Title: Front End - Classical Mechanics 5

Date: Aug 17, 2016 10:30 AM

URL: http://pirsa.org/16080067

Abstract:

e) CONSTRAINTS

- LET NOT ALL q^i, p_i BE INDEPENDENT

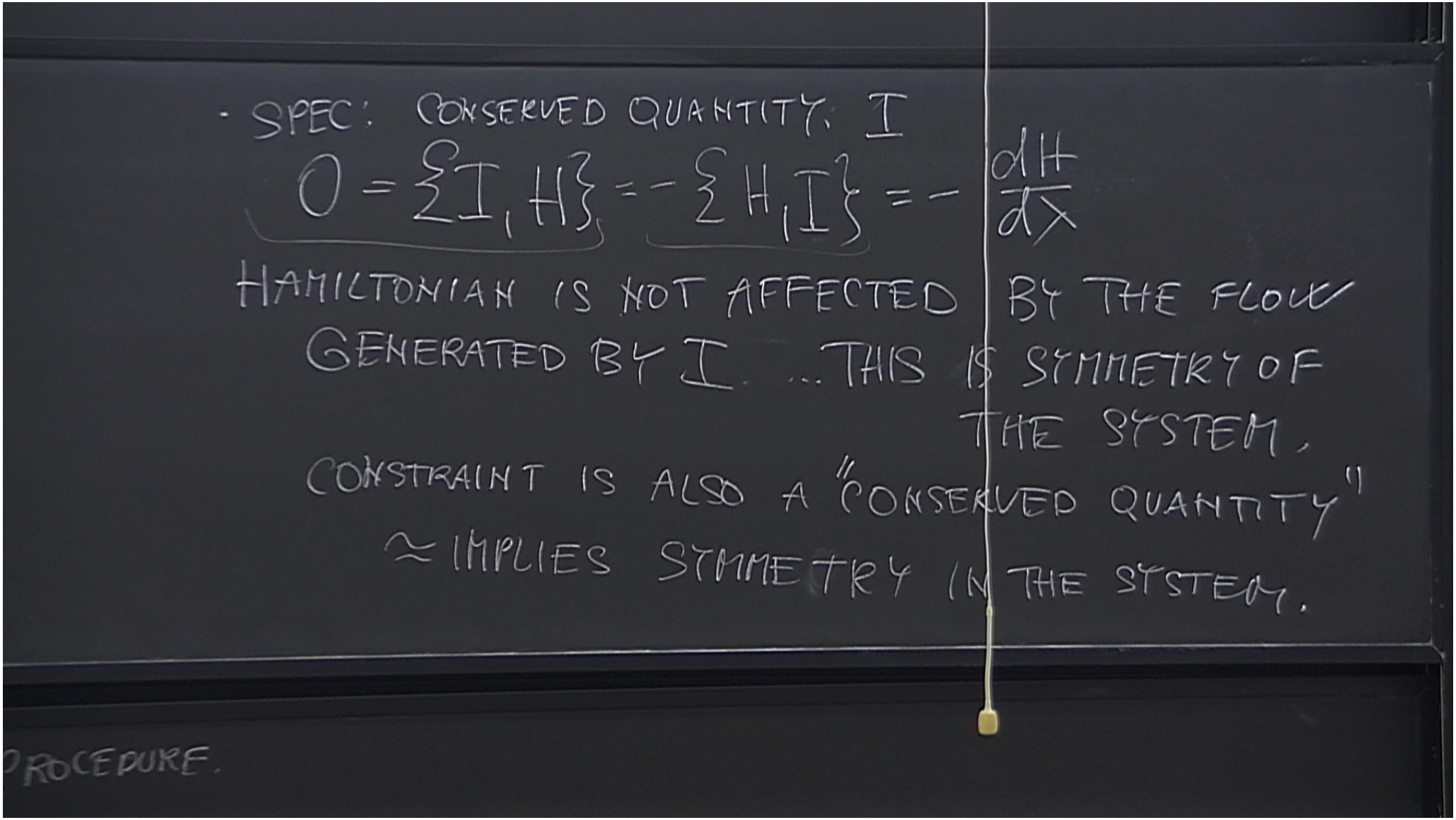
$$\boxed{\phi(q^i, p_i) = 0}$$ CONSTRAINT $\forall t$

- REMARK: ANY $f \equiv H$ "HAMILTONIAN"

$$\frac{dg}{d\lambda} = \{g, f\}$$

λ ↑ FICTITIOUS TIME

- TO RECOGNIZE A CONSTRAINT.- USE DIRAC'S PROCEDURE


· SPEC: CONSERVED QUANTITY: I
$0 = \{I, H\} = -\{H, I\} = -\frac{dH}{dx}$
HAMILTONIAN IS NOT AFFECTED BY THE FLOW GENERATED BY I ... THIS IS SYMMETRY OF THE SYSTEM.
CONSTRAINT IS ALSO A "CONSERVED QUANTITY"
≃ IMPLIES SYMMETRY IN THE SYSTEM.
ROCEDURE.

- TO RECOGNIZE A CONSTRAINT.- USE DIRAC'S PROCEDURE.
- TO DEAL WITH CONSTRAINTS USE THE METHOD OF LAGRANGE MULTIPLIERS.

$$H_T = H + \sum_{i=1}^{m} \lambda_i \phi_i$$

H_T: TOTAL HAMILT. H: HAM

LAGRANGE MULTIPLIERS

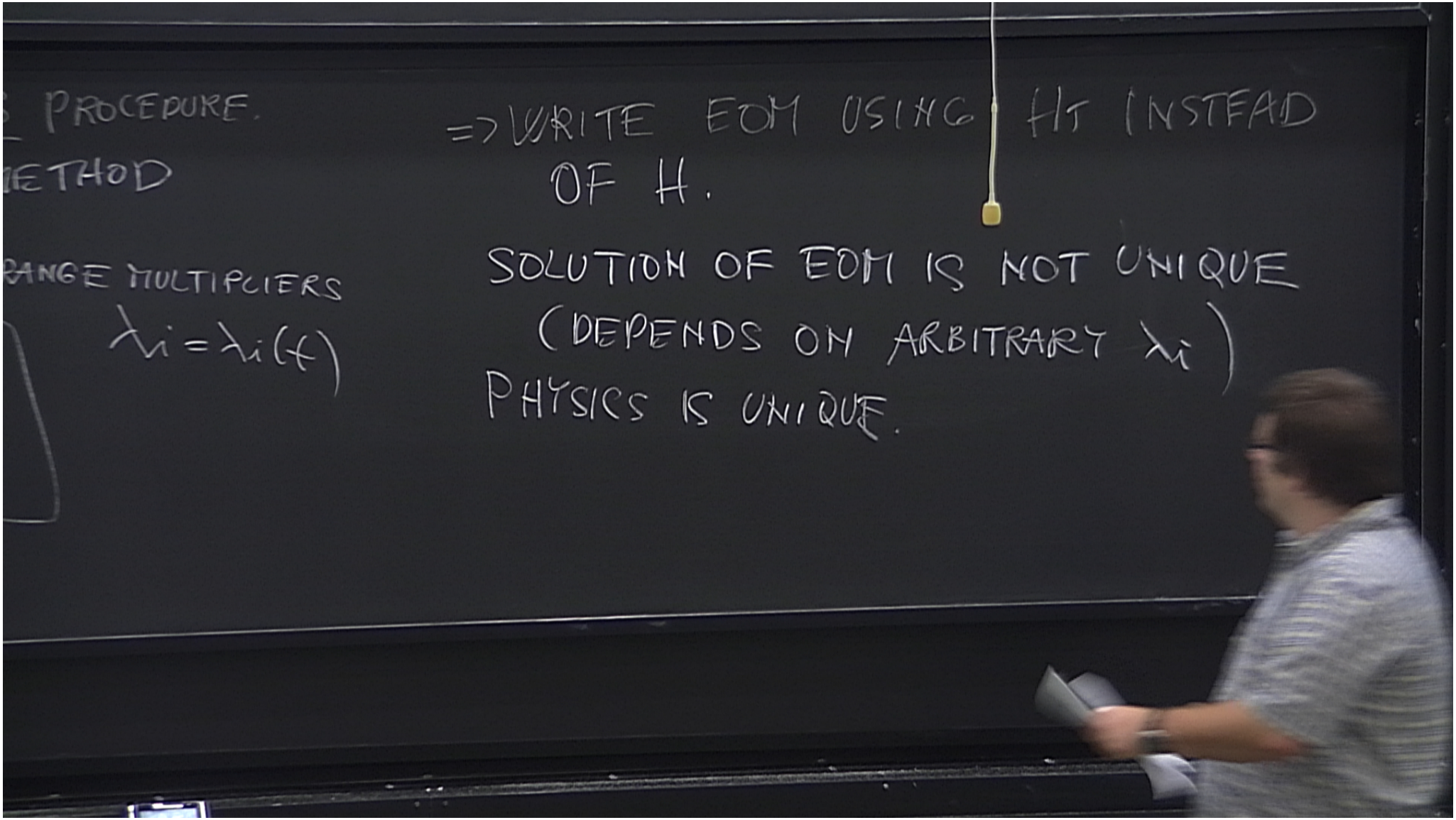
$\lambda_i = \lambda_i(t)$

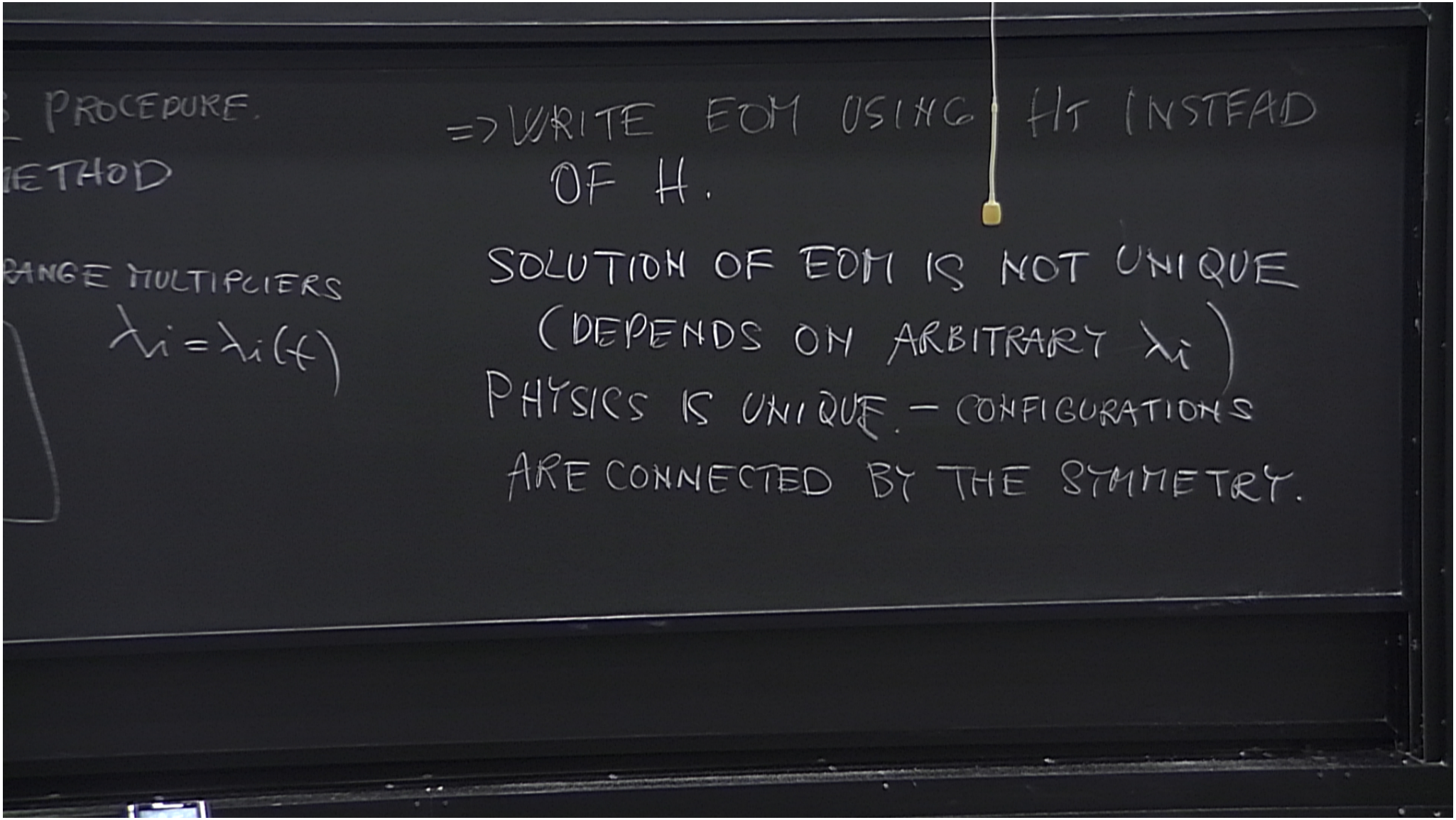
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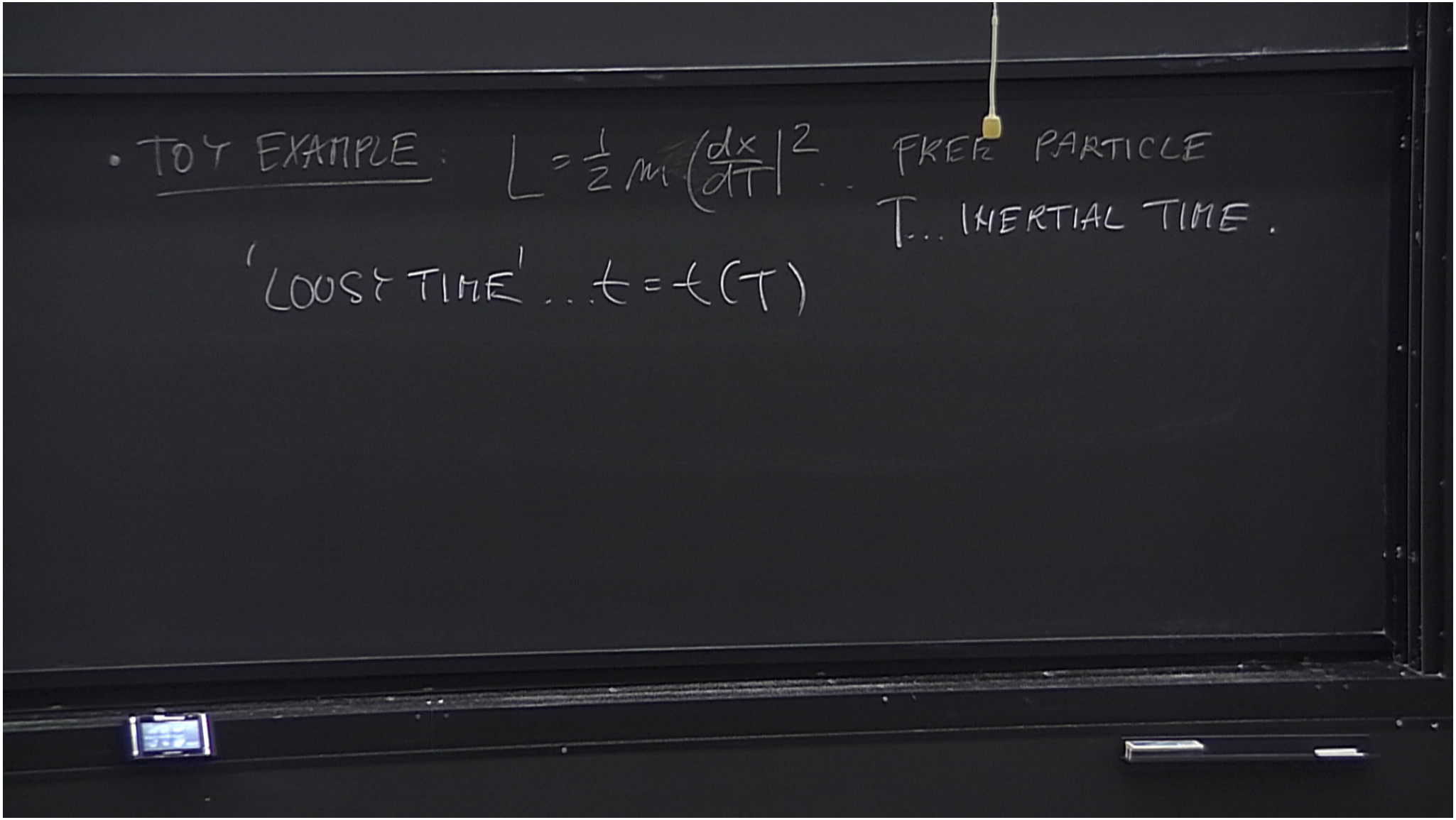
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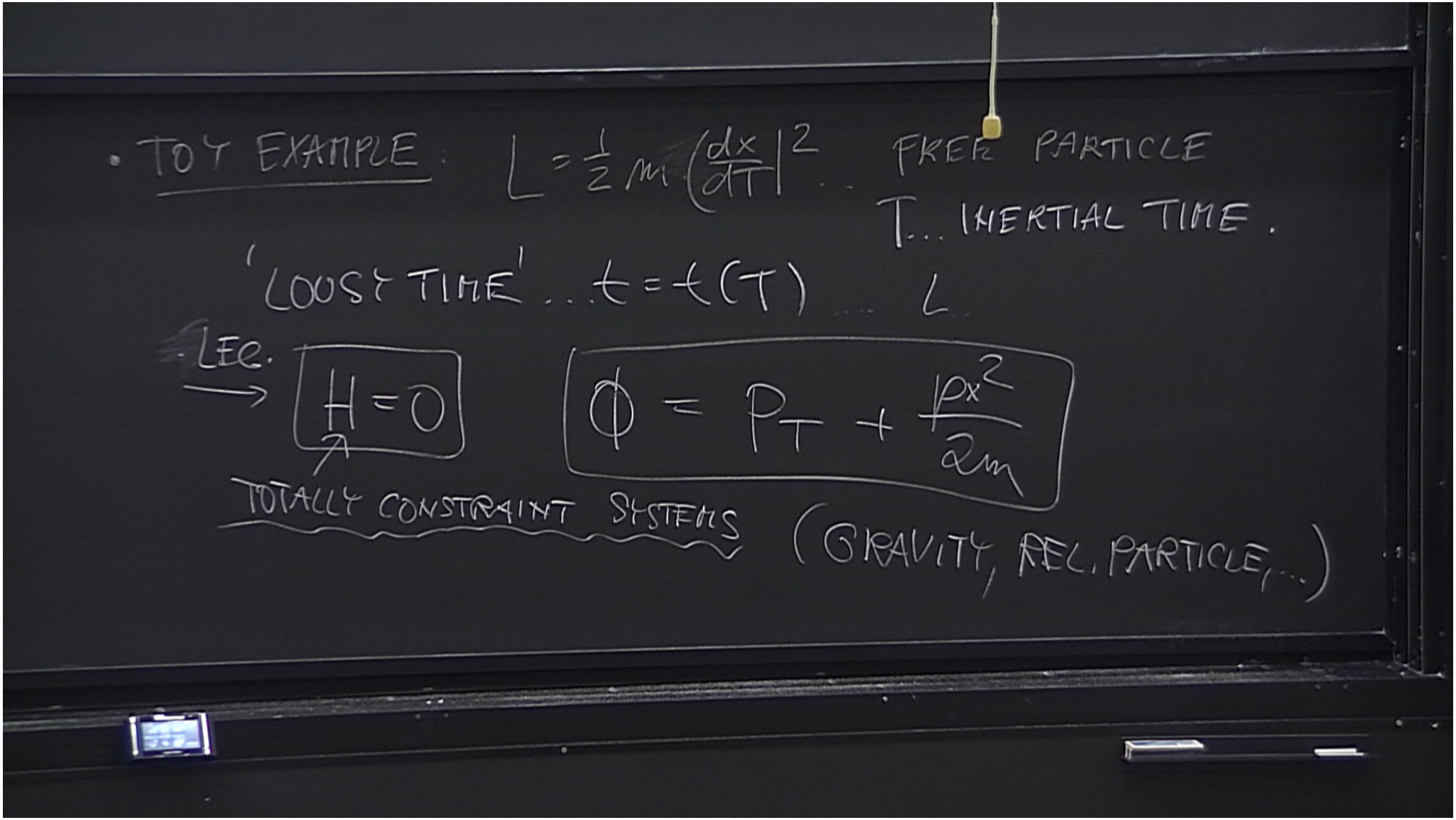
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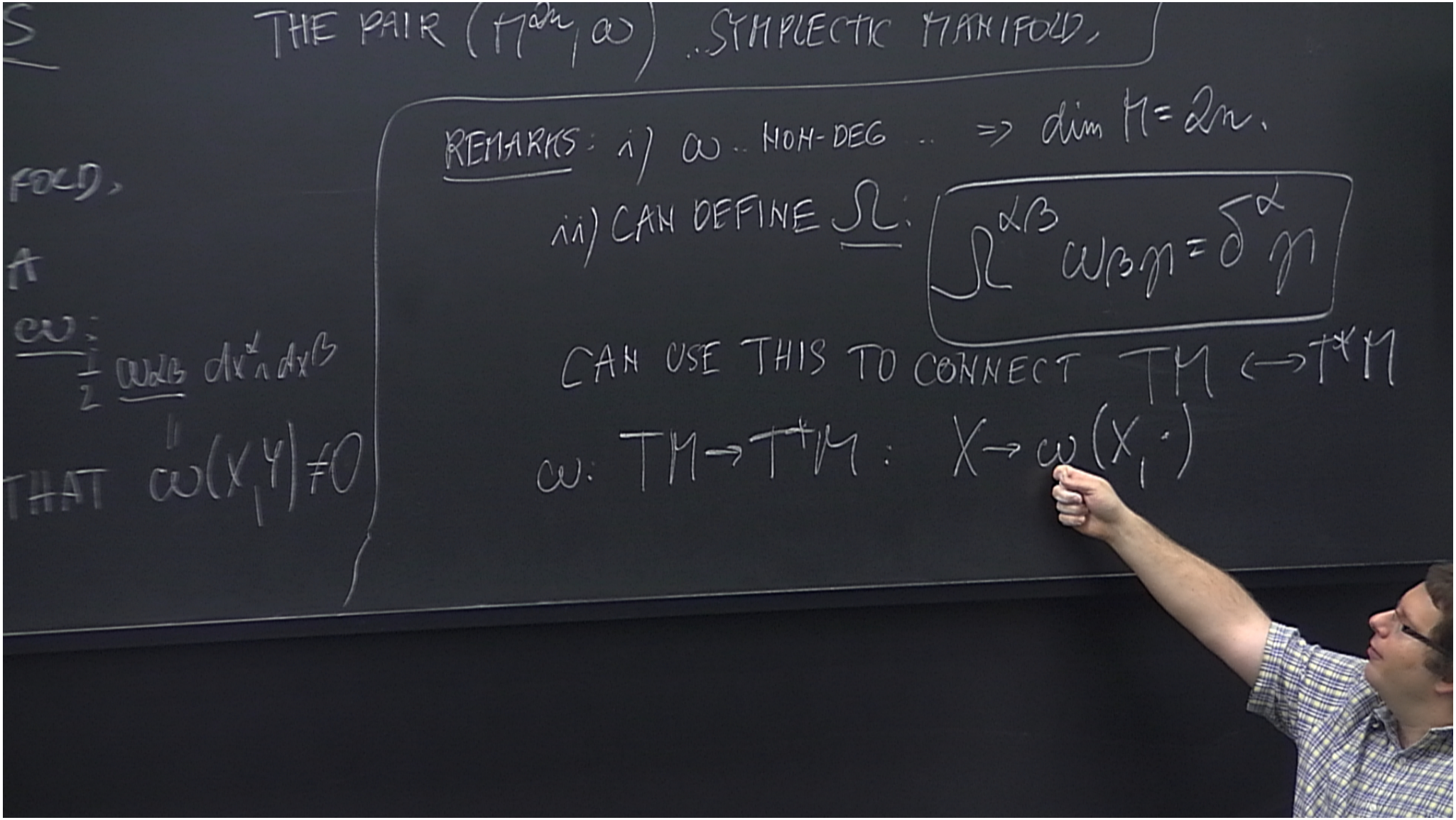
LAGRANGE MULTIPLIERS $\lambda_i = \lambda_i(t)$


PROCEDURE.
METHOD
RANGE MULTIPLIERS
$\lambda_i = \lambda_i(t)$
=> WRITE EOM USING H_T INSTEAD OF H.
SOLUTION OF EOM IS NOT UNIQUE
(DEPENDS ON ARBITRARY λ_i)
PHYSICS IS UNIQUE.


PROCEDURE.
METHOD
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$\lambda_i = \lambda_i(t)$
=> WRITE EOM USING H_T INSTEAD OF H.
SOLUTION OF EOM IS NOT UNIQUE
(DEPENDS ON ARBITRARY λ_i)
PHYSICS IS UNIQUE. – CONFIGURATIONS ARE CONNECTED BY THE SYMMETRY.


• TOY EXAMPLE: $L = \frac{1}{2} m \left(\frac{dx}{dT}\right)^2$ .. FREE PARTICLE
T .. INERTIAL TIME.
'LOCAL TIME' .. $t = t(T)$


• TOY EXAMPLE: $L = \frac{1}{2} m \left(\frac{dx}{dT}\right)^2$ .. FREE PARTICLE
T .. INERTIAL TIME.
'LOOSE TIME' .. $t = t(T)$ .. L
LEG.
$H = 0$
$\phi = p_T + \frac{p_x^2}{2m}$
TOTALLY CONSTRAINT SYSTEMS (GRAVITY, REL. PARTICLE, ...)


THE PAIR (M^{2n}, ω) .. SYMPLECTIC MANIFOLD.
REMARKS: i) ω .. NON-DEG .. $\Rightarrow \dim M = 2n$.
ii) CAN DEFINE Ω: $\Omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^{\alpha}_{\gamma}$
CAN USE THIS TO CONNECT $TM \leftrightarrow T^*M$
$\omega: TM \to T^*M : X \to \omega(X, \cdot)$
FOLD.
A
ω:
$\frac{1}{2}\omega_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta}$
THAT $\omega(X, Y) \neq 0$

REMARKS: i) ω .. NON-DEG .. $\Rightarrow$ dim M even

ii) CAN DEFINE Ω: $\boxed{\Omega^{\alpha\beta}\omega_{\beta\gamma} = \delta^{\alpha}_{\gamma}}$

CAN USE THIS TO CONNECT $TM \longleftrightarrow T^*M$

$\omega: TM \to T^*M : X \to \omega(X, \cdot) \in T^*M$

$X^{\alpha} \to X_{\beta} = X^{\alpha}\omega_{\alpha\beta}$

$\omega_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta}$

$\omega(X, Y) \neq 0$

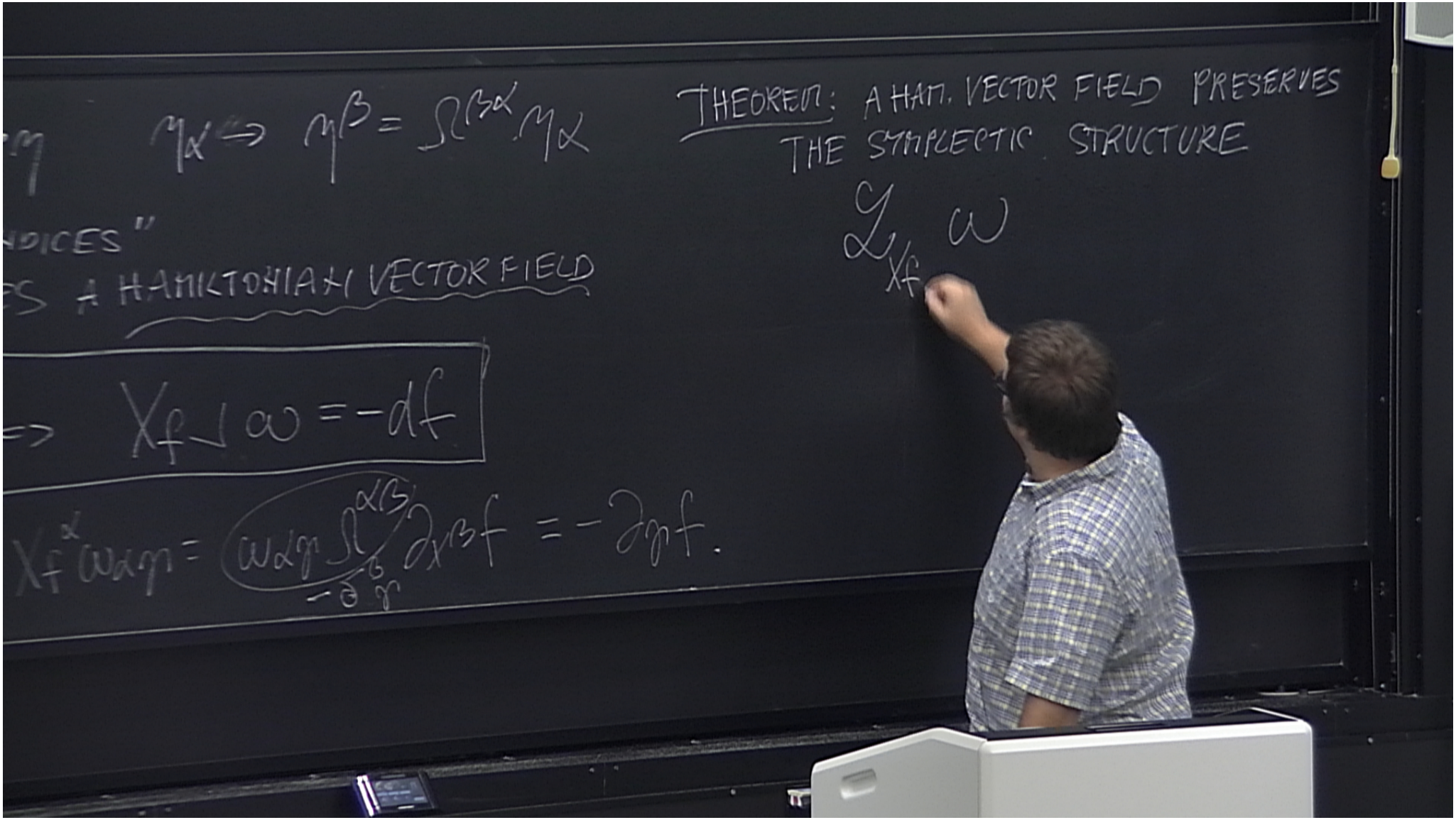
$\Omega : T^*M \to TM \qquad \eta \to \Omega \cdot \eta \qquad \eta_\alpha \to \eta^\beta = \Omega^{\beta\alpha}\eta_\alpha$

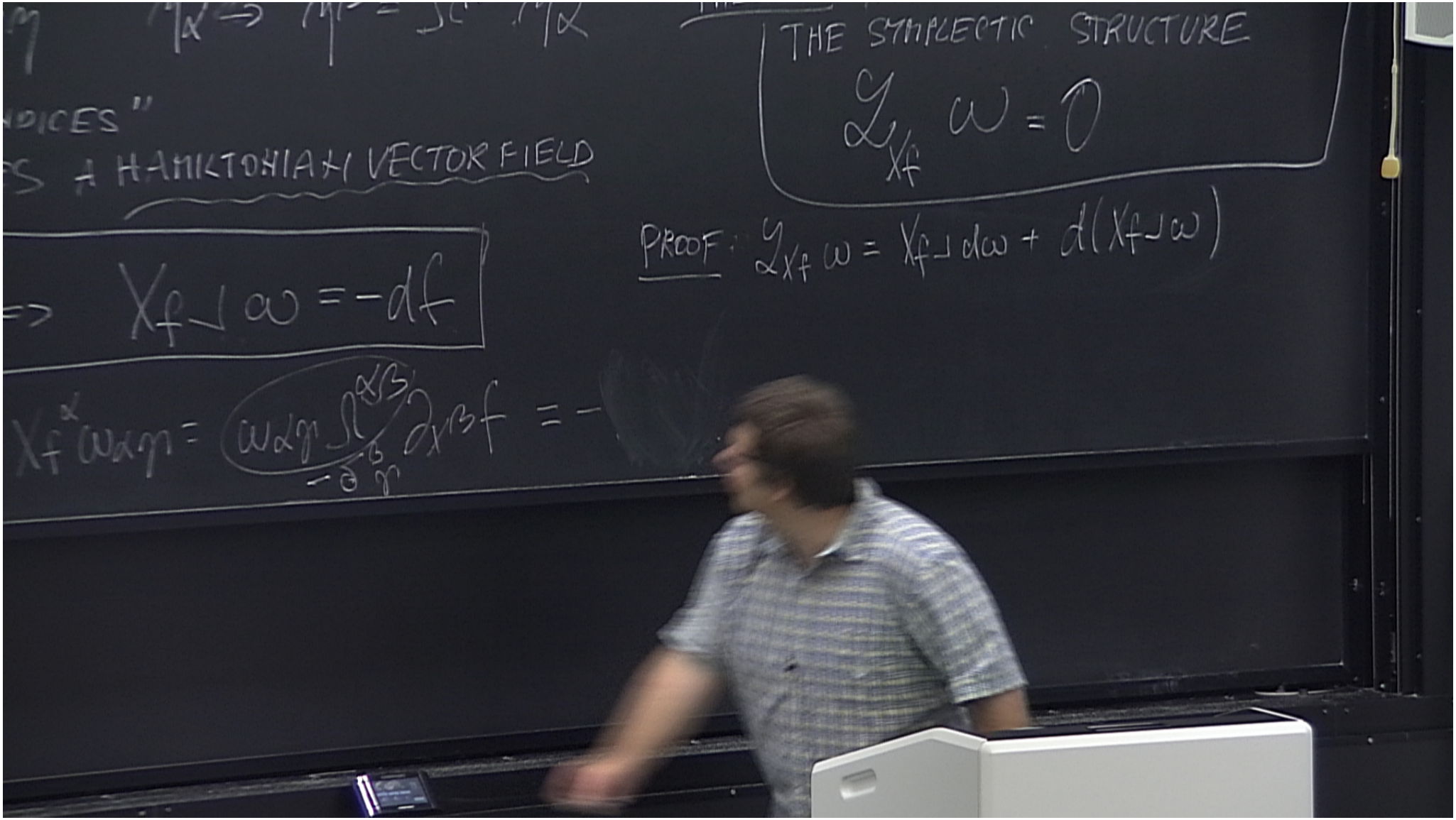
"RAISING & LOWERING INDICES"

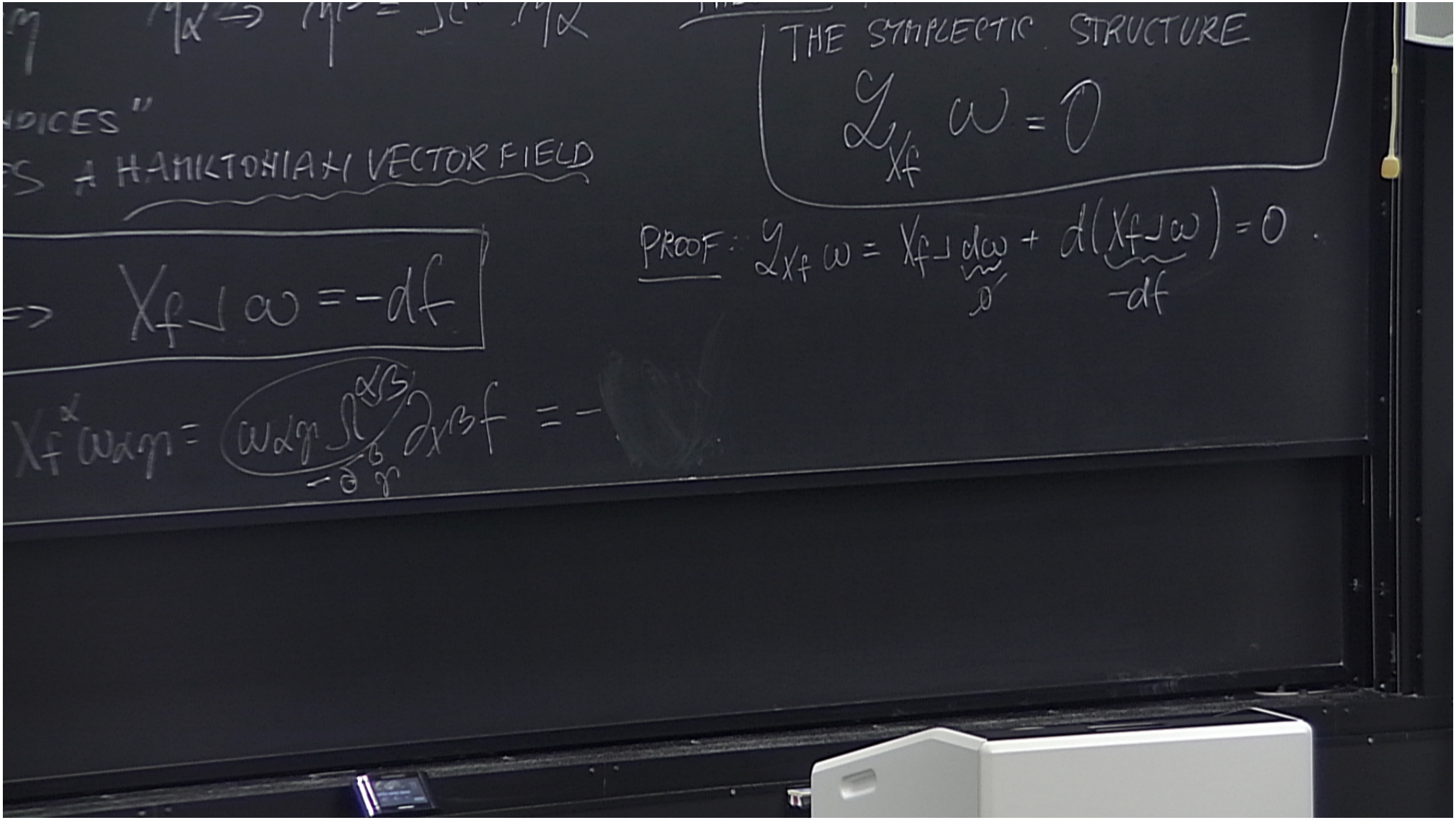
iii) GIVEN A FUNCTION f, ω DEFINES A HAMILTONIAN VECTOR FIELD

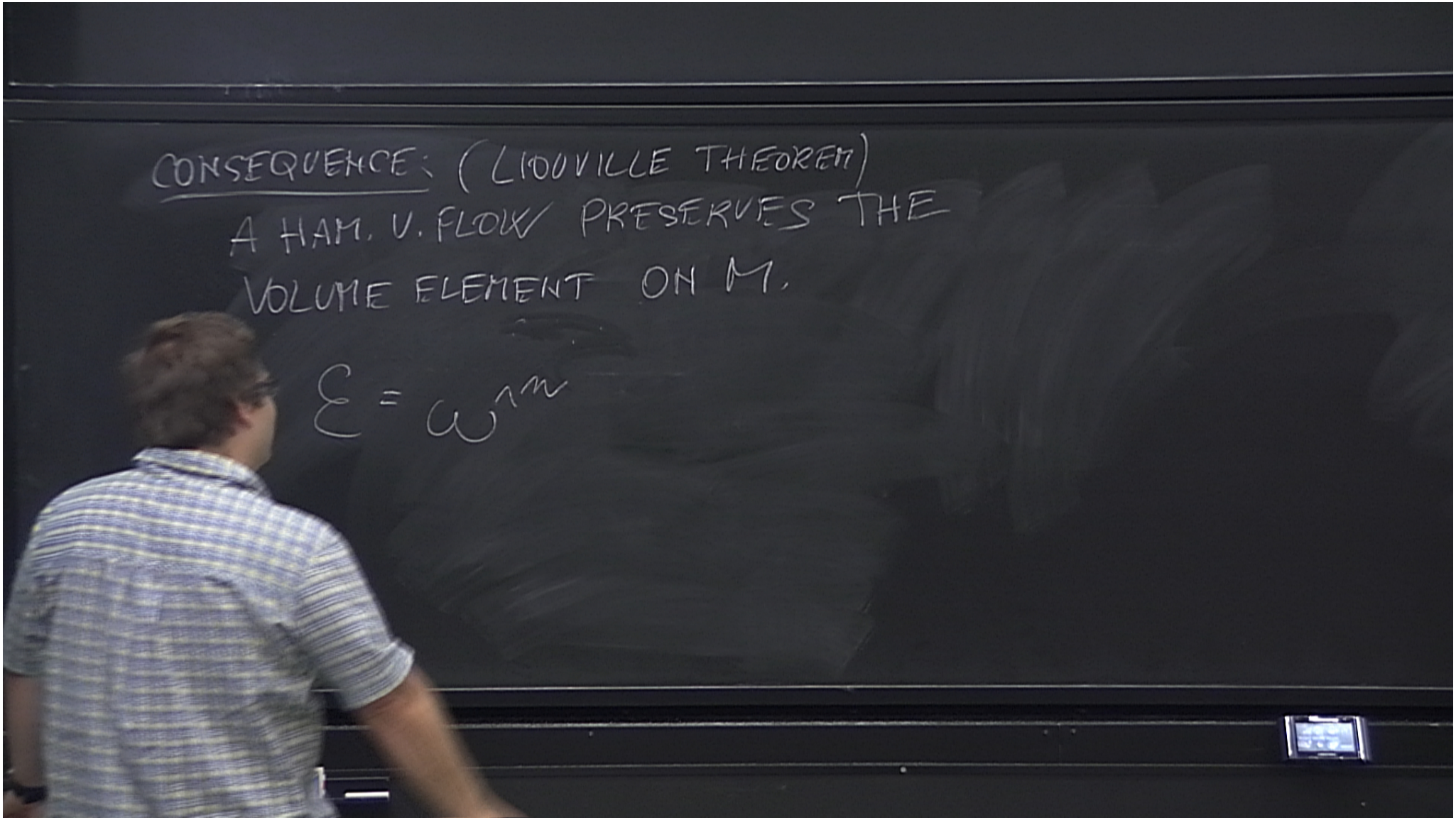
$$X_f = \Omega \cdot df \iff X_f \lrcorner \omega = -df$$

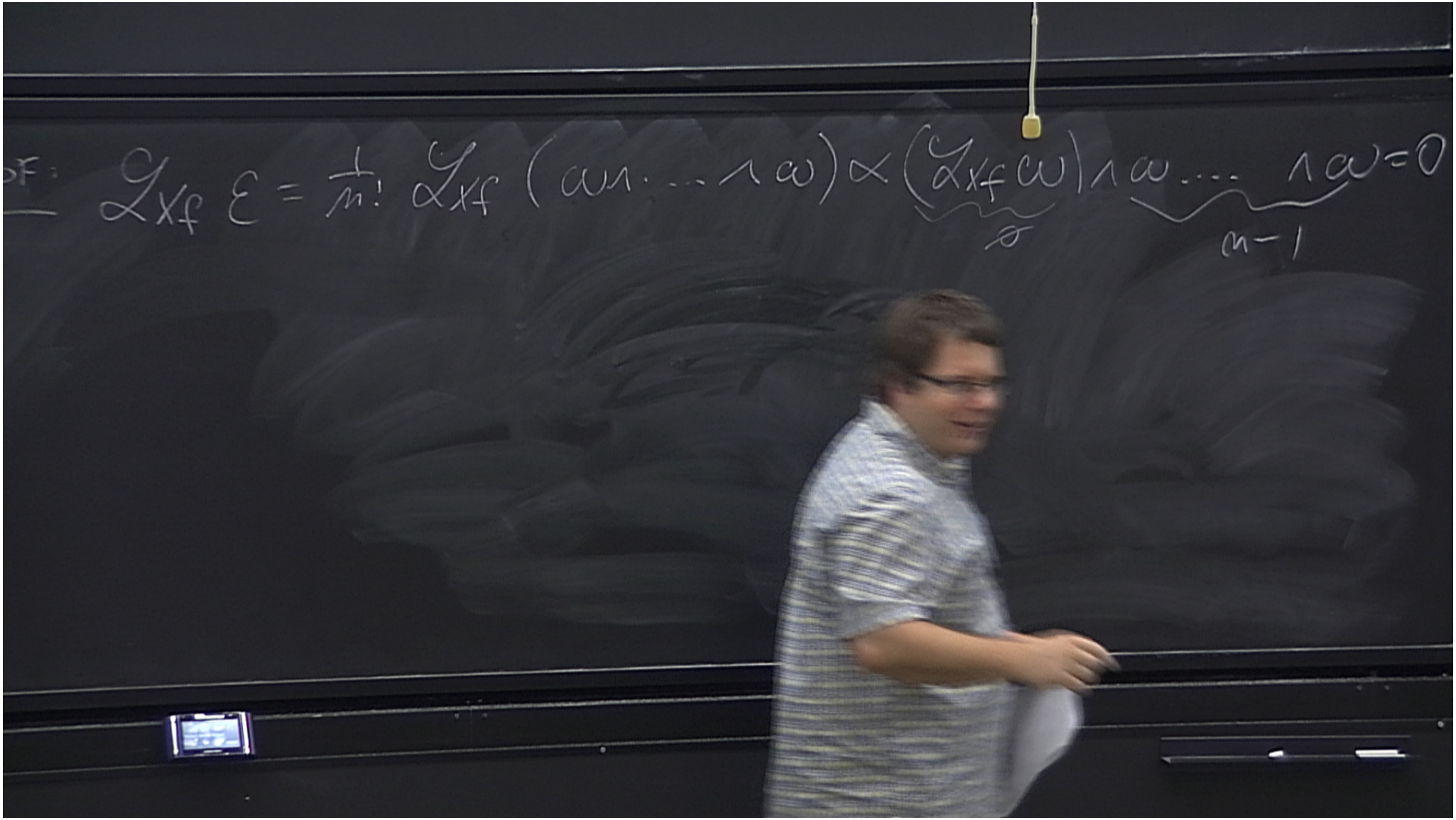
$$X_f^\alpha = \Omega^{\alpha\beta}\partial_{x^\beta} f \qquad X_f^\alpha \omega_{\alpha\gamma} = \omega_{\alpha\gamma}\Omega^{\alpha\beta}\partial_{x^\beta} f$$

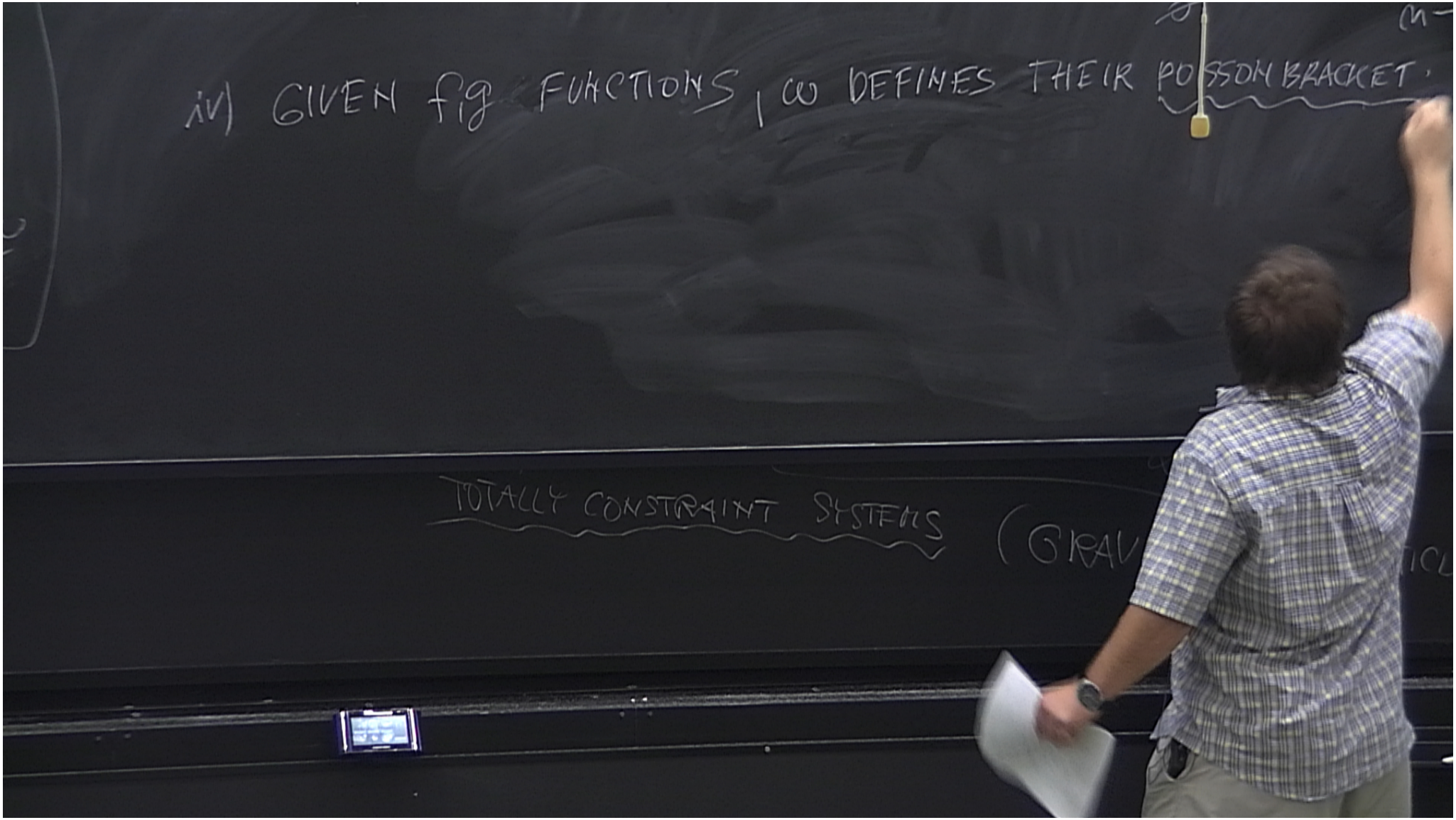

$\eta_\alpha \Rightarrow \eta^\beta = \Omega^{\beta\alpha}\eta_\alpha$
INDICES"
ES A HAMILTONIAN VECTOR FIELD
THEOREM: A HAM. VECTOR FIELD PRESERVES THE SYMPLECTIC STRUCTURE
$\mathcal{L}_{X_f}\omega$
$\Rightarrow X_f \lrcorner \omega = -df$
$X_f^\alpha \omega_{\alpha\eta} = \omega_{\alpha\eta}\Omega^{\alpha\beta}\partial_\beta f = -\partial_\eta f.$
$-\partial_\eta f$


THE SYMPLECTIC STRUCTURE
$\mathcal{L}_{X_f}\omega = 0$
PROOF: $\mathcal{L}_{X_f}\omega = X_f \lrcorner d\omega + d(X_f \lrcorner \omega)$
HAMILTONIAN VECTOR FIELD
$X_f \lrcorner \omega = -df$


THE SYMPLECTIC STRUCTURE
$\mathcal{L}_{X_f}\omega = 0$
PROOF: $\mathcal{L}_{X_f}\omega = X_f \lrcorner d\omega + d(X_f \lrcorner \omega) = 0$
$X_f \lrcorner \omega = -df$
HAMILTONIAN VECTOR FIELD


CONSEQUENCE: (LIOUVILLE THEOREM)
A HAM. V. FLOW PRESERVES THE
VOLUME ELEMENT ON M.
$\mathcal{E} = \omega^{\wedge m}$


PF: $\mathcal{L}_{X_f}\mathcal{E} = \frac{1}{n!}\mathcal{L}_{X_f}(\omega_1 \wedge \dots \wedge \omega) \propto (\underbrace{\mathcal{L}_{X_f}\omega}_{\sigma}) \wedge \underbrace{\omega \wedge \dots \wedge \omega}_{n-1} = 0$


iv) GIVEN fig FUNCTIONS, ω DEFINES THEIR POISSON BRACKET.
TOTALLY CONSTRAINT SYSTEMS

iv) GIVEN f,g FUNCTIONS, ω DEFINES THEIR POISSON BRACKET:

$$\{f,g\} = df \cdot \Omega \cdot dg = -\omega(X_f, X_g)$$

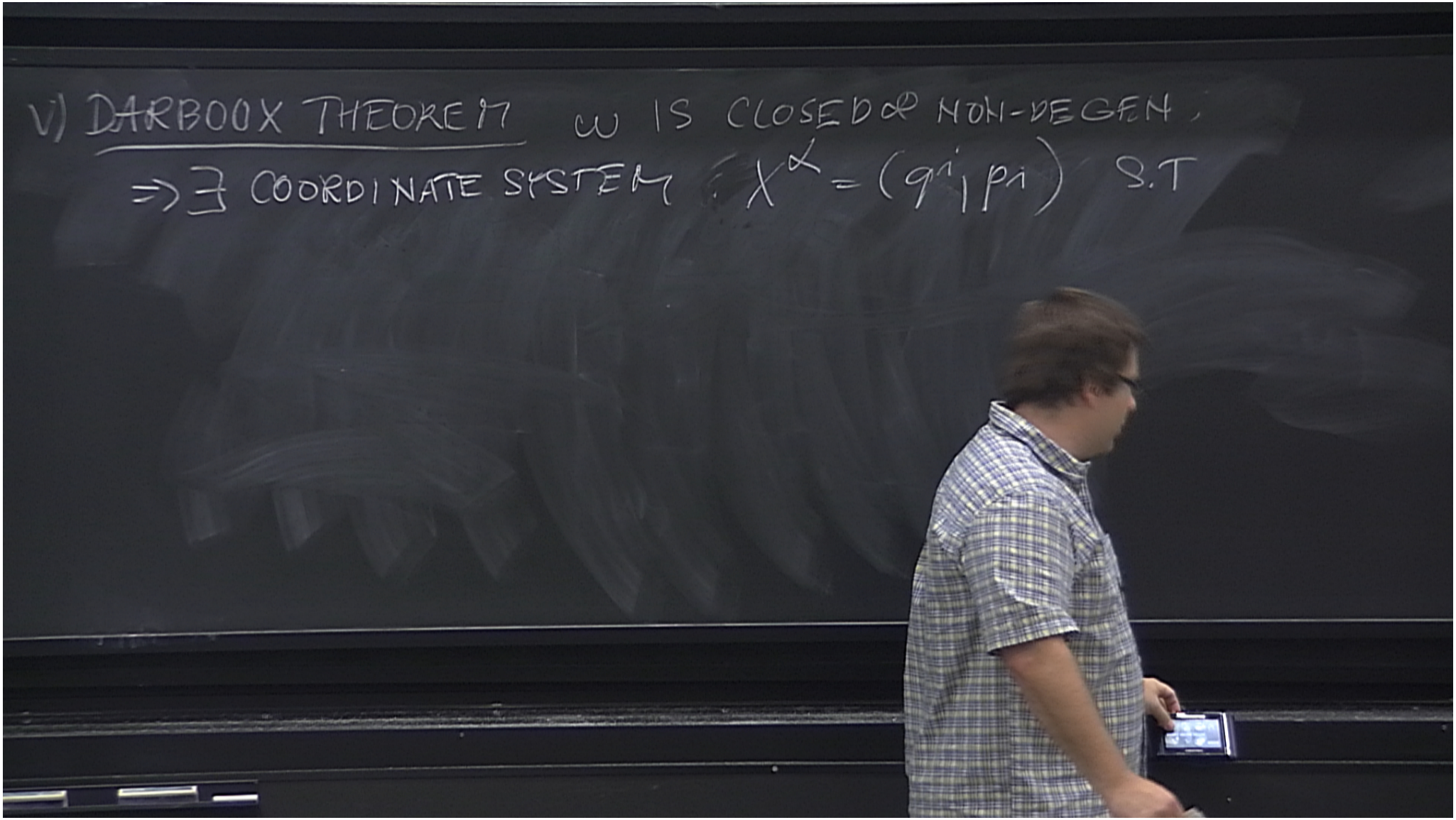
TOTALLY CONSTRAINT SYSTEMS (GRAVITY, REL. PARTICLE, ...)

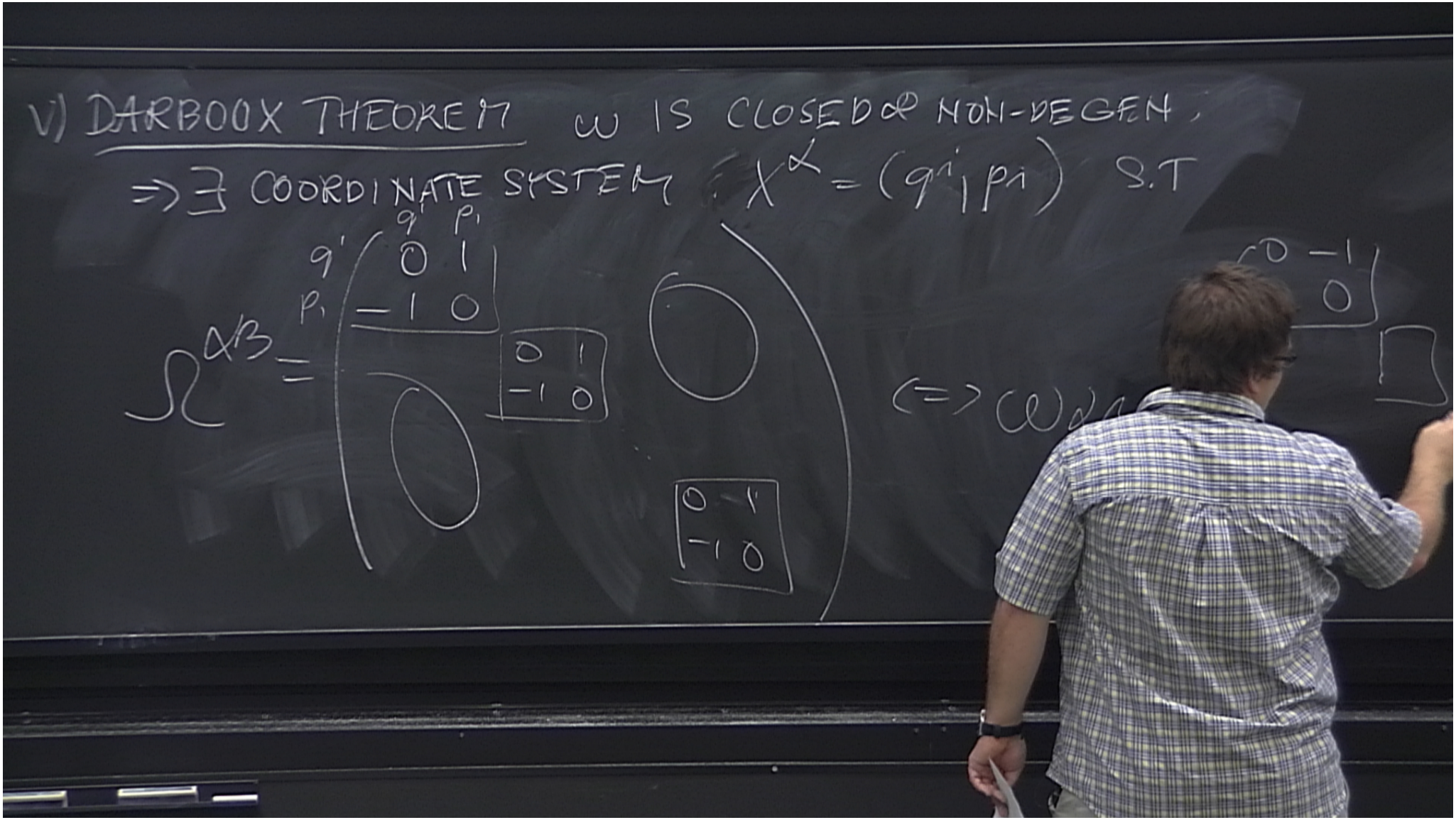
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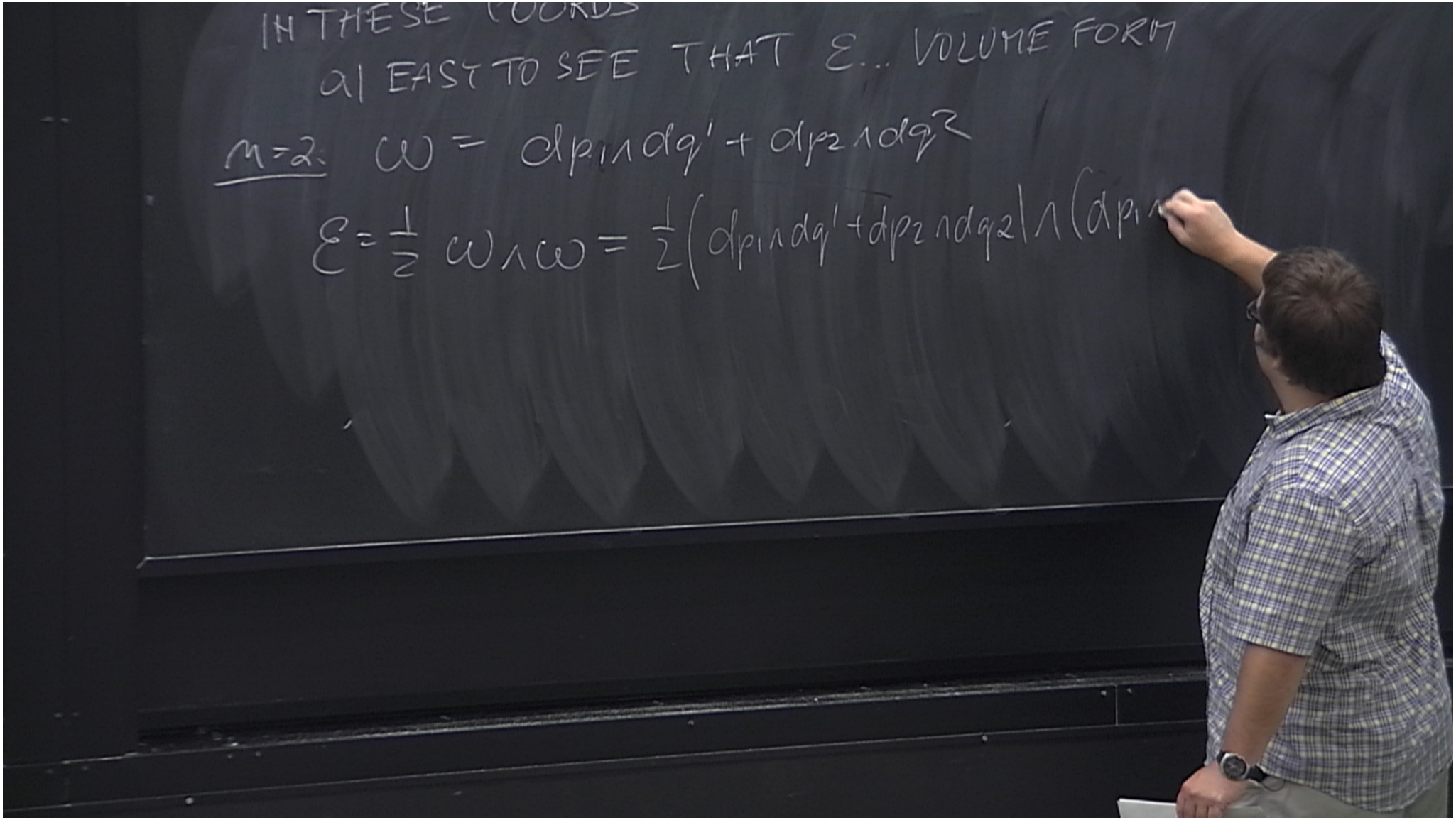
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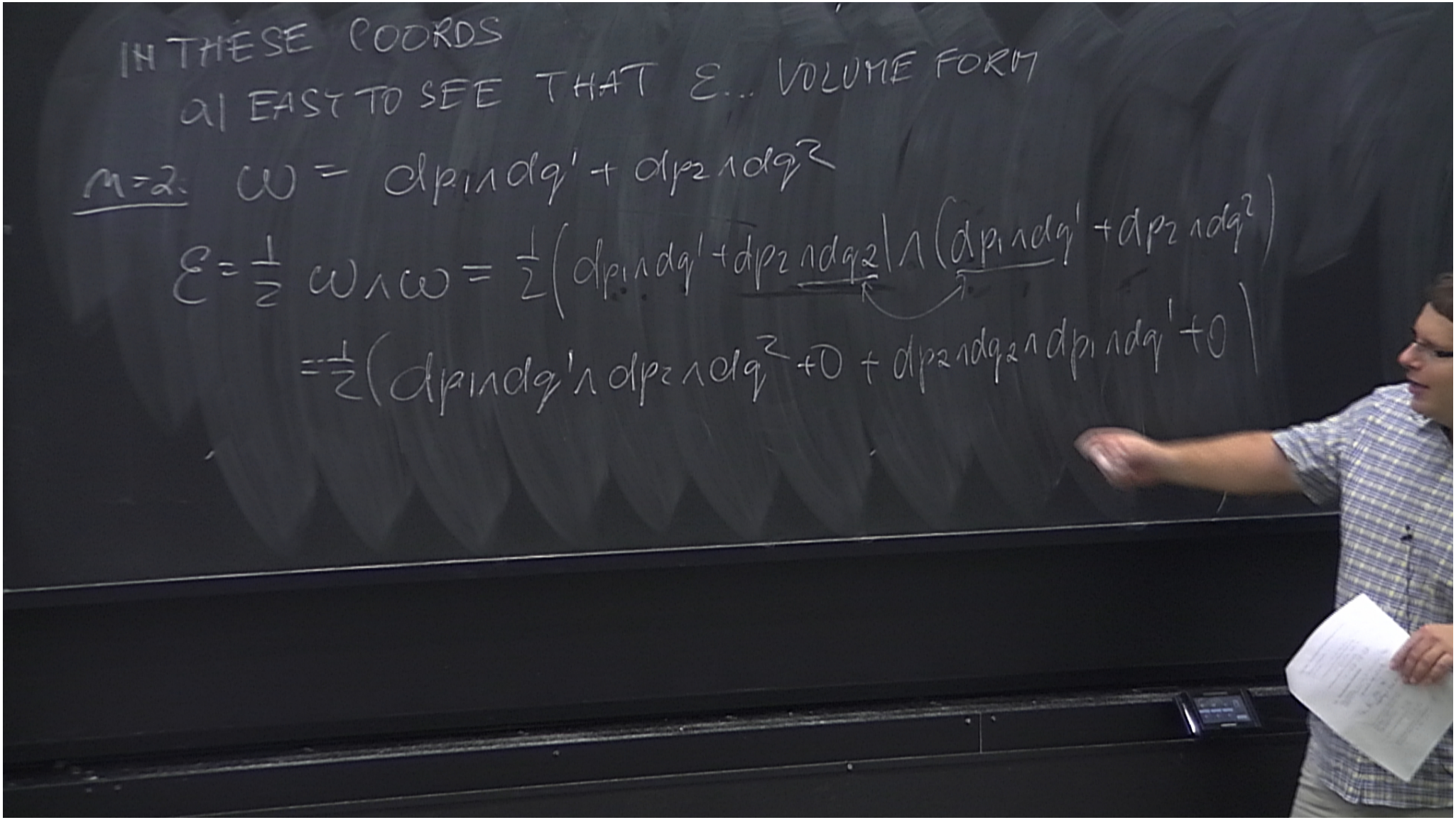
- LINEAR, ANTISYMM., LEIBNITZ, JACOBI IDENTITY

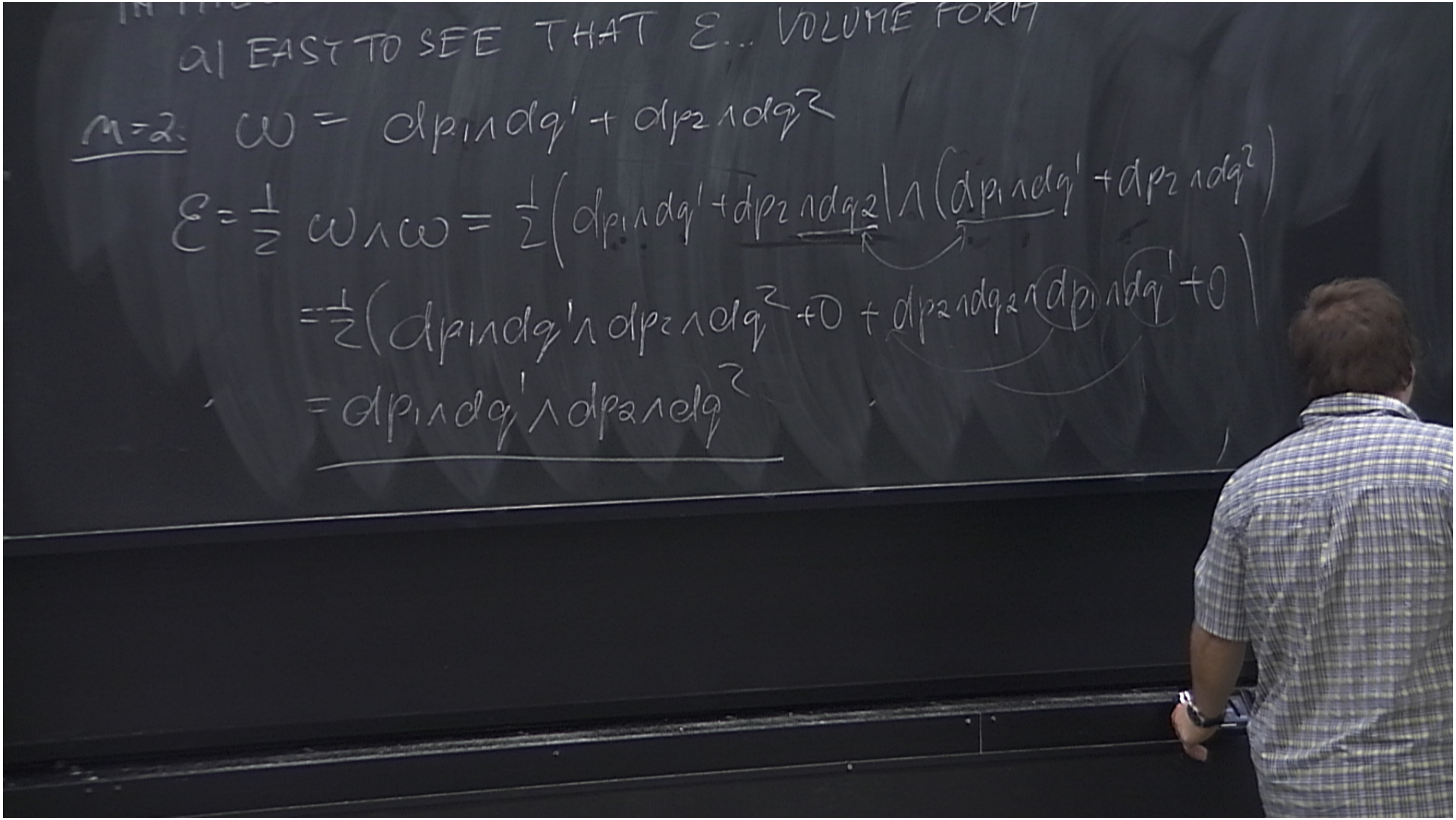
TOTALLY CONSTRAINT SYSTEMS (GRAVITY, REL. PARTICLE, ...)


V) DARBOUX THEOREM ω IS CLOSED & NON-DEGEN.
⇒ ∃ COORDINATE SYSTEM : $X^\alpha = (q^i, p_i)$ S.T


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⇒ ∃ COORDINATE SYSTEM $X^\alpha = (q^i, p_i)$ S.T
$\Omega^{\alpha\beta}$ =
q^i p_i
q^i 0 1
p_i −1 0
0 1
−1 0
0 −1
1 0
0 1
−1 0
⇔ ω


IN THESE COORDS
a) EASY TO SEE THAT $\mathcal{E}$ .. VOLUME FORM
$n=2$: $\omega = dp_1 \wedge dq^1 + dp_2 \wedge dq^2$
$\mathcal{E} = \frac{1}{2}\omega \wedge \omega = \frac{1}{2}(dp_1 \wedge dq^1 + dp_2 \wedge dq_2) \wedge (dp_1$


IN THESE COORDS
IT IS EASY TO SEE THAT $\mathcal{E}$ .. VOLUME FORM
$n=2$: $\omega = dp_1 \wedge dq^1 + dp_2 \wedge dq^2$
$\mathcal{E} = \frac{1}{2}\omega \wedge \omega = \frac{1}{2}(dp_1 \wedge dq^1 + dp_2 \wedge dq^2) \wedge (dp_1 \wedge dq^1 + dp_2 \wedge dq^2)$
$= \frac{1}{2}(dp_1 \wedge dq^1 \wedge dp_2 \wedge dq^2 + 0 + dp_2 \wedge dq^2 \wedge dp_1 \wedge dq^1 + 0)$


a) EASY TO SEE THAT Σ... VOLUME FORM
n=2: ω = dp1∧dq1 + dp2∧dq2
Σ = 1/2 ω∧ω = 1/2 (dp1∧dq1 + dp2∧dq2 ∧ (dp1∧dq1 + dp2∧dq2)
= 1/2 (dp1∧dq1∧dp2∧dq2 + 0 + dp2∧dq2∧dp1∧dq1 + 0)
= dp1∧dq1∧dp2∧dq2

IN THESE COORDS

a) EASY TO SEE THAT $\mathcal{E}$ .. VOLUME FORM

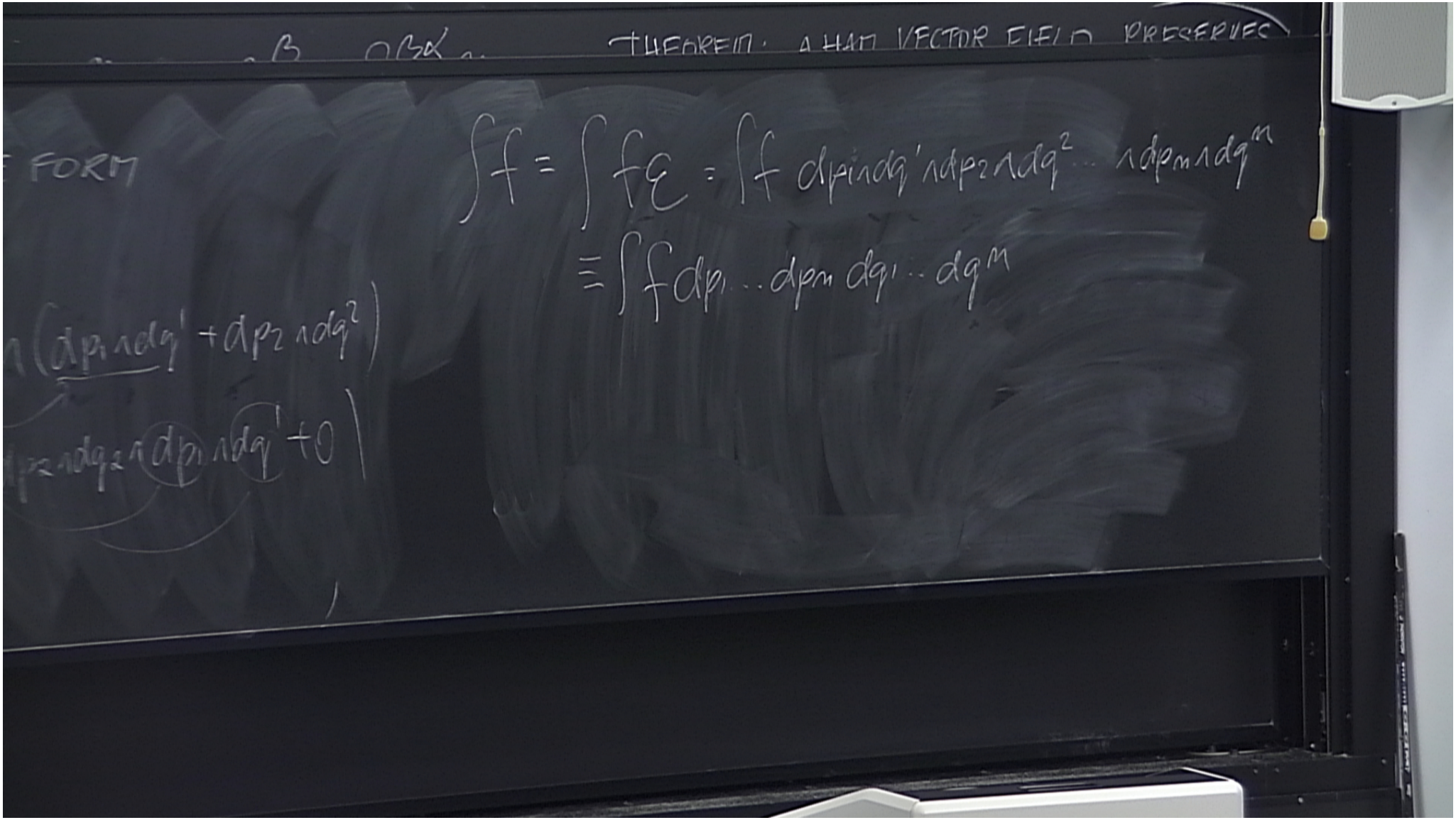
$\int f =$

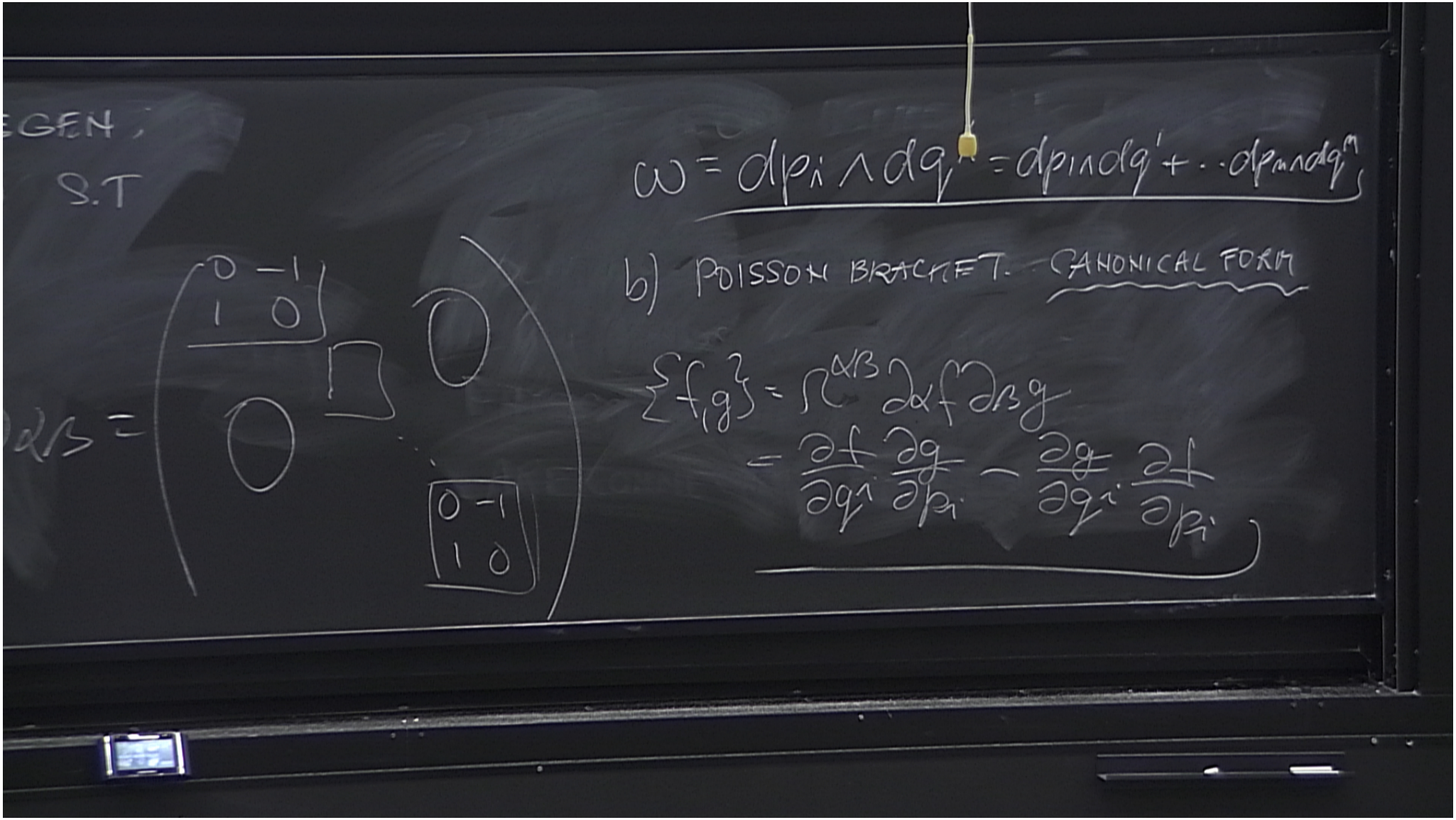
$n=2$: $\omega = dp_1 \wedge dq^1 + dp_2 \wedge dq^2$

$$\mathcal{E} = \frac{1}{2}\omega \wedge \omega = \frac{1}{2}\left(dp_1 \wedge dq^1 + dp_2 \wedge dq^2\right) \wedge \left(dp_1 \wedge dq^1 + dp_2 \wedge dq^2\right)$$

$$= \frac{1}{2}\left(dp_1 \wedge dq^1 \wedge dp_2 \wedge dq^2 + 0 + dp_2 \wedge dq^2 \wedge dp_1 \wedge dq^1 + 0\right)$$

$$= dp_1 \wedge dq^1 \wedge dp_2 \wedge dq^2$$


THEOREM: A HAM VECTOR FIELD PRESERVES
FORM
$\int f = \int f\varepsilon = \int f\, dp_1 \wedge dq^1 \wedge dp_2 \wedge dq^2 \ldots \wedge dp_m \wedge dq^m$
$\equiv \int f\, dp_1 \ldots dp_m\, dq^1 \ldots dq^m$
$(dp_1 \wedge dq^1 + dp_2 \wedge dq^2)$
$dp_2 \wedge dq_2 \wedge dp_1 \wedge dq^1 + 0$

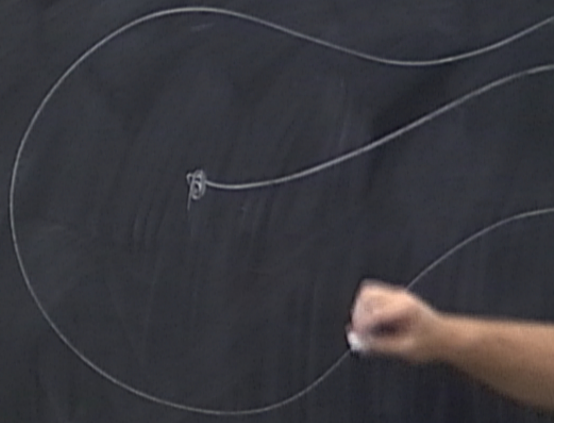

EGEN;
S.T
$\omega = dp_i \wedge dq^i = dp_1 \wedge dq^1 + \cdots dp_m \wedge dq^m$
b) POISSON BRACKET. CANONICAL FORM
$\{f,g\} = \Omega^{\alpha\beta}\partial_\alpha f \partial_\beta g$
$= \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}$

b) HAMILTONIAN MECHANICS REVISED

- BASIC DEFINITIONS

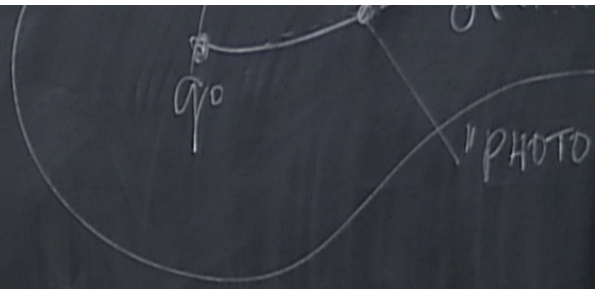
CONFIGURATION SPACE C - FOR A SYSTEM WITH n DOF .. n-dim MANIFOLD

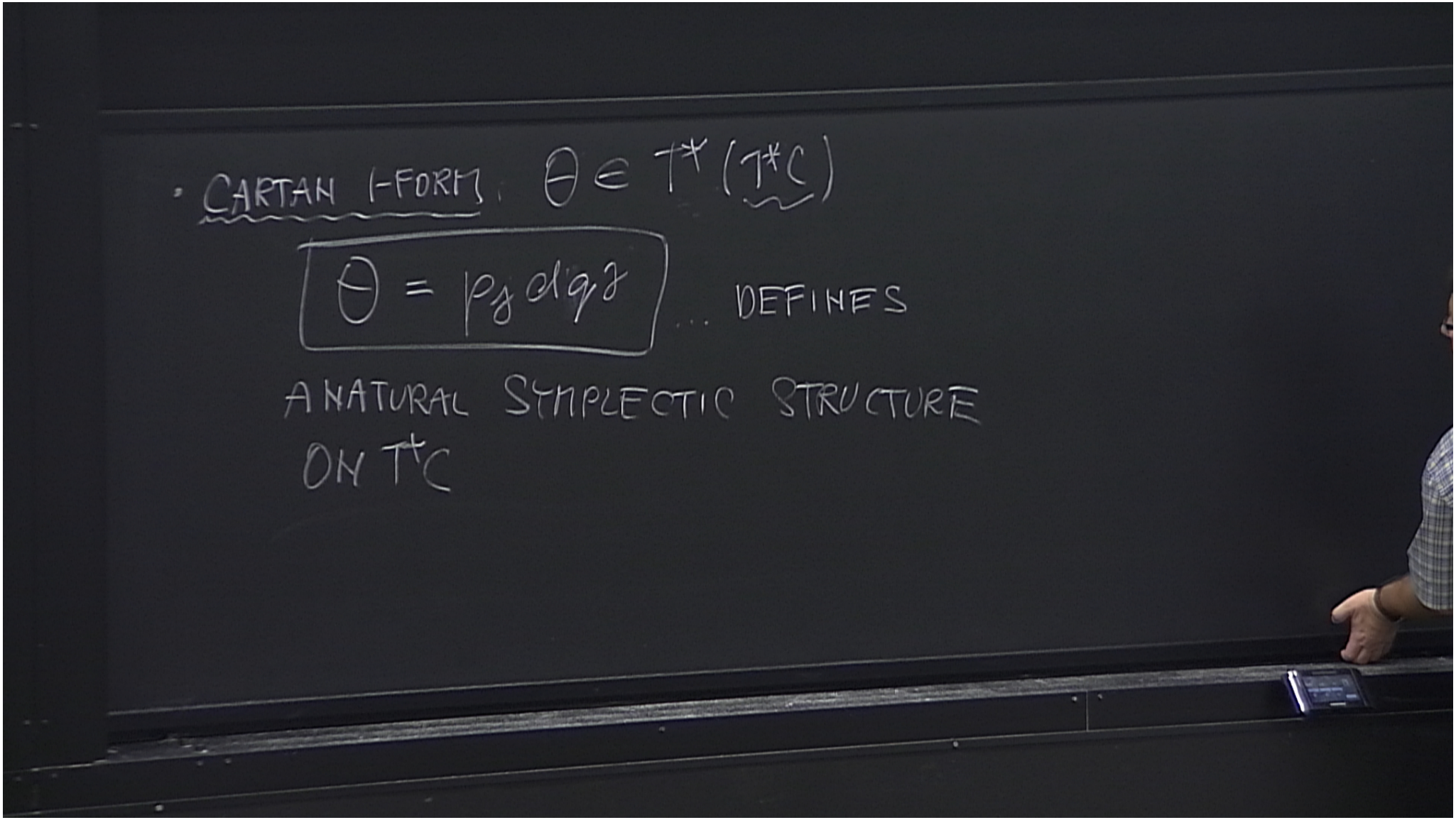
.. COORDINATES $(q^1 \ldots, q^n)$

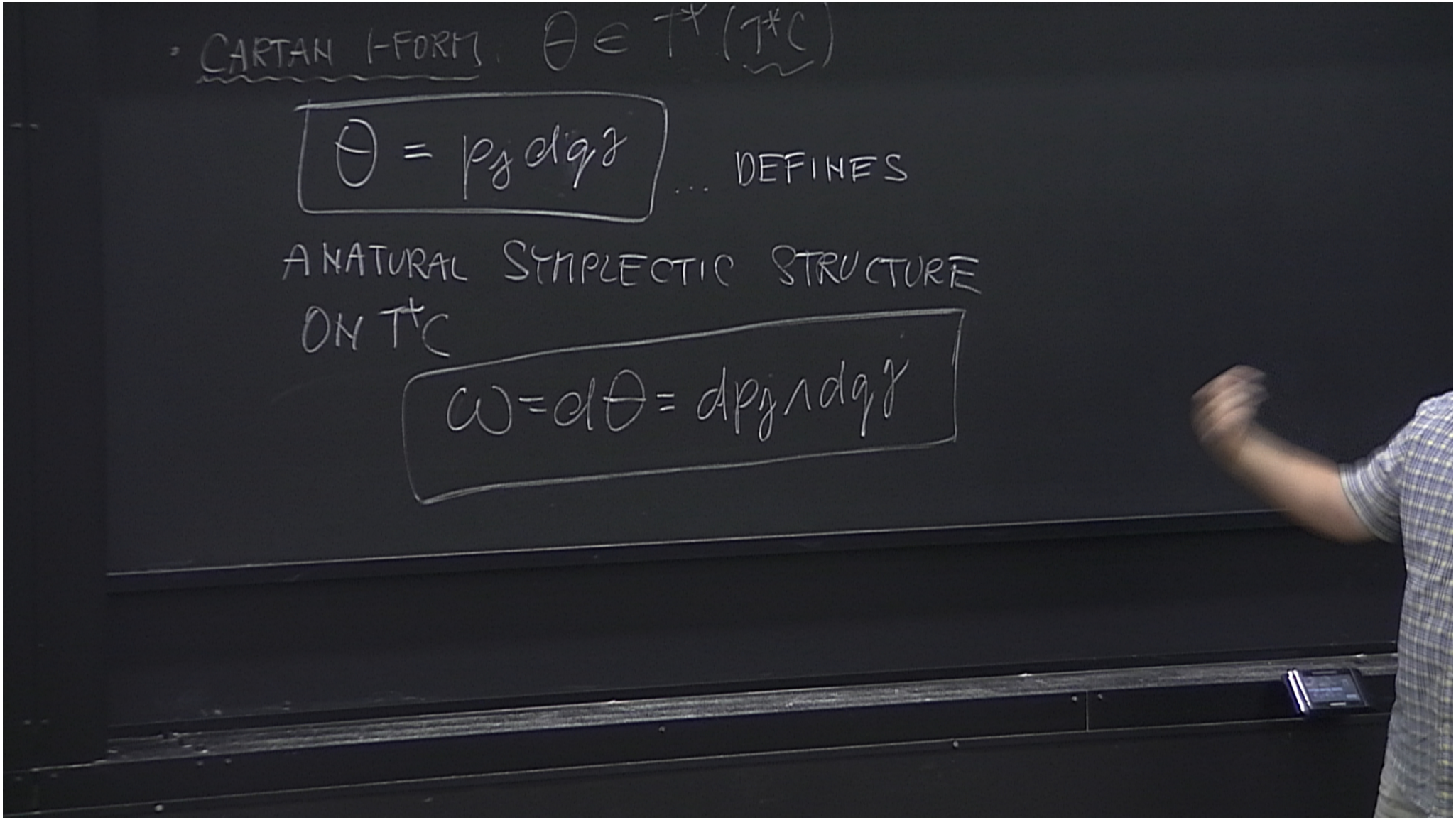


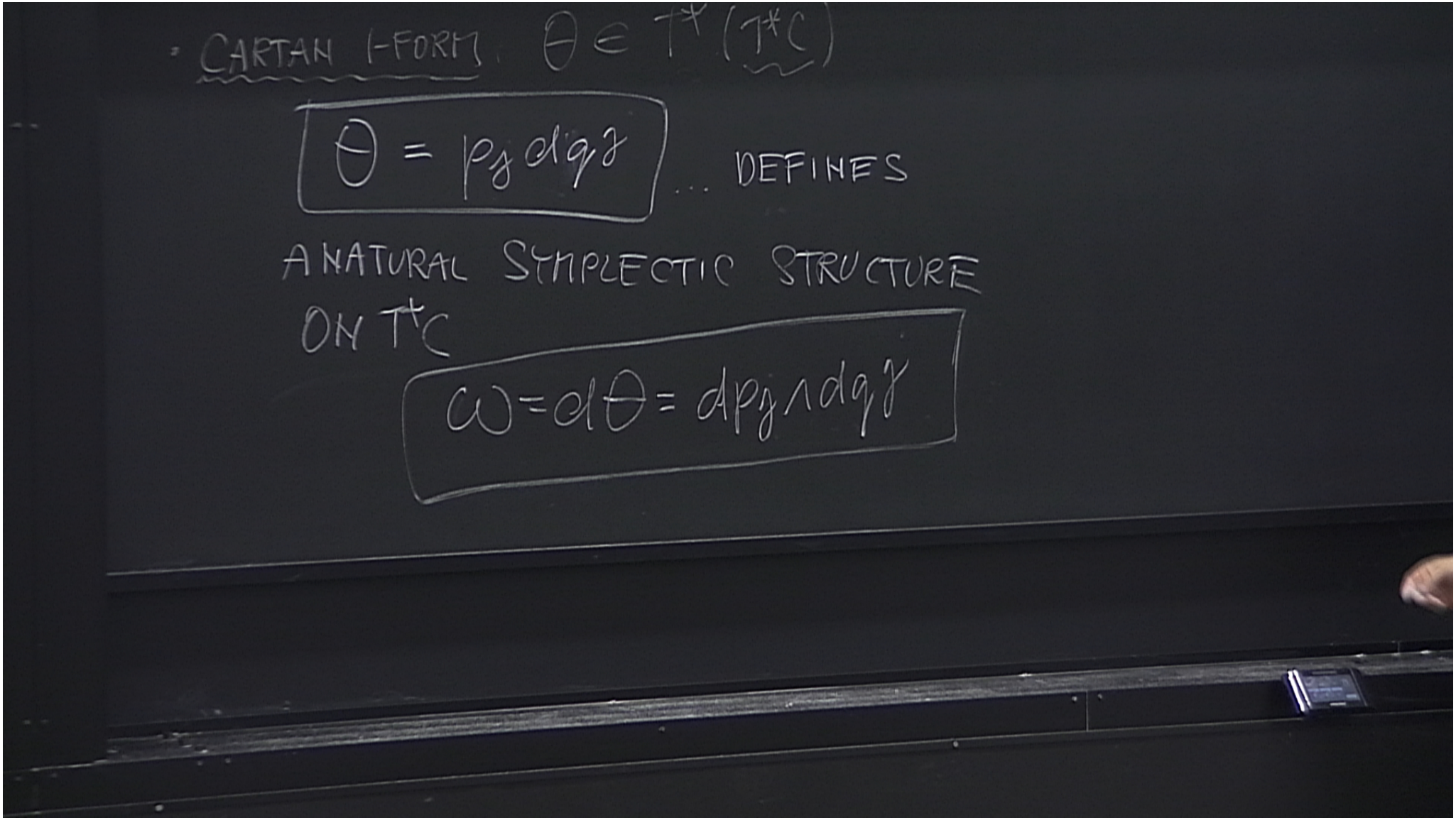
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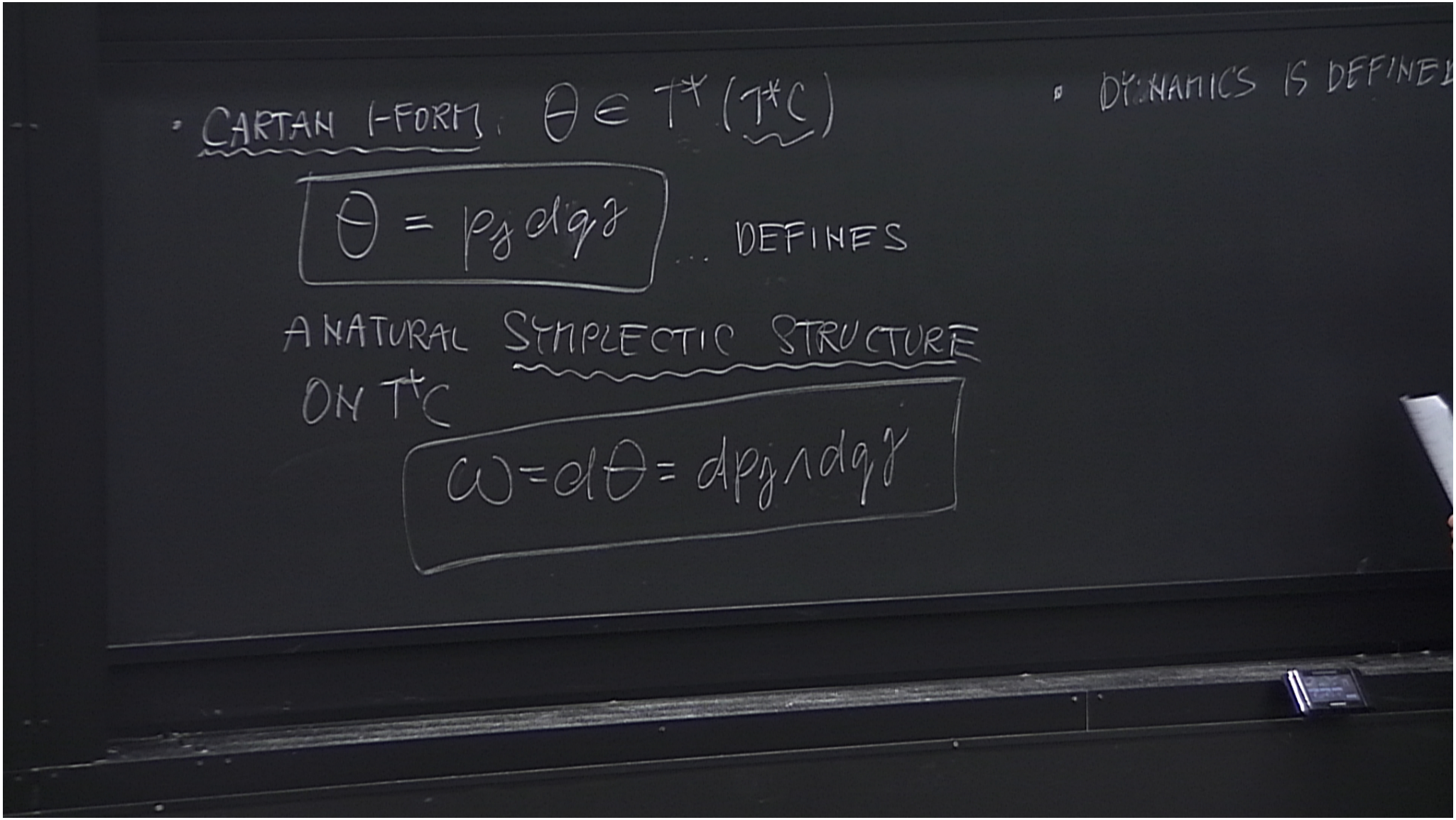
- PHASE SPACE

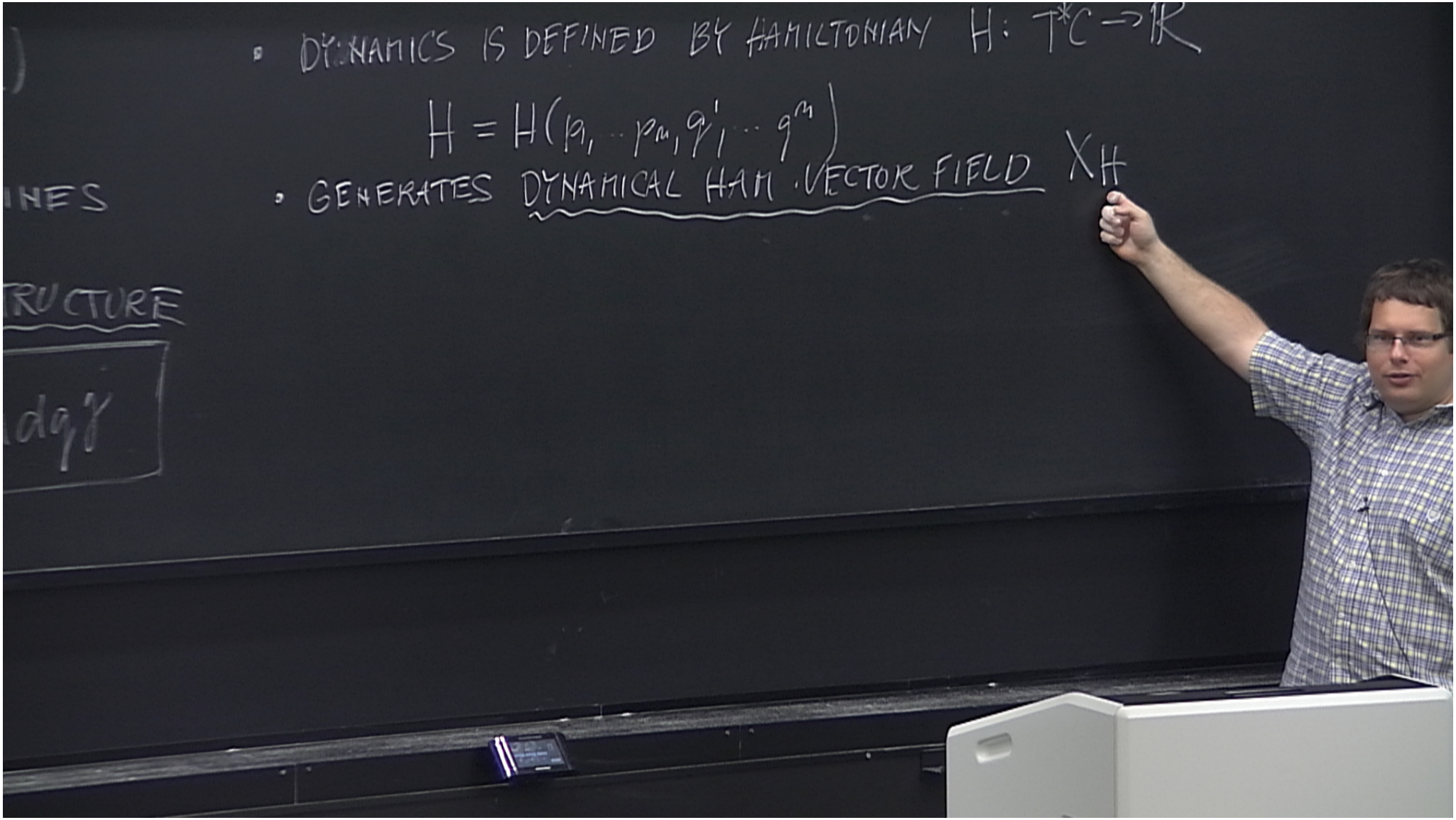


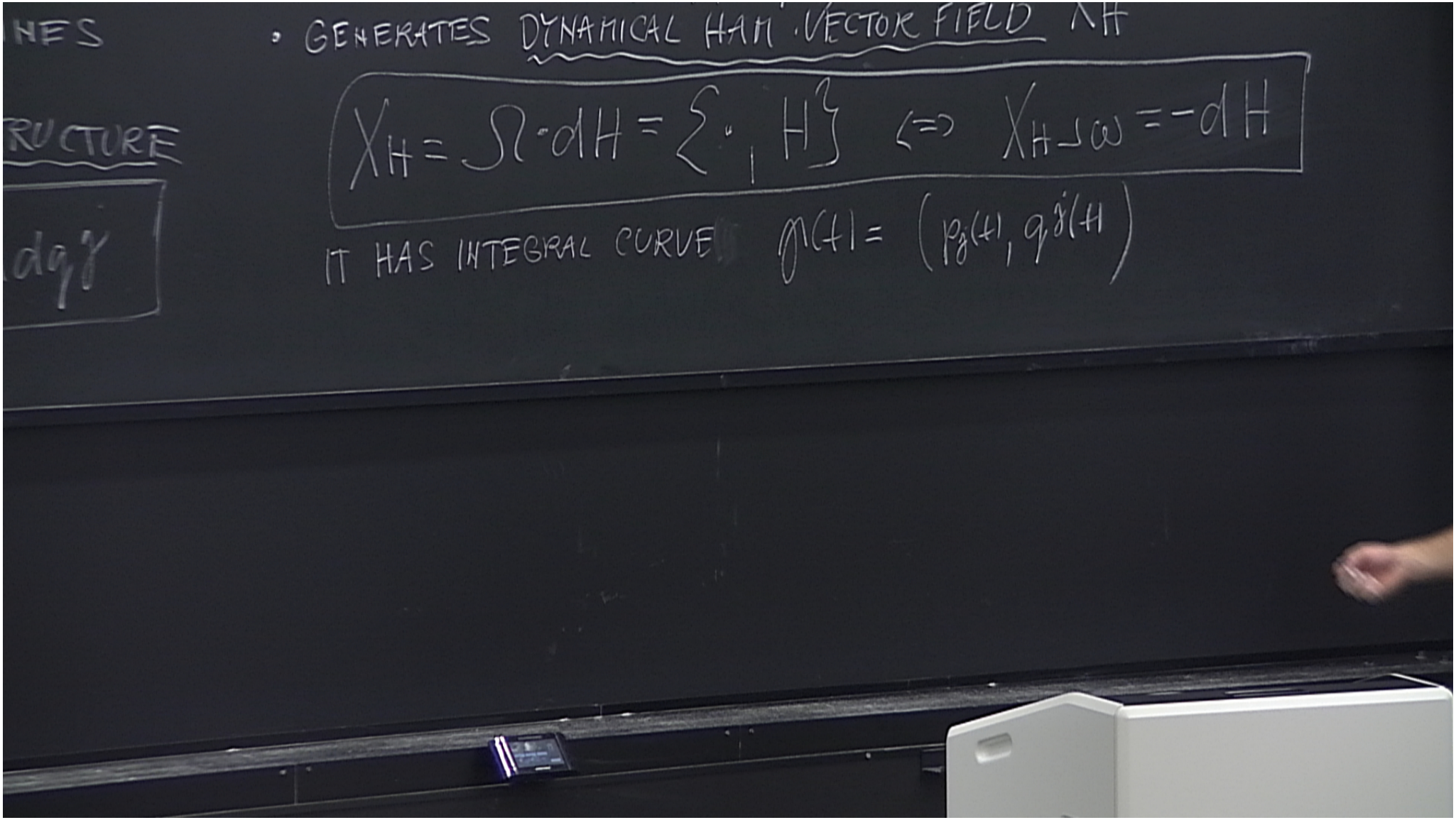

• CARTAN 1-FORM: $\theta \in T^*(T^*C)$
$\theta = p_j dq^j$ ... DEFINES
A NATURAL SYMPLECTIC STRUCTURE
ON T^*C


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$\theta = p_j dq^j$ ... DEFINES
A NATURAL SYMPLECTIC STRUCTURE ON T^*C
$\omega = d\theta = dp_j \wedge dq^j$
• DYNAMICS IS DEFINED


• DYNAMICS IS DEFINED BY HAMILTONIAN H: T*C → ℝ
H = H(p1, ... pm, q1, ... qm)
• GENERATES DYNAMICAL HAM. VECTOR FIELD XH
INES
TRUCTURE
dqγ


NES
RUCTURE
dqγ
• GENERATES DYNAMICAL HAM. VECTOR FIELD XH
$X_H = \Omega \cdot dH = \{\cdot, H\} \iff X_H \lrcorner \omega = -dH$
IT HAS INTEGRAL CURVE $\gamma(t) = (p_i(t), q^i(t))$

$\omega = d\theta = dp_j \wedge dq^j$

IT HAS INTEGRAL CURVE

$X_H = \frac{d}{dt} = \frac{dq^j}{dt}\frac{\partial}{\partial q^j} + \frac{dp_j}{dt}\frac{\partial}{\partial p_j}$ ... INTEGRAL CURVE.

$\overset{\text{DEF}}{=} \{\cdot, H\} \overset{\text{CAN D.}}{=} \overset{\text{COORDS}}{=}$

$\frac{\partial H}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j}$

$\boxed{\frac{\partial H}{\partial p_j} = \dot{q}^j}$ $\boxed{\dot{p}_j = -\frac{\partial H}{\partial q^j}}$