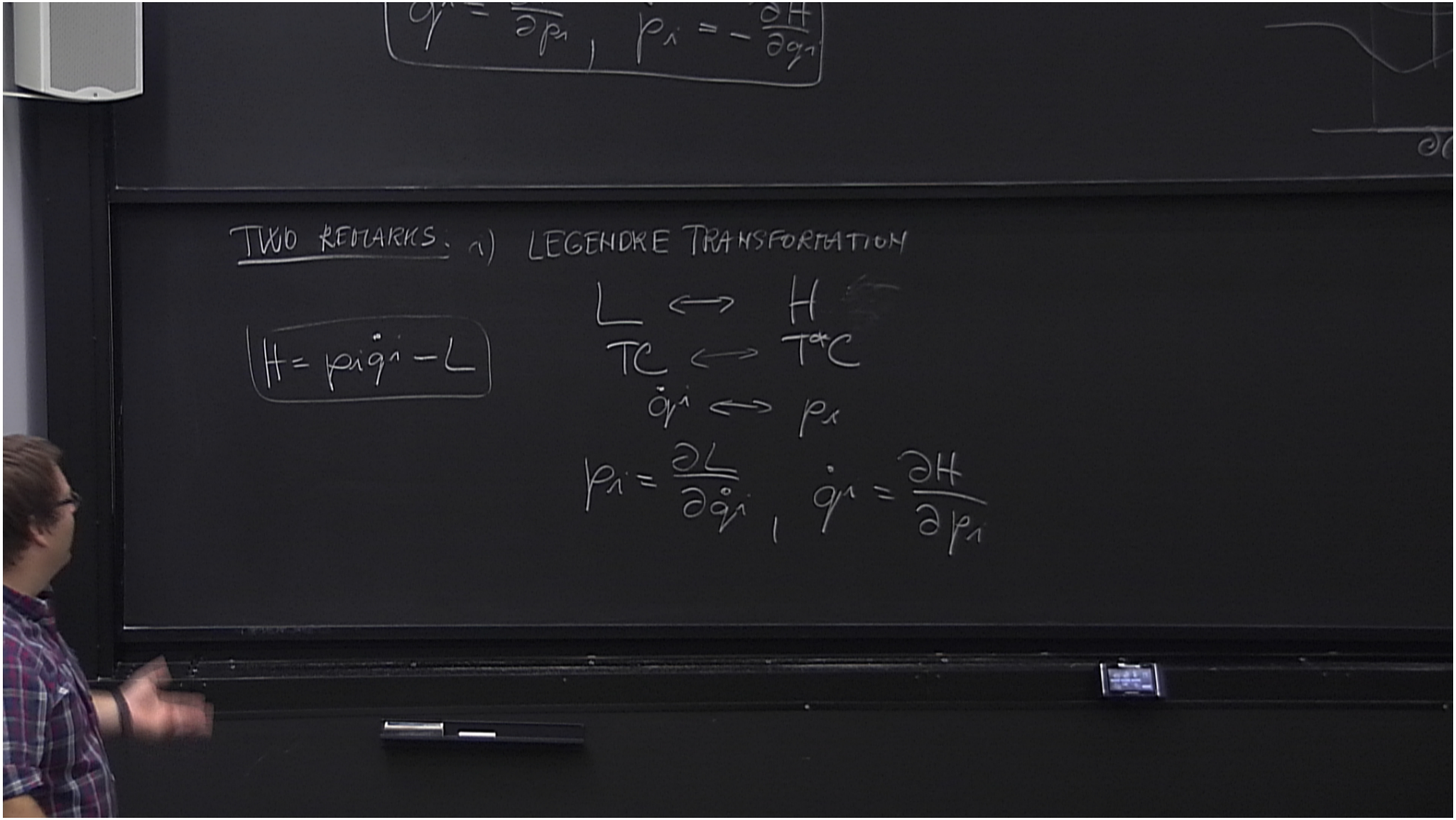
$\frac{\partial H}{\partial q^i}$

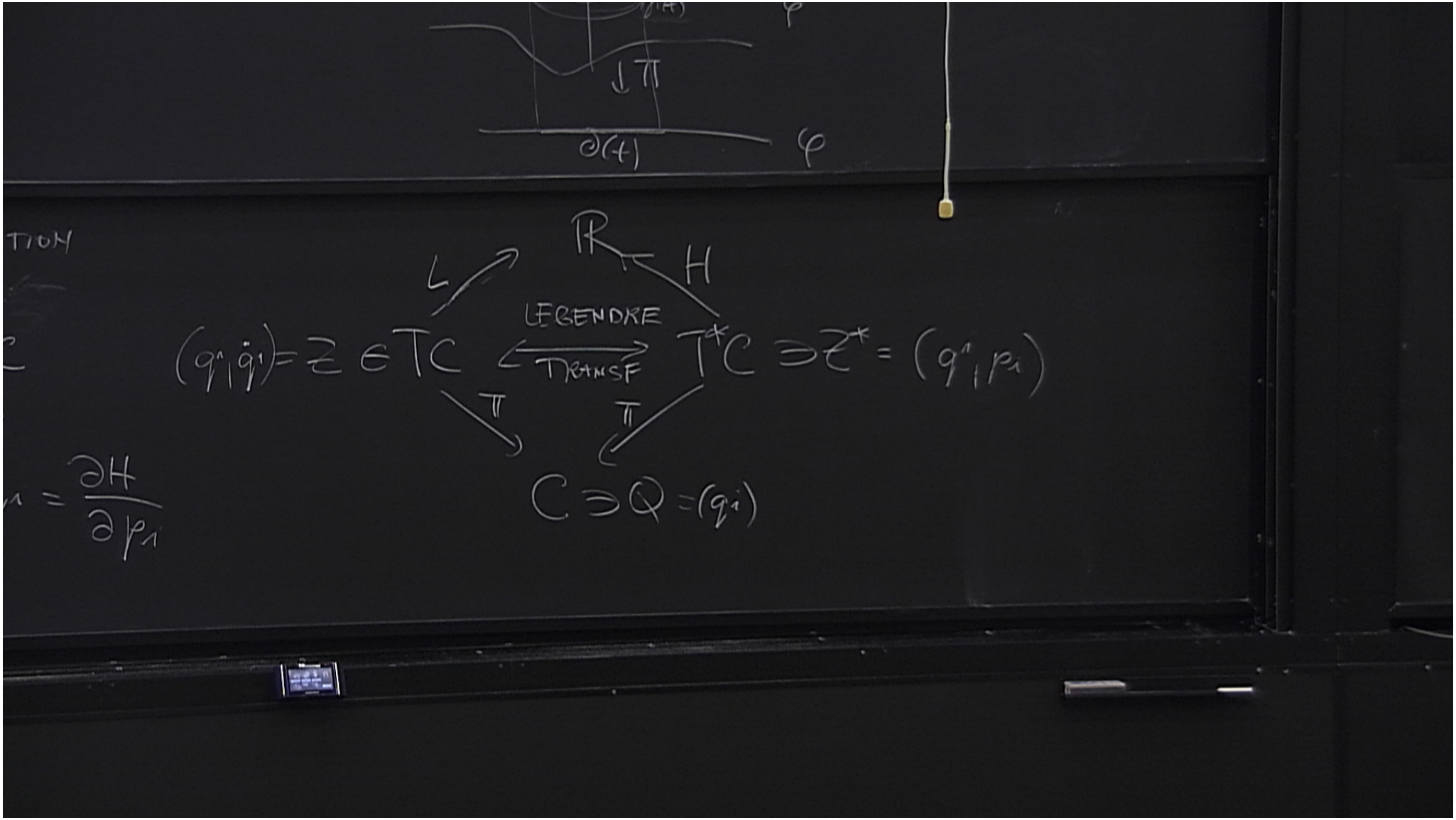
TO FIND TRAJECTORY, USE PROJECTION ON $\gamma(t)$

$$\delta(t) = \pi(\gamma(t))$$

EX: MATHEMATICAL PENDULUM




$\dot{q}^i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q^i}$
TWO REMARKS: 1) LEGENDRE TRANSFORMATION
$H = p_i \dot{q}^i - L$
$L \leftrightarrow H$
$TC \leftrightarrow T^*C$
$\dot{q}^i \leftrightarrow p_i$
$p_i = \frac{\partial L}{\partial \dot{q}^i}$, $\dot{q}^i = \frac{\partial H}{\partial p_i}$


π
$\mathcal{C}(t)$
φ
TION
$\mathbb{R}$
L
H
LEGENDRE
TRANSF
$(q^i, \dot{q}^i) = \mathcal{Z} \in TC$
$T^*C \ni \mathcal{Z}^* = (q^i, p_i)$
π
π
$C \ni Q = (q^i)$
$= \frac{\partial H}{\partial p_i}$

TWO REMARKS: 1) LEGENDRE TRANSFORMATION

$$H = p_i \dot{q}^i - L$$

$L \longleftrightarrow H$

$TC \longleftrightarrow T^*C$ $\qquad (q^i, \dot{q}^i) =$

$\dot{q}^i \longleftrightarrow p_i$

$$p = \frac{\partial f}{\partial \dot{q}^i} dq^i \qquad p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}$$

iii) X_H ... HAM. VECTOR FIELD $\Rightarrow \mathcal{L}_{X_H}\omega = 0$

$\Rightarrow$ LIOUVILLE THEOREM

$\mathcal{L}_{X_H} \ldots$ $\mathcal{L}_{X_H}\omega = 0$

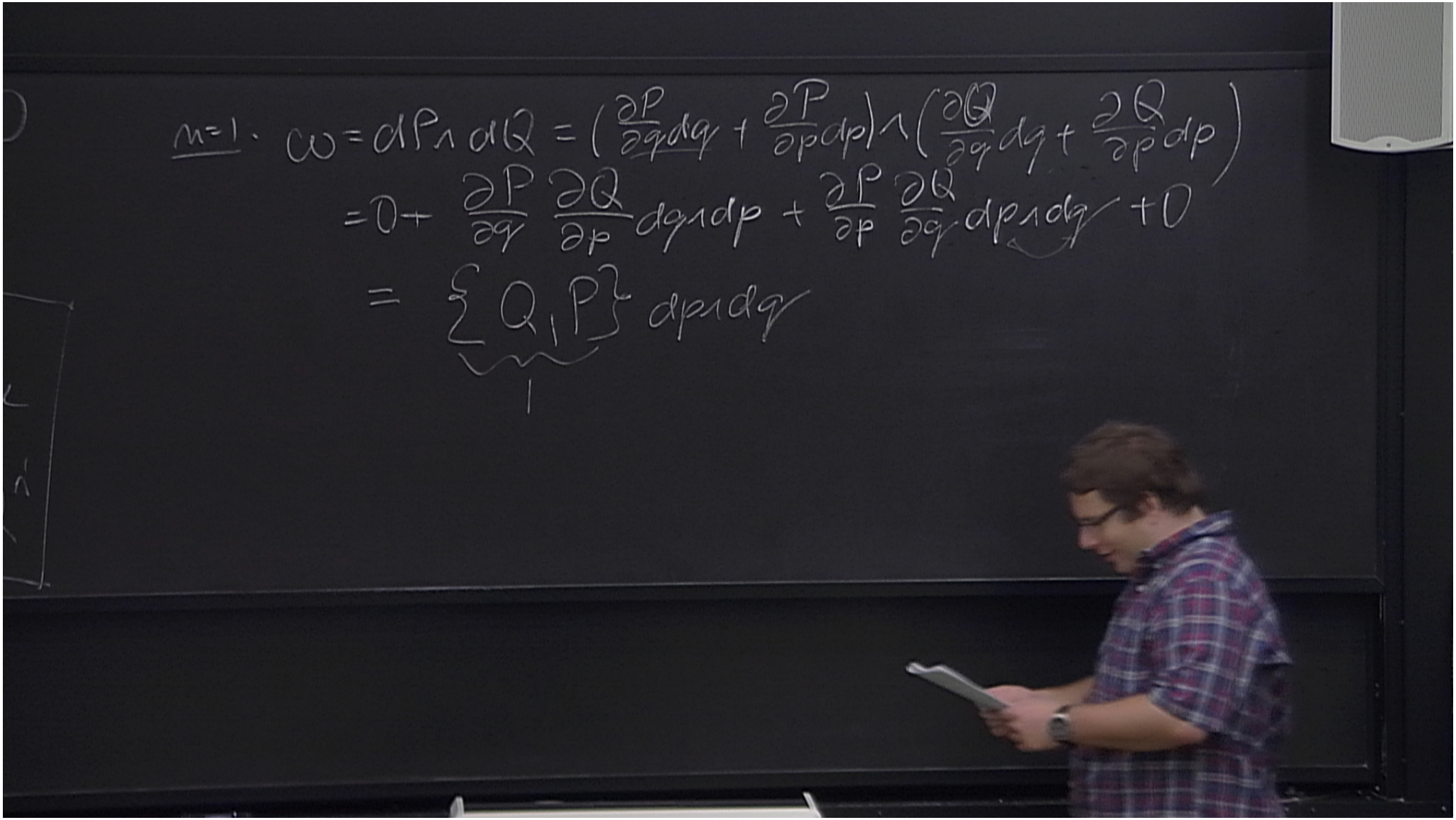
$\Rightarrow$ LIOUVILLE THEOREM

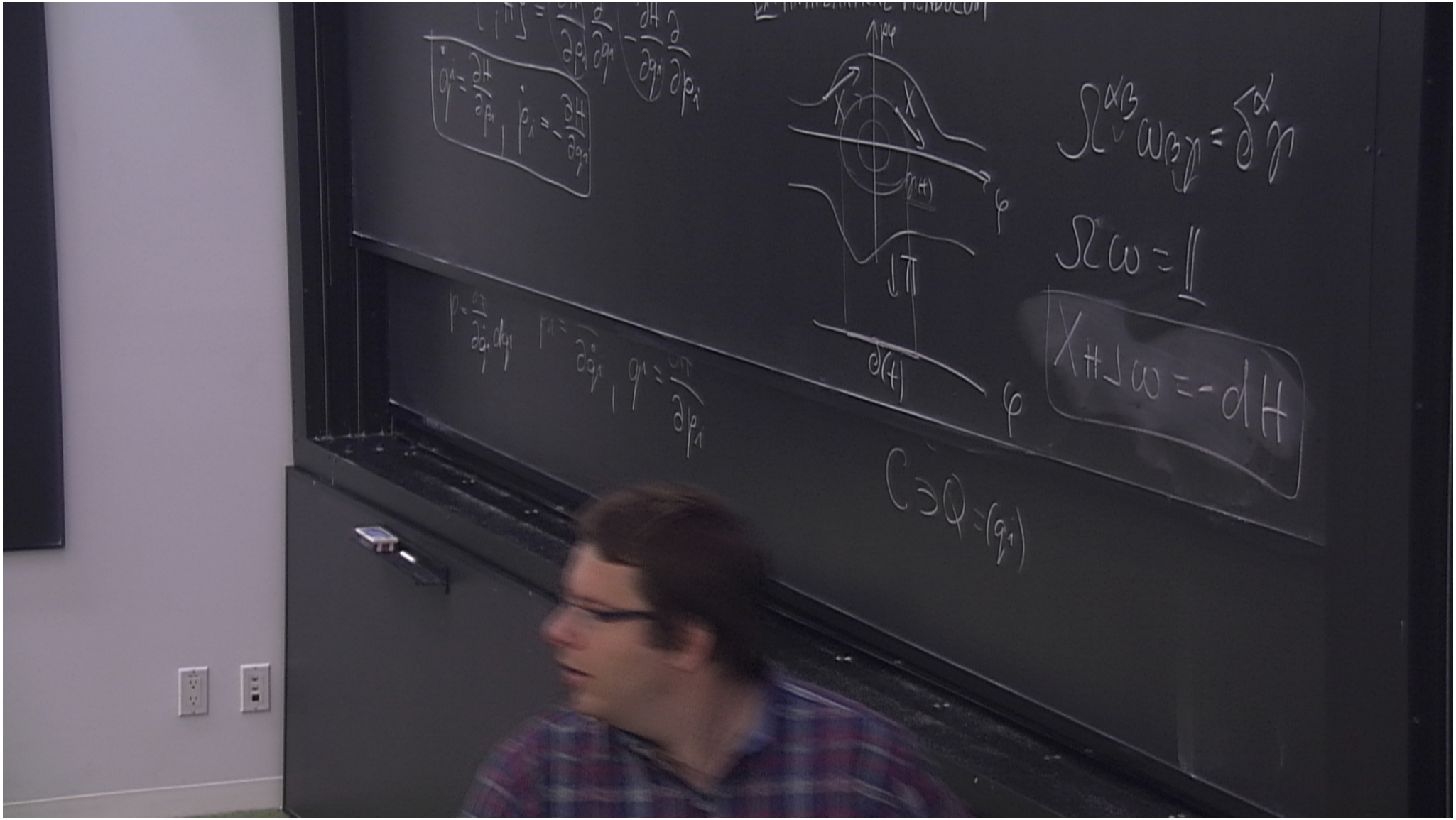
. CANONICAL TRANSFORMATIONS

DEF: $Q^j = Q^j(q^i, p_i)$, $P_j = P_j(q^i, p_i)$

IS CANONICAL IF IT PRESERVES "CANONICAL FORM" OF ω.

$$\omega = dp_i \wedge dq^i = dP_i \wedge dQ^i$$


n=1. $\omega = dP \wedge dQ = \left(\frac{\partial P}{\partial q}dq + \frac{\partial P}{\partial p}dp\right) \wedge \left(\frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp\right)$
$= 0 + \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}dq \wedge dp + \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q}dp \wedge dq + 0$
$= \{Q, P\}\, dp \wedge dq$
1


$\dot{q}^i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q^i}$
$\Omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^\alpha_\gamma$
$\Omega\omega = \mathbb{1}$
$X_H \lrcorner \omega = -dH$
$C \ni Q = (q^i)$

FORM" OF ω.

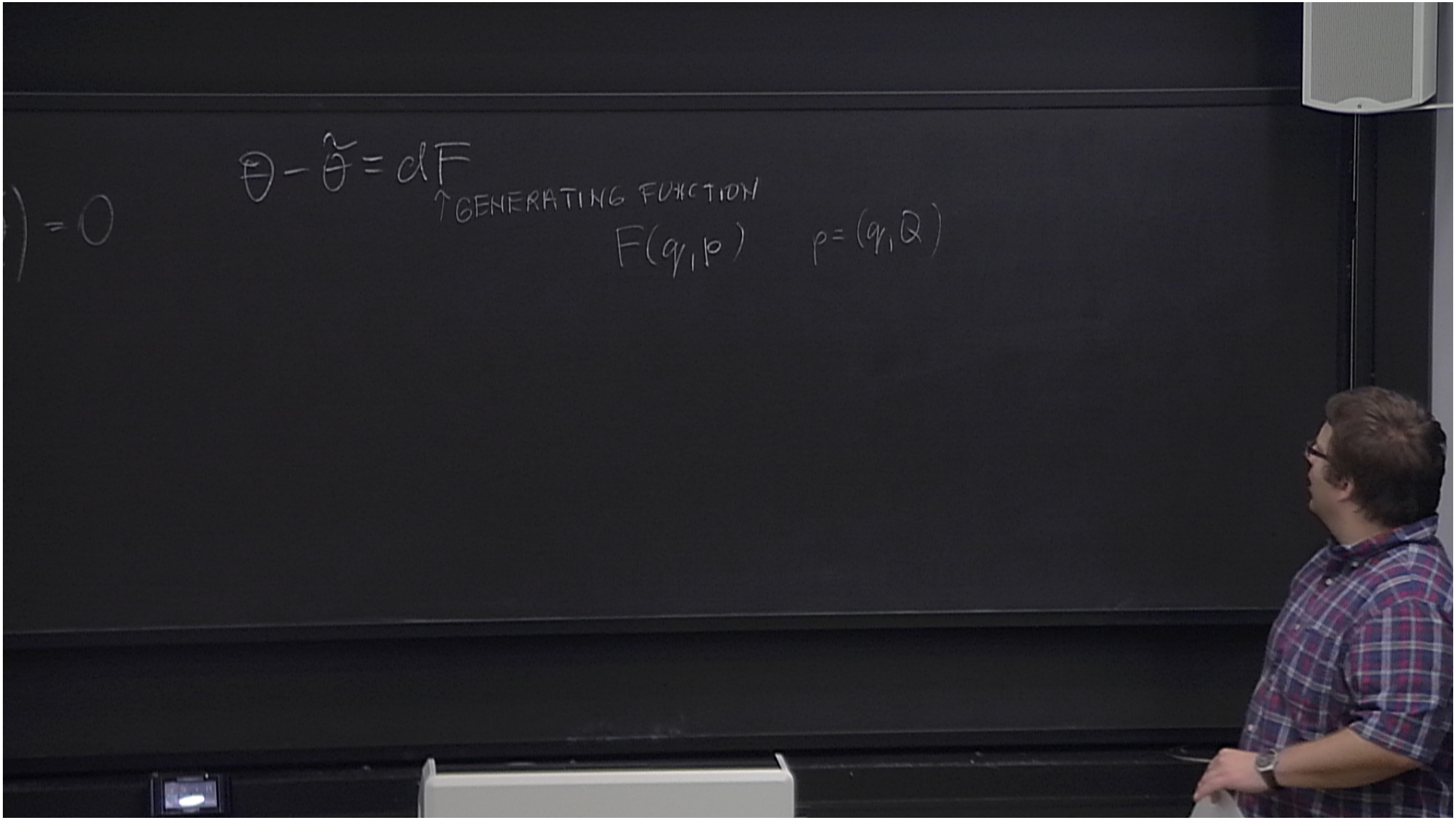
$$\omega = dp_i \wedge dq^i = dP_i \wedge dQ^i$$

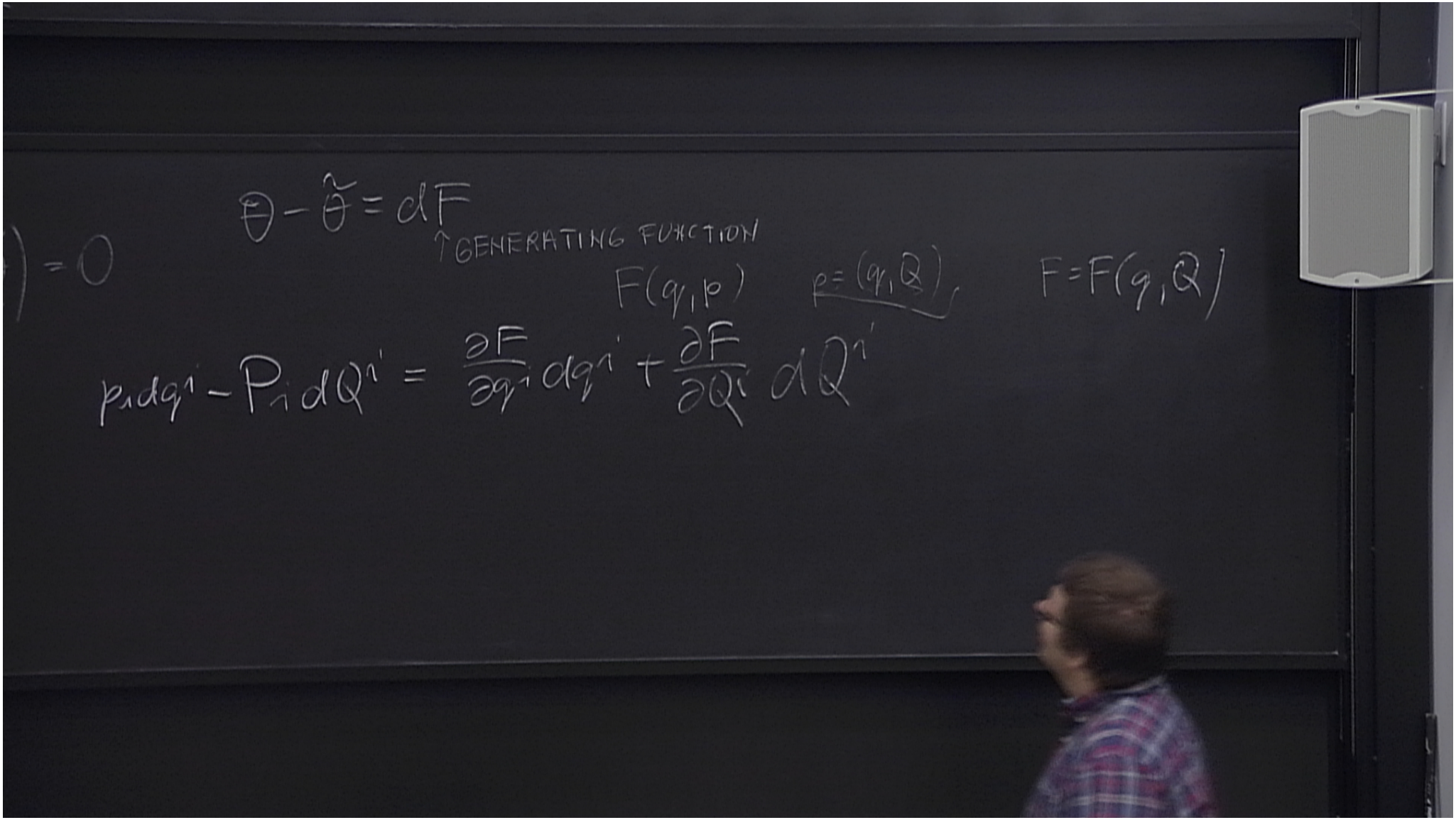
- HAVING TWO CARTAN FORMS.

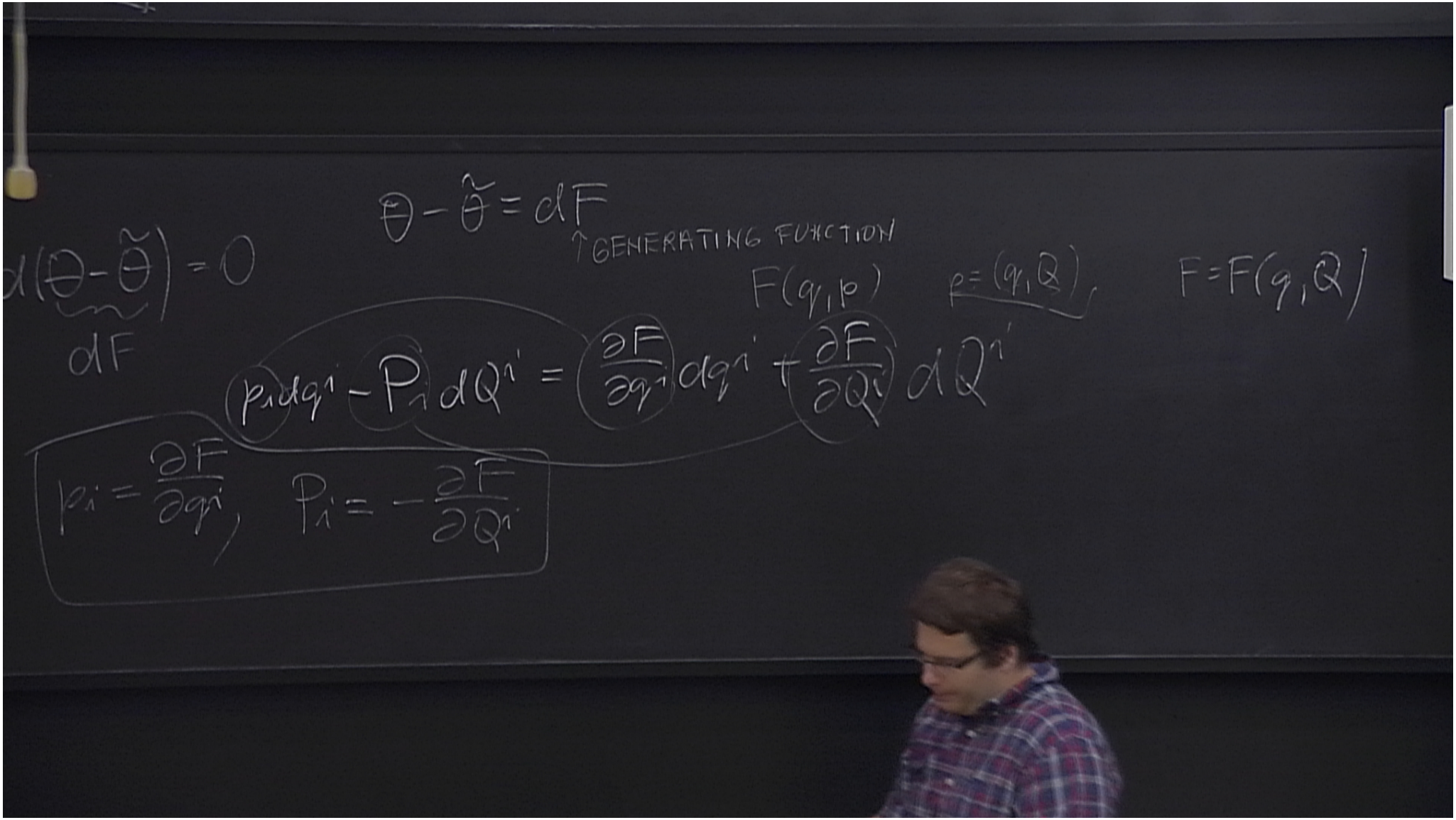
$$\left.\begin{aligned} \Theta &= p_i dq^i & \omega &= d\Theta \\ \tilde{\Theta} &= P_i dQ^i & \omega &= d\tilde{\Theta} \end{aligned}\right\} \quad d(\Theta - \tilde{\Theta}) = 0$$

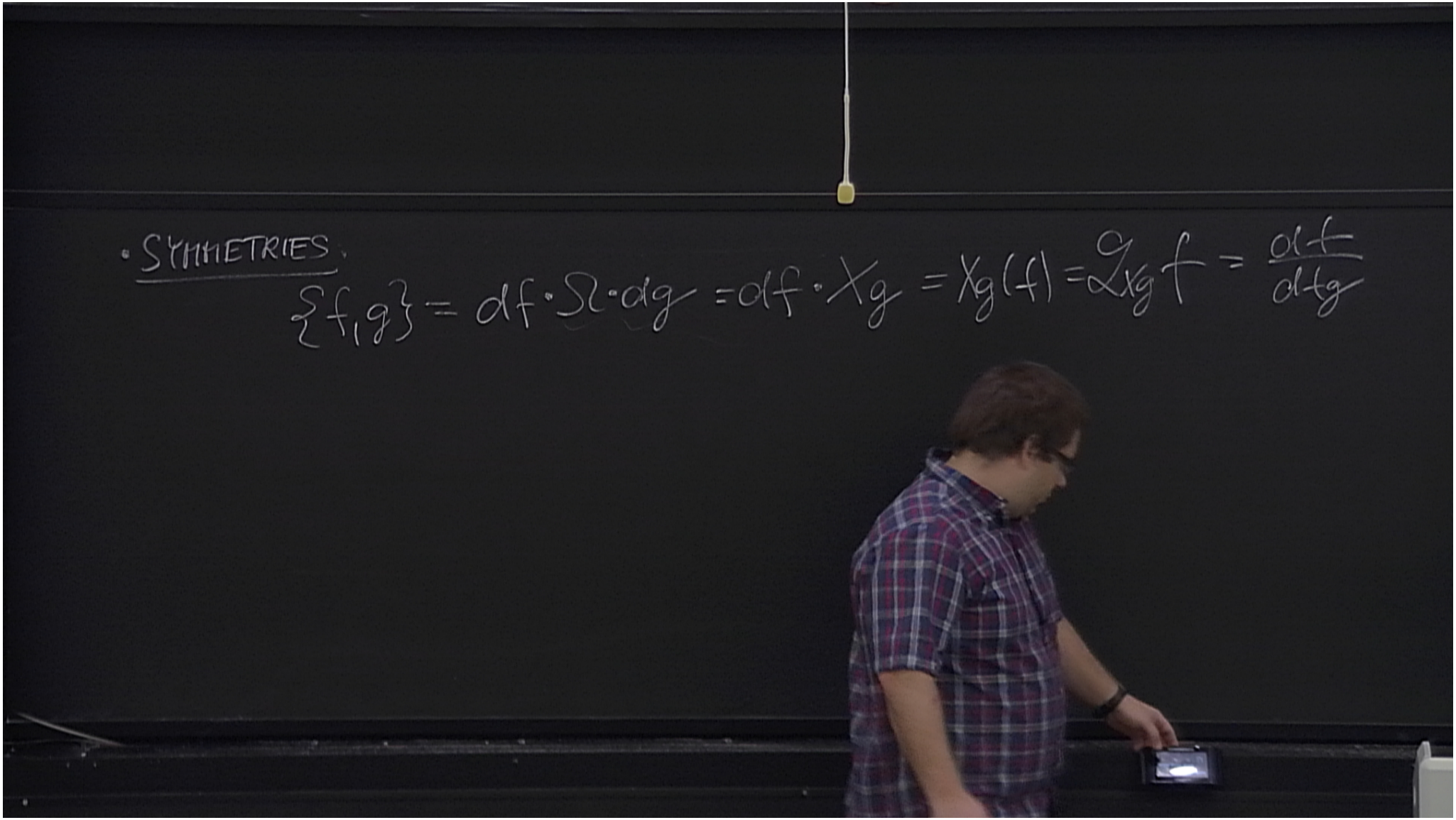
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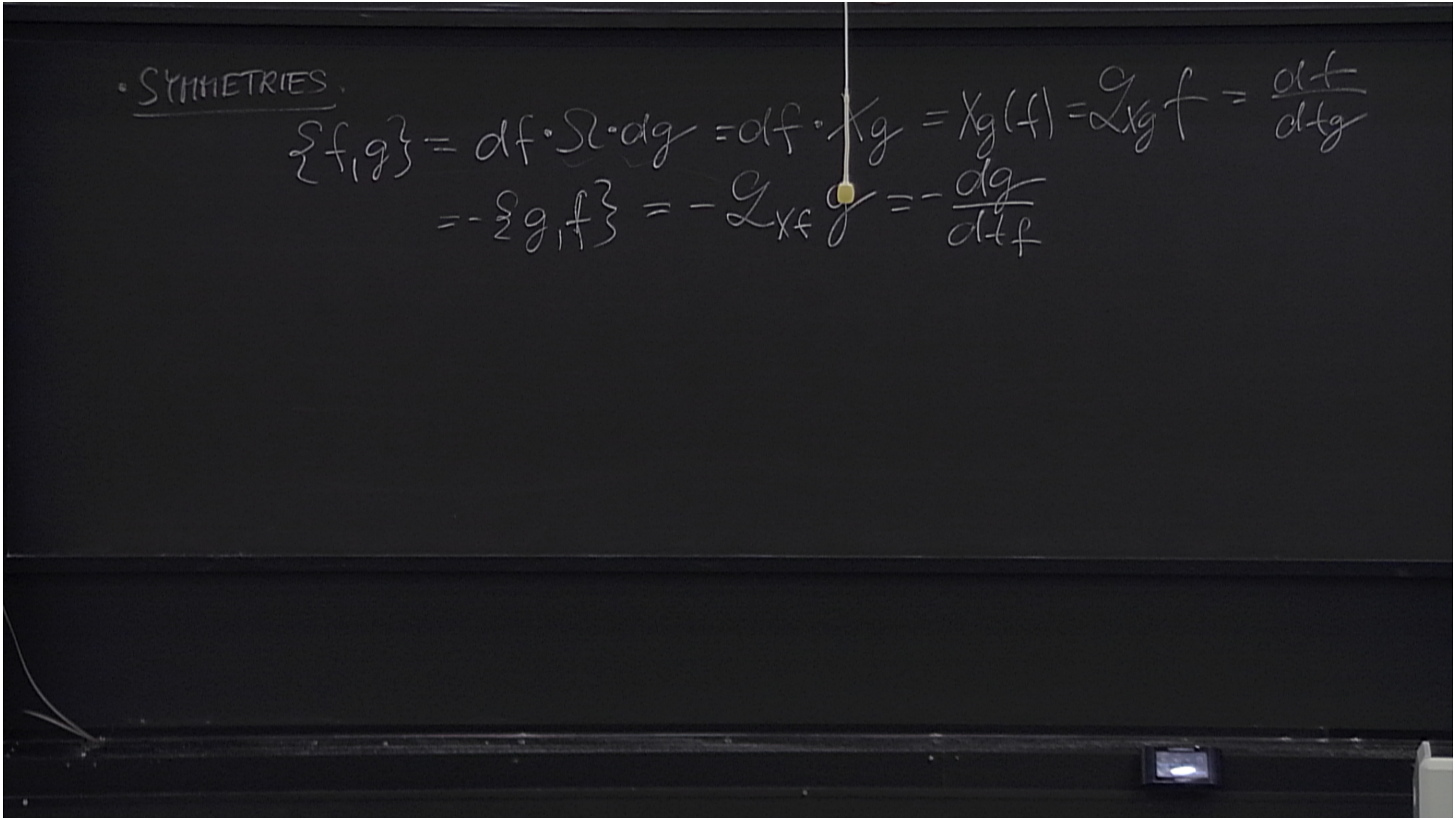
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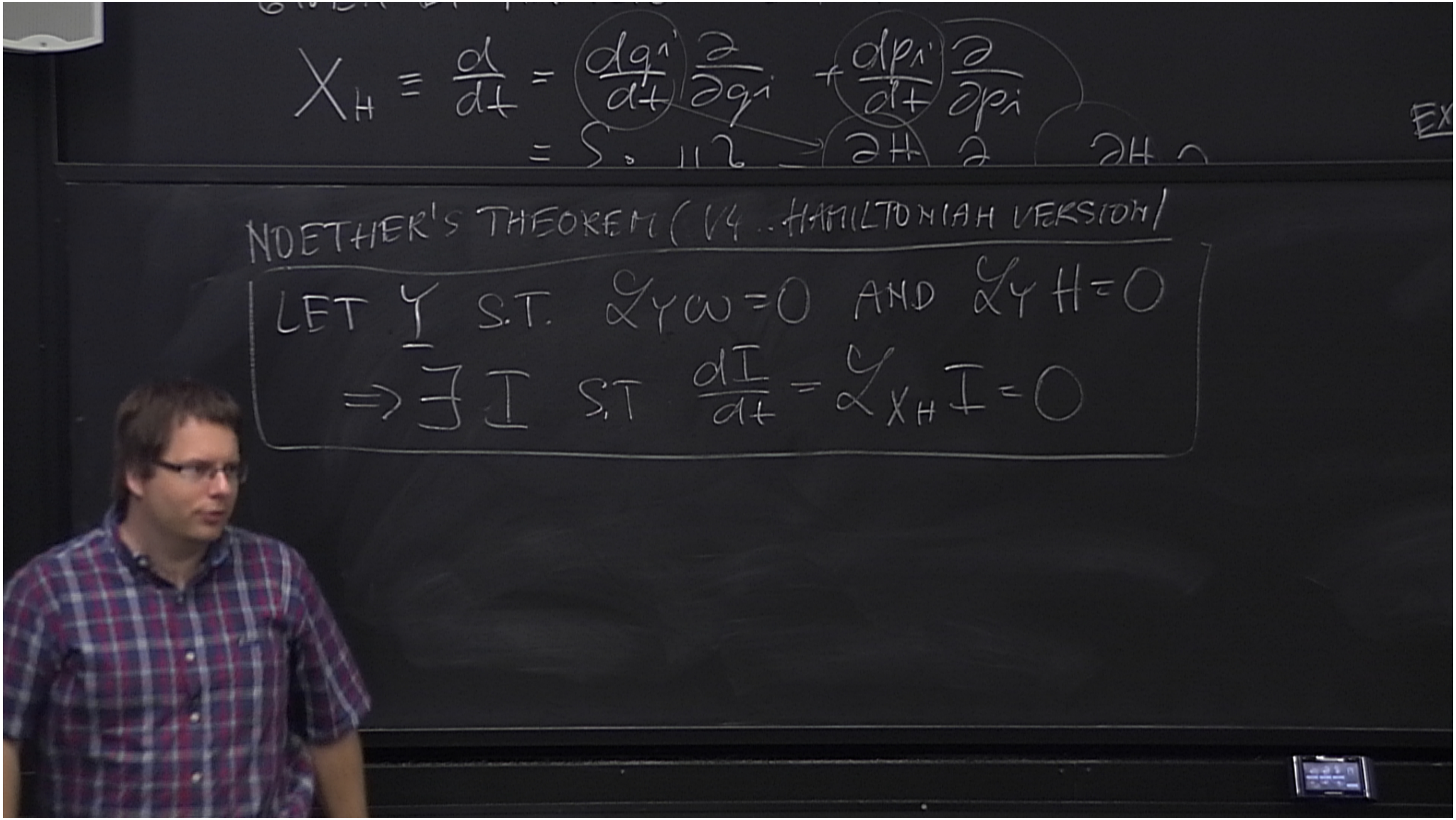

$\Theta - \tilde{\Theta} = dF$
↑GENERATING FUNCTION
$F(q,P)$ $P=(q,Q)$

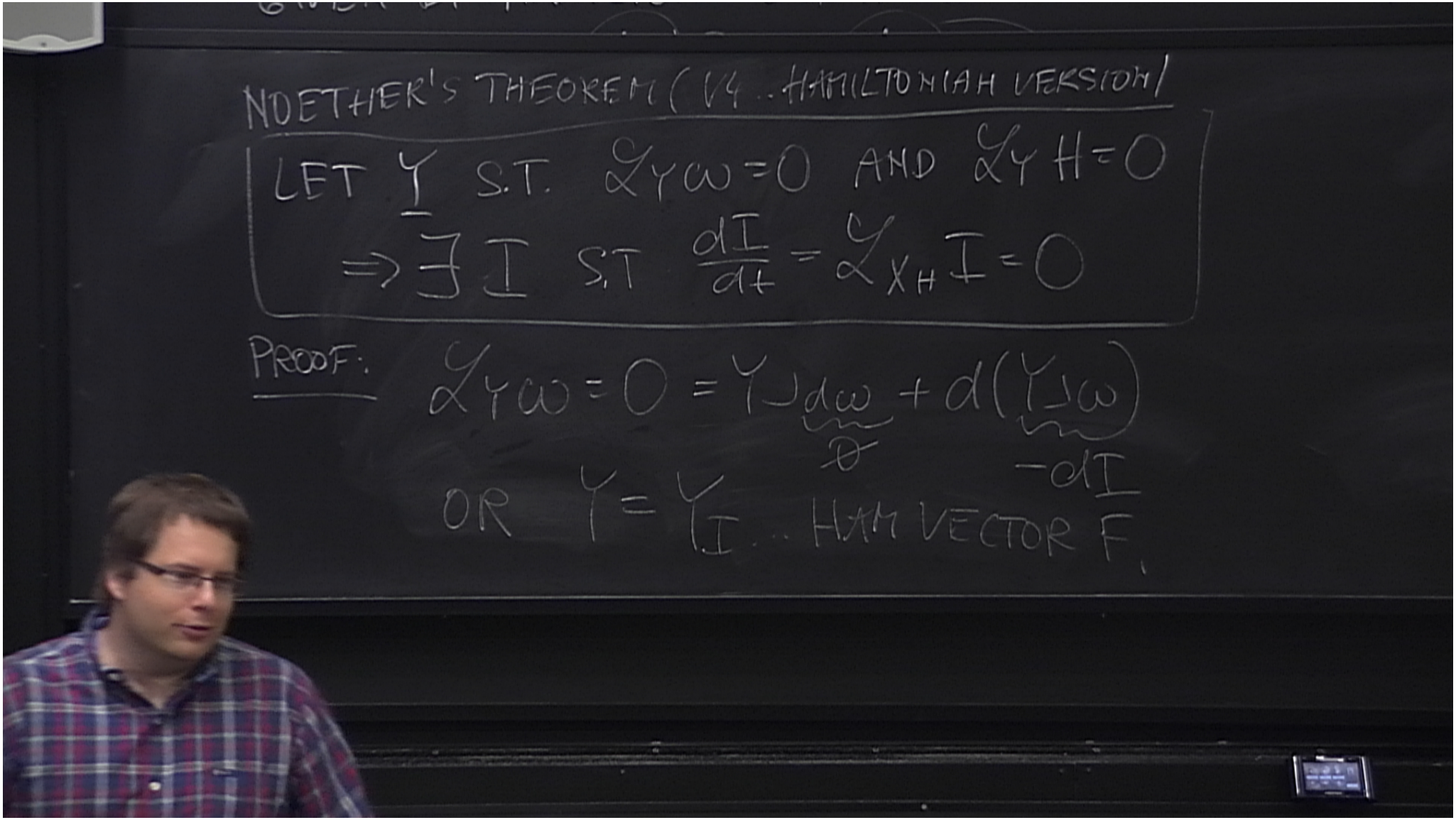

$= 0$
$\Theta - \tilde{\Theta} = dF$
↑GENERATING FUNCTION
$F(q,P)$ $P=(q,Q)$ $F=F(q,Q)$
$p_i dq^i - P_i dQ^i = \frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial Q^i} dQ^i$

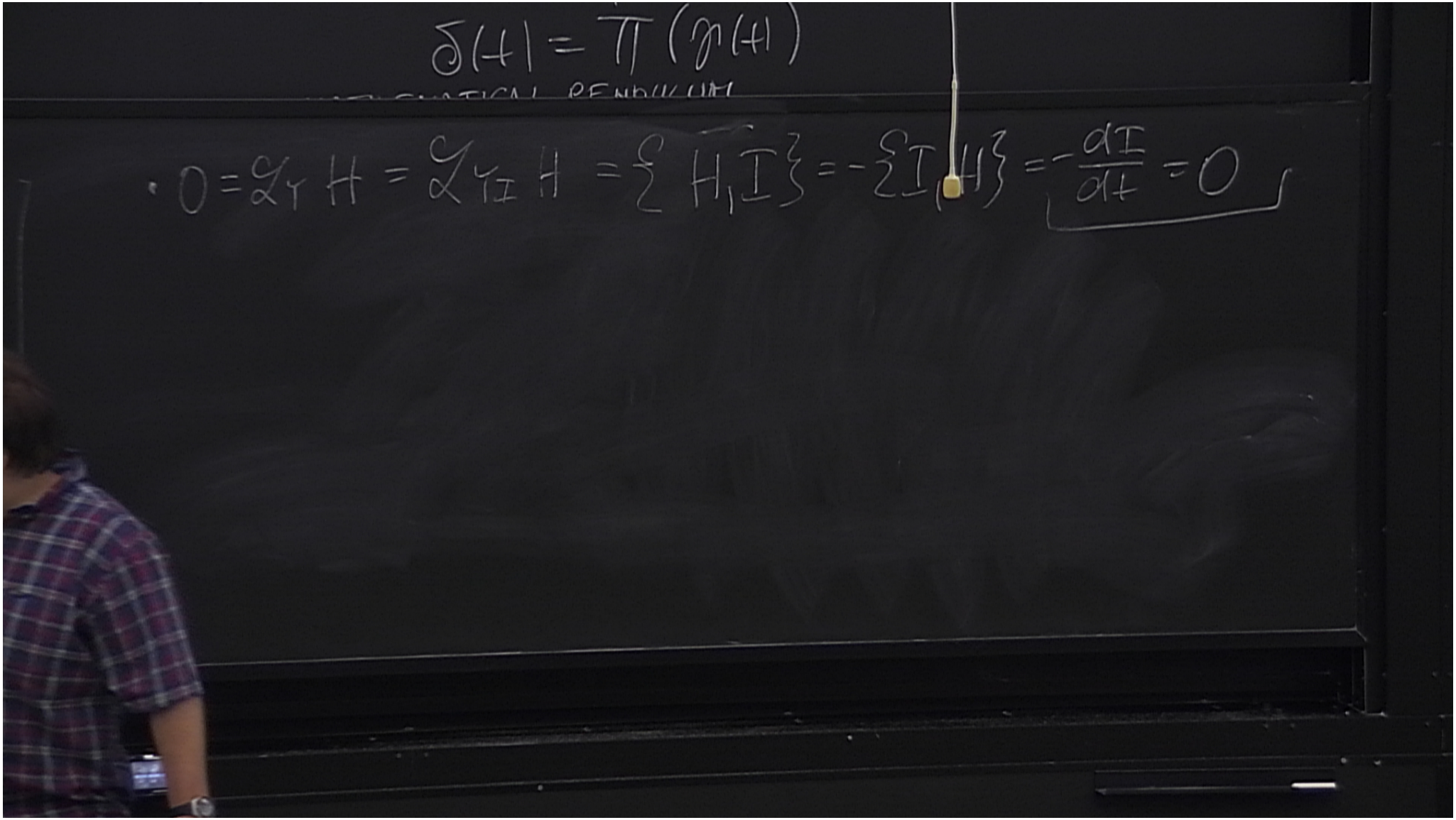

$\Theta - \tilde{\Theta} = dF$
↑GENERATING FUNCTION
$d(\Theta - \tilde{\Theta}) = 0$
dF
$F(q,P)$ $P=(q,Q)$, $F=F(q,Q)$
$p_i dq^i - P_i dQ^i = \frac{\partial F}{\partial q^i} dq^i + \frac{\partial F}{\partial Q^i} dQ^i$
$p_i = \frac{\partial F}{\partial q^i}$, $P_i = -\frac{\partial F}{\partial Q^i}$

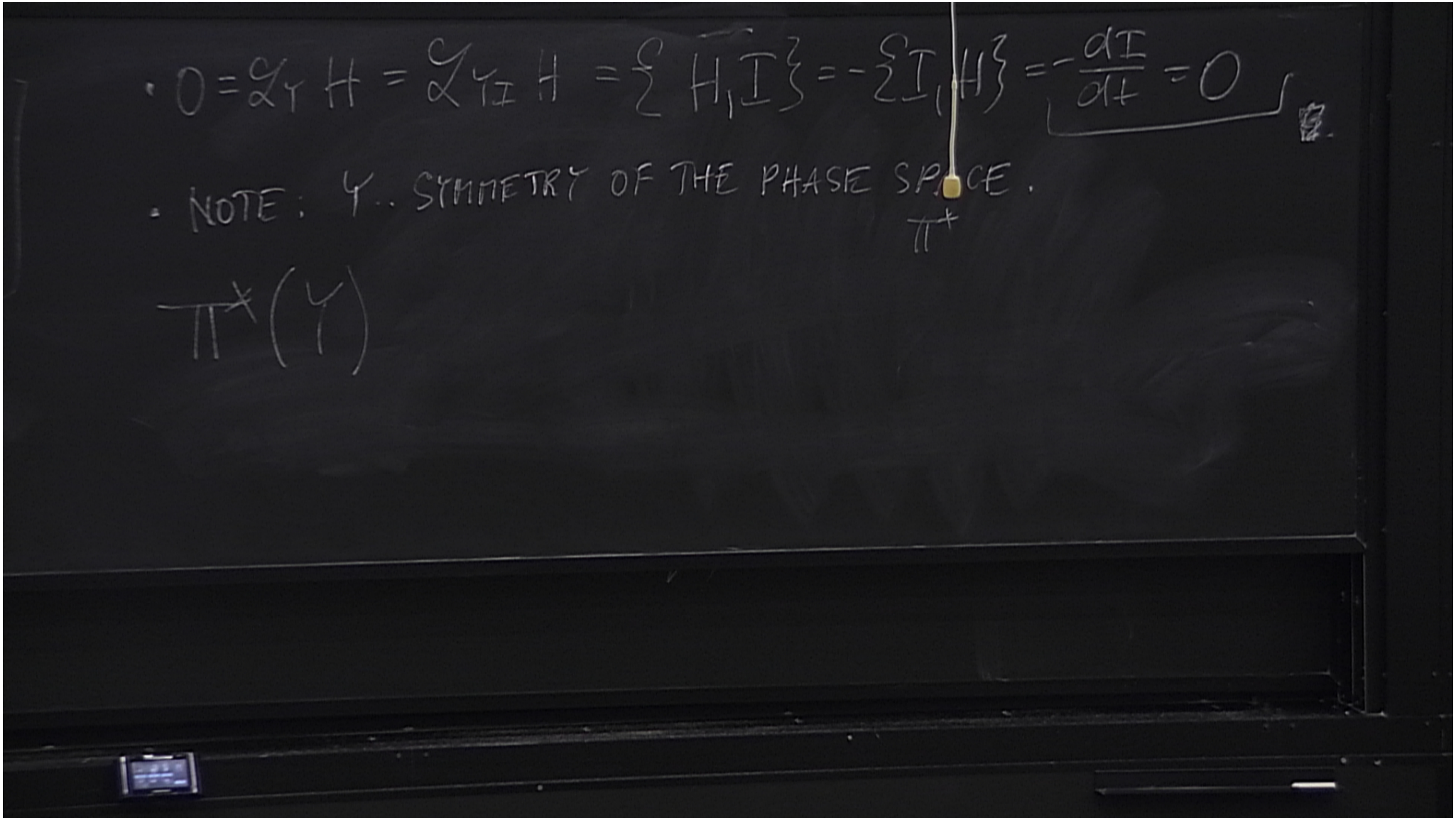

• SYMMETRIES
$\{f,g\} = df \cdot \Omega \cdot dg = df \cdot X_g = X_g(f) = \mathcal{L}_{X_g} f = \frac{df}{dt_g}$

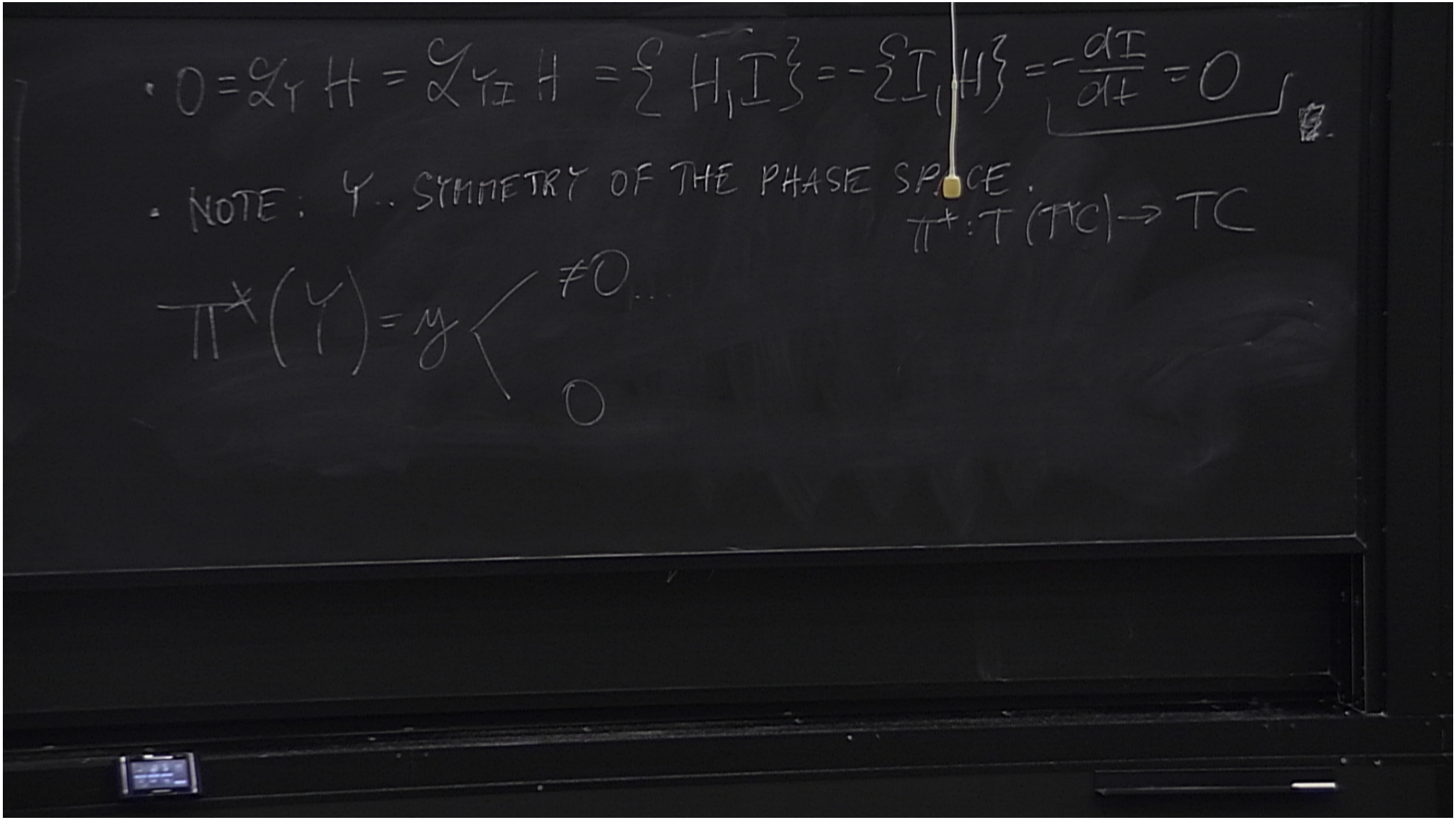

• SYMMETRIES.
$\{f,g\} = df \cdot \Omega \cdot dg = df \cdot X_g = X_g(f) = \mathcal{L}_{X_g} f = \frac{df}{dt_g}$
$= -\{g,f\} = -\mathcal{L}_{X_f} g = -\frac{dg}{dt_f}$

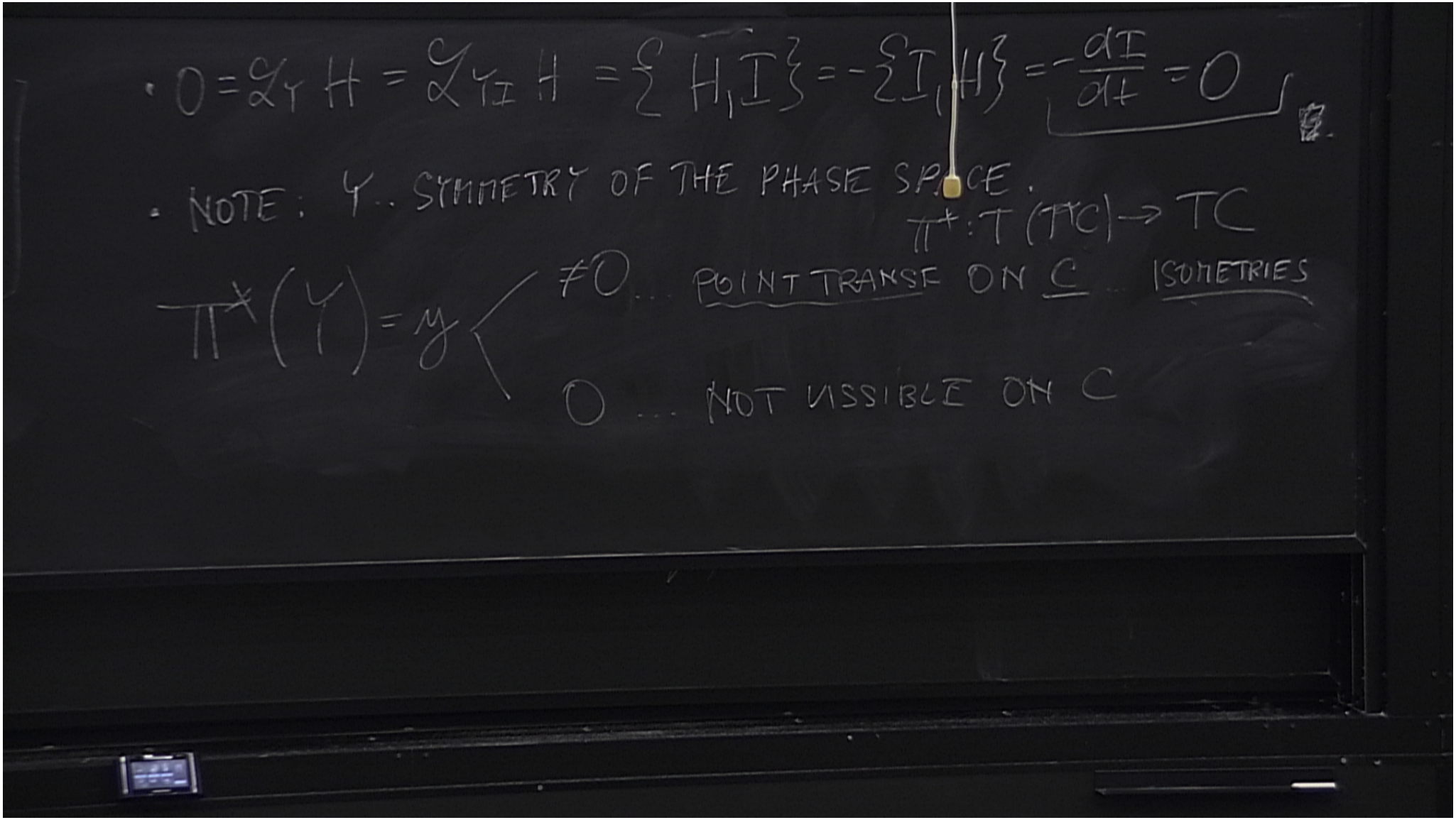

$X_H \equiv \frac{d}{dt} = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i}$
$= \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$
NOETHER'S THEOREM (V4 - HAMILTONIAN VERSION)
LET Y S.T. $\mathcal{L}_Y \omega = 0$ AND $\mathcal{L}_Y H = 0$
$\Rightarrow \exists I$ S.T $\frac{dI}{dt} = \mathcal{L}_{X_H} I = 0$

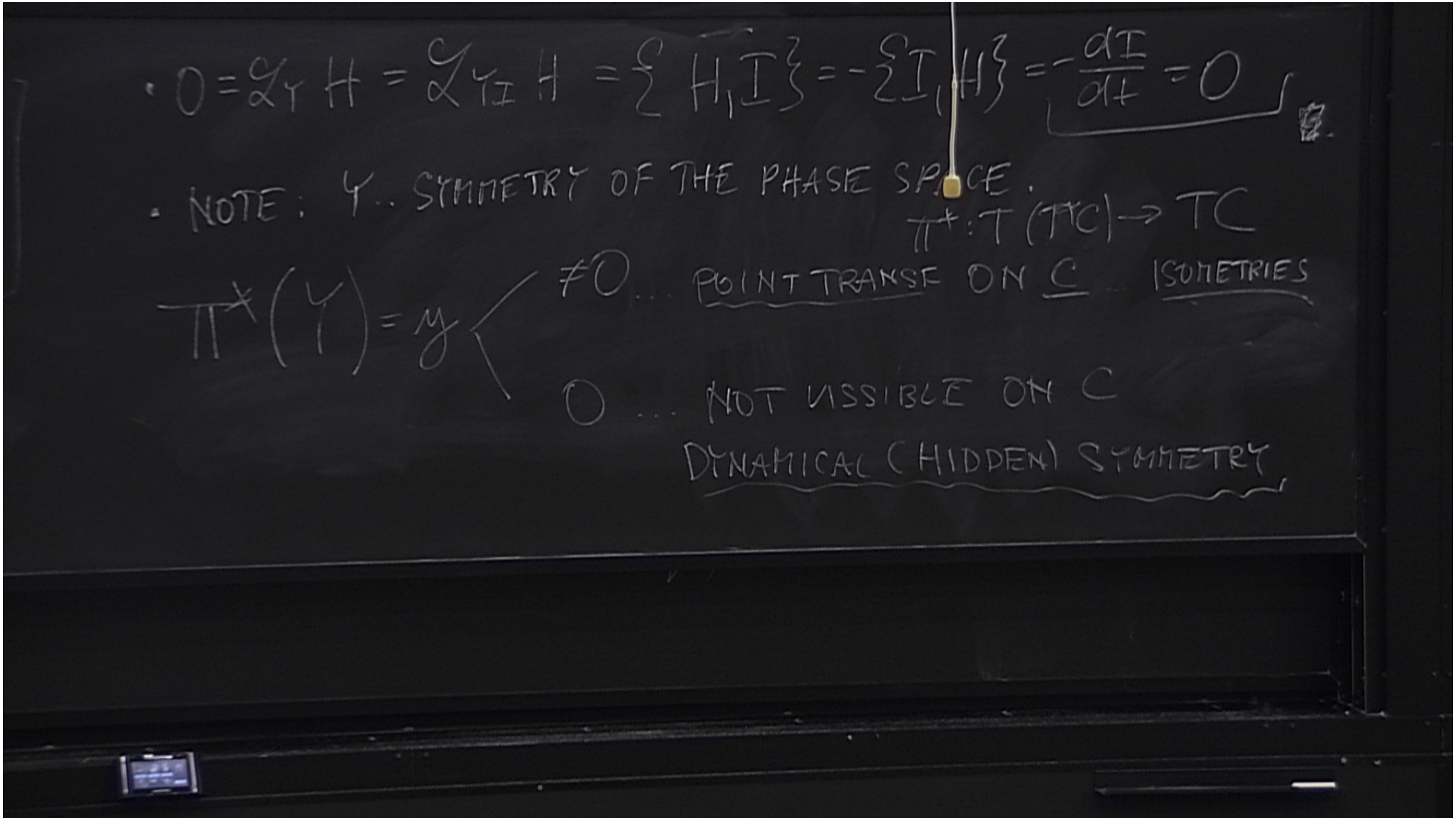

NOETHER'S THEOREM (V4 .. HAMILTONIAN VERSION)
LET Y S.T. $\mathcal{L}_Y \omega = 0$ AND $\mathcal{L}_Y H = 0$
$\Rightarrow \exists I$ S.T $\frac{dI}{dt} = \mathcal{L}_{X_H} I = 0$
PROOF: $\mathcal{L}_Y \omega = 0 = Y \lrcorner d\omega + d(Y \lrcorner \omega)$
$d\omega = 0$
$Y \lrcorner \omega = -dI$
OR $Y = Y_I$ .. HAM VECTOR F.

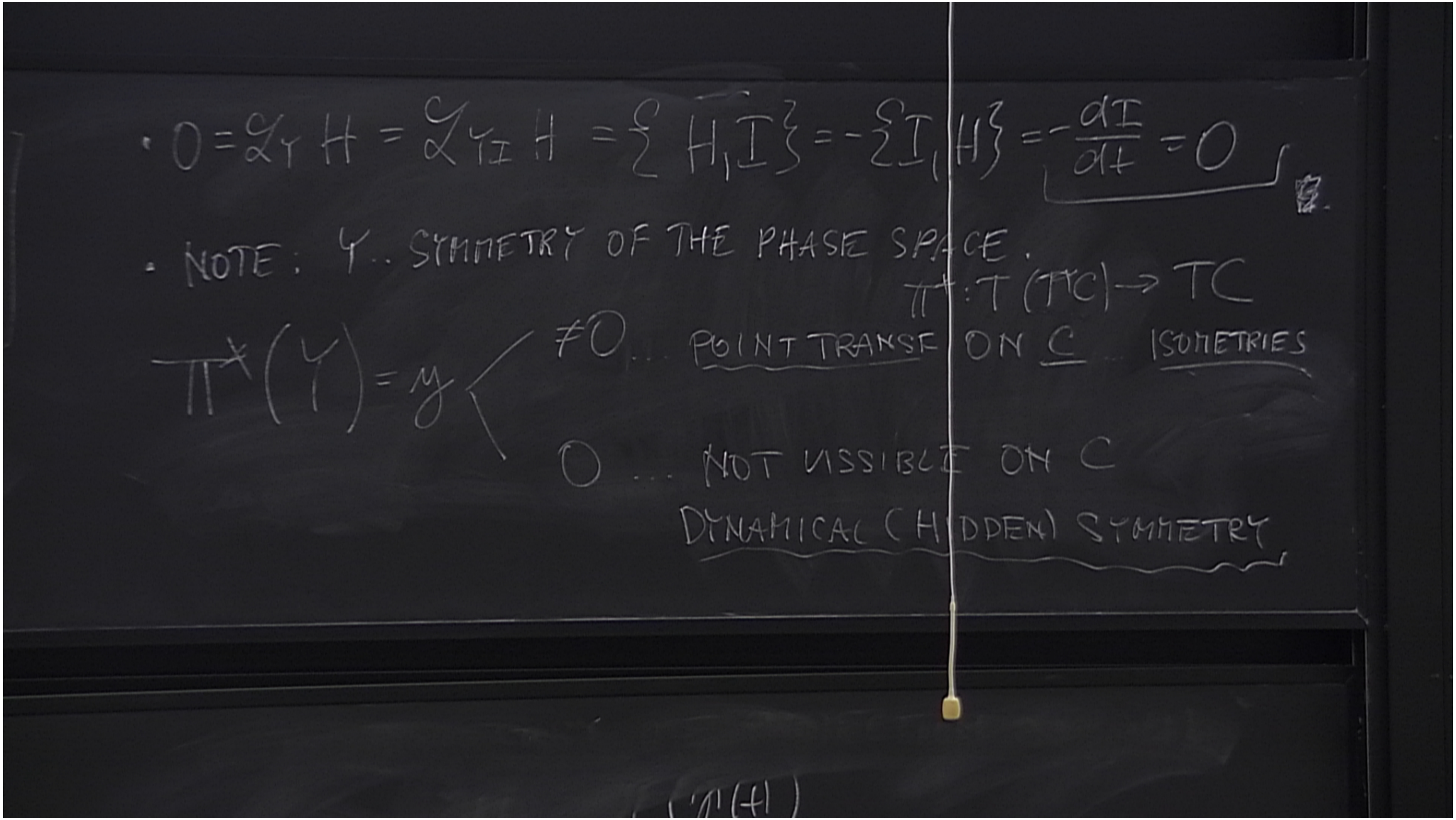

$\delta(t) = \pi(\gamma(t))$
$0 = \mathcal{L}_Y H = \mathcal{L}_{Y_I} H = \{H, I\} = -\{I, H\} = -\frac{dI}{dt} = 0$

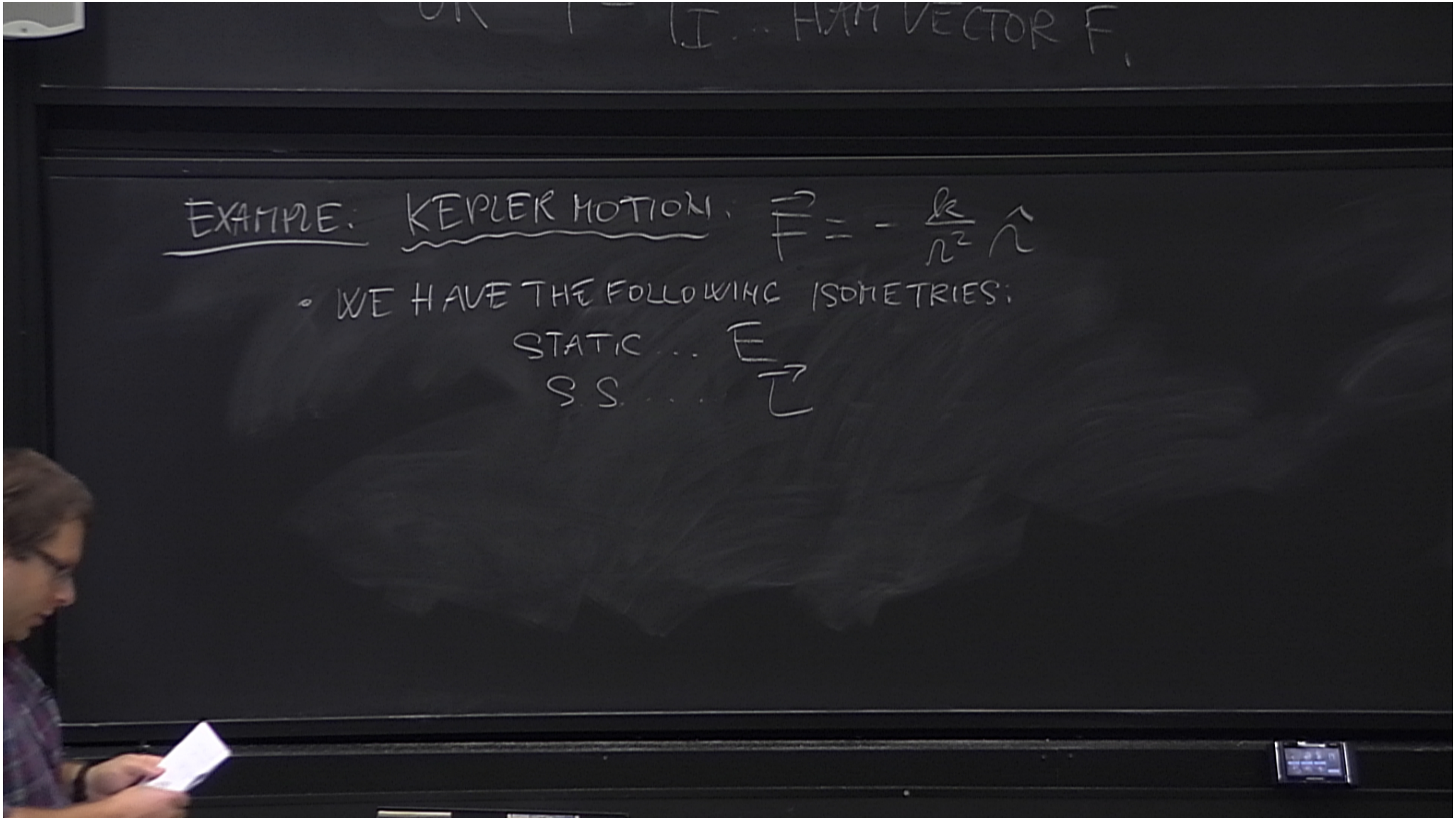

$0 = \mathcal{L}_Y H = \mathcal{L}_{Y_I} H = \{H, I\} = -\{I, H\} = -\frac{dI}{dt} = 0$
NOTE: Y .. SYMMETRY OF THE PHASE SPACE.
π^*
$\pi^*(Y)$

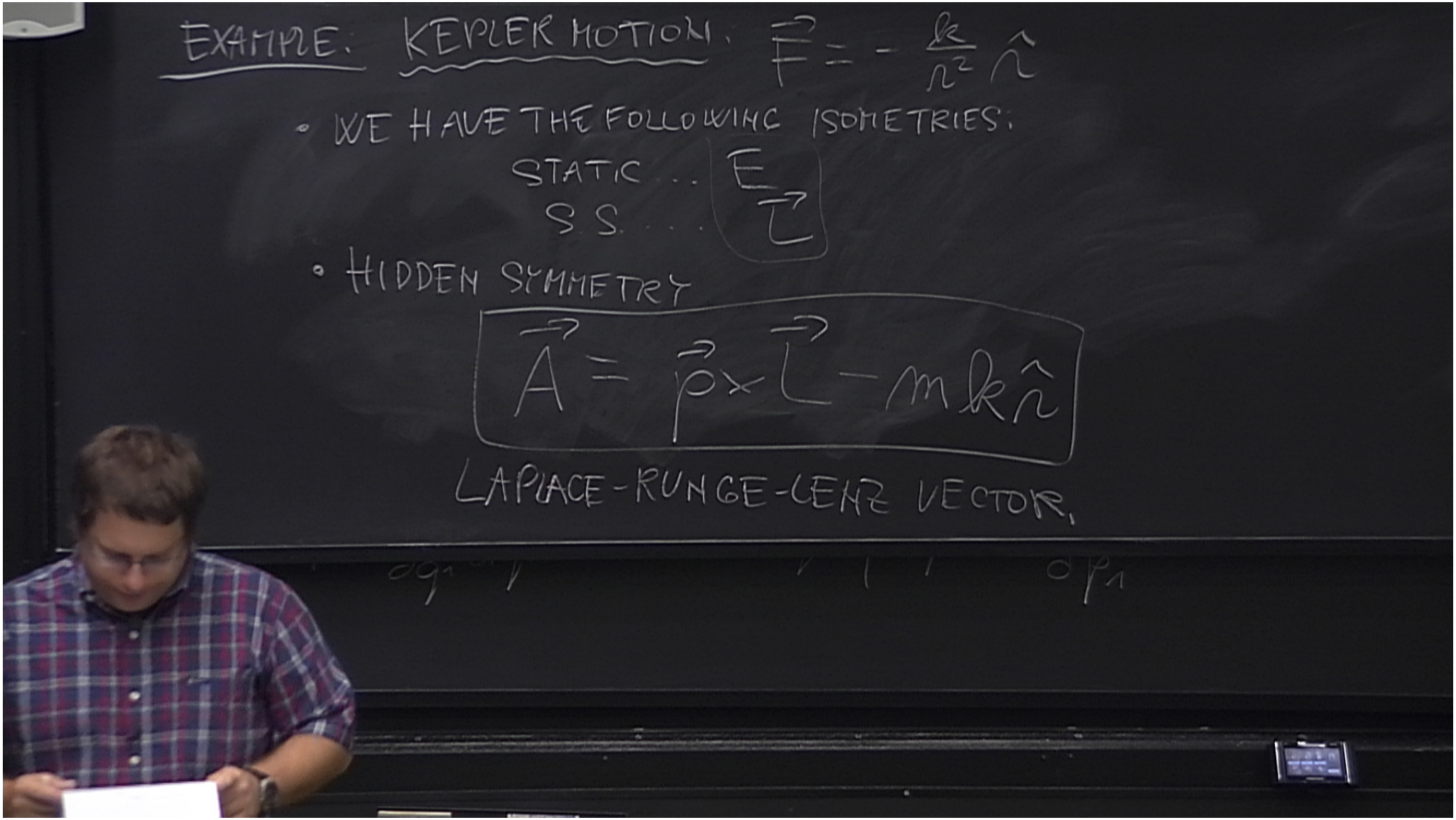

$0 = \mathcal{L}_Y H = \mathcal{L}_{Y_I} H = \{H, I\} = -\{I, H\} = -\frac{dI}{dt} = 0$
NOTE: Y.. SYMMETRY OF THE PHASE SPACE.
$\pi^*: T(TC) \to TC$
$\pi^*(Y) = y \begin{cases} \neq 0 \\ 0 \end{cases}$

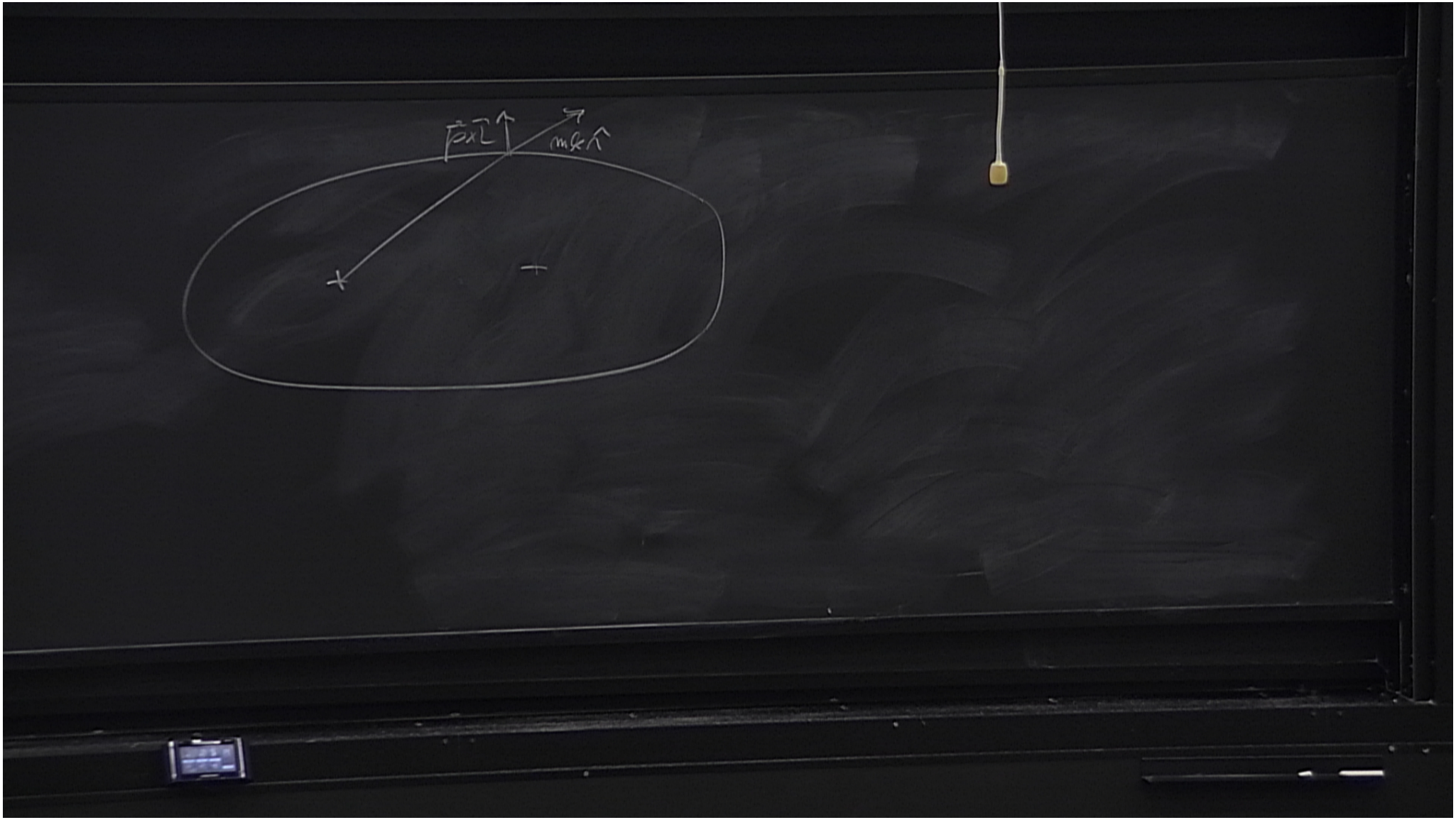

$0 = \mathcal{L}_Y H = \mathcal{L}_{Y_I} H = \{H, I\} = -\{I, H\} = -\frac{dI}{dt} = 0$
NOTE: Y.. SYMMETRY OF THE PHASE SPACE.
$\pi^*: T(T^*C) \to TC$
$\pi^*(Y) = y \begin{cases} \neq 0 \ldots \text{POINT TRANSF ON } C \ldots \text{ISOMETRIES} \\ 0 \ldots \text{NOT VISIBLE ON } C \end{cases}$

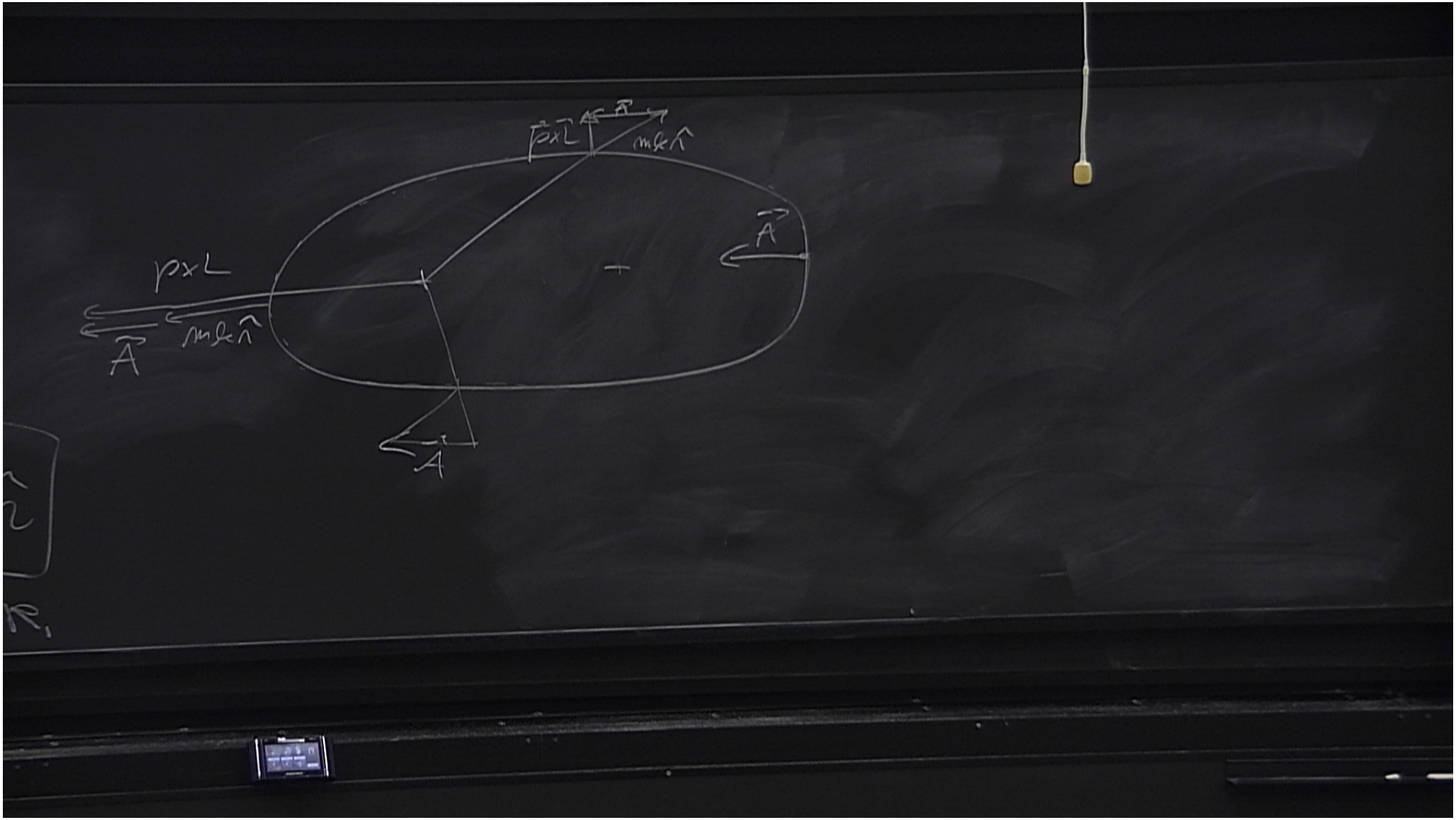

$0 = \mathcal{L}_Y H = \mathcal{L}_{Y_I} H = \{H, I\} = -\{I, H\} = -\frac{dI}{dt} = 0$
NOTE: Y.. SYMMETRY OF THE PHASE SPACE.
$\pi^*: T(T^*C) \to TC$
$\pi^*(Y) = y \begin{cases} \neq 0 \\ 0 \end{cases}$
.. POINT TRANSF ON C .. ISOMETRIES
.. NOT VISSIBLE ON C
DYNAMICAL (HIDDEN) SYMMETRY

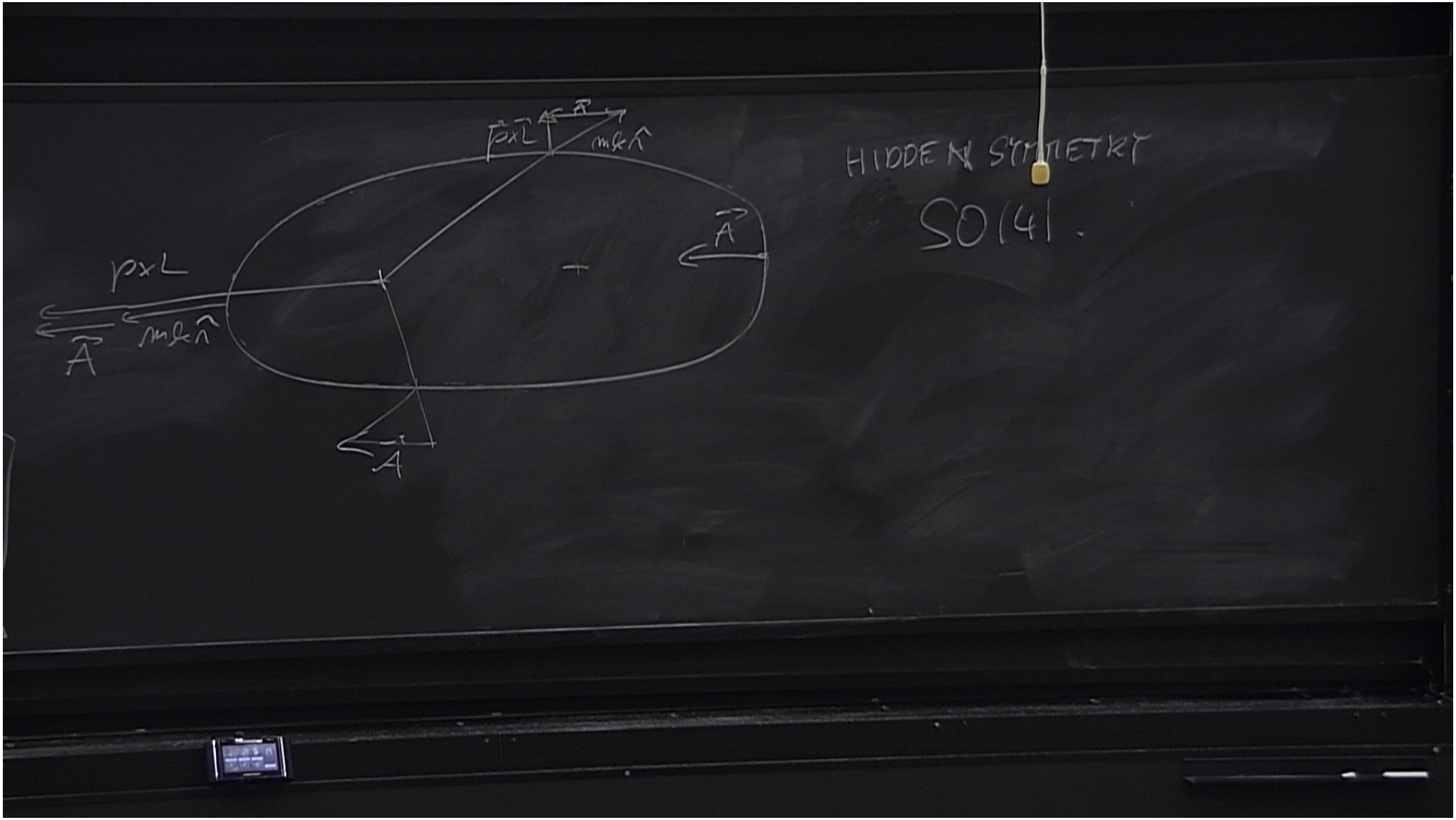

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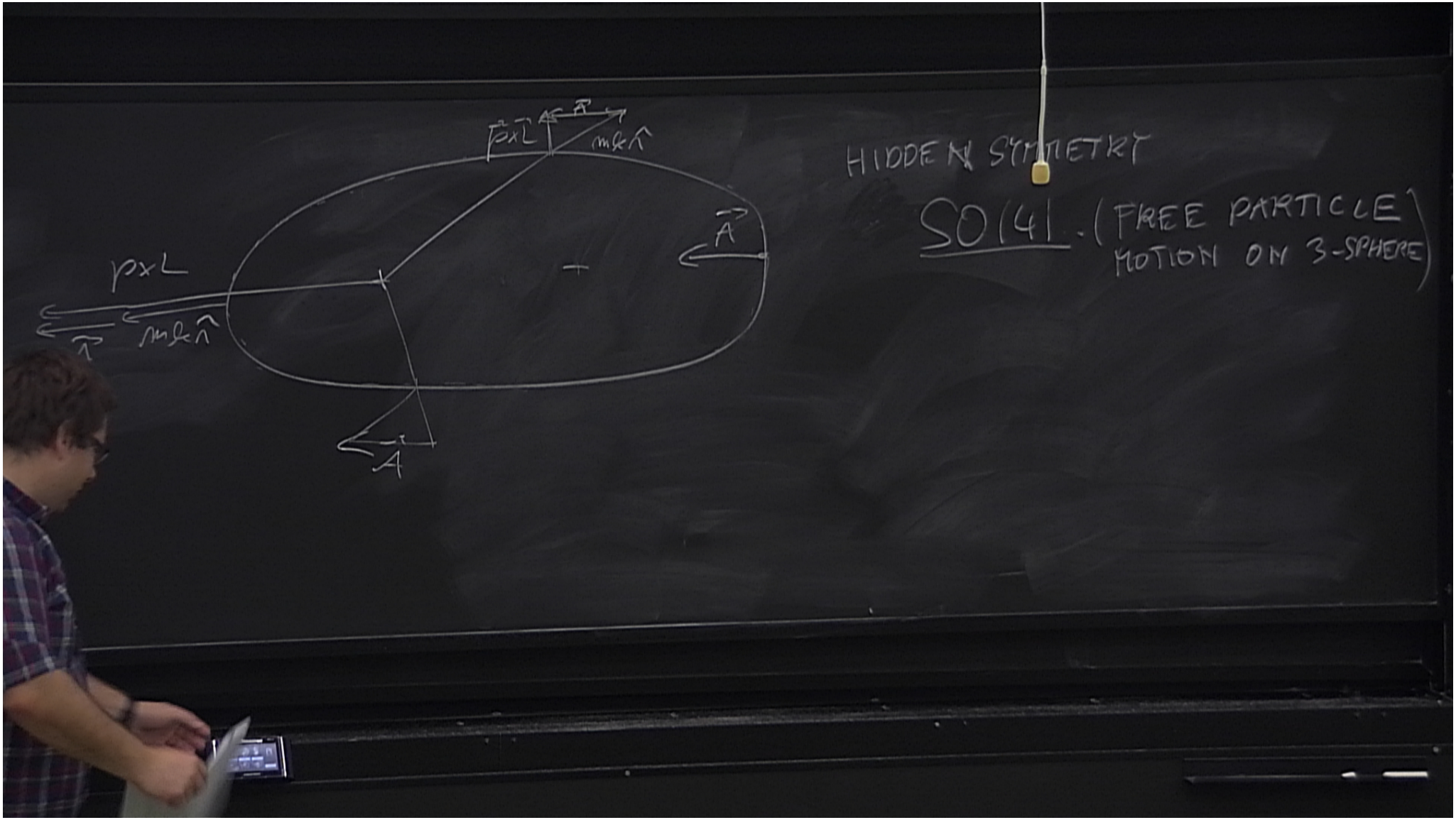

I.. HAM VECTOR F.
EXAMPLE: KEPLER MOTION. $\vec{F} = -\frac{k}{r^2}\hat{r}$
• WE HAVE THE FOLLOWING ISOMETRIES:
STATIC .. E
S S .. $\vec{L}$

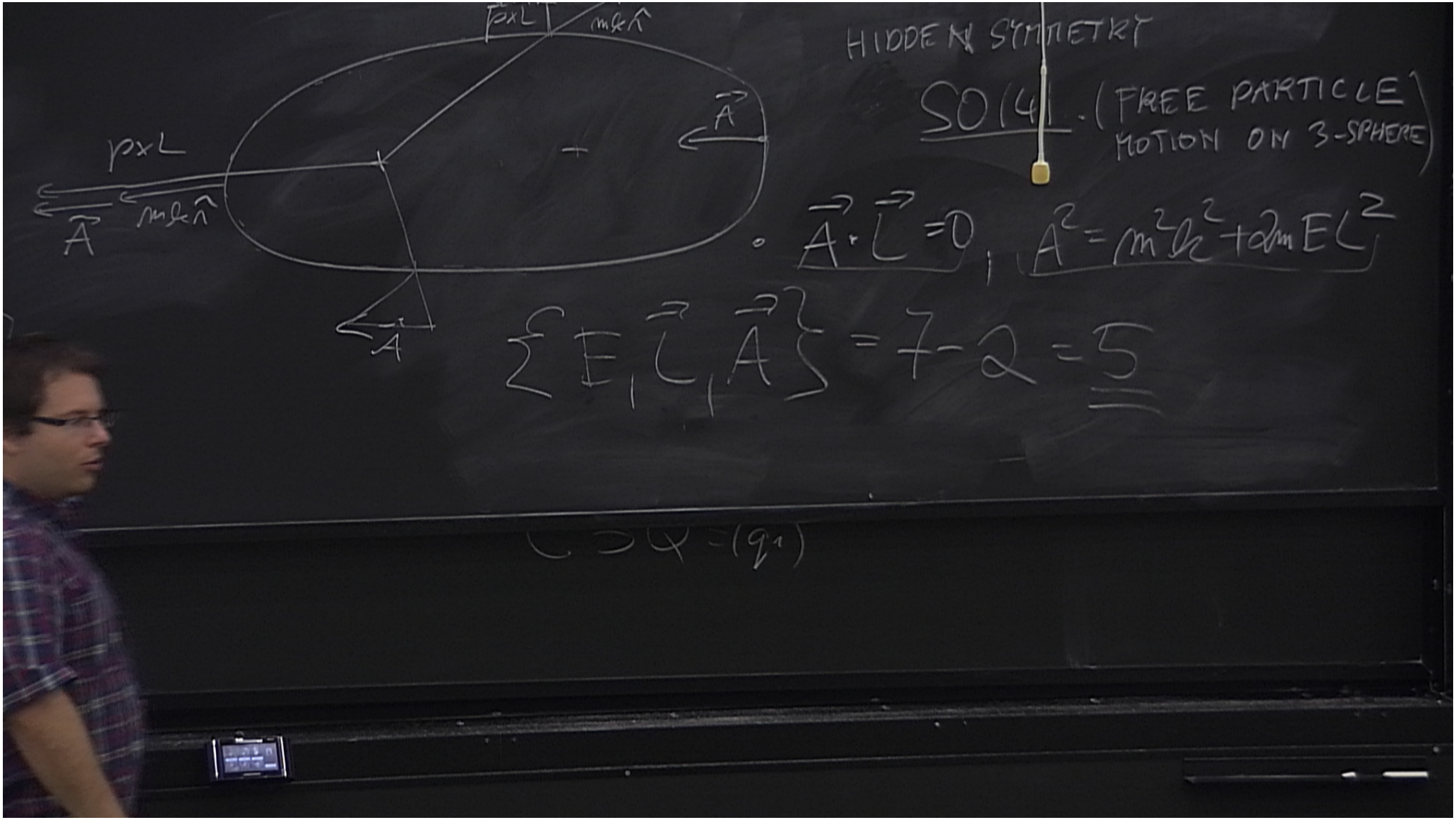

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S S ... $\vec{L}$
• HIDDEN SYMMETRY
$\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}$
LAPLACE-RUNGE-LENZ VECTOR.

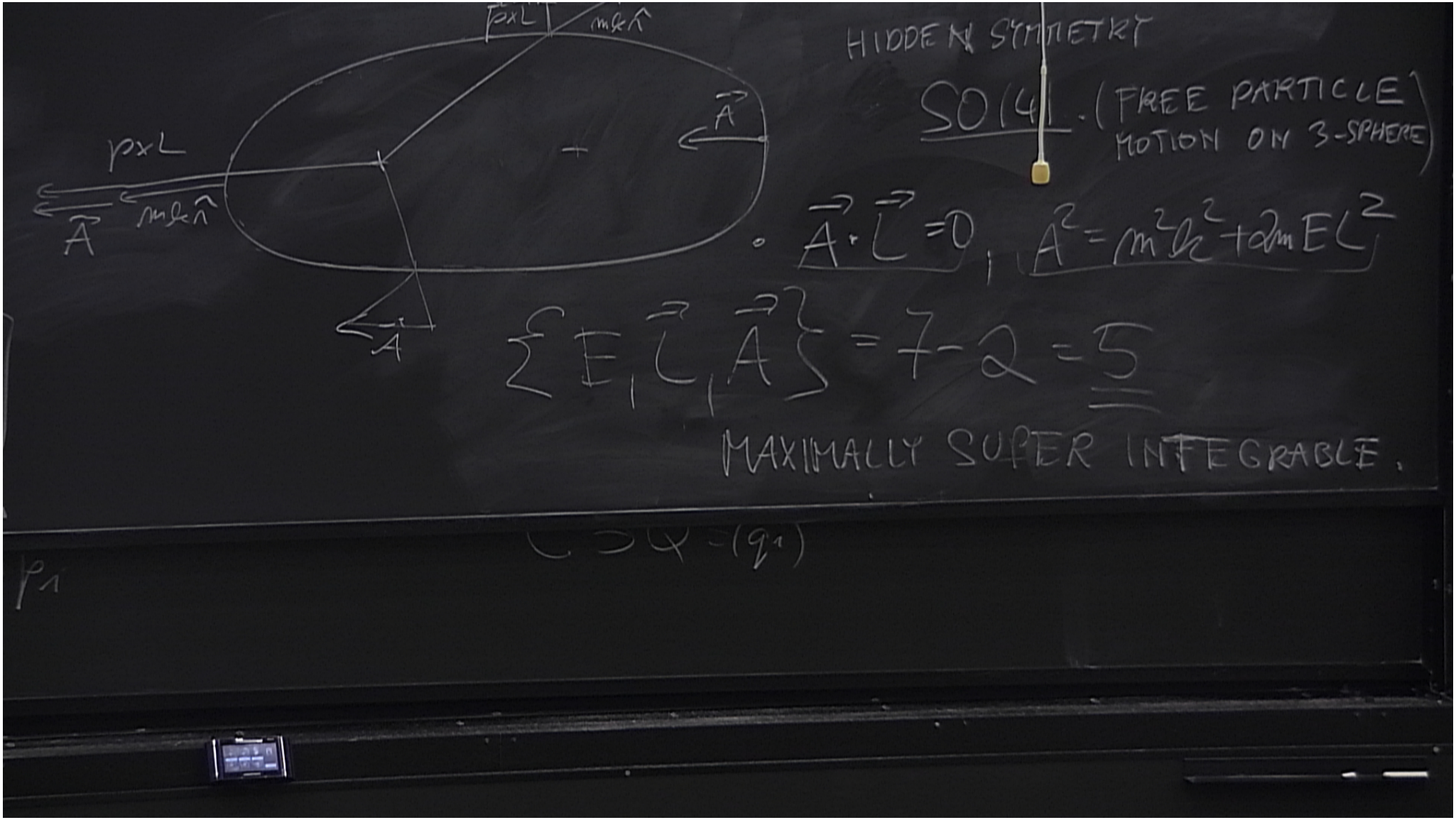

$\vec{F} \times \vec{L}$
$m\ell\hat{\wedge}$


p×L
mℓn̂
A
p×L
mℓn̂
A
A


p×L
mkr̂
A
HIDDEN SYMMETRY
SO(4).


HIDDEN SYMMETRY
SO(4). (FREE PARTICLE MOTION ON 3-SPHERE)


HIDDEN SYMMETRY
SO(4) (FREE PARTICLE MOTION ON 3-SPHERE)
$\vec{A}\cdot\vec{L}=0$; $A^2=m^2k^2+2mEL^2$
$\{E,\vec{L},\vec{A}\}=7-2=5$



HIDDEN SYMMETRY

SO(4). (FREE PARTICLE MOTION ON 3-SPHERE)

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MAXIMALLY SUPER INTEGRABLE.

$(2n-1)$



HIDDEN SYMMETRY

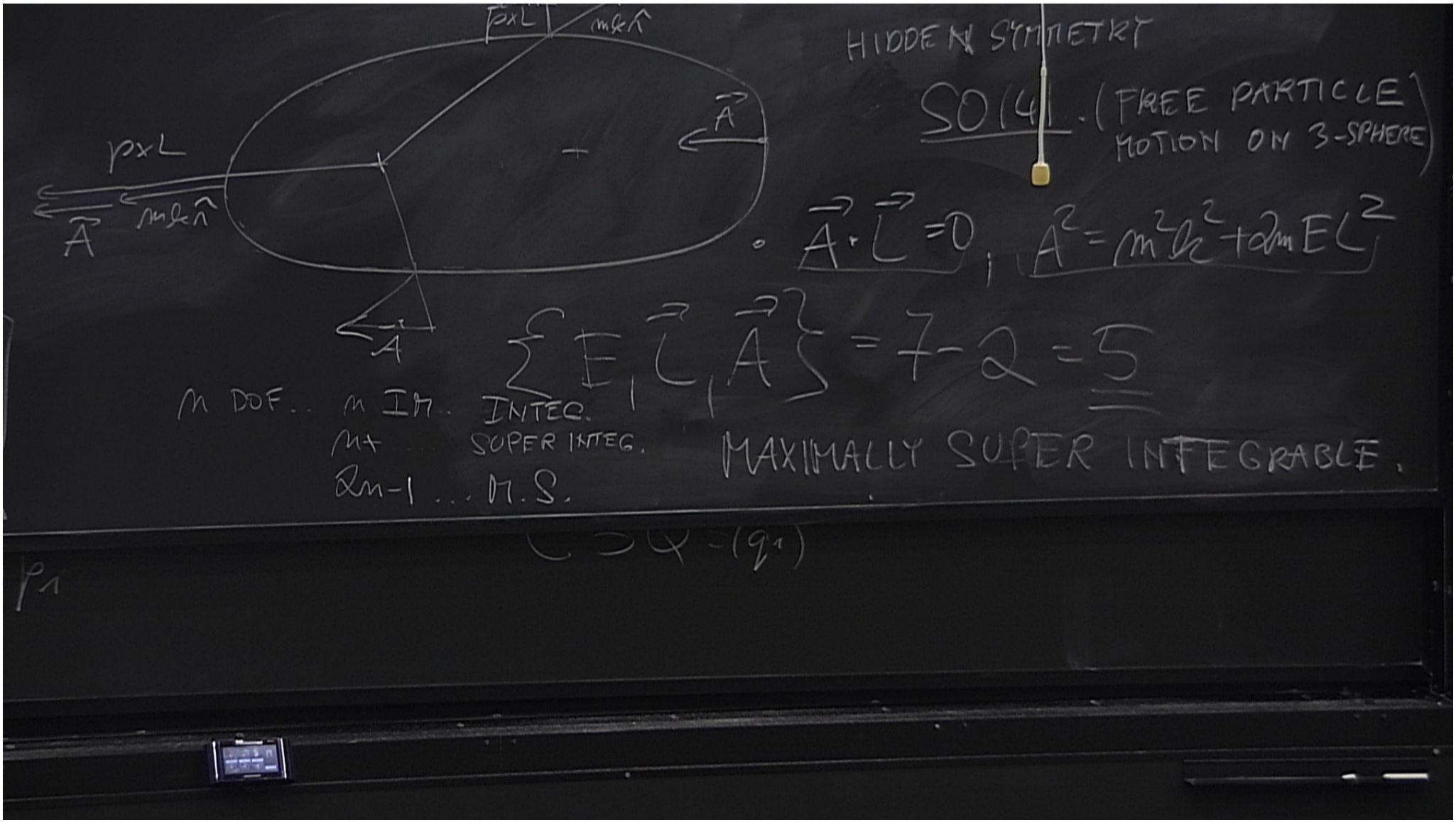
SO(4). (FREE PARTICLE MOTION ON 3-SPHERE)

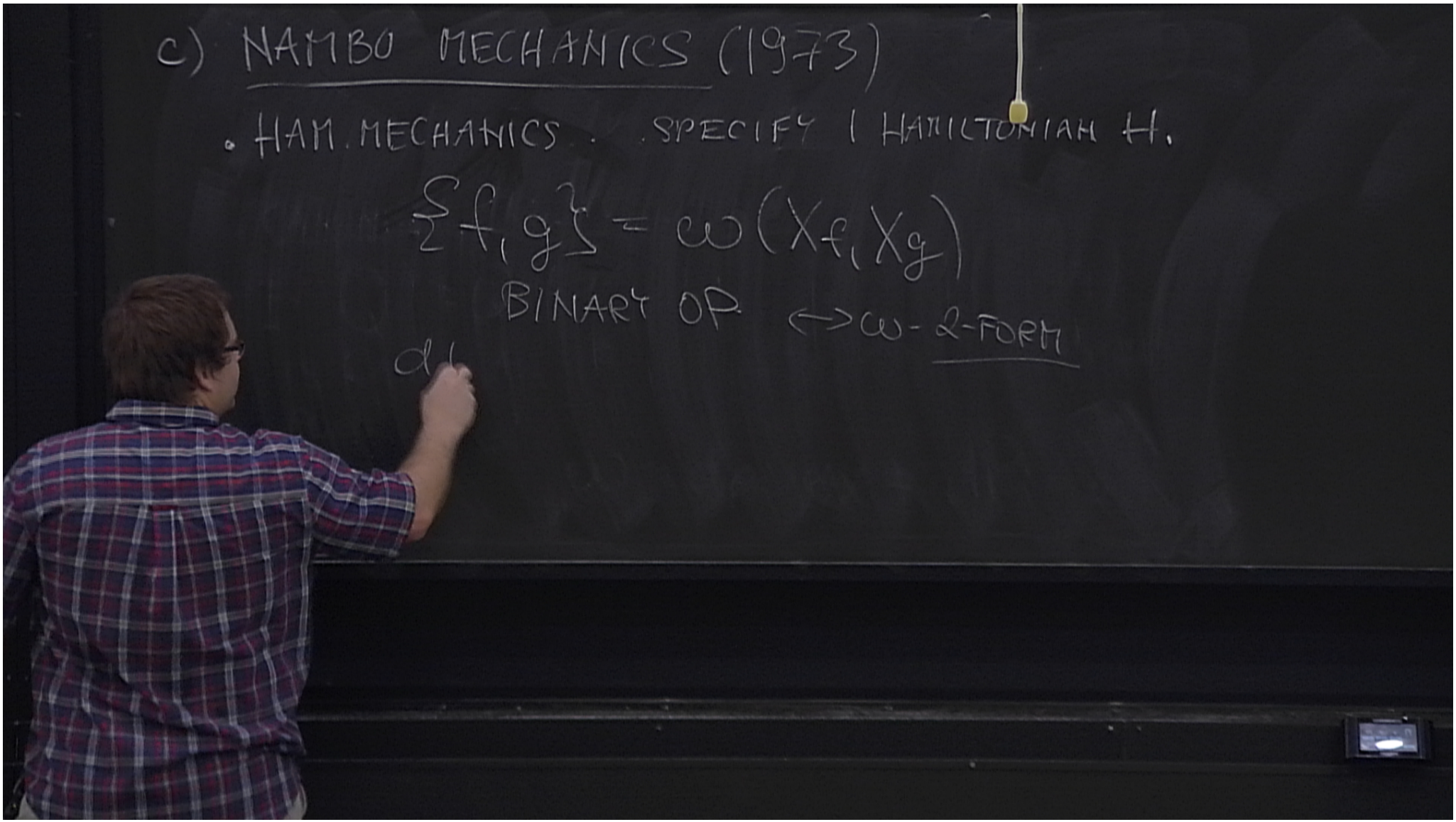
$\vec{A}\cdot\vec{L}=0$; $A^2=m^2k^2+2mEL^2$

n DOF. n IM. INTEG. $\{E, \vec{L}, \vec{A}\} = 7-2=5$

MAXIMALLY SUPER INTEGRABLE.

(q_1, p_1)


p×L
mk̂
p×L
mk̂
$\vec{A}$
$\vec{A}$
$\vec{A}$
HIDDEN SYMMETRY
SO(4). (FREE PARTICLE MOTION ON 3-SPHERE)
$\vec{A}\cdot\vec{L}=0$, $A^2=m^2k^2+2mEL^2$
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n DOF.. n IM. INTEG.
n+ SUPER INTEG.
2n-1 ... M.S.
MAXIMALLY SUPER INTEGRABLE.
$L=\gamma\times(q_1)$
p_1


c) NAMBO MECHANICS (1973)
• HAM. MECHANICS . SPECIFY (HAMILTONIAN H.
$\{f,g\} = \omega(X_f, X_g)$
BINARY OP $\leftrightarrow$ ω-2-FORM
$d\omega$

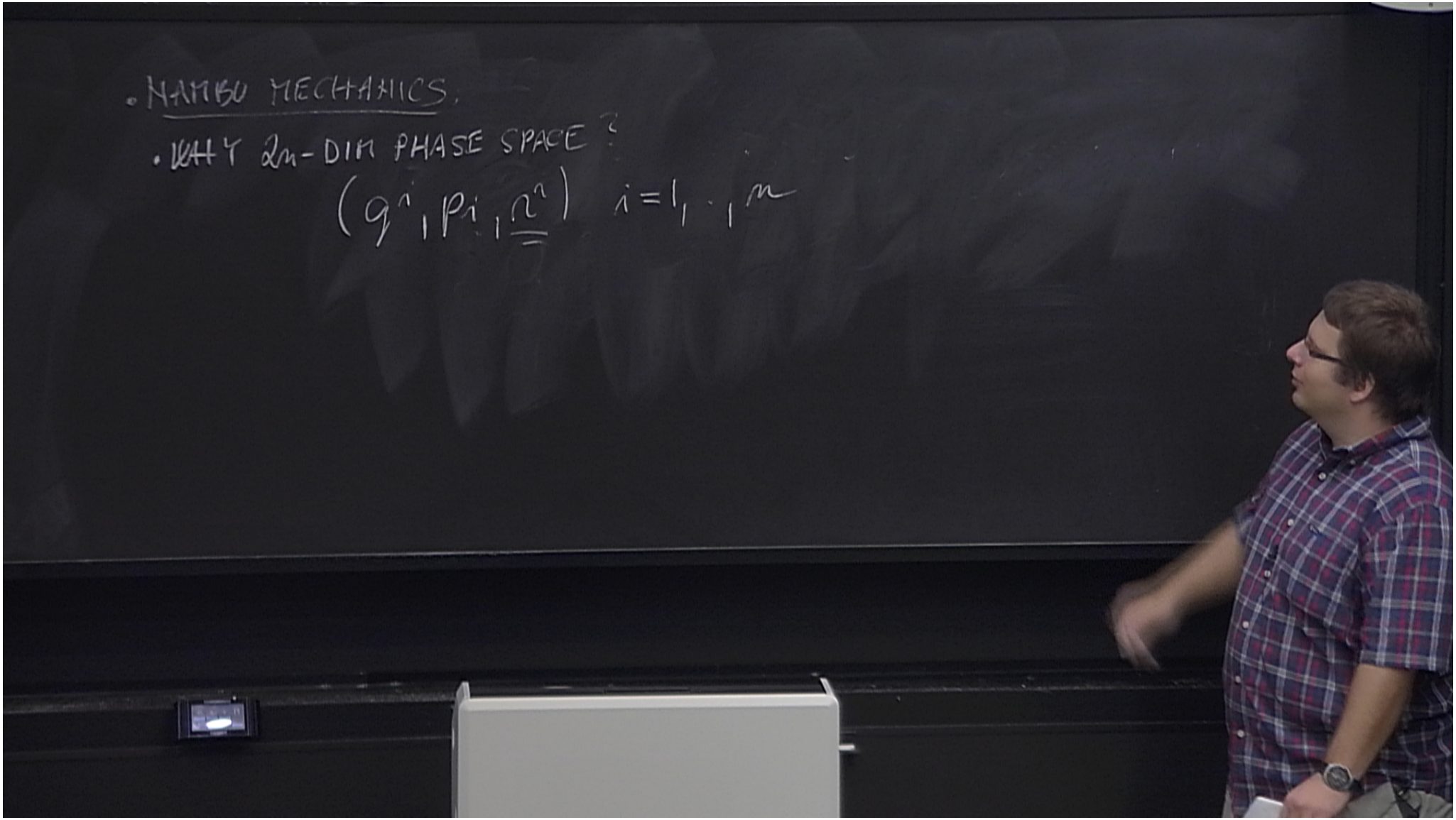
c) NAMBU MECHANICS (1973)

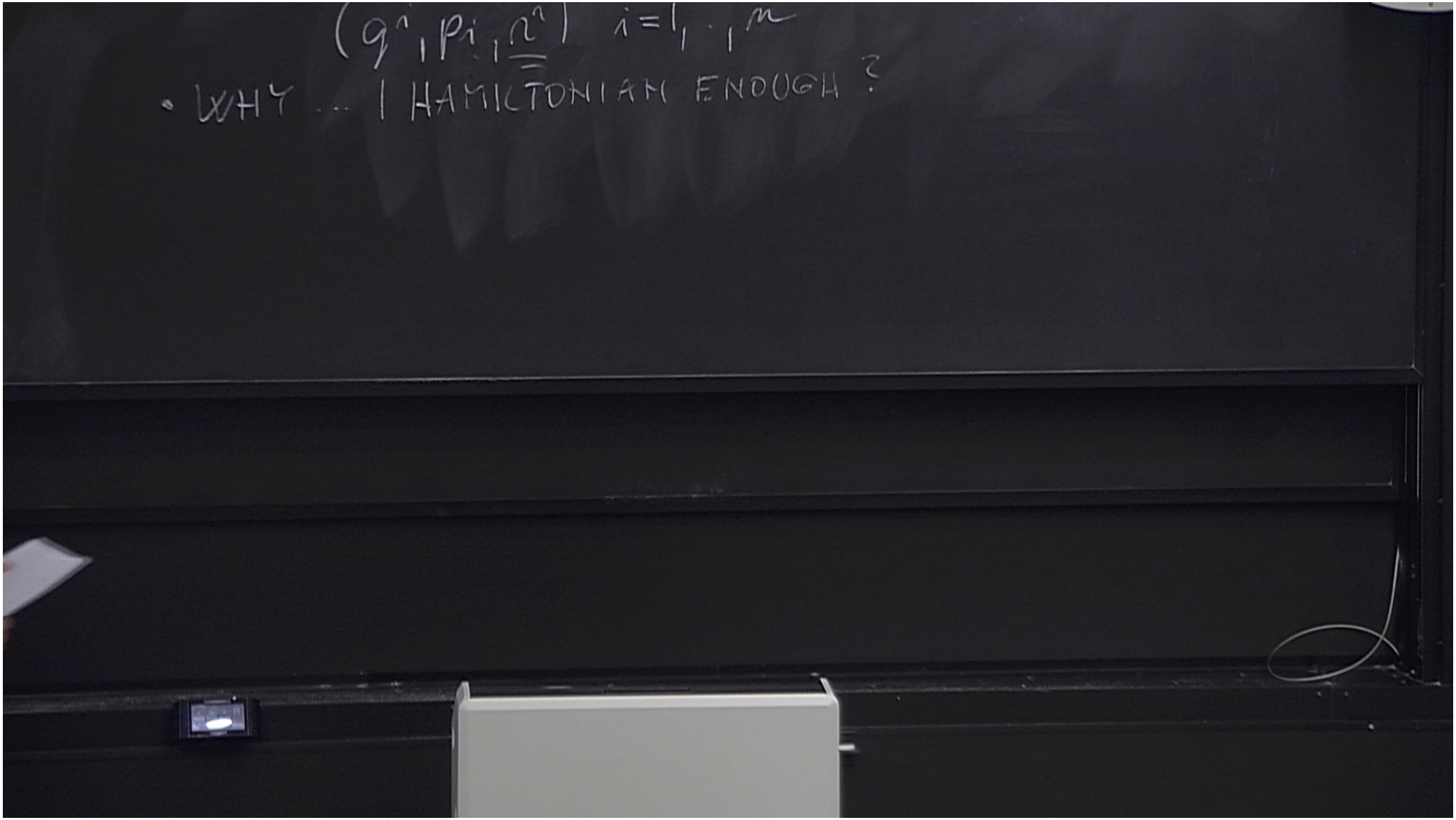
- HAM. MECHANICS . SPECIFY 1 HAMILTONIAN H.

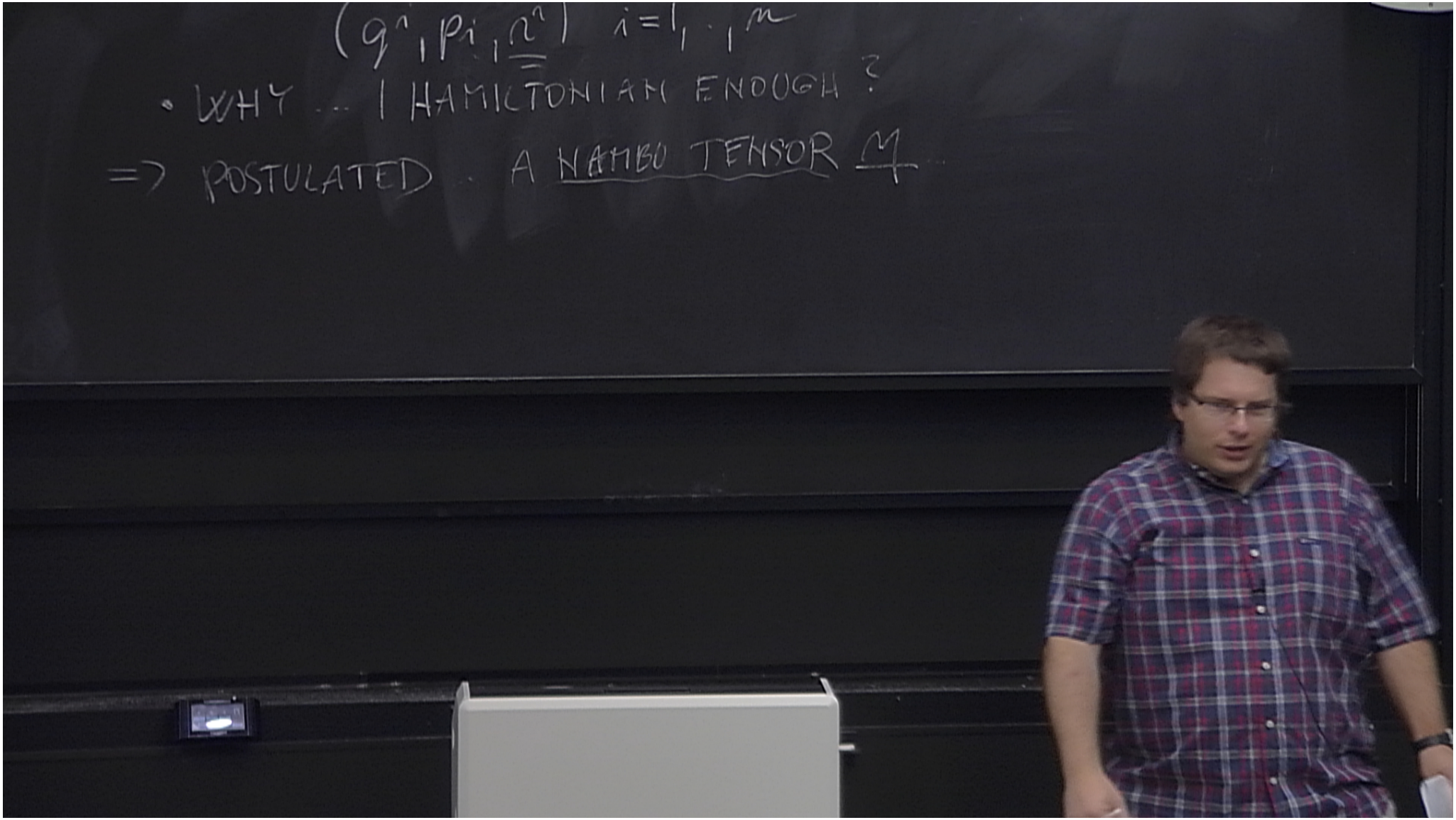
$$\{f, g\} = \omega(X_f, X_g)$$

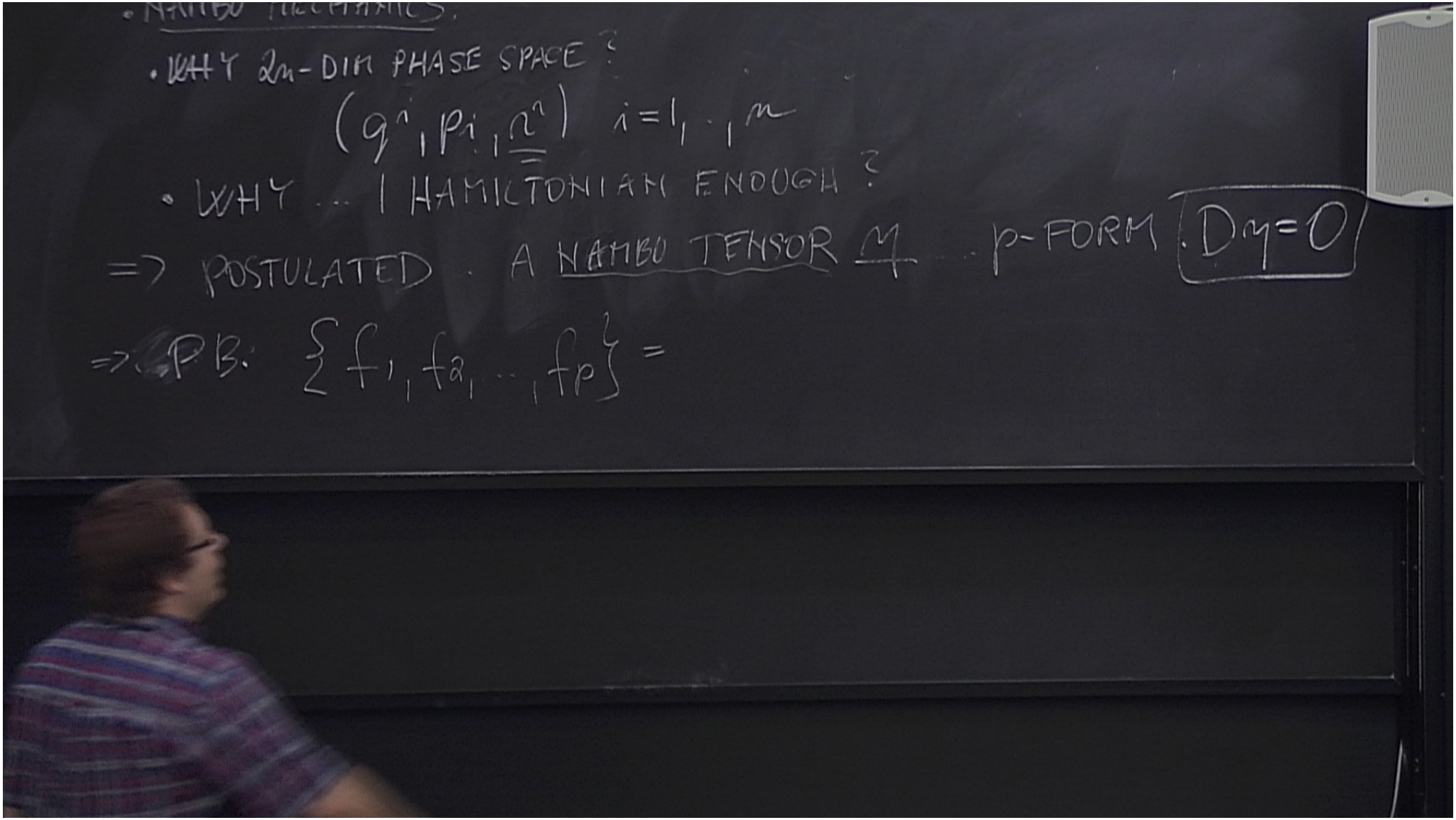
BINARY OP $\leftrightarrow$ ω - 2-FORM

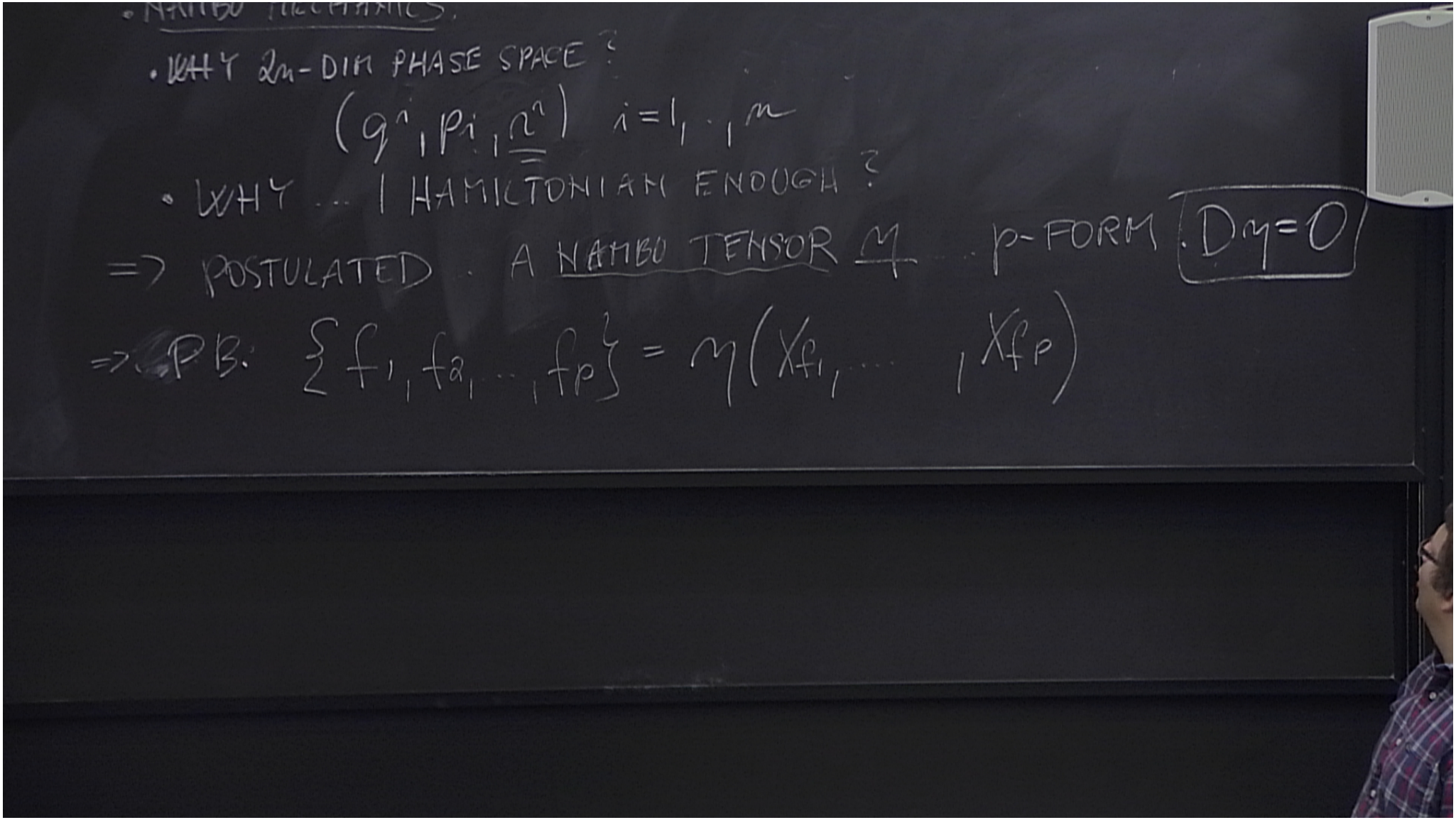
$$\frac{df}{dt} = \{f, H\}$$

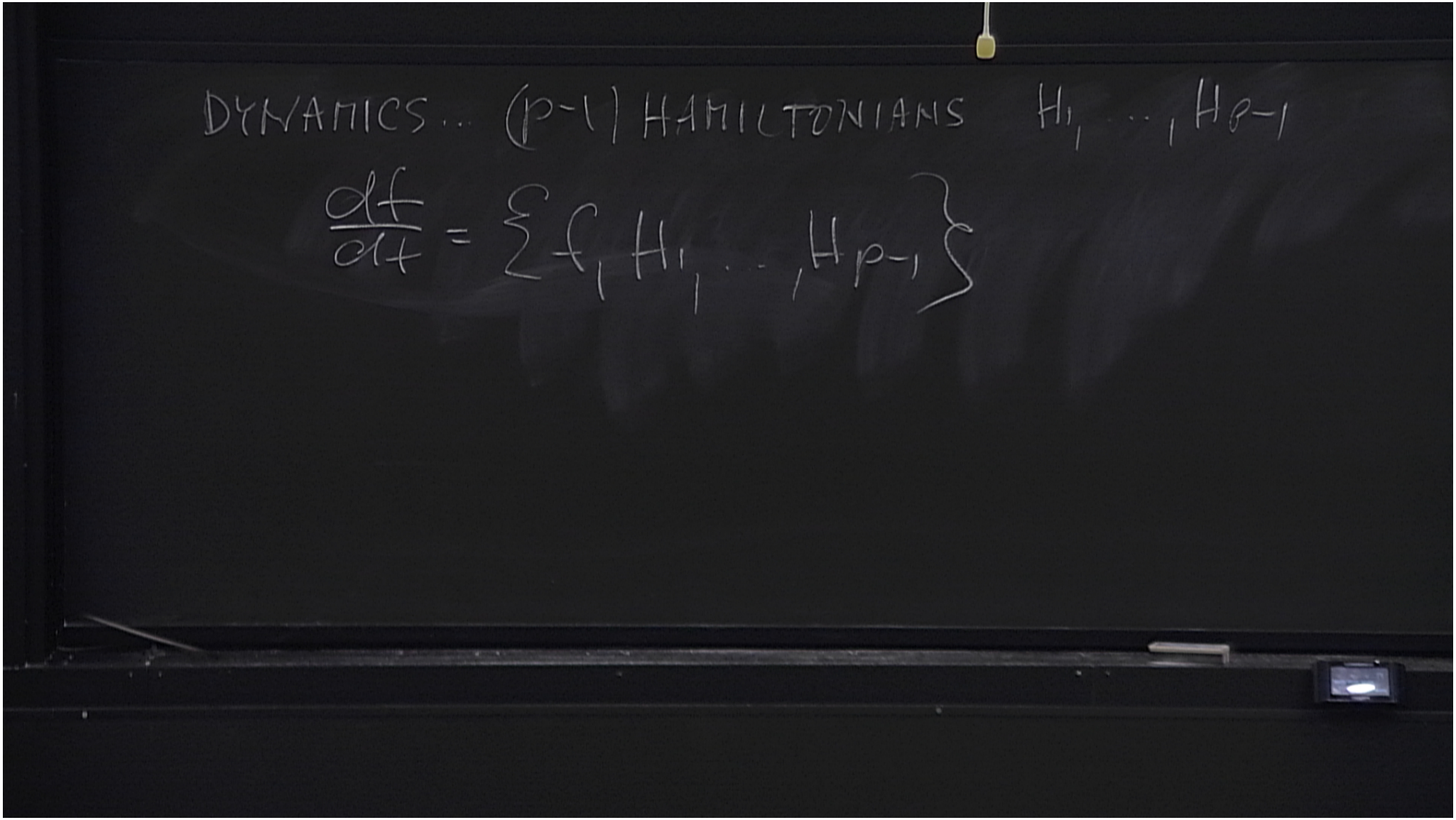

• NAMBU MECHANICS.
• WHY 2n-DIM PHASE SPACE?
(q^i, p_i, Ω^i) $i = 1, \ldots, n$


$(q_i, p_i, \underline{\underline{\Omega}}^2)$ $i = 1, \ldots, n$
• WHY ... | HAMILTONIAN ENOUGH ?


$(q^i, p_i, \underline{\underline{\Omega^i}})$ $i = 1, \ldots, n$
• WHY ... | HAMILTONIAN ENOUGH ?
=> POSTULATED ... A NAMBU TENSOR $\underline{M}$


• NAMBU MECHANICS:
• WHY 2n-DIM PHASE SPACE?
$(q^i, p_i, \underline{\underline{x}}^a)$ $i = 1, \ldots, n$
• WHY ... 1 HAMILTONIAN ENOUGH?
=> POSTULATED: A NAMBU TENSOR $\underline{\underline{M}}$ ... p-FORM $D_M = 0$
=> PB: $\{f_1, f_2, \ldots, f_p\} =$


• NAMBU MECHANICS
• WHY 2n-DIM PHASE SPACE ?
$(q^i, p_i, \underline{\underline{x}}^a)$ $i = 1, \ldots, n$
• WHY ... 1 HAMILTONIAN ENOUGH ?
=> POSTULATED A NAMBU TENSOR M ... p-FORM $Dm = 0$
=> PB: $\{f_1, f_2, \ldots, f_p\} = m(X_{f_1}, \ldots, X_{f_p})$


DYNAMICS .. (p-1) HAMILTONIANS $H_1, \ldots, H_{p-1}$
$\frac{df}{dt} = \{f, H_1, \ldots, H_{p-1}\}$

DYNAMICS .. $(p-1)$ HAMILTONIANS $H_1, \dots, H_{p-1}$

$$\frac{df}{dt} = \{f, H_1, \dots, H_{p-1}\}$$

EX. 3d SPACE (x^1, x^2, x^3) $\eta = \varepsilon$ LEVI-CIVITA

$$\{f,g,h\} = \varepsilon^{ijk} \partial_i f \partial_j g \partial_k h$$

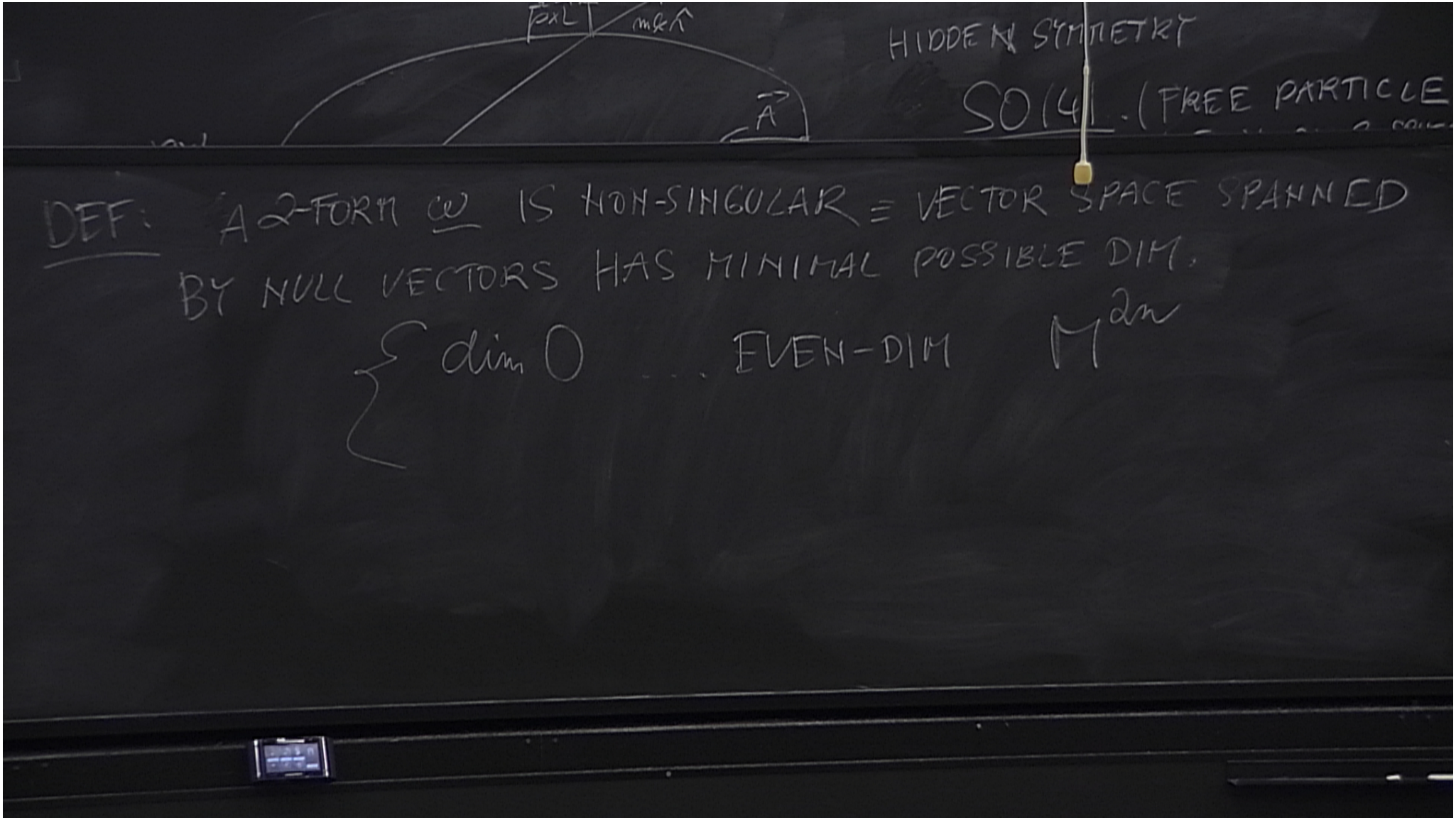
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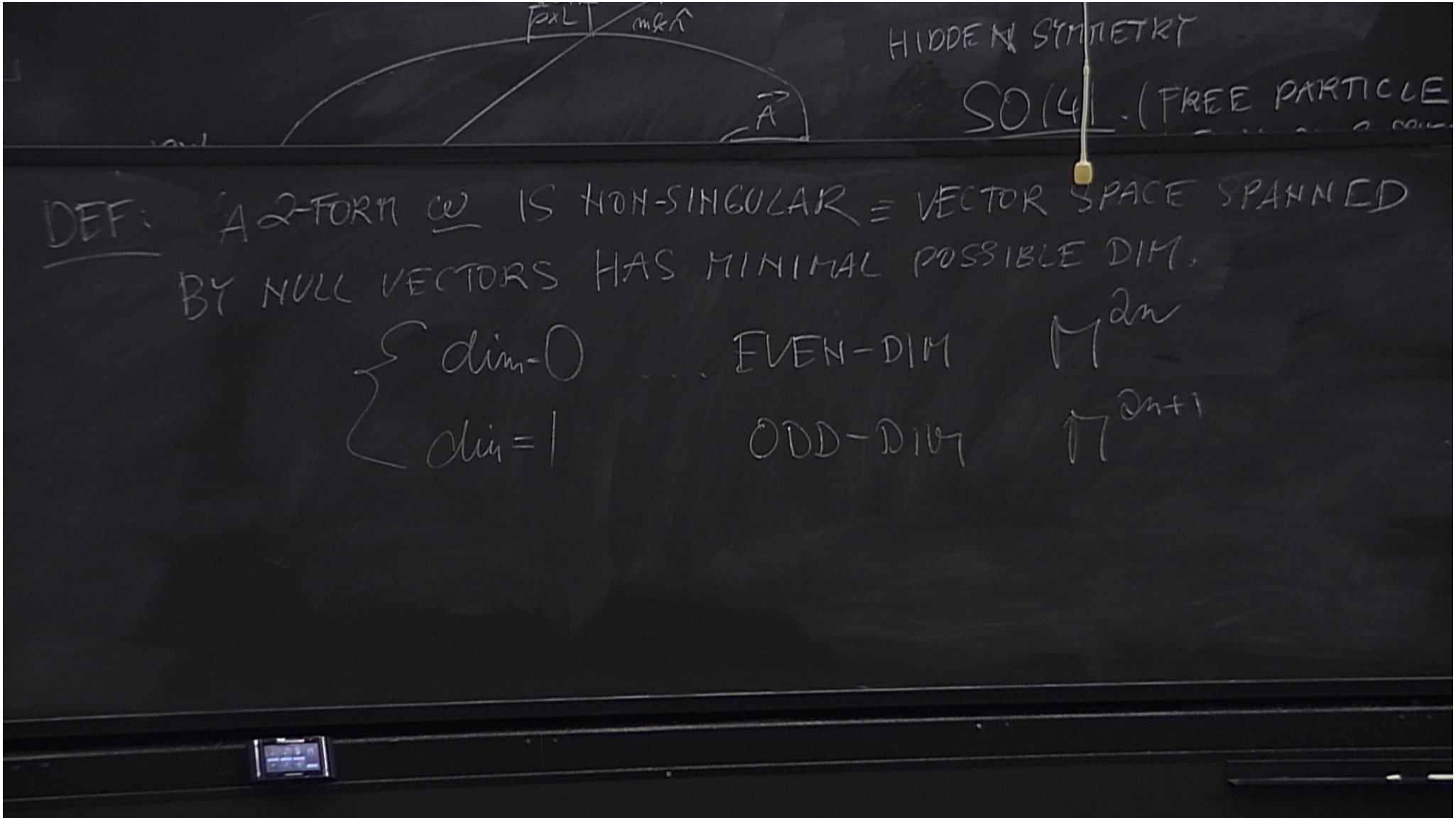
- WE HAVE THE FOLLOWING ISOMETRIES:

a) CONTACT GEOMETRY

DEF: A VECTOR X FOR WHICH $\omega(X,Y) = 0 \ \forall Y$ IS A NULL VECTOR OF 2-FORM ω.




HIDDEN SYMMETRY
SO(4,1) (FREE PARTICLE
DEF: A 2-FORM ω IS NON-SINGULAR = VECTOR SPACE SPANNED
BY NULL VECTORS HAS MINIMAL POSSIBLE DIM.
dim 0 ... EVEN-DIM M^{2n}


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dim=0 ... EVEN-DIM M^{2n}
dim=1 ODD-DIM M^{2n+1}

BY NULL VECTORS HAS MINIMAL POSSIBLE DIM.

$\begin{cases} \dim = 0 & \ldots \text{EVEN-DIM} \\ \dim = 1 & \text{ODD-DIM} \end{cases}$

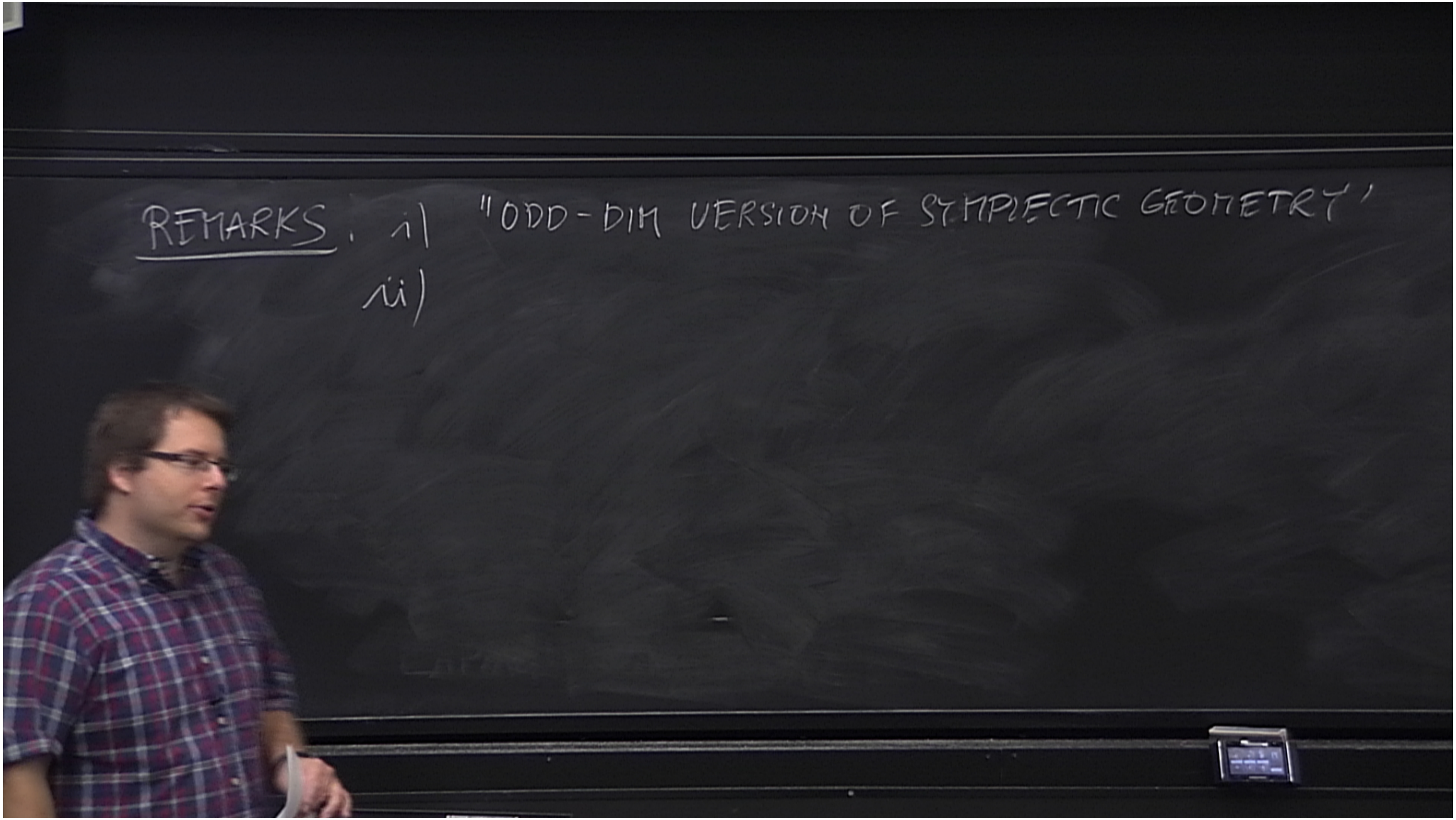
M^{2n}

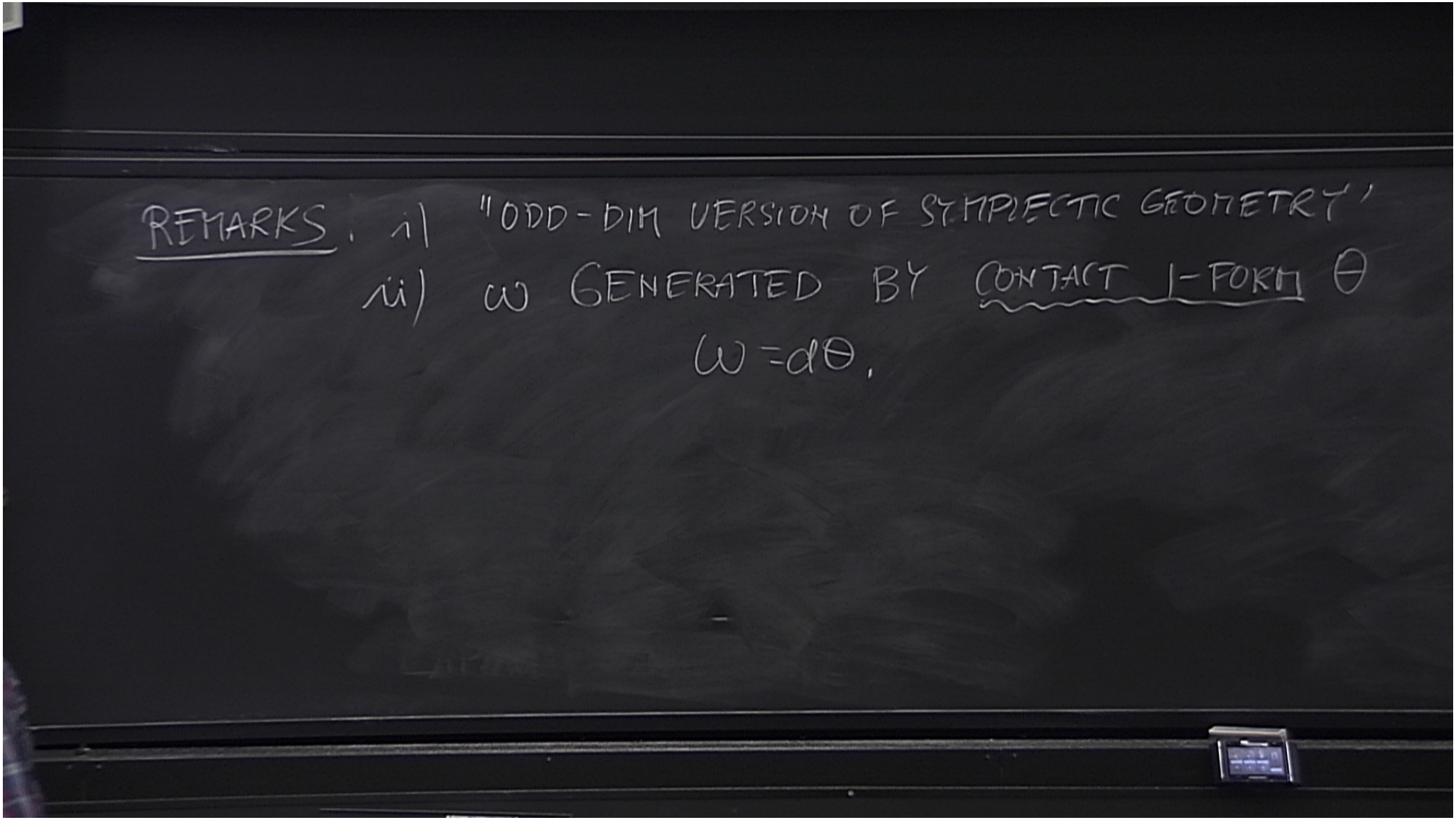
M^{2n+1}

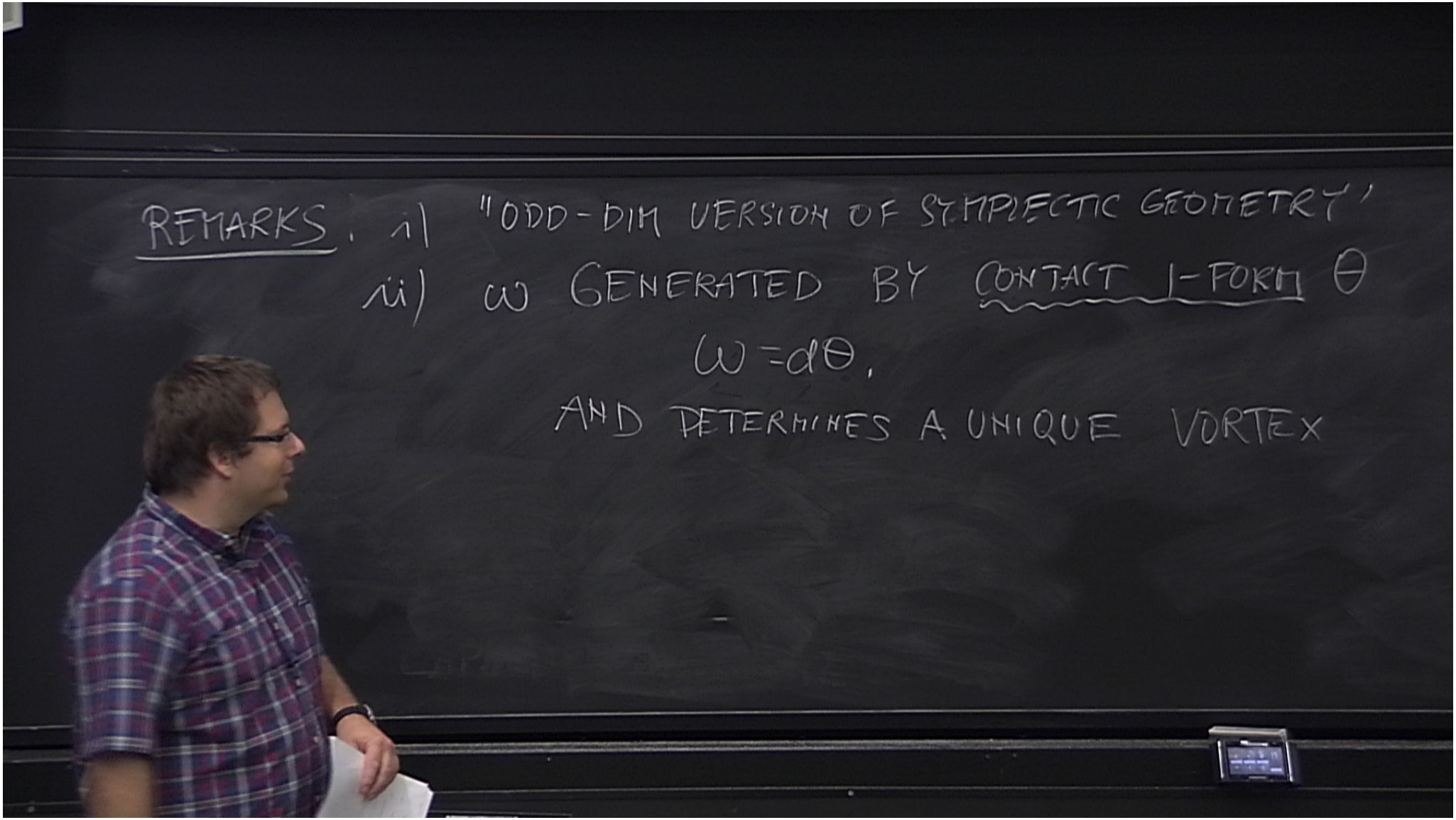
DEF: WHEN ω IS CLOSED & NON-SINGULAR

(M^{2n+1}, ω) ... CONTACT MANIFOLD

$\partial/\partial q_1$


REMARKS: i) "ODD-DIM VERSION OF SYMPLECTIC GEOMETRY"
ii)


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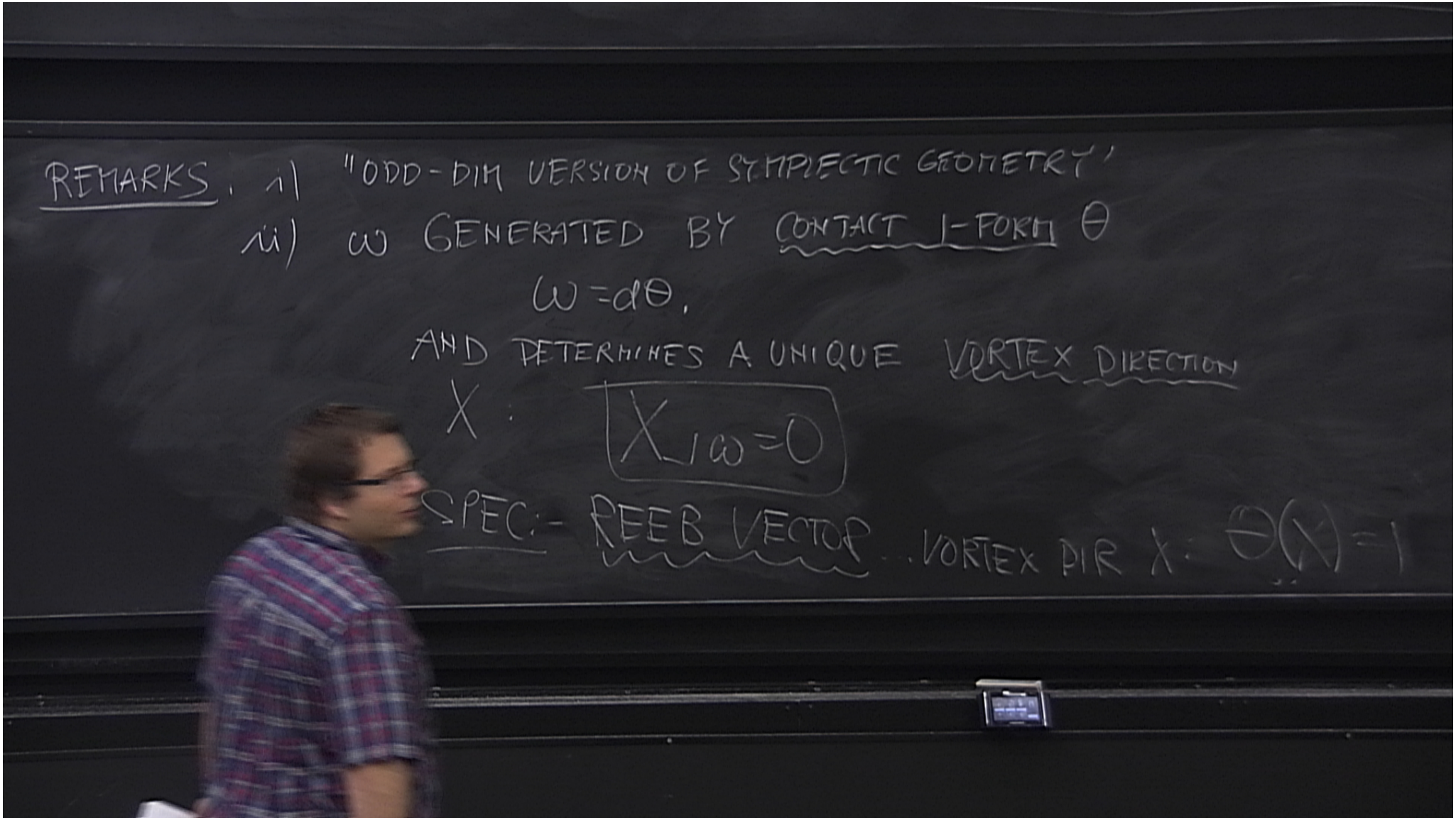
ii) ω GENERATED BY CONTACT 1-FORM θ

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X:

$$\boxed{X \lrcorner \omega = 0}$$


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ii) ω GENERATED BY CONTACT 1-FORM θ
ω = dθ.
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X : $\boxed{X \lrcorner \omega = 0}$
SPEC:- REEB VECTOR . VORTEX PAIR X : θ(X) = 1