Title: Front End - Classical Mechanics 4

Date: Aug 16, 2016 10:30 AM

URL: http://pirsa.org/16080066

Abstract:

FREEDOM IN DESCRIPTION: CAN TRANSF.

$$Q^j = Q^j(q^n, p_n, t), \quad P_j = P_j(q^n, p_n, t)$$

SO THAT $\exists H'$

$$\frac{\partial H'}{\partial Q^j} = -\dot{P}_j, \quad \frac{\partial H'}{\partial P_j} = \dot{Q}^j$$

GENERATED FROM GENERATING FUNCTIONS

$\dots, P_{n-1}, t)$

$$\frac{\partial H'}{\partial P_j} = \dot{Q}_j$$

...NCTIONS

<u>EX.</u> $F(q, Q, t)$

$$S = \int \left(p\dot{q} - H - \frac{dF}{dt}\right) dt = \int \left(P\dot{Q} - H'\right) dt$$

$$= \int \left[\dot{q} \underbrace{\left(p - \frac{\partial F}{\partial q}\right)}_{0} - \dot{Q} \underbrace{\frac{\partial F}{\partial Q}}_{-P} - \underbrace{\left(H + \frac{\partial F}{\partial t}\right)}_{H'} \right] dt$$

$$p_j = \frac{\partial F}{\partial q_j}, \quad P_j = -\frac{\partial F}{\partial Q_j}, \quad H' = H + \frac{\partial F}{\partial t}$$

$$\frac{\partial H'}{\partial P_j} = \dot{Q}_j, \quad \frac{\partial H'}{\partial Q_j} = -\dot{P}_j$$

$\ldots, t)$

$\frac{\partial H'}{\partial P_j} = \dot{Q}_j$

...TIONS

EX: $F(q, Q, t)$

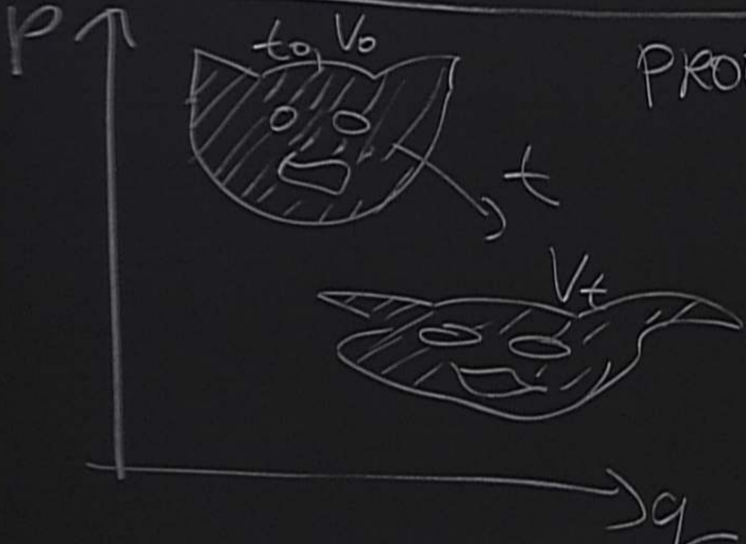
$$S = \int \left(p\dot{q} - H - \frac{dF}{dt}\right) dt = \int \left(P\dot{Q} - H'\right) dt$$

$$= \int \left[\dot{q} \underbrace{\left(p - \frac{\partial F}{\partial q}\right)}_{0} - \dot{Q} \underbrace{\frac{\partial F}{\partial Q}}_{-P} - \underbrace{\left(H + \frac{\partial F}{\partial t}\right)}_{H'}\right] dt$$

TIME EVOL ... CAN TR $\boxed{F = -S(q_0, t_0, q, t)}$

HAMILTON'S F.

LIOUVILLE'S THEOREM. VOLUME OF THE PHASE SPACE REMAINS INVARIANT UNDER CAN. TRANSF (TIME EVOLUTION IN PARTICULAR)



PROOF ($n=1$)

$$V_t = \int dQ\,dP = \int \underbrace{|\mathrm{Jac}|}\, dq\,dp$$

$$= \int dq\,dp = V_0$$

$$1 = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \{Q, P\}_{q,p}$$

INGENIOUS IDEA: LET'S FIND SPECIAL $F=S$

. S.T $\boxed{H'=0}$ $\Rightarrow$ $\boxed{\begin{array}{l} Q^j = \alpha^j = \text{CONST} \\ P_j = -\beta_j = \text{CONST} \end{array}}$

SOL $\boxed{q = q(\alpha, \beta, t)}$

BY INVERSION

. $P_j = -\dfrac{\partial S}{\partial \alpha^j} = -\beta_j \Leftrightarrow \boxed{\beta_j = \dfrac{\partial S(q^i, \alpha^i, t)}{\partial \alpha^j}}$

· TO FIND MIRACOLOUS GEN. F. $\underline{S}$

$$p_j = \frac{\partial S}{\partial q_j}, \quad \underbrace{H' = 0 = H + \frac{\partial S}{\partial t}}$$

$$\boxed{0 = H\left(q_j, \frac{\partial S}{\partial q_j}, t\right) + \frac{\partial S}{\partial t}}$$

1 PDE

HAMILTON–JACOBI EQ.

SOL. $q = q(\alpha, \beta, t)$

(q^i, α^i, t) BY INVERSION

$\partial \alpha_j$

HOW TO SOLVE THIS? — COMPLETE INTEGRAL $S = S(q, \alpha, t)$
— GENERAL INTEGRAL . (FUNCTION) ..NOT INTEREST

TRY SEPARATION OF VARIABLES

$$H \neq H(t) \Rightarrow S(q,t) = S_0(q) - Et$$

$$H \neq H(q_c) \Rightarrow S(q,t) = \alpha_c q_c + S(q^1, \ldots, \hat{q}^c, \ldots q^n, t)$$

TRY: COMPLETE SEPARATION OF VARIABLES

$$\boxed{S(q^j, t) = S_1(q^1) + S_2(q^2) \ldots + S_n(q^n) - Et.}$$

EX: FREE FALL IN HOM. GRAV. FIELD

$$H = \frac{p^2}{2m} + V, \quad V = -mgx$$

H-J FOR $S = S(x,t)$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - mgx + \frac{\partial S}{\partial t} = 0$$

- $H \neq H(t)$ $\quad S = S_0(x) - Et$

$$\frac{1}{2m}\left(\frac{dS_0}{dx}\right)^2 - mgx - E = 0$$

ELD

gX

$$= S = \int \sqrt{(E+mgx)\,2m}\; dx - Et = \frac{1}{3m^2 g}\left[2m\left(E+mgx\right)\right]^{3/2} - Et$$

$$\frac{\partial S}{\partial E} = \beta = \frac{1}{mg}\sqrt{2m\left(E+mgx\right)} - t$$

$$\Rightarrow \boxed{x = \frac{g}{2}\left(t+\beta\right)^2 - E/mg}$$

$$\frac{1}{2m}\left(\frac{dS}{dx}\right)^2 - mgx - E = 0$$

- CONNECTION TO QM:
- SCHROD. EQ: $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$, $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$, $\hat{p} = -i\hbar\nabla$
- ANSATZ: $\psi = \psi_0\, e^{\frac{i}{\hbar} S(x,t)}$ $\quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi =$

WKB (GEOM. OPTICS) APPROX.

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x) \, , \quad \hat{p} = -i\hbar\nabla$$

t)

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi = \underbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi}$$

$$\left(\frac{(\nabla S)^2}{2m} - \frac{i\hbar}{2m}\nabla^2 S + V\right)\psi$$

ROX

SCHROD. EQ: $i\hbar \frac{\partial \psi}{\partial t} = H\psi, \quad H = \frac{p^2}{2m} + V(x)$

ANSATZ: $\psi = \psi_0 \, e^{\frac{i}{\hbar} S(x,t)}$

$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 \ldots \right.$

$\frac{|\nabla S|^2}{2m}$

WKB (GEOM. OPTICS) APPROX

$\hbar \to 0$

$$\boxed{\frac{\partial S}{\partial t} = \frac{|\nabla S|^2}{2m} + V}$$

$mgx - E = 0$

QM:

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi, \quad \hat{H} = \frac{\hat{p}^2}{2m} + V(x), \quad \hat{p} = -i\hbar\nabla$$

$$\psi = \psi_0\, e^{\frac{i}{\hbar}S(x,t)}$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi = \underbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi}_{\left(\frac{|\nabla S|^2}{2m} - \underset{\downarrow\, 0}{\frac{i\hbar}{2m}}\nabla^2 S + V\right)\psi}$$

. OPTICS) APPROX

$$\boxed{-\frac{\partial S}{\partial t} = \frac{|\nabla S|^2}{2m} + V}$$

.. H-J

e) INTEGRABLE SYSTEMS

DEF: A SYSTEM WITH n DOF IS

COMPLETELY INTEGRABLE (LIOUVILLE)

IF IT ADMITS n INDEPENDENT

INTEGRALS OF MOTION $F_i(q,p) = f_i$

$\{F_i, H\} = 0$ THAT ARE IN INVOLUTION

$\{F_i, F_j\} = 0 \quad \forall i,j$

$\frac{\partial \ldots}{\partial \alpha_j} = -\beta_j \iff \beta_j = \frac{\partial S(q^i, \alpha^i, t)}{\partial \alpha_j}$

LIOUVILLE THEOREM (2) THE SOL OF EOM OF A COMPLETELY INTEGRABL SYSTEM CAN BE OBTAINED BY "QUADRATURE" (FINITE # OF INTEGRALS & ALGEBRAIC OPERATIONS)

RAMARKS: 1). TO INTEGRATE n LAGRANGE EOM

$\Rightarrow$ SOL ($\underline{2n}$ CONSTANTS)

"FOR CANONICAL SYSTEMS WE NEED ONLY HALF!"

EACH INTEGRAL HAS TO BE OSED TWICE. REDOCE THE ORDER OF THE SYSTEM BY TWO.

ED

ANALOGOUS STATEMENT IN QFT. IN GAUGE THEORY.

$$\boxed{\#\,(\text{TRUE DOF}) = \#\,(\text{APPARENT DOF}) - 2\times(\text{DOF OF GAUGE F.})}$$

EX: EM. $(\varphi, \vec{A})$ ... POTENTIALS ... 4 COMPTS = 4 AP. DOF.

GAUGE TRANSF.

$$\left.\boxed{\begin{aligned} \varphi &\to \varphi + \partial_t \Lambda \\ \vec{A} &\to \vec{A} + \nabla\Lambda \end{aligned}}\right\}$$ LEAVE $\vec{E}$, $\vec{B}$ INVARIANT

Λ : 1 COMPT.

$\#$ TRUE DOF $= 4 - 2\times 1 = \underline{\underline{2}}$ POLARIZATI[ONS] OF PHOTON

$\varphi = \varphi(\alpha, \beta, t)$

$(\ldots, \alpha^i, t)$ BY INVERSION

HAMILTON-JACOBI EQ.

TWICE ... REDUCE THE ORDER OF THE SYSTEM BY TWO.

$$\varphi \to \varphi + \partial_t \Lambda$$
$$\vec{A} \to \vec{A} + \nabla \Lambda$$

<u>EX2</u>. GRAVITY .. POTENTIAL. $g_{\mu\nu}$ .. 4x4 SYM. .. 10 APPAR. DOF.

GAUGE TRANSF. $\quad g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ LEAVES FIELD STRENGTH INVAR.

GAUGE F. ξ_μ ... 4 COMPTS.

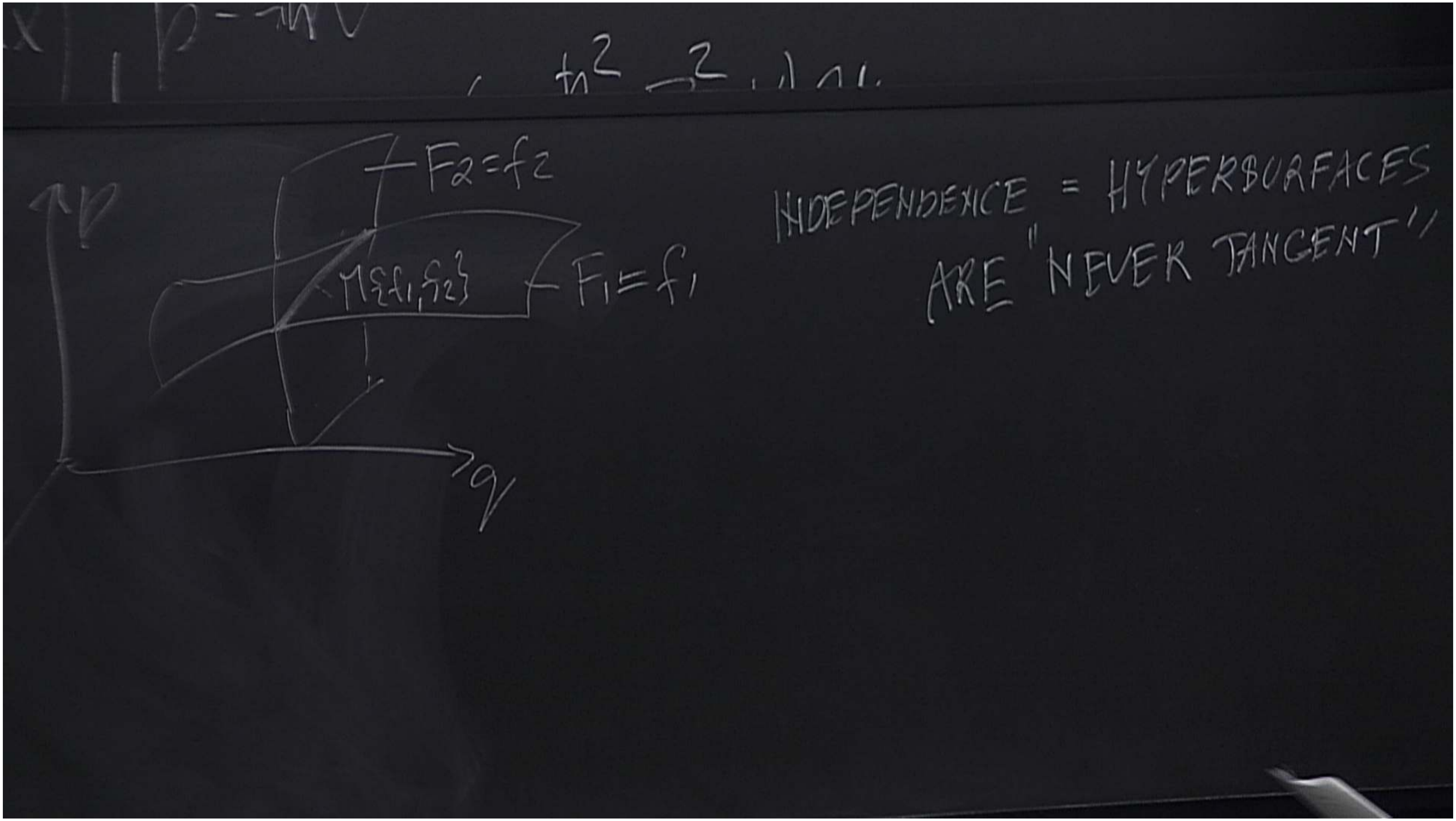
TRUE DOF $= 10 - 2 \times 4 = 2$ <u>POLAR. OF GRAVITON</u>

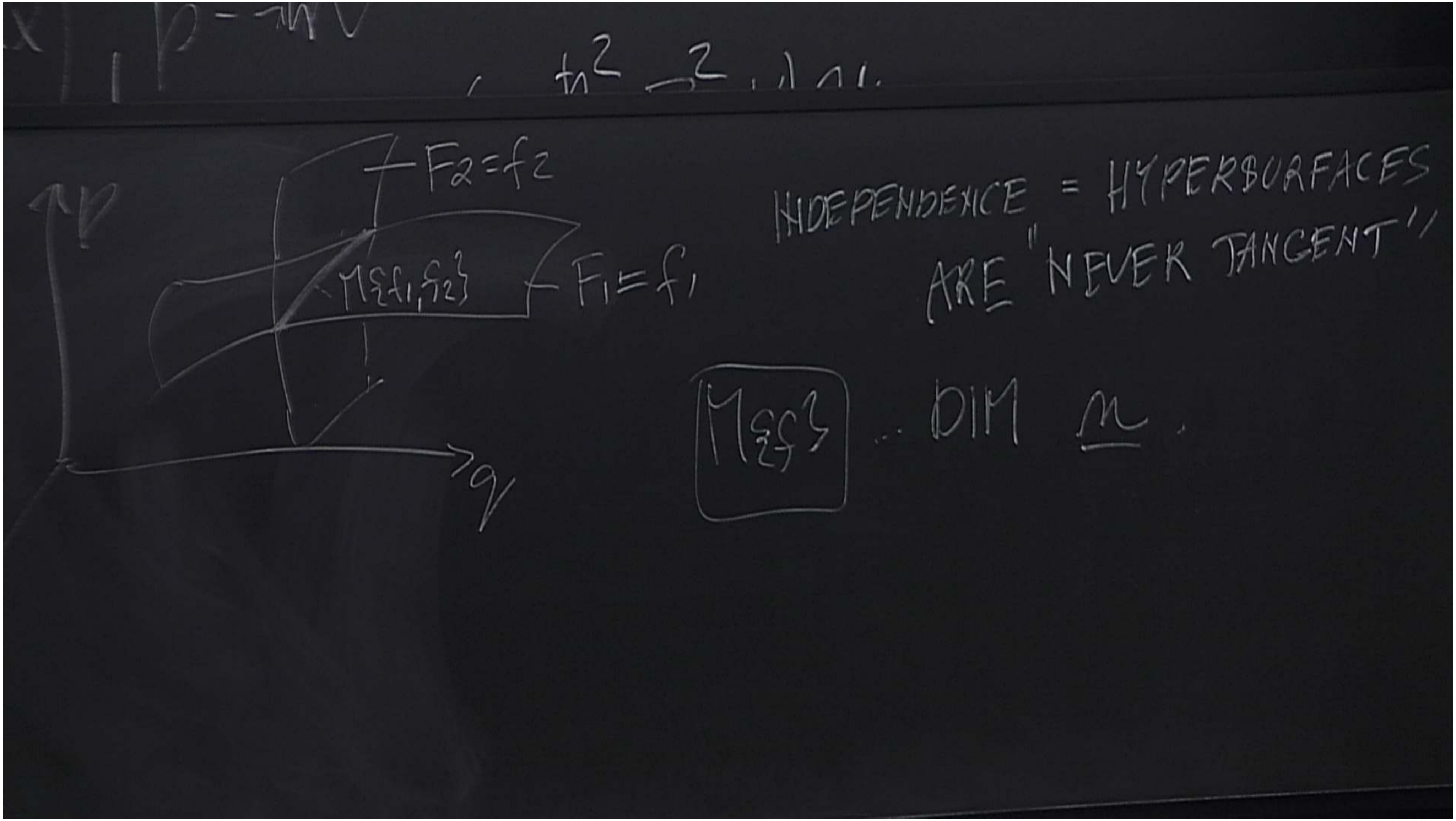
• ANSATZ: $\frac{i}{\hbar} S(x,t)$

ii) INDEPENDENCE: EACH INTEGRAL DEFINES A HYPERSURFACE IN PHASE SPACE

$$F_i(q,p) = f_i$$

MOTION HAS TO HAPPEN IN THAT HYPERSURFACE


F2=f2
F1=f1
γ{f1,f2}
p
q
INDEPENDENCE = HYPERSURFACES
ARE "NEVER TANGENT"



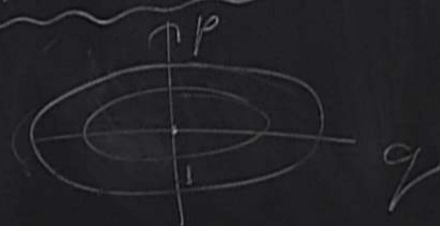
F2=f2
p
M{f1, f2}
F1=f1
q
INDEPENDENCE = HYPERSURFACES
ARE "NEVER TANGENT"
M{f} ... DIM n

iii) ONE CANNOT HAVE MORE THAN n INTEGRALS IN INVOLUTION, OTHERWISE POISS. BRACKET DEGENERATE.

$$\boxed{H = H(F_1 \ldots)}$$

iv) UNDER GLOBAL HYPOTHESIS: $M_{\{f\}}$ IS AN n-DIM T^n.

EX: HARMONIC OSCILLATOR $H = \frac{1}{2}(p^2 + \omega^2 q^2)$



PHASE SPACE "FIBRED" INTO T_1 ELLIPSES $H = E$.

...ATE.

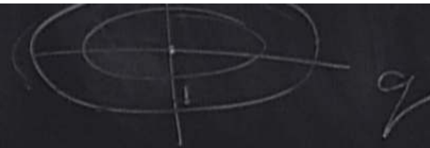
INTRODUCE $p = \rho \cos\theta$, $q = \frac{\rho}{\omega} \sin\theta$

$\Rightarrow$ MOTION: $= \{ \rho = \sqrt{2E}, \; \theta = \omega t + \theta_0 \}$

ACTION-ANGLE VARIABLES

... I_a.

...NTO: T_I



ELLIPSES $H = E$.

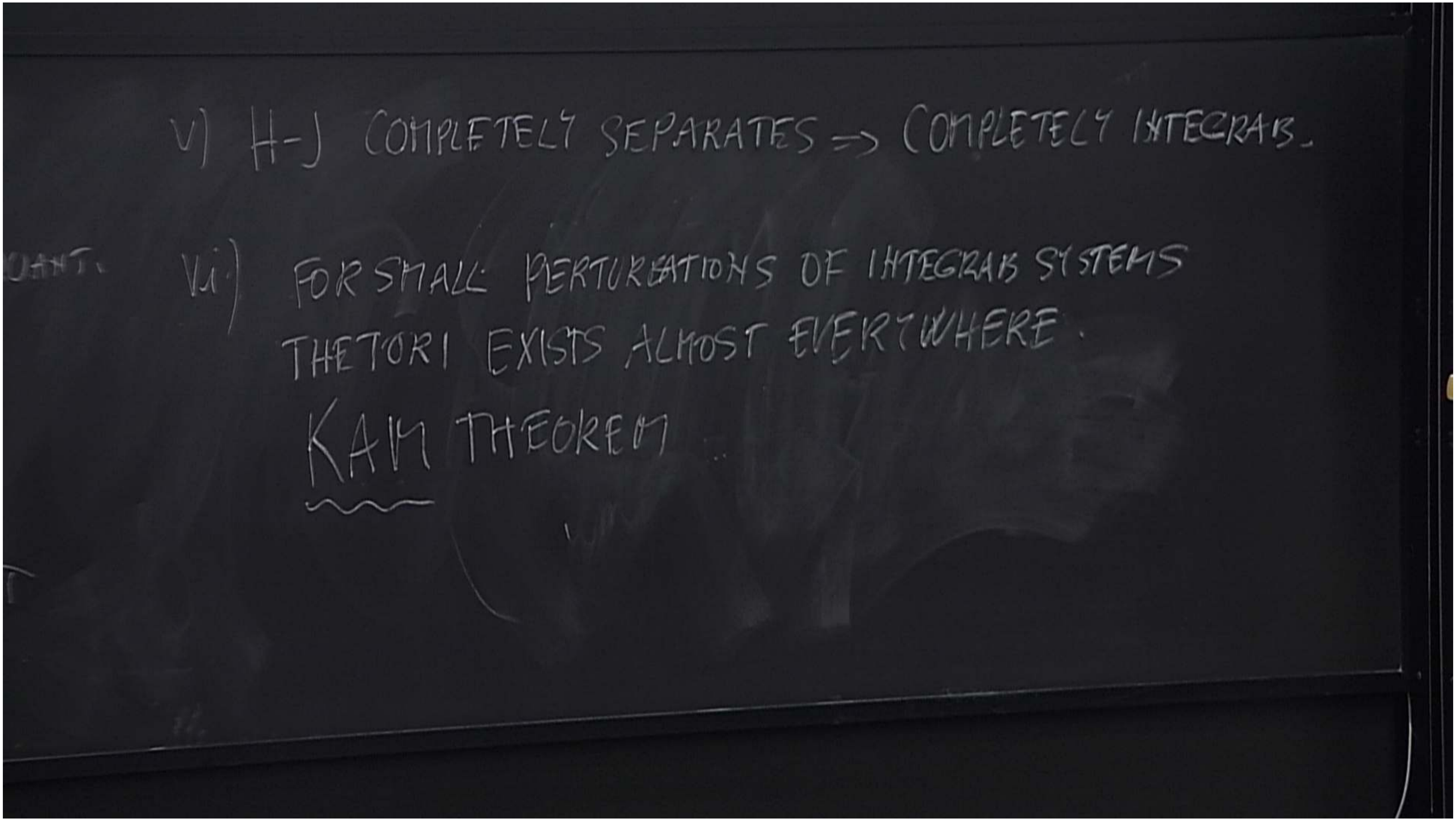
v) H-J

• GENERALIZATION $H = \frac{1}{2} \sum_i (\underbrace{p_i^2 + \omega_i^2 q_i^2}_{H_i})$ H_i .. CONS. QUANT.

INTEGRAB.

$M_{\{f\}} = \{ H_i = E_i \}$ ... T_n

ANY INTEGRABLE SYSTEM "CAN BE BROUGHT INTO THIS FORM"


V) H-J COMPLETELY SEPARATES ⇒ COMPLETELY INTEGRAB.
Vi) FOR SMALL PERTURBATIONS OF INTEGRAB SYSTEMS
THE TORI EXISTS ALMOST EVERYWHERE.
KAM THEOREM