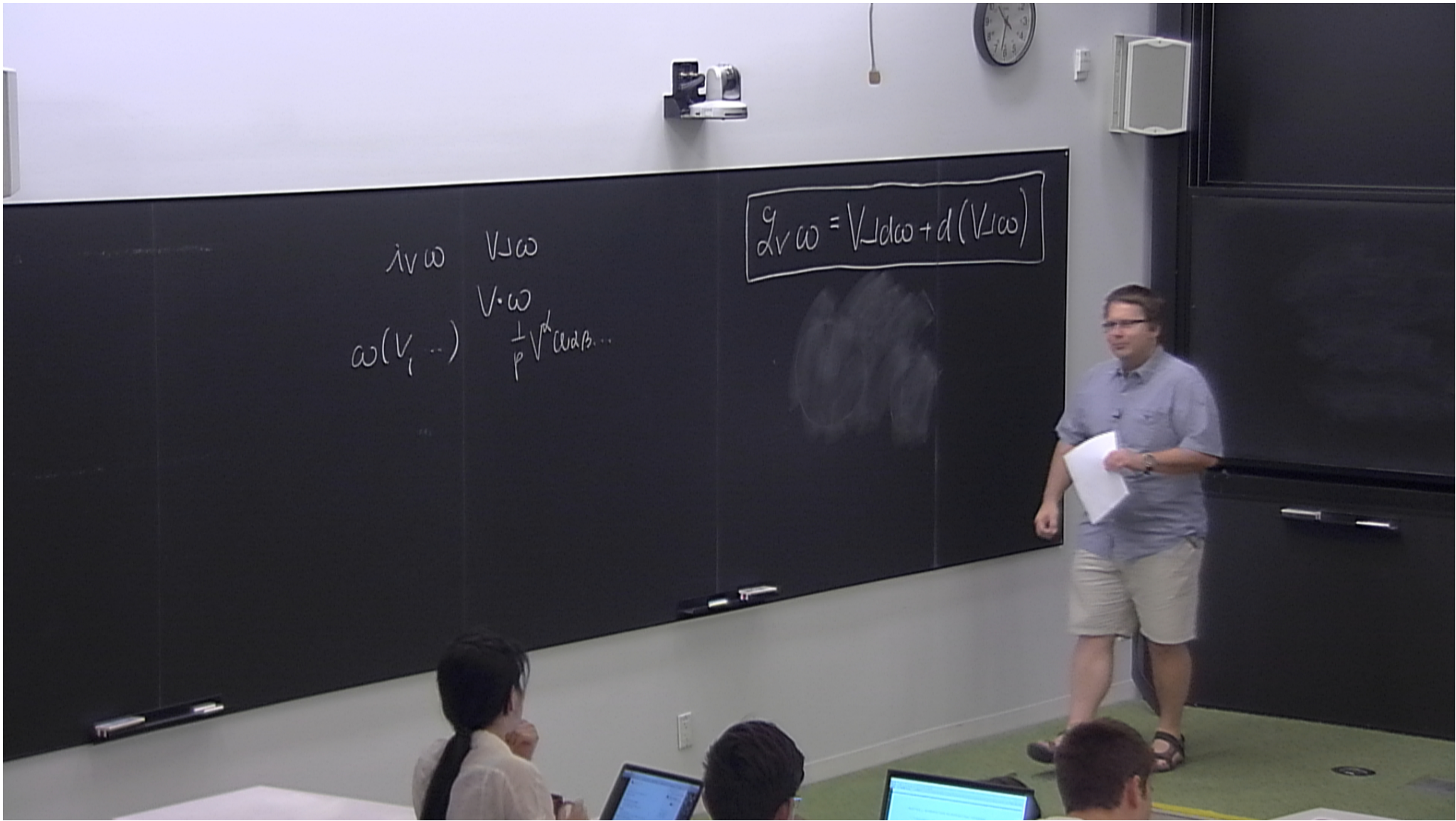


Title: Front End - Classical Mechanics 2

Date: Aug 11, 2016 10:30 AM

URL: <http://pirsa.org/16080063>

Abstract:



2) LAGRANGIAN MECHANICS

a) HAMILTON'S PRINCIPLE OF LEAST ACTION

MOTIONS OF THE MECHANICAL SYSTEM IN
TIME INTERVAL $t \in (t_1, t_2)$ COINCIDE
WITH THE EXTREMALS OF THE ACTION
FUNCTIONAL

$$S = S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$$

2) LAGRANGIAN MECHANICS

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→ LAGRANGIAN

MECHANICS

PRINCIPLE OF LEAST ACTION

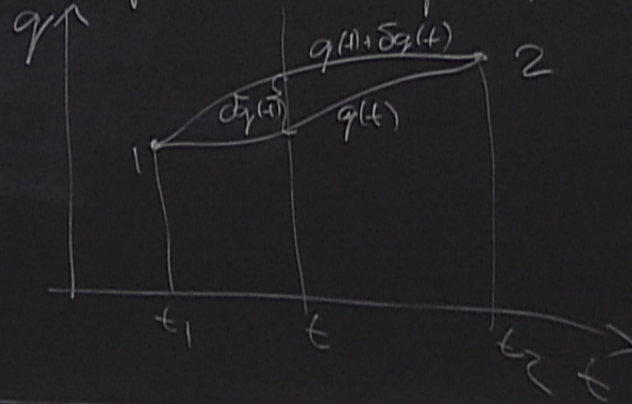
MECHANICAL SYSTEM IN
FIXED END POINTS
OF THE ACTION

$$S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$$

→ LAGRANGIAN

• TO DERIVE EOM CONSIDER FIXED END POINTS

$$\delta q(t_1) = 0 = \delta q(t_2)$$



MECHANICS

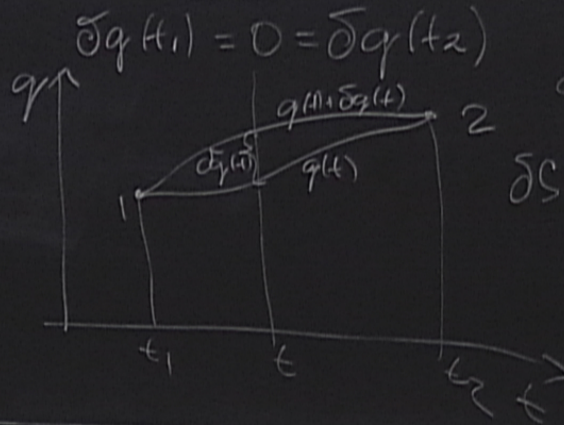
PRINCIPLE OF LEAST ACTION

A MECHANICAL SYSTEM IN
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LACKRANGIAN

• TO DERIVE EOM CONSIDER FIXED END POINTS



SEEK EXTREMUM

$$\delta S = S[q + \delta q] - S[q] = 0$$

$$S = S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$$

$$\delta S = \int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \underbrace{\delta \dot{q}}_{\frac{d}{dt} \delta q} \right) dt \stackrel{\text{BY PARTS}}{=} \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q = 0$$

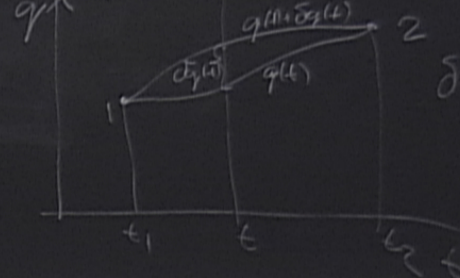
a) HAMILTON'S PRINCIPLE OF LEAST ACTION

MOTIONS OF THE MECHANICAL SYSTEM IN TIME INTERVAL $t \in (t_1, t_2)$ COINCIDE WITH THE EXTREMALS OF THE ACTION FUNCTIONAL

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LAGRANGIAN

$$\delta q(t_1) = 0 = \delta q(t_2)$$



SEEK EXTREMUM

$$\delta S = S[q + \delta q] - S[q] = 0$$

FIXED

$$\delta \frac{d}{dt} = \frac{d}{dt} \delta$$

$$\delta S = \int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q \right) dt$$

$$+ \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q = 0$$

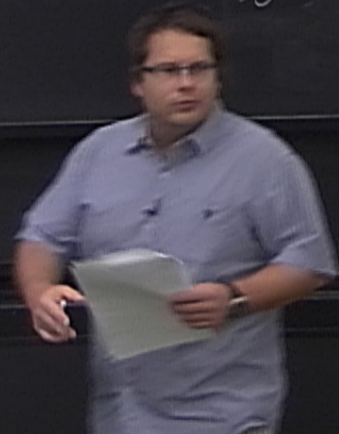
BY PARTS

$$\left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2}$$

0 - FIXED ENDPOINTS

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0$$

EULER-LAGRANGE (E-L)



3 REMARKS:

- $L = L(q_i, \dot{q}_i, t) \Rightarrow$ (E-L) 2ND-ORDER. EOM,
- CAN SHOW FOR CONSERVATIVE SYSTEMS

$$L = T - V \quad (L \propto \dot{L})$$

- FREEDOM:

$$L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{d\Lambda(q_i, t)}{dt}$$

\Rightarrow SAME (E-L)

\Rightarrow (E-L) 2ND-ORDER. EOM,
OR CONSERVATIVE SYSTEMS

$-V$ (L & L)

$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d\Lambda(q, t)}{dt}$
E (E-L)

PROOF: $S' = \int L' dt = \int L dt + \int \frac{d\Lambda}{dt} dt$
 $= S + \Lambda|_{t_1}^{t_2}$

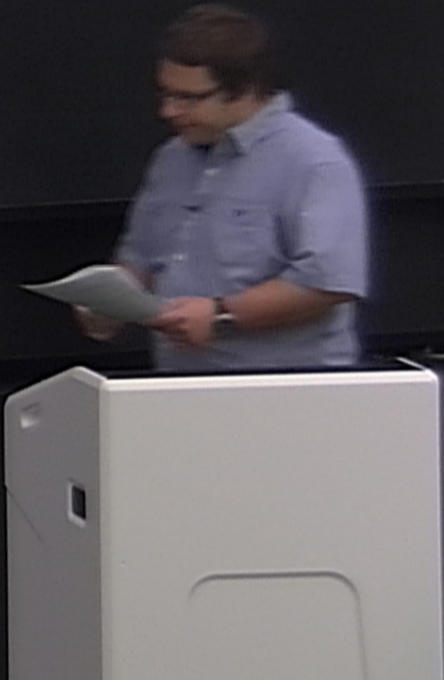
$\delta S' = \delta S + \underbrace{\left[\frac{\partial \Lambda}{\partial q} \delta q \right]}_{\theta} \Big|_{t_1}^{t_2}$

NOETHER'S THEOREM (VERSION 1). FOR EVERY
GLOBAL CONTINUOUS SYMMETRY OF THE
SYSTEM, THERE IS A CORRESPONDING
INTEGRAL OF MOTION.

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EXS: $L \neq L(t) \Rightarrow E = \frac{\partial L}{\partial \dot{q}^I} \dot{q}^I - L$ GENERALIZED ENERGY

$\frac{dE}{dt} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} - \frac{\partial L}{\partial q} \dot{q} - \frac{\partial L}{\partial q} \ddot{q} = 0$. PROVIDED $(L-E)$ ✓



NOETHER'S THEOREM (VERSION 1). FOR EVERY GLOBAL CONTINUOUS SYMMETRY OF THE SYSTEM, THERE IS A CORRESPONDING INTEGRAL OF MOTION.

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$L \neq L(q, \dot{q}) \xrightarrow{(E-L)}$ $p = \frac{\partial L}{\partial \dot{q}}$ GENERALIZED MOMENTUM.
CYCLIC

2 REMARKS ON TERMINOLOGY

- VALID ON SHELL ... PROVIDED EOM ARE SATISFIED
- OFF SHELL - DON'T NEED EOM.

• TYPES OF SYMMETRIES

$I = 1, \dots, n \dots m \dots$ DOF.

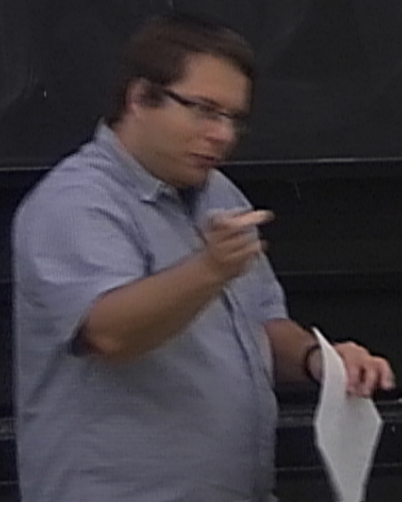
PROVIDED EOM
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FOR CONCRETENESS

$$t \rightarrow t' = t + \delta t, \quad q \rightarrow q'(t') = q(t) + \delta q(t)$$

$$\delta q_j = \epsilon^k \delta_k q_j, \quad \delta t = \epsilon^k \delta_k t$$



FOR CONCRETENESS

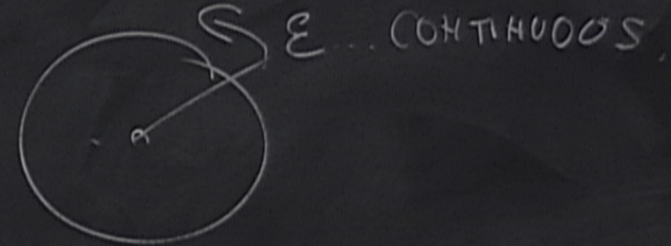
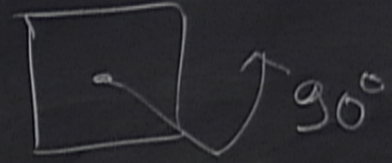
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"GENERATOR" (DEFINES ACTION)
PARAMETER (HOW BIG THE ACTION IS)

SYMMETRIES

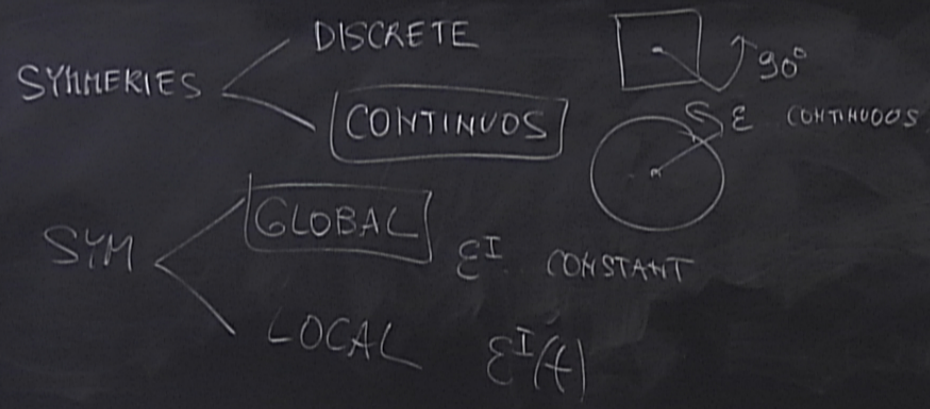
- DISCRETE
- CONTINUOUS



SYM

- GLOBAL E^I CONSTANT
- LOCAL $E^I(x)$

"GENERATOR" (DEFINES ACTION)
 PARAMETER (HOW BIG THE ACTION IS)

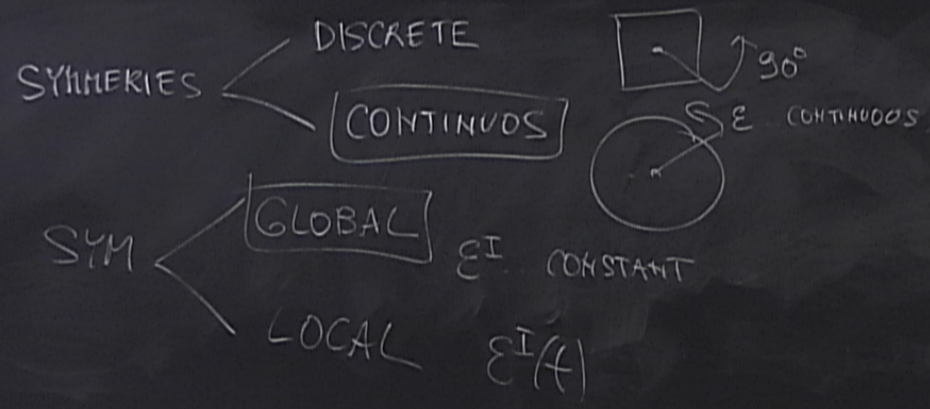


NOETHER'S THEOREM (VERSION 2 - UGLY VERSION)
 LET $\tilde{\mathcal{L}}$ BE A GLOBAL CONT. SYMMETRY,
 THAT IS OFF-SHELL WE FIND $\tilde{\delta}t, \tilde{\delta}q$
 S.T. $\tilde{\delta}S = 0 \Rightarrow I = \frac{\partial \mathcal{L}}{\partial q} \tilde{\delta}q$



SYMMETRIES

"GENERATOR" (DEFINES ACTION)
 PARAMETER (HOW BIG THE ACTION IS)

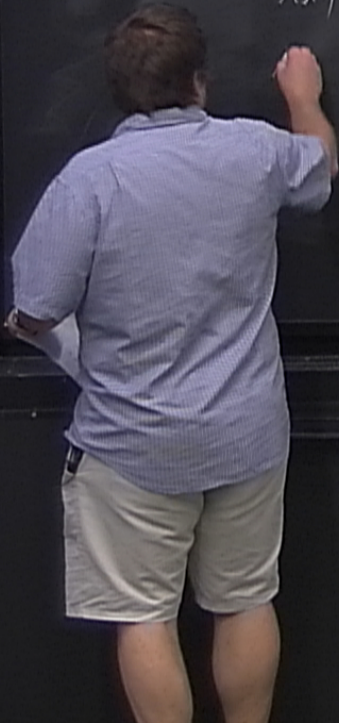


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 S.T. $\tilde{\delta}S = 0 \Rightarrow I = \frac{\partial L}{\partial \dot{q}} \tilde{\delta}q + (L - \dot{q} \frac{\partial L}{\partial \dot{q}}) \tilde{\delta}t$
 IS AN ON SHELL INTEGRAL OF MOTION.

SYM $\left\{ \begin{array}{l} \text{GLOBAL } \epsilon^I \text{ CONSTANT} \\ \text{LOCAL } \epsilon^I(x) \end{array} \right.$

S.T $\frac{\delta S}{\delta \psi} = 0 \Rightarrow I = \frac{\delta S}{\delta q} \delta q$
 IS AN ON SHELL INTEGRAL

EX: $i) (\vec{\delta}_t, \vec{\delta}_q) = \epsilon(1, 0)$... TIME TRANSLATION $\Rightarrow I$ ENERGY
 $ii) (\vec{\delta}_t, \vec{\delta}_q) = \epsilon(0, 1)$... SPACE TRANSL $\Rightarrow I$ MOMENTUM } SPACETIME HOMOGENEITY



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iii) $(\tilde{\delta}_t, \tilde{\delta}_\varphi) = \epsilon(0, 1)$	ROTATION $\Rightarrow I$ ANG. MOM	ISOTROPY OF SPACE

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- } SPACETIME HOMOGENEITY
 ISOTROPY OF SPACE

NOETHER'S THEOREM (VERSION 1) FOR EVERY

GLOBAL SYMMETRY OF THE

PROOF:

$$\delta \leftrightarrow \tilde{\delta}$$

$$\tilde{\delta} dt = dt' - dt = d\tilde{\delta}t = \frac{d\tilde{\delta}t}{dt} dt$$

$$\tilde{\delta} q_j(t) = q_j'(t') - q_j(t) = \underbrace{q_j'(t)} + \tilde{\delta}t \frac{dq_j'(t)}{dt} + \dots - q_j(t)$$

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$$= \delta q_j(t) + \tilde{\delta}t \frac{dq_j(t)}{dt} + \dots$$

$$= \left(\delta + \tilde{\nu} + \frac{d}{dt} \right) q(t) + \dots$$

$$\begin{aligned} \delta S &= \int \left(\tilde{\nu} L dt + L \tilde{\nu} dt \right) = 0 \\ &= \int \left(\delta L + \tilde{\nu} \frac{dL}{dt} + L \frac{d\tilde{\nu}}{dt} \right) dt \\ &= \int \frac{d}{dt} (\tilde{\nu} L) dt \end{aligned}$$

$$\delta L = \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right)$$

$$\delta S = \int (\delta L + \frac{d}{dt} \delta t + L \frac{\delta t}{dt}) dt$$

$$\delta L = \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q + \frac{\partial L}{\partial q} \delta q$$

$$= \int \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q + L \frac{\delta t}{dt} + \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta q \right] dt$$

$$I(t_1) = I(t_2)$$

I

0 ON SHELL

LOCAL $\mathcal{E}^I(\mathcal{F})$

NOETHER THEOREM (V3 - SNEAKY RECIPE)

- i) OBSERVE THAT S IS INVAR. UNDER A GLOBAL CONT. TRANSFORM. (\mathcal{E}^I CONSTANT) : $\tilde{\delta}S = 0$
- ii)

NOETHER THEOREM (V3 - SNEAKY RECIPE)

i) OBSERVE THAT S IS INVAR. UNDER A GLOBAL CONT. TRANSFORM. (ϵ^I CONSTANT) : $\delta S = 0$

ii) PROMOTE ϵ^I TO $\epsilon^I(t)$ MIXED ENDPOINT COND.

$$\Rightarrow \delta S = \int dt \epsilon^I \underbrace{\left(\frac{\delta S}{\delta \epsilon^I} \right)}_{\text{NOETHER INTEGRAL}}$$

EVERYTHING

$$\epsilon^I(t)$$

IS AN ON-SHELL INTEGRAL OF MOTION.

(V3 - SNEAKY RECIPE)

S IS INVARI. UNDER A GLOBAL
M. (ϵ^I CONSTANT) : $\delta S = 0$

iii) TO SEE THIS (INTEGRATE BY PARTS)

$$\delta S = - \int dt \frac{dI^I}{dt} \epsilon^I = 0$$

ON-SHELL
"S" "0"

ϵ^I TO $\epsilon^I(t)$ MIXED ENDPOINT

$$\delta S = \int dt \dot{\epsilon}^I \left(\overset{\text{TOGETHER}}{\text{INTEGRAL}} \right) \left(\underset{\text{EVERYTHING}}{\text{I}} \right)$$