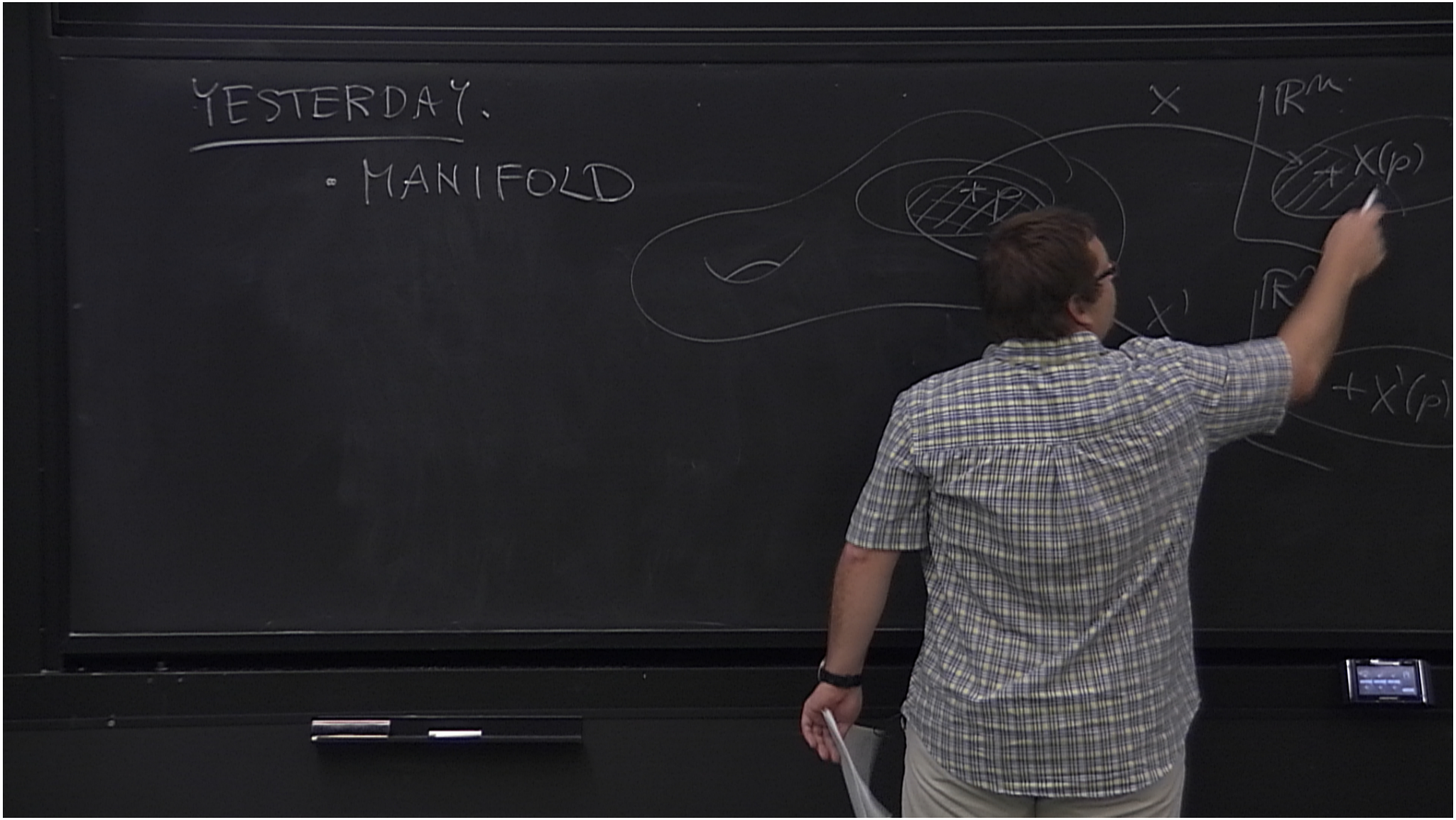
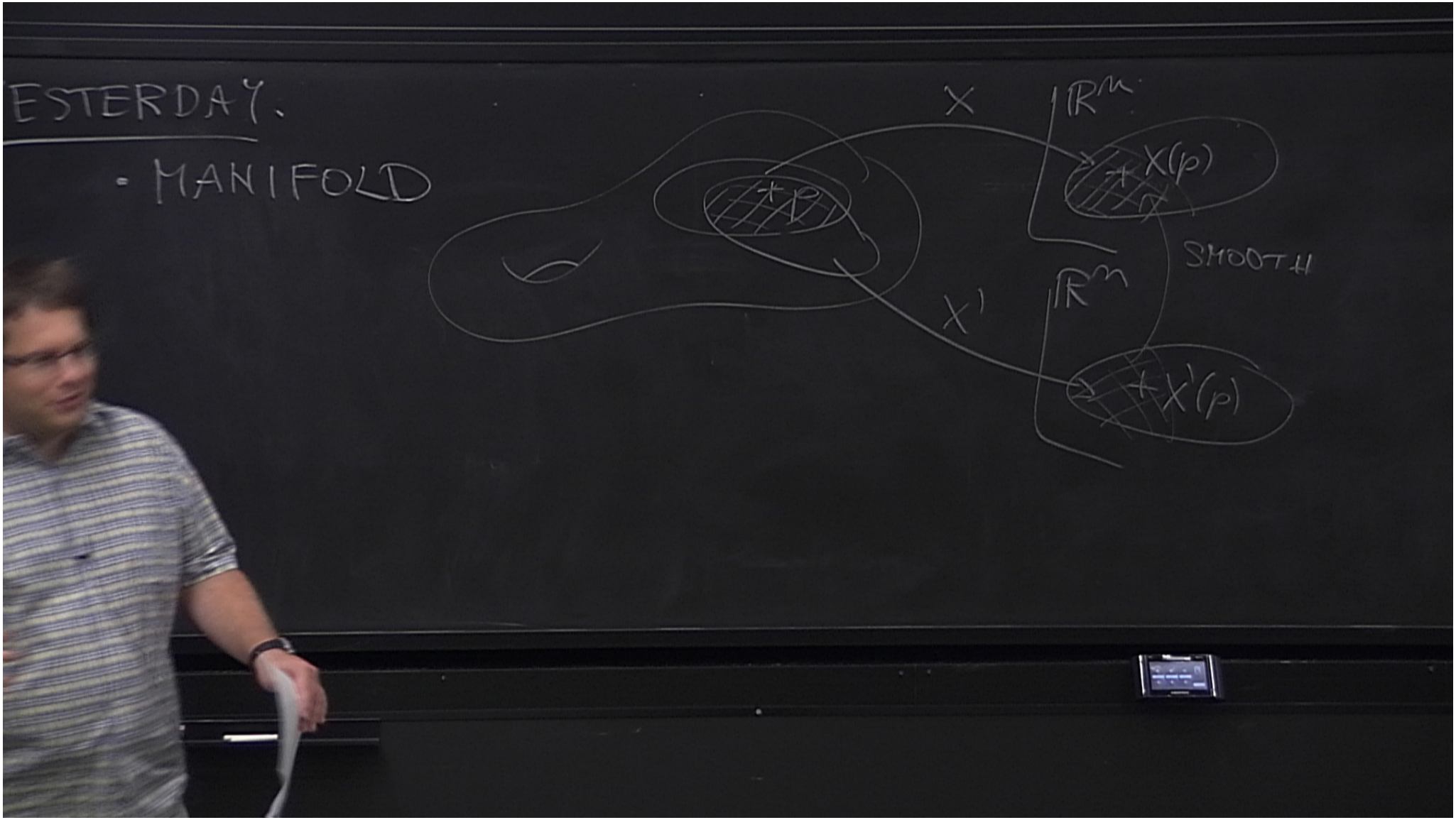
Title: Front End - Classical Mechanics 1

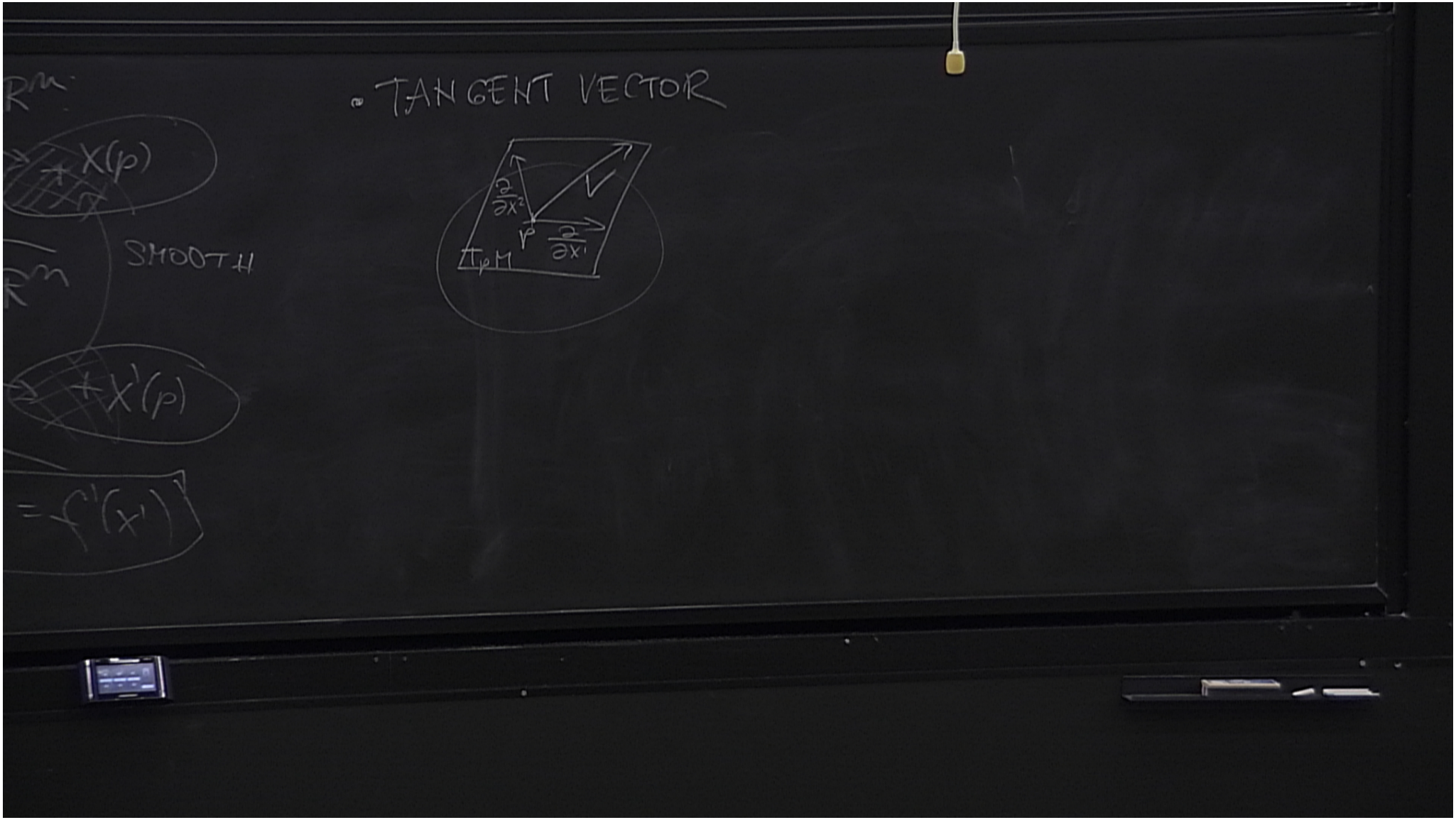
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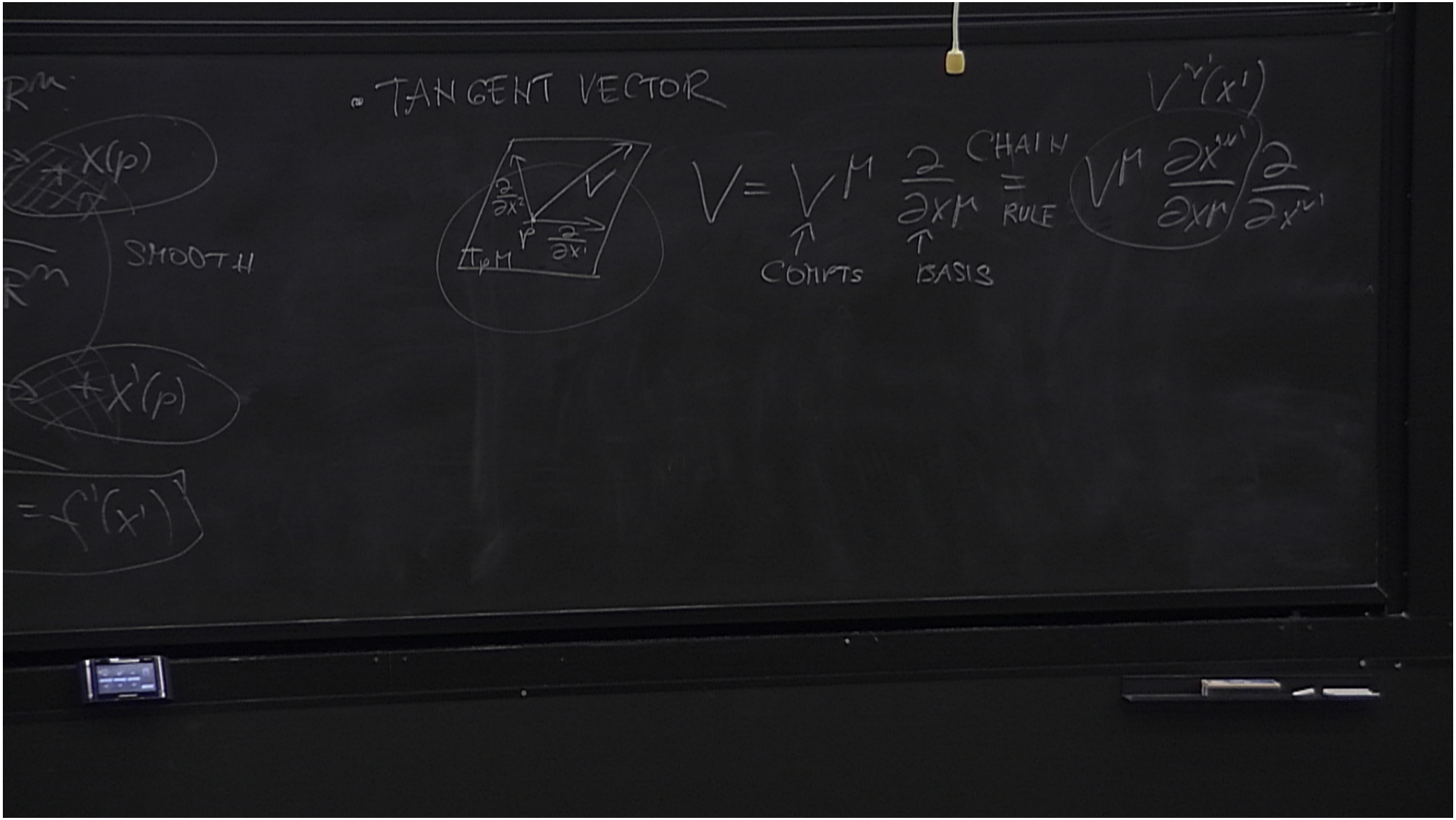
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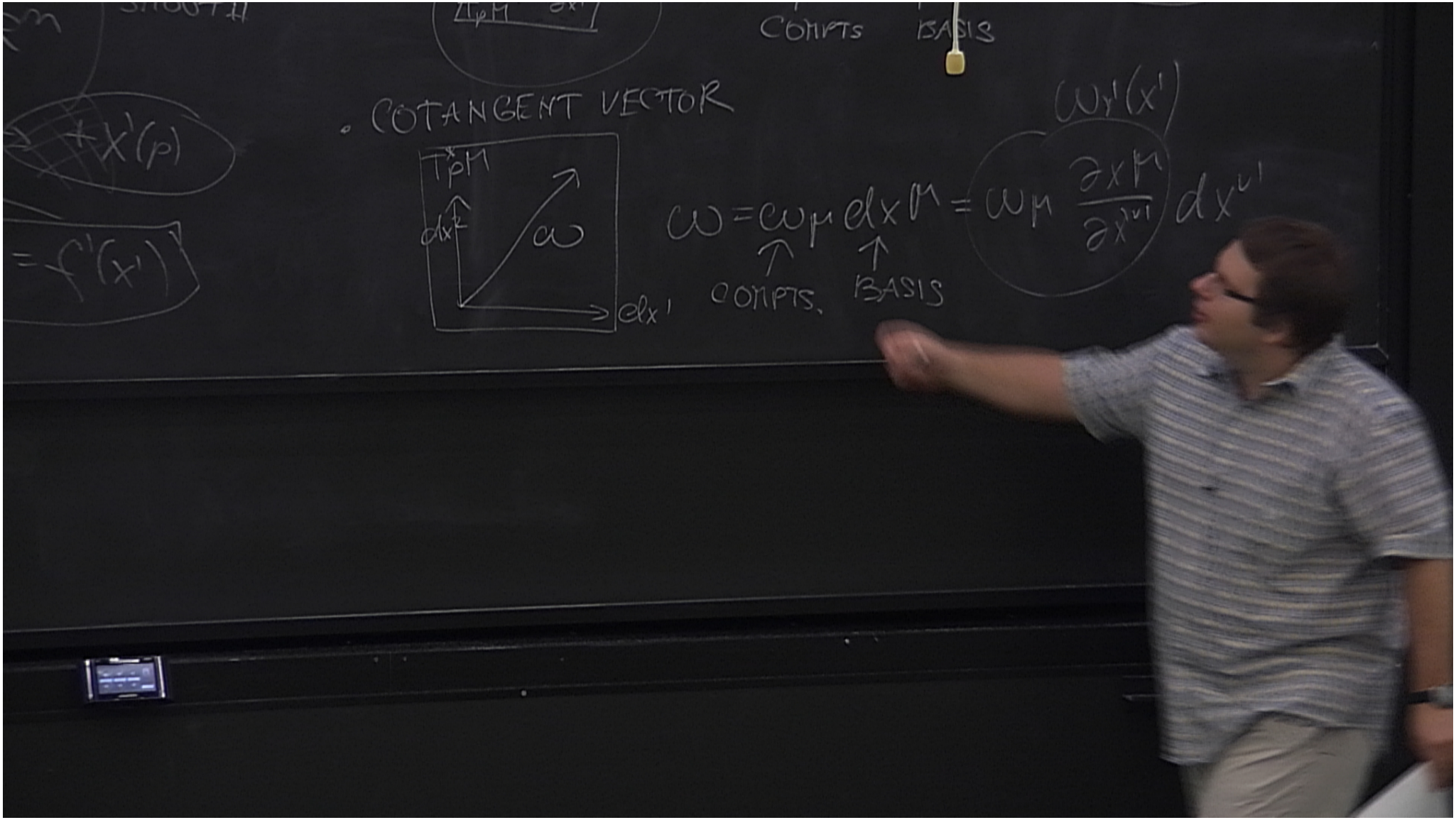
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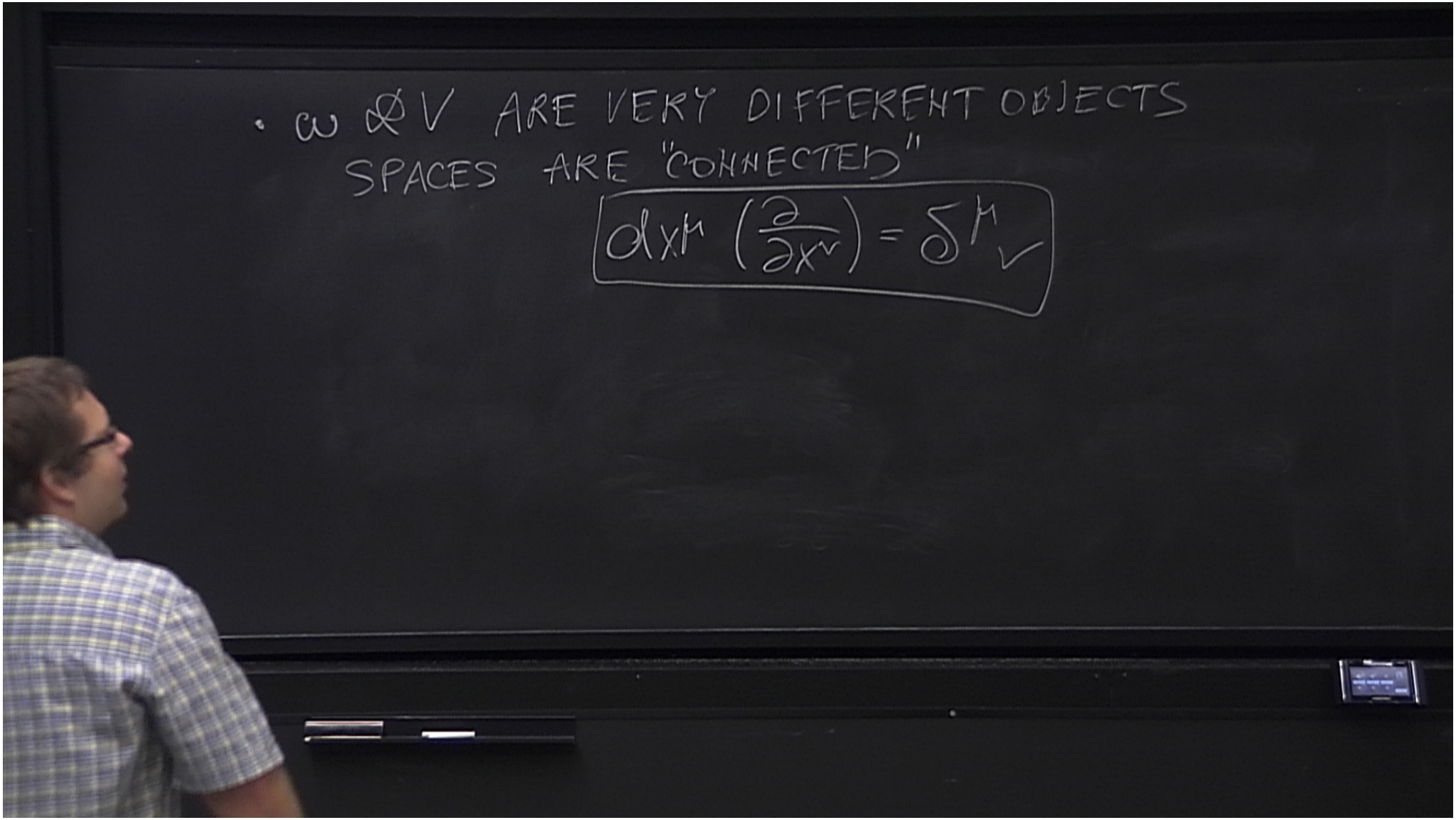

YESTERDAY.
· MANIFOLD
X
$\mathbb{R}^m$
$+X(p)$
$+X'(p)$

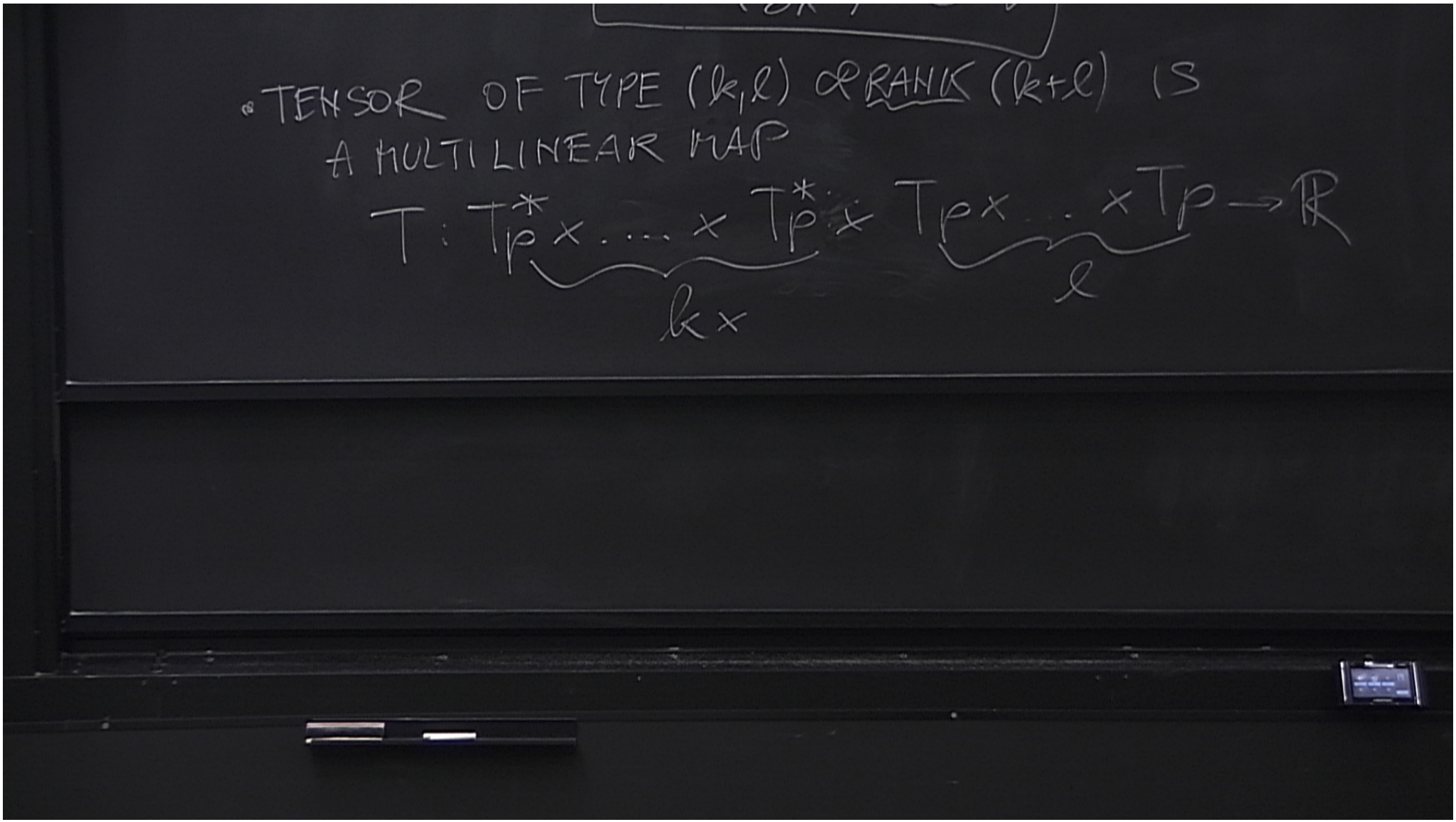

ESTERDAY.
- MANIFOLD
X
$\mathbb{R}^m$
p
X(p)
SMOOTH
X'
$\mathbb{R}^n$
X'(p)

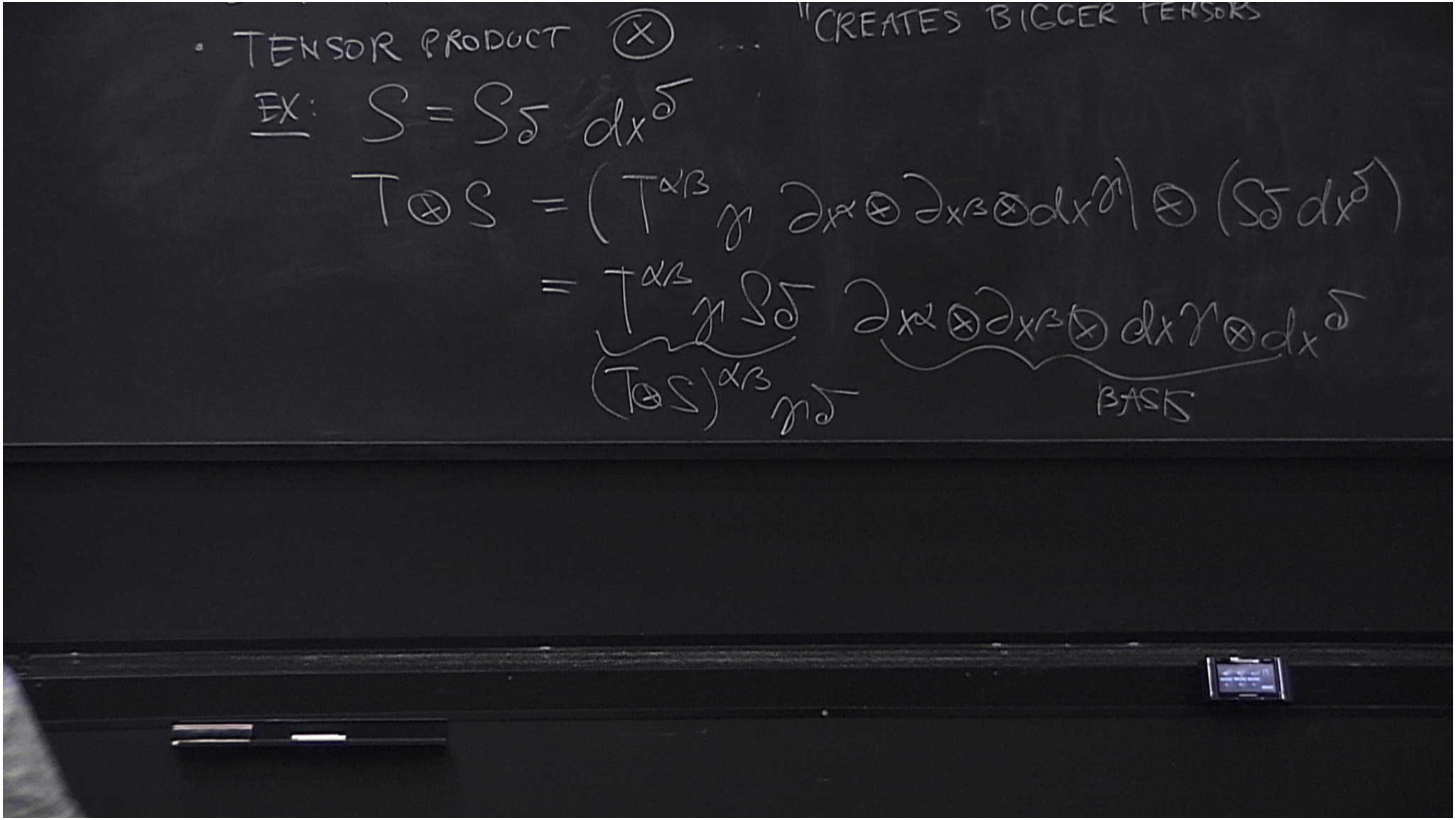

TANGENT VECTOR
X(p)
SMOOTH
X'(p)
f'(x')
V
p
T_pM

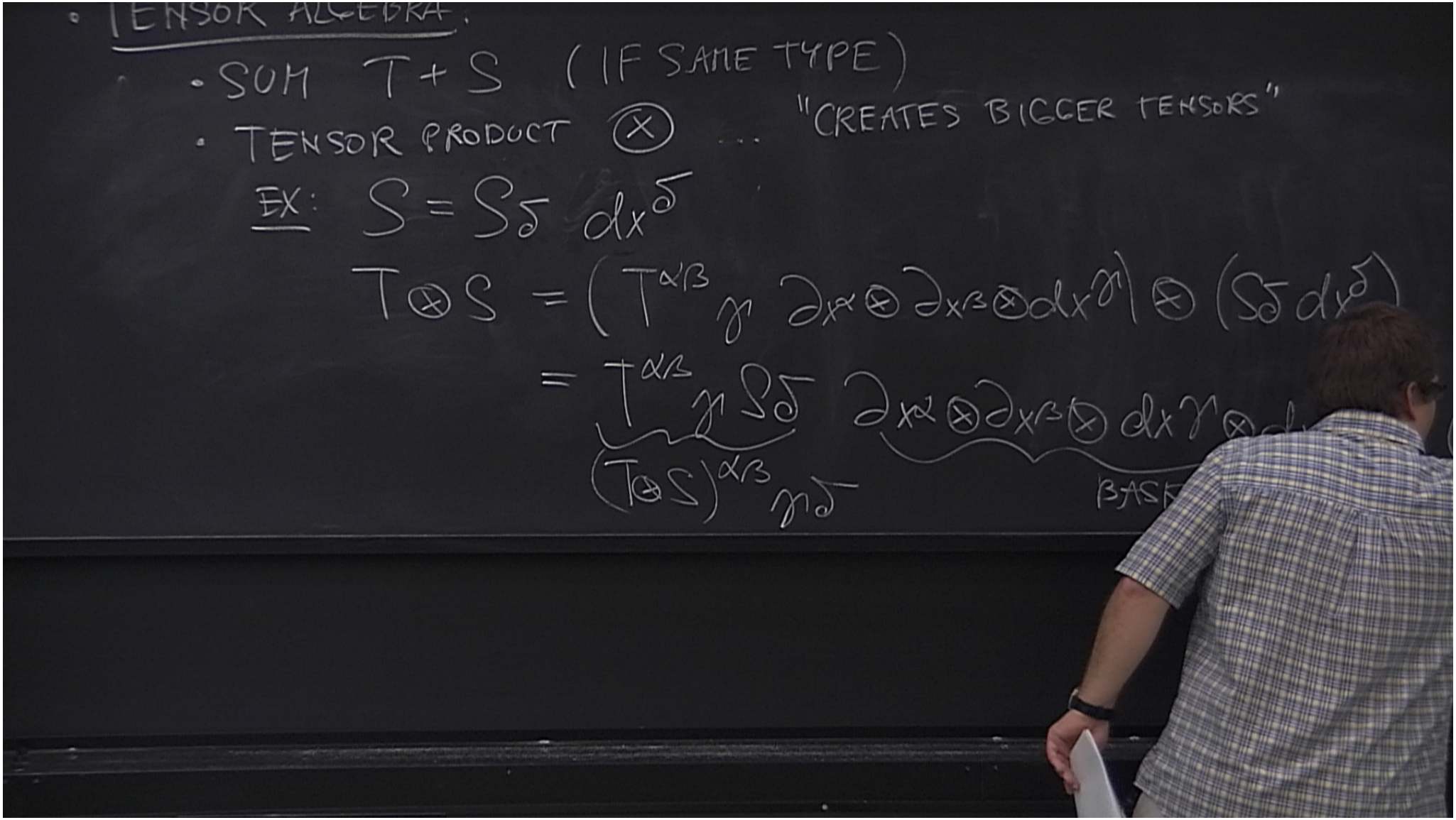

TANGENT VECTOR
SMOOTH
CHAIN RULE
COMPTS
BASIS

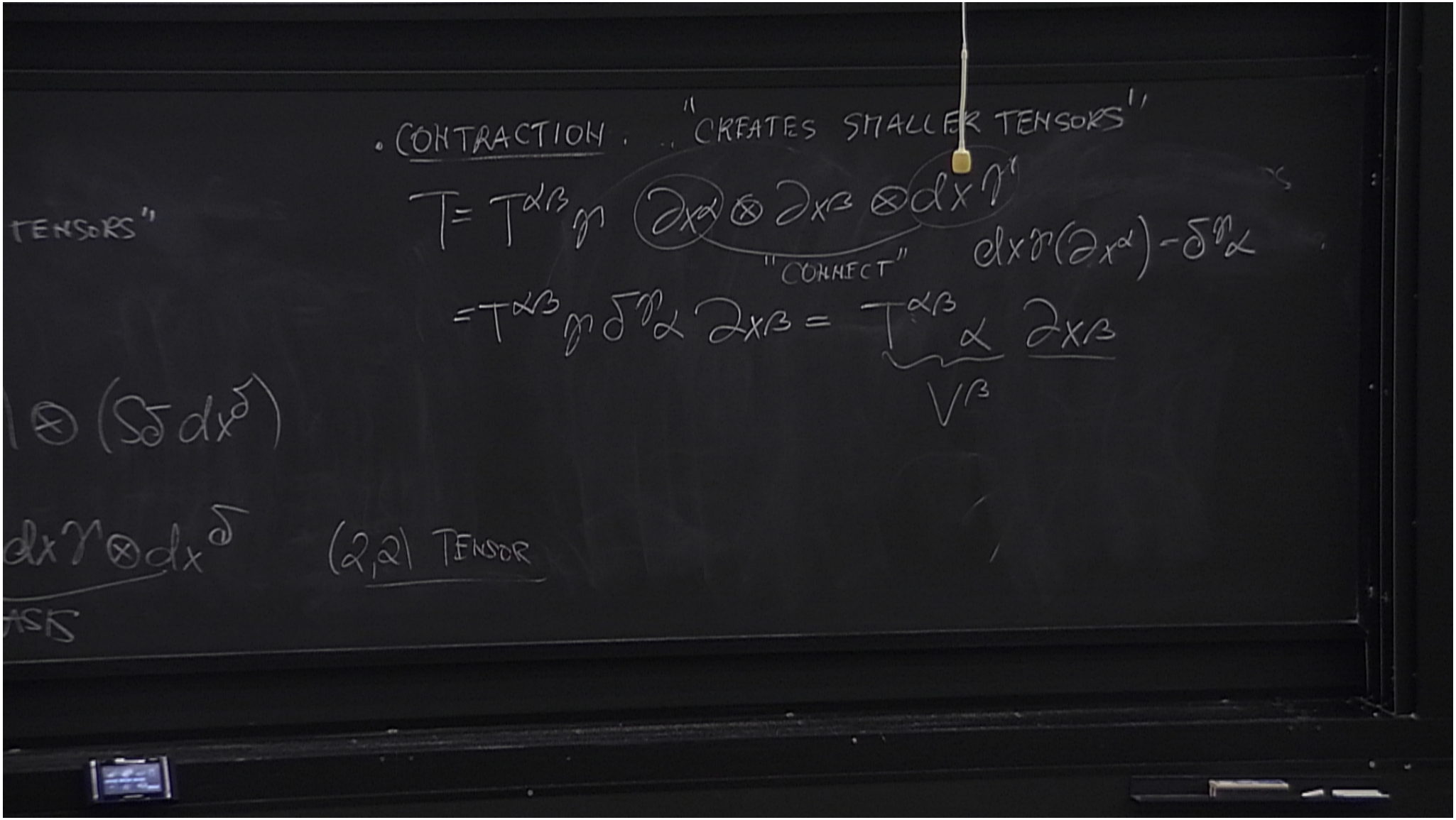

COMPTS
BASIS
COTANGENT VECTOR
T^*_pM
dx^2
ω
dx^1
$\omega = \omega_\mu dx^\mu = \omega_\mu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$
COMPTS.
BASIS
$\omega_{x'}(x')$

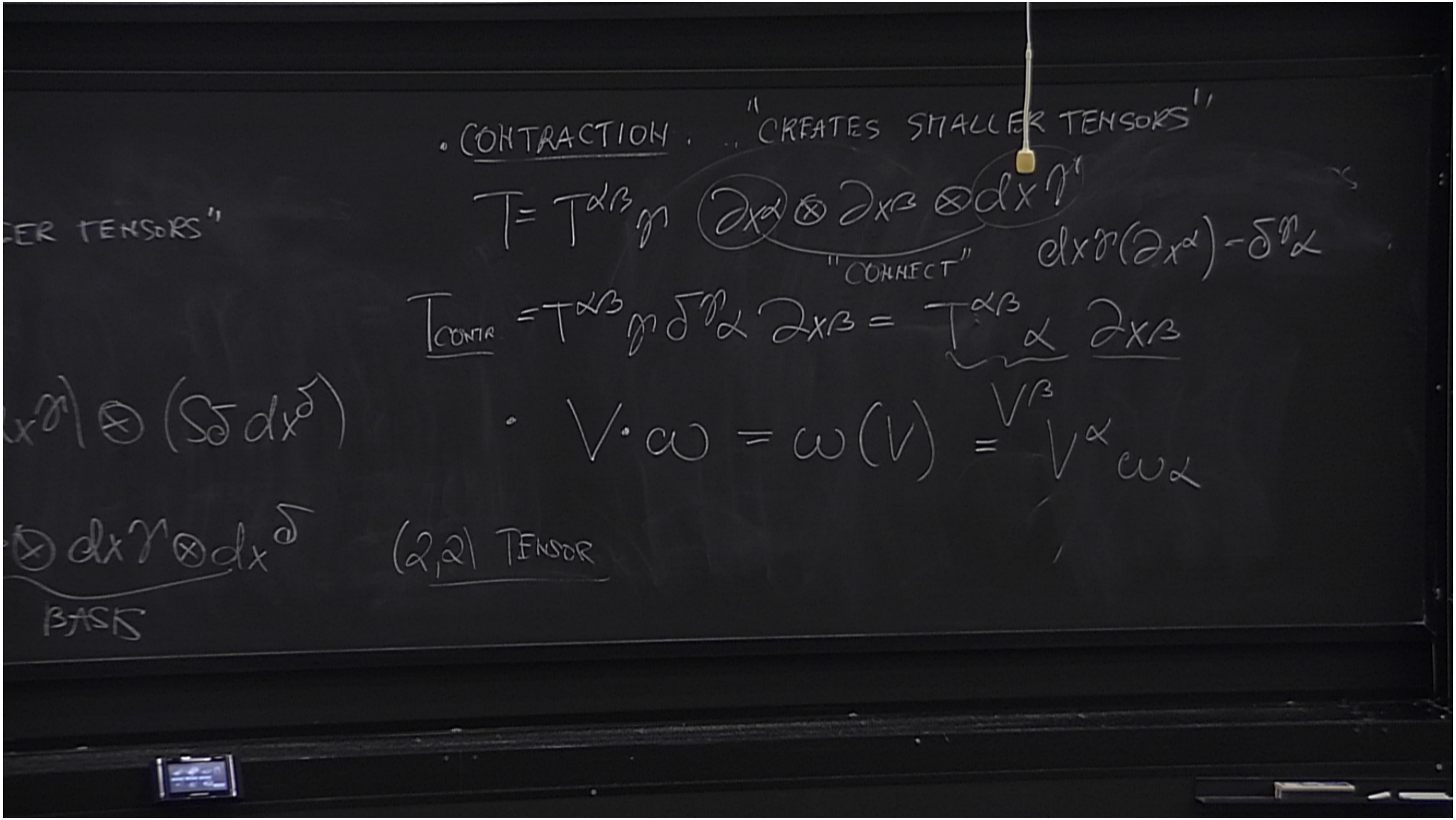

• ω & V ARE VERY DIFFERENT OBJECTS
SPACES ARE "CONNECTED"
$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu$

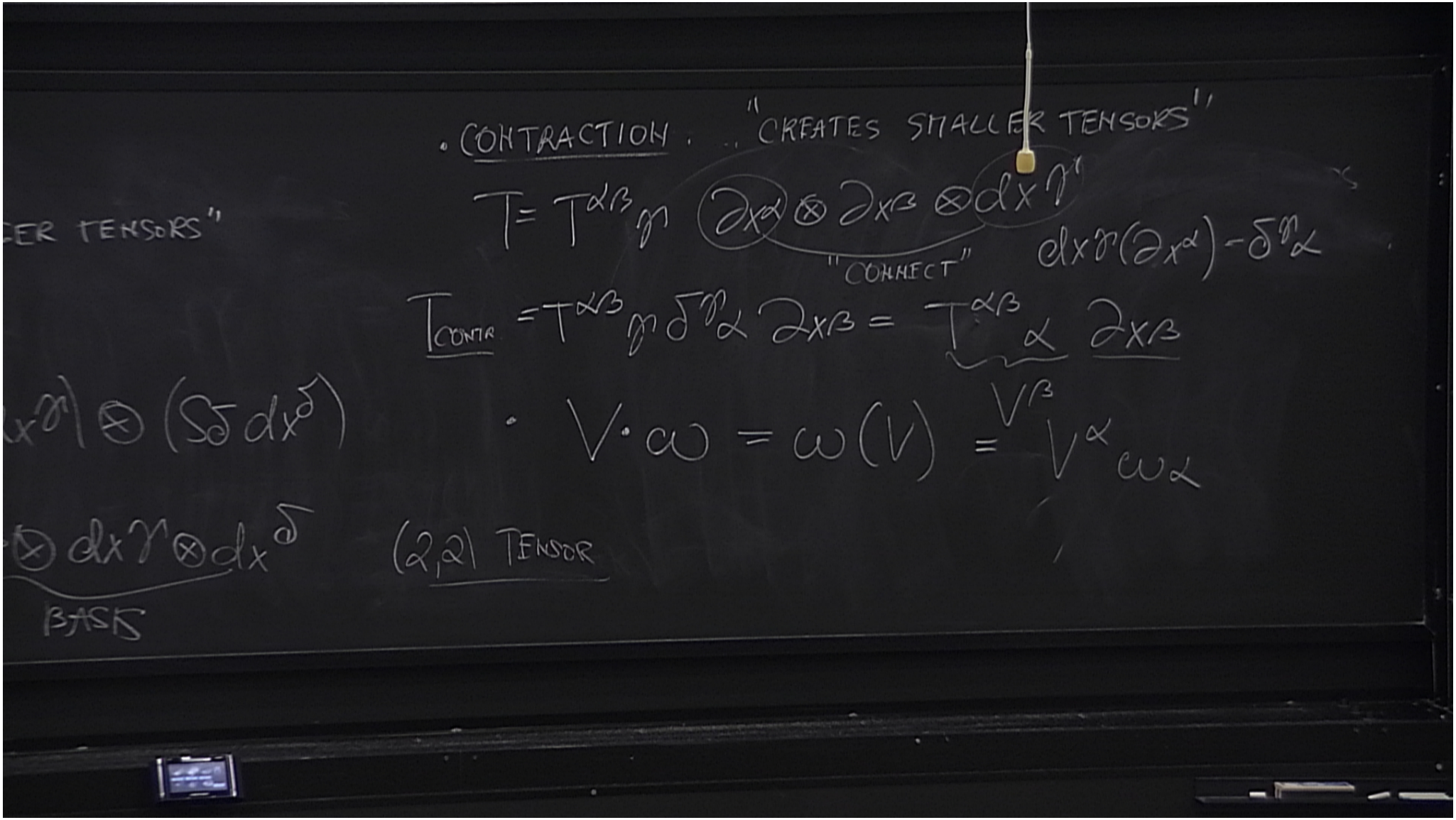

• TENSOR OF TYPE (k,ℓ) & RANK (k+ℓ) IS A MULTILINEAR MAP
$T : \underbrace{T_p^* \times \cdots \times T_p^*}_{k} \times \underbrace{T_p \times \cdots \times T_p}_{\ell} \to \mathbb{R}$

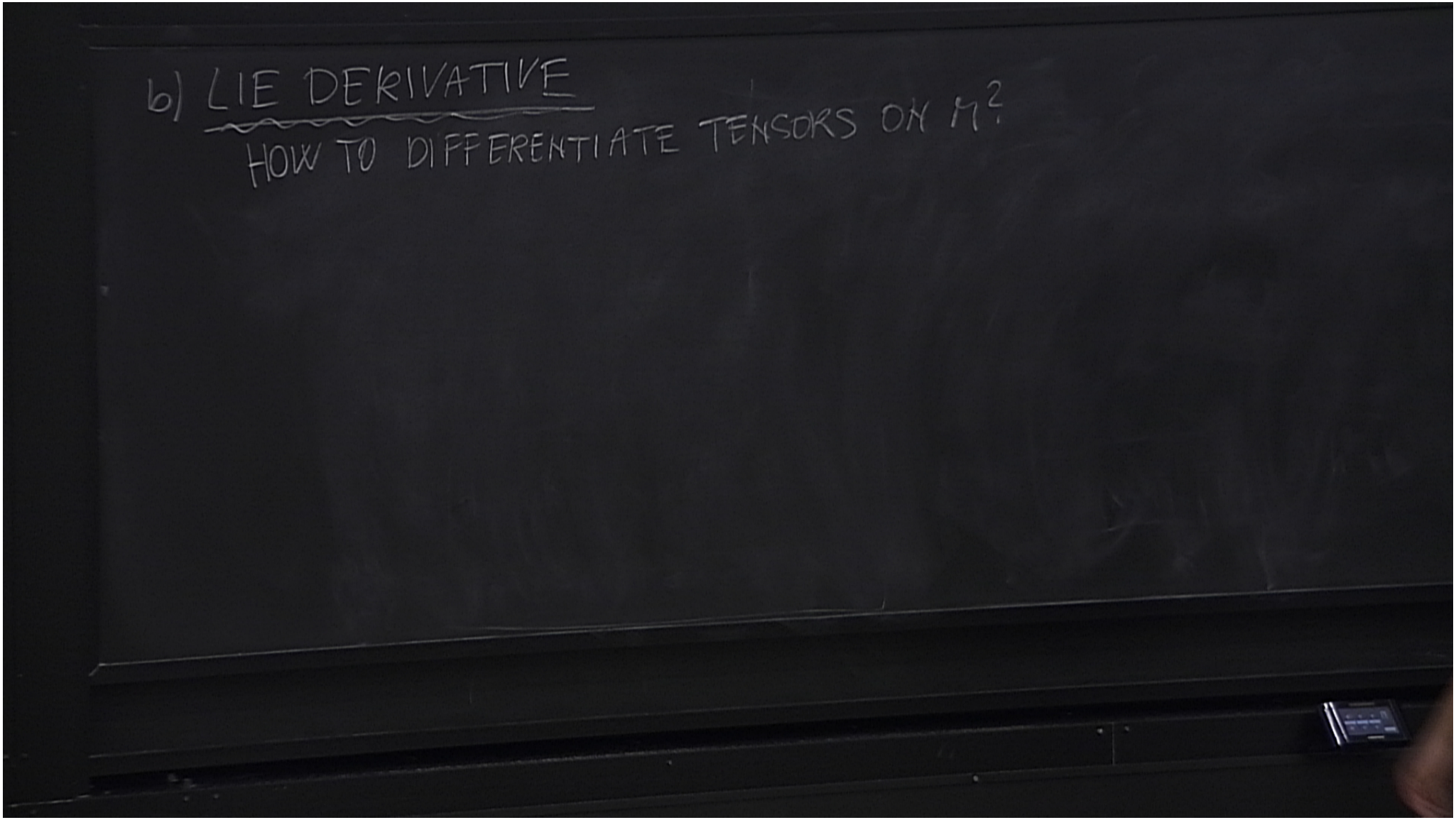

• TENSOR PRODUCT ⊗ ... "CREATES BIGGER TENSORS
EX: $S = S_\delta \, dx^\delta$
$T \otimes S = (T^{\alpha\beta}{}_\gamma \, \partial_{x^\alpha} \otimes \partial_{x^\beta} \otimes dx^\gamma) \otimes (S_\delta \, dx^\delta)$
$= T^{\alpha\beta}{}_\gamma S_\delta \, \partial_{x^\alpha} \otimes \partial_{x^\beta} \otimes dx^\gamma \otimes dx^\delta$
$(T \otimes S)^{\alpha\beta}{}_{\gamma\delta}$
BASIS

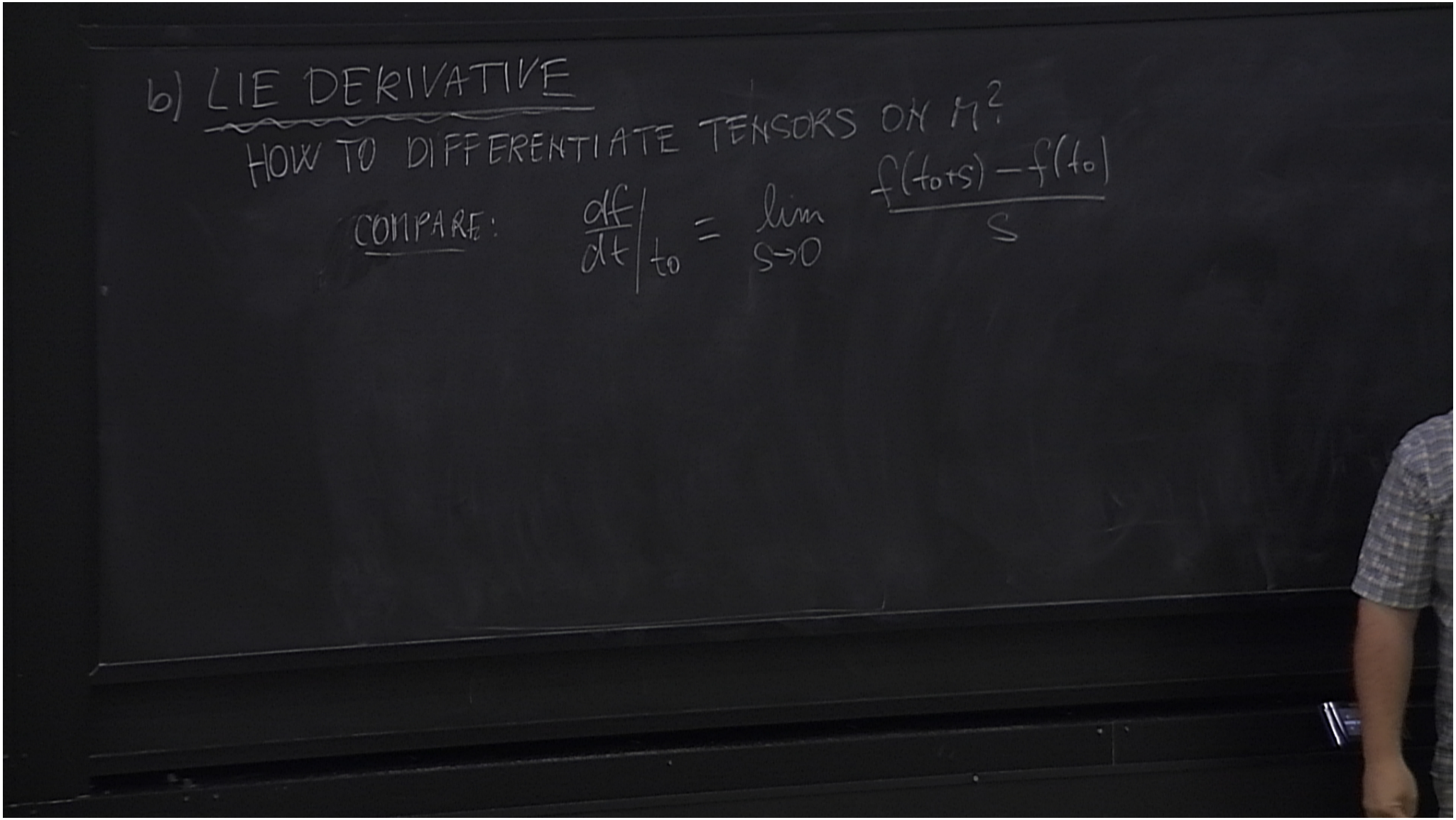

TENSOR ALGEBRA:
• SUM T+S (IF SAME TYPE)
• TENSOR PRODUCT ⊗ ... "CREATES BIGGER TENSORS"
EX: S = S_δ dx^δ
T⊗S = (T^{αβ}_γ ∂x^α ⊗ ∂x^β ⊗ dx^γ) ⊗ (S_δ dx^δ)
= T^{αβ}_γ S_δ ∂x^α ⊗ ∂x^β ⊗ dx^γ ⊗
(T⊗S)^{αβ}_{γδ}

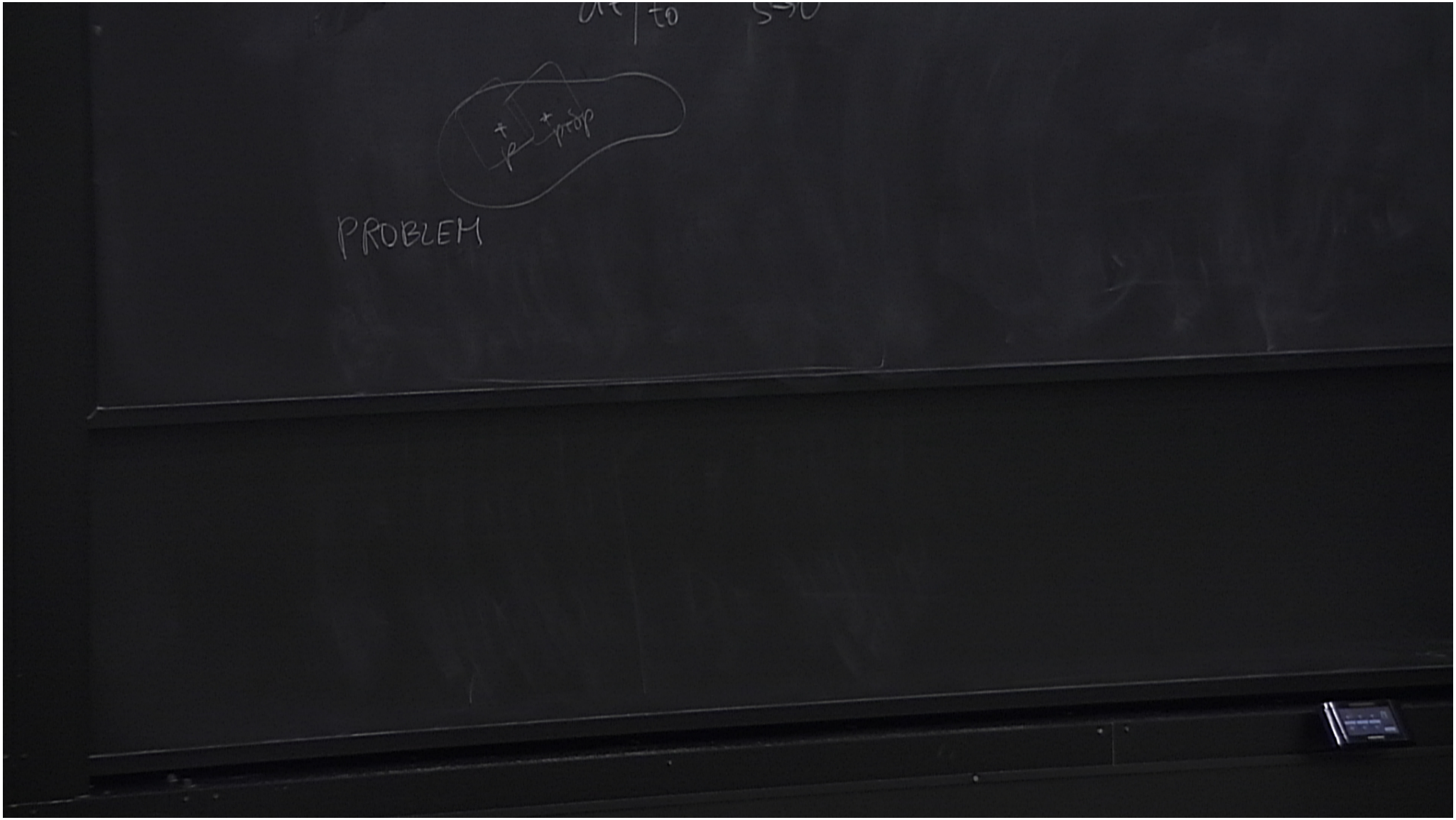

• CONTRACTION . . . "CREATES SMALLER TENSORS"
$T = T^{\alpha\beta}{}_{\gamma}\, \partial x^{\alpha} \otimes \partial x^{\beta} \otimes dx^{\gamma}$
"CONTRACT"
$dx^{\gamma}(\partial x^{\alpha}) = \delta^{\gamma}_{\alpha}$
$= T^{\alpha\beta}{}_{\gamma}\, \delta^{\gamma}_{\alpha}\, \partial x^{\beta} = T^{\alpha\beta}{}_{\alpha}\, \partial x^{\beta}$
V^{β}
TENSORS"
$\otimes (S\, dx^{\delta})$
$dx^{\gamma} \otimes dx^{\delta}$
(2,2) TENSOR


• CONTRACTION . . . "CREATES SMALLER TENSORS"
$T = T^{\alpha\beta}{}_{\gamma}\, \partial x^{\alpha} \otimes \partial x^{\beta} \otimes dx^{\gamma}$
"CONNECT"
$dx^{\gamma}(\partial x^{\alpha}) = \delta^{\gamma}{}_{\alpha}$
$T_{CONTR} = T^{\alpha\beta}{}_{\gamma}\, \delta^{\gamma}{}_{\alpha}\, \partial x^{\beta} = T^{\alpha\beta}{}_{\alpha}\, \partial x^{\beta}$
$V \cdot \omega = \omega(V) = V^{\alpha}\omega_{\alpha}$
(2,2) TENSOR
BASIS


• CONTRACTION . . . "CREATES SMALLER TENSORS"
$T = T^{\alpha\beta}{}_{\gamma}\ \partial x^\alpha \otimes \partial x^\beta \otimes dx^\gamma$
"CONNECT"
$dx^\gamma(\partial x^\alpha) = \delta^\gamma{}_\alpha$
$T_{CONTR} = T^{\alpha\beta}{}_{\gamma}\,\delta^\gamma{}_\alpha\,\partial x^\beta = T^{\alpha\beta}{}_{\alpha}\,\partial x^\beta$
$V \cdot \omega = \omega(V) = V^\beta \omega_\alpha$
(2,2) TENSOR
BASIS


b) LIE DERIVATIVE
HOW TO DIFFERENTIATE TENSORS ON M^2

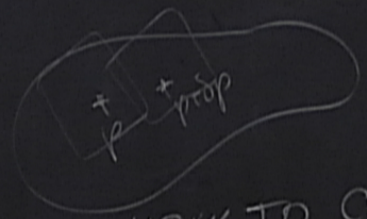

b) LIE DERIVATIVE
HOW TO DIFFERENTIATE TENSORS ON M^2?
COMPARE: $\left.\frac{df}{dt}\right|_{t_0} = \lim_{s\to 0} \frac{f(t_0+s)-f(t_0)}{s}$


PROBLEM

b) LIE DERIVATIVE

HOW TO DIFFERENTIATE TENSORS ON M^2

COMPARE: $\left.\frac{df}{dt}\right|_{t_0} = \lim_{s \to 0} \frac{f(t_0+s) - f(t_0)}{s}$



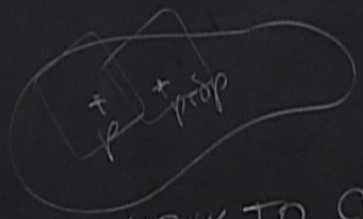
PROBLEM HOW TO COMPARE A VECTOR AT $p+\delta p$ WITH A VECTOR AT p.

b) LIE DERIVATIVE

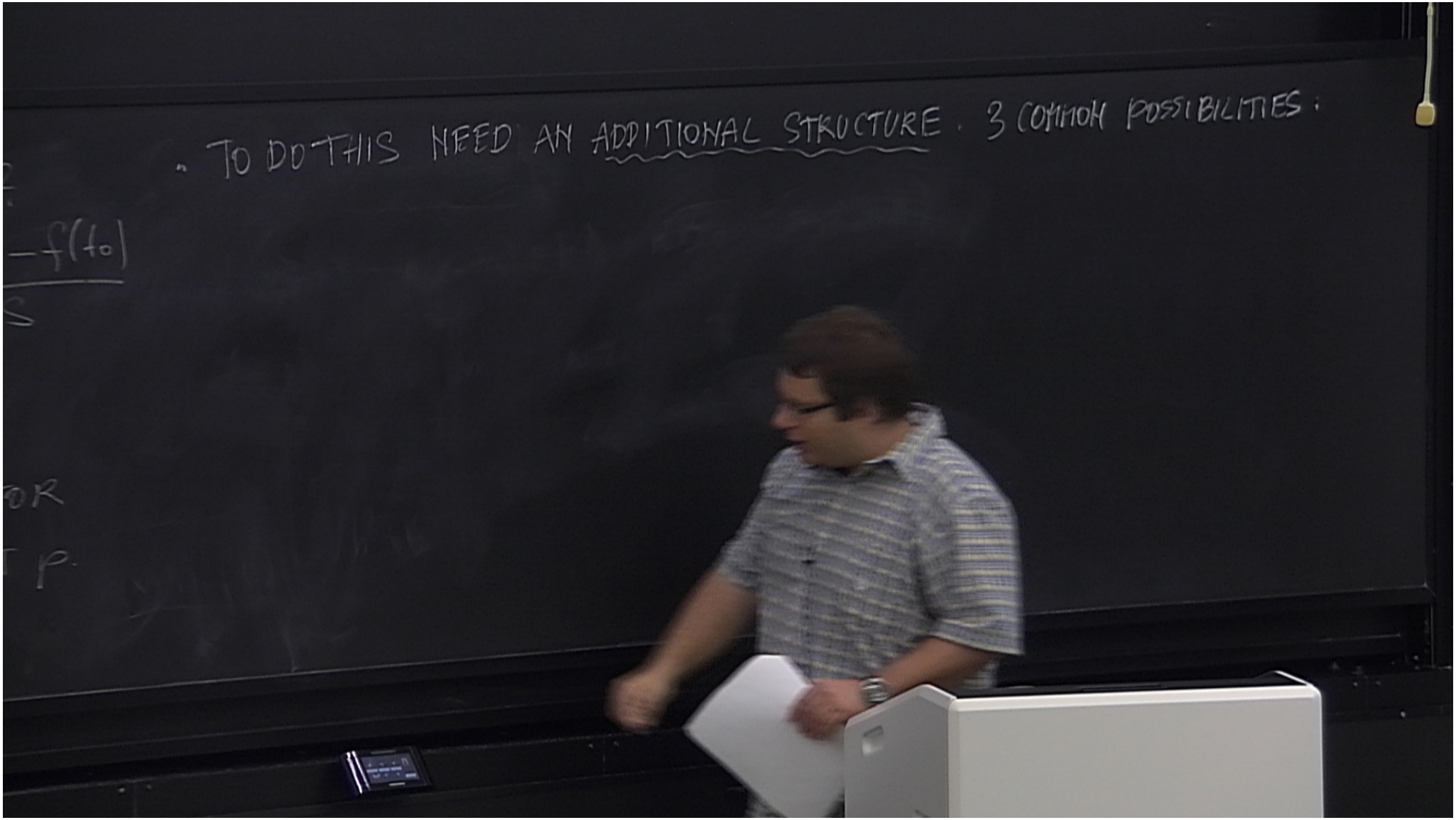
HOW TO DIFFERENTIATE TENSORS ON M^2

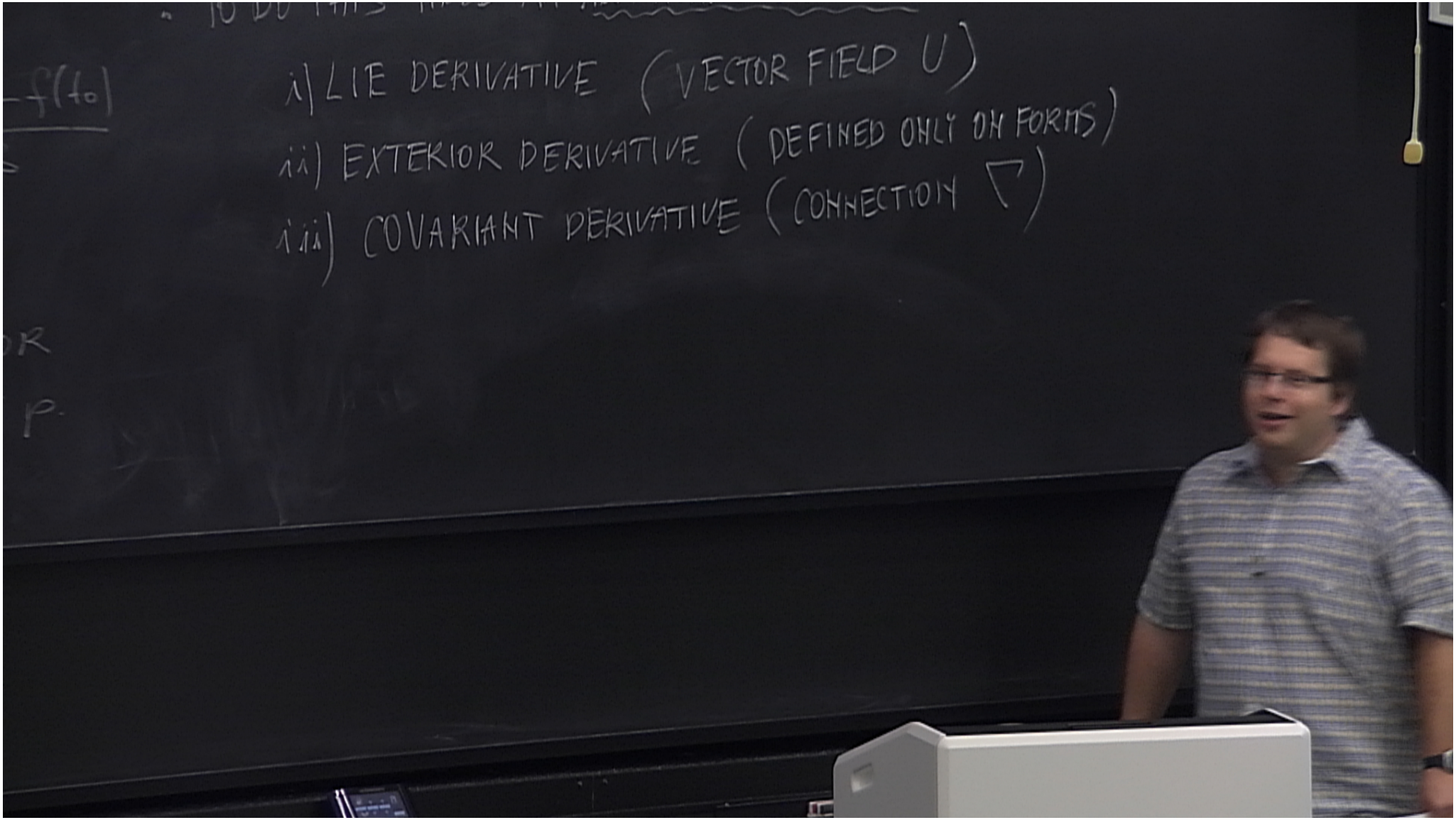
. TO DO THIS NE

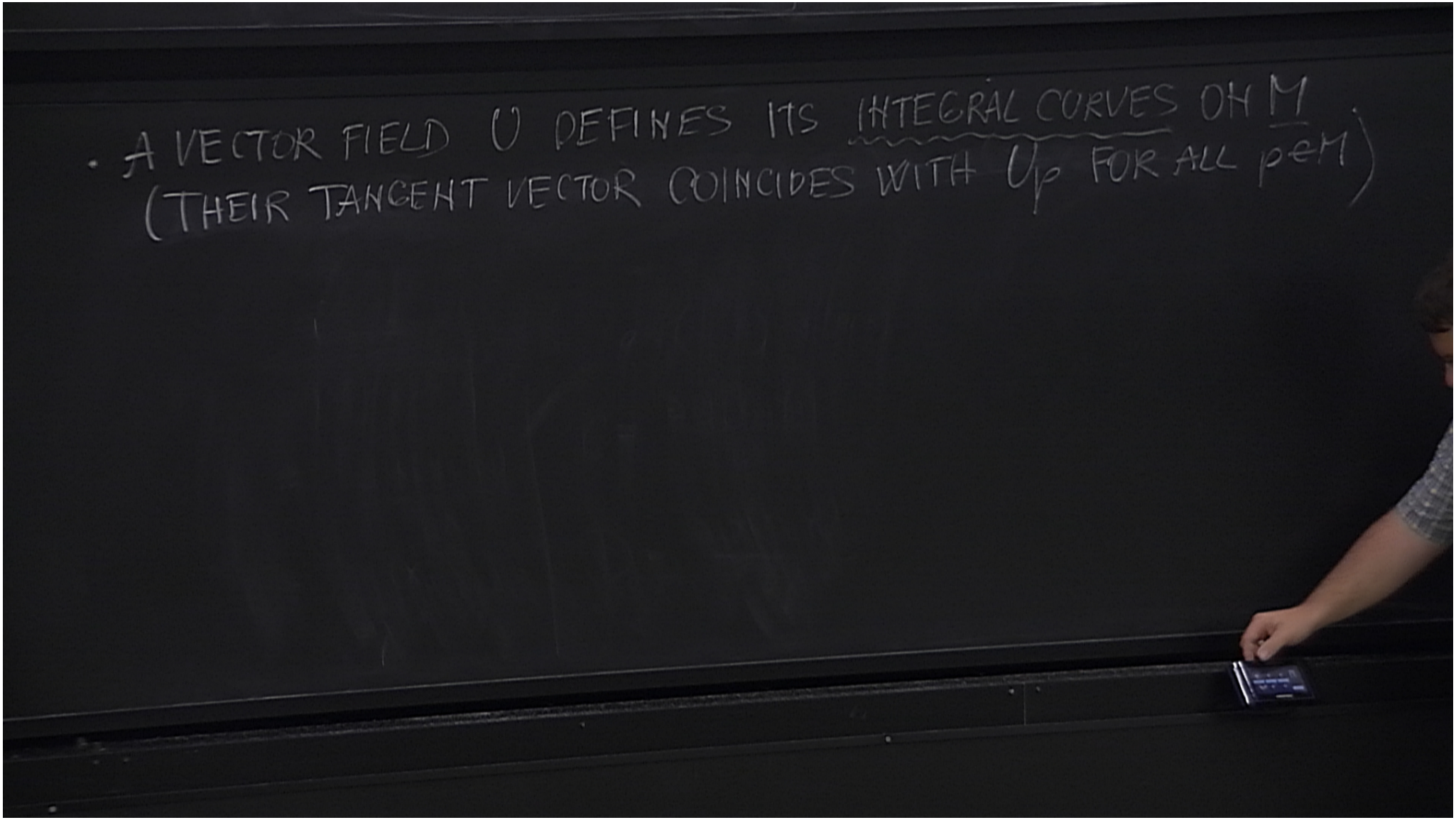
COMPARE: $\left.\frac{df}{dt}\right|_{t_0} = \lim_{s\to 0} \frac{f(t_0+s)-f(t_0)}{s}$

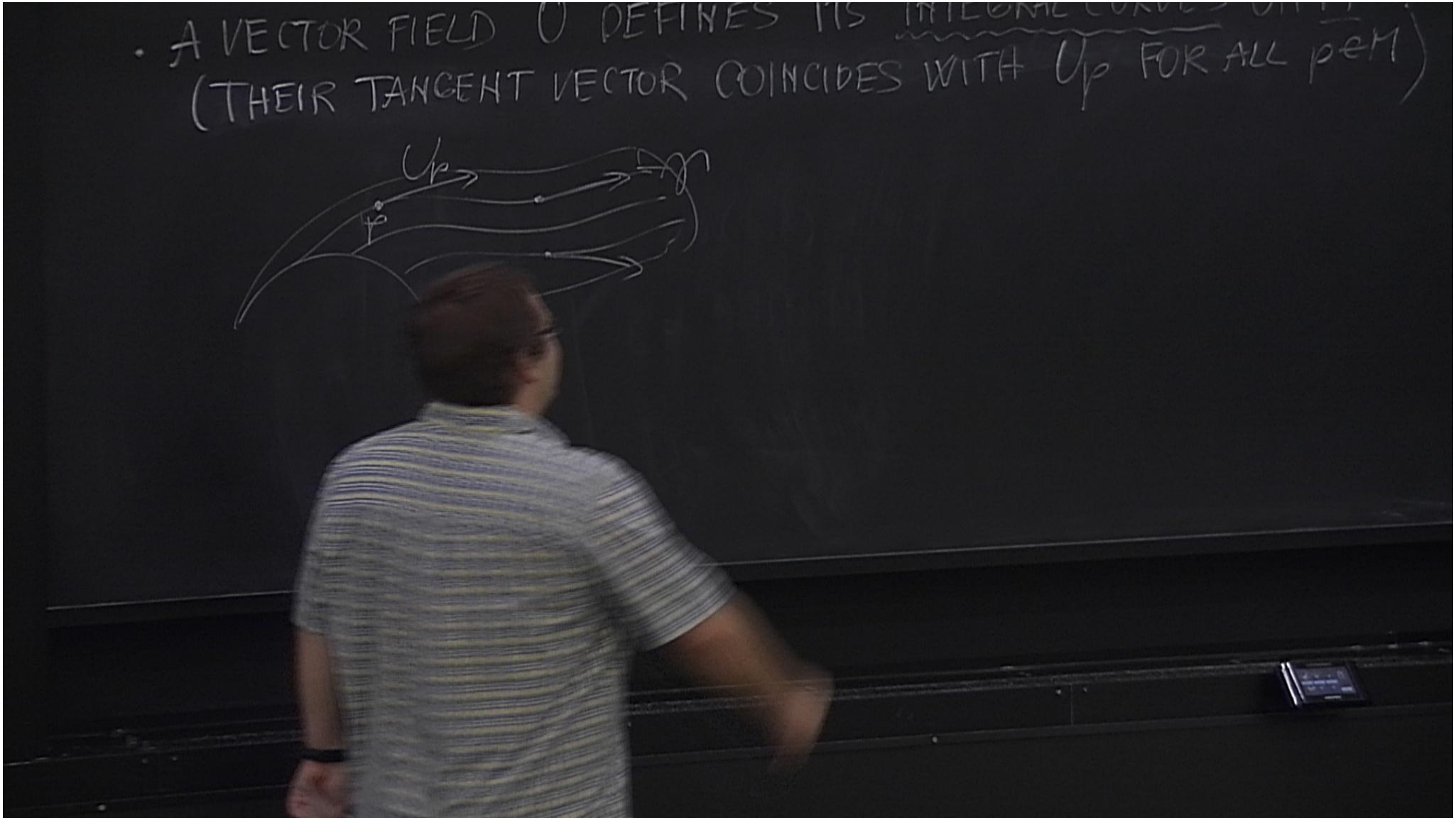


PROBLEM HOW TO COMPARE A VECTOR AT $p+\delta p$ WITH A VECTOR AT p.

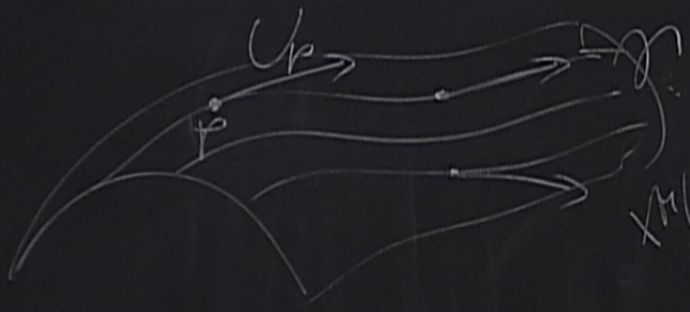

TO DO THIS NEED AN ADDITIONAL STRUCTURE. 3 COMMON POSSIBILITIES:


i] LIE DERIVATIVE (VECTOR FIELD U)
ii) EXTERIOR DERIVATIVE (DEFINED ONLY ON FORMS)
iii) COVARIANT DERIVATIVE (CONNECTION ∇)


• A VECTOR FIELD U DEFINES ITS INTEGRAL CURVES ON M
(THEIR TANGENT VECTOR COINCIDES WITH U_p FOR ALL $p \in M$)


A VECTOR FIELD V DEFINES ITS INTEGRAL CURVES
(THEIR TANGENT VECTOR COINCIDES WITH V_p FOR ALL $p \in M$)
V_p
p

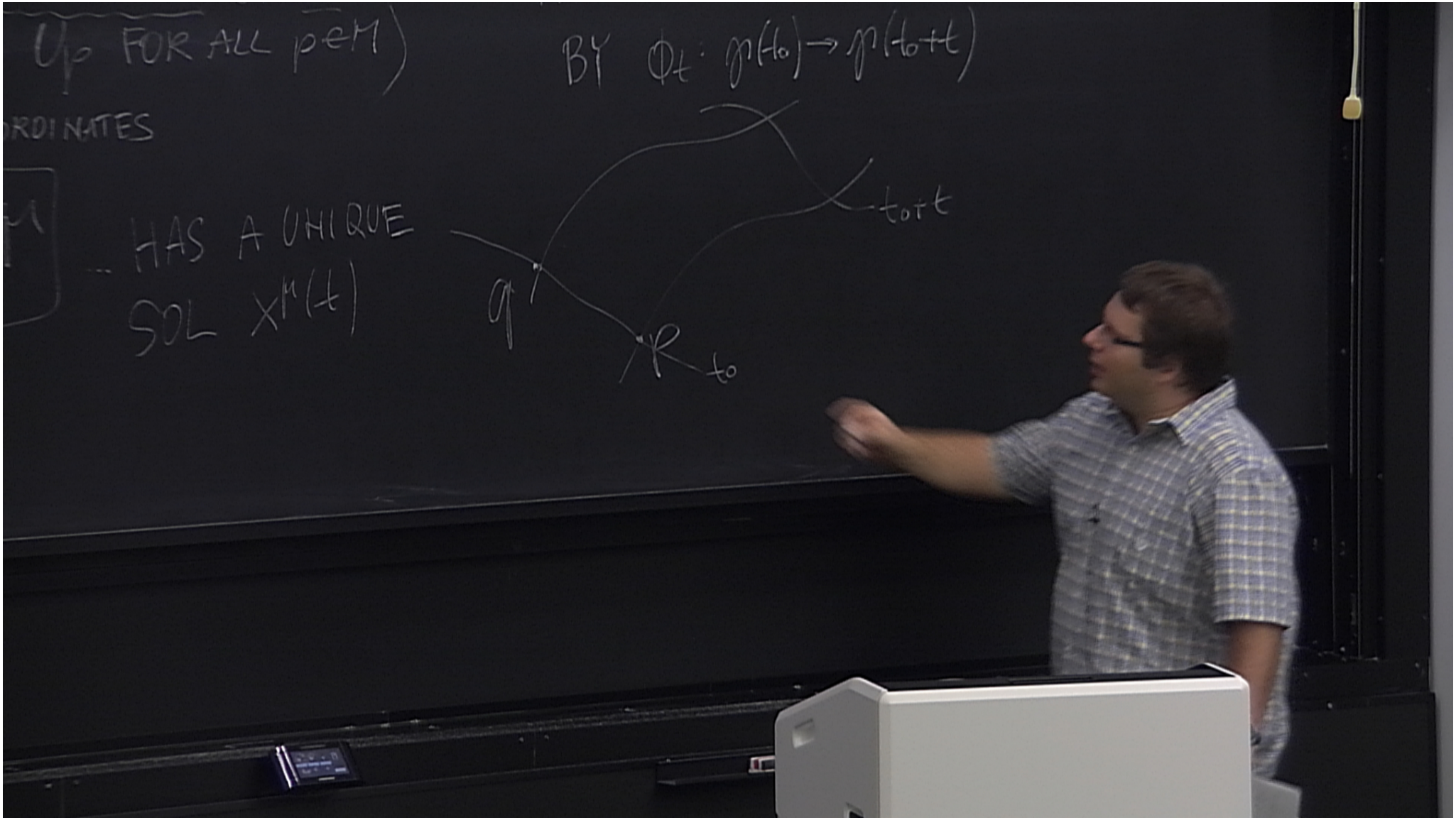
- A VECTOR FIELD U DEFINES ITS INTEGRAL CURVES ON M
  (THEIR TANGENT VECTOR COINCIDES WITH U_p FOR ALL $p \in M$)



PROOF. IN COORDINATES

$$\frac{dx^\mu}{dt} = U^\mu$$

... HAS A UNIQUE SOL $x^\mu(t)$


FOR ALL p∈M)
BY $\phi_t : \gamma(t_0) \to \gamma(t_0+t)$
HAS A UNIQUE
SOL $X^M(t)$
t_0+t
t_0

ϕ_t · t .. CONTINUOS

• $\phi_0 = Id$, $\phi_{t+s} = \phi_t \circ \phi_s$, $\phi_{-t} = \phi_t^{-1}$

ϕ_t : t .. CONTINUOS

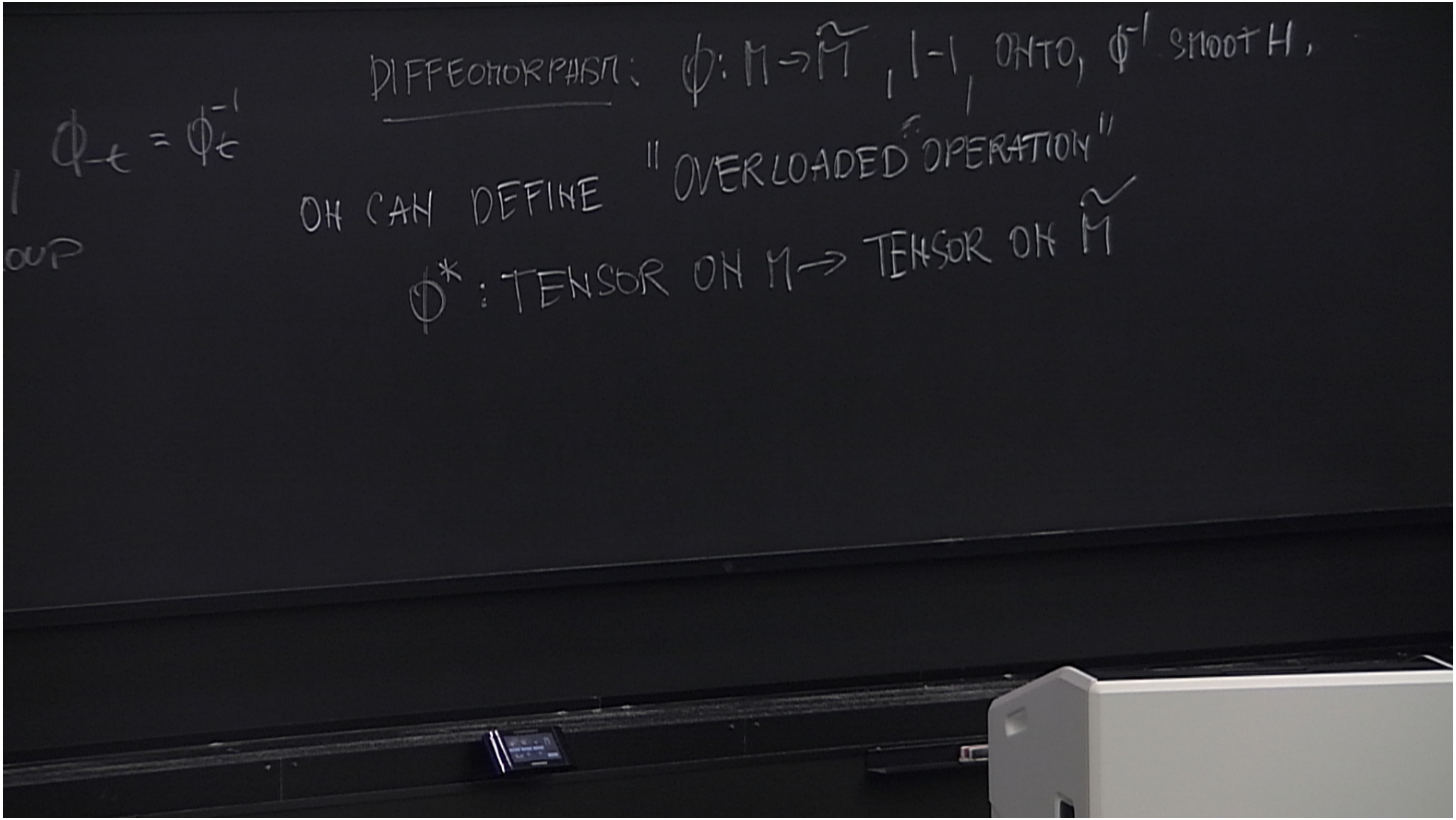
• $\phi_0 = Id$, $\phi_{t+s} = \phi_t \circ \phi_s$, $\phi_{-t} = \phi_t^{-1}$

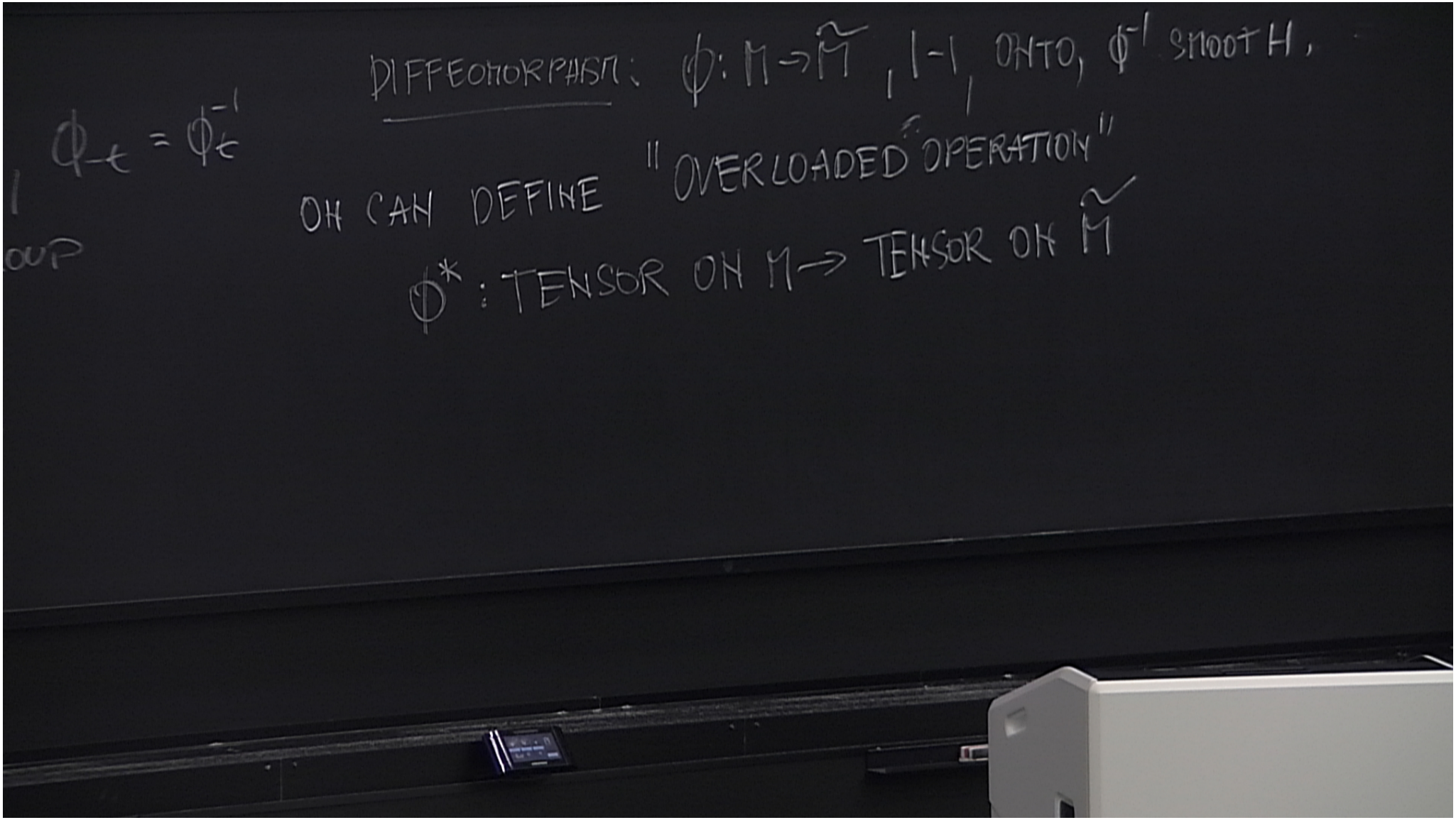
1-PARAMETRIC (LIE) GROUP

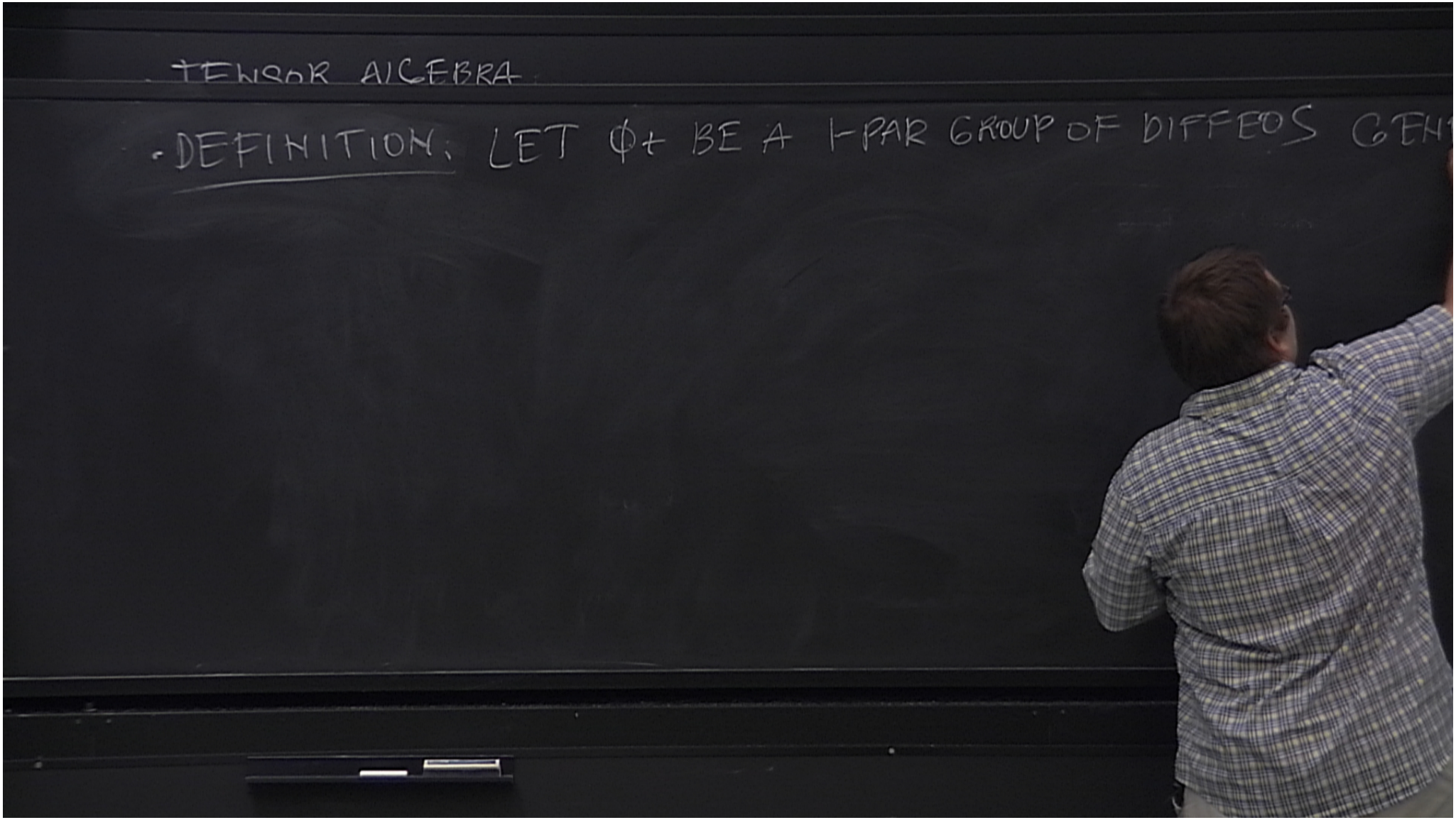
OF DIFFEOMORPHISMS

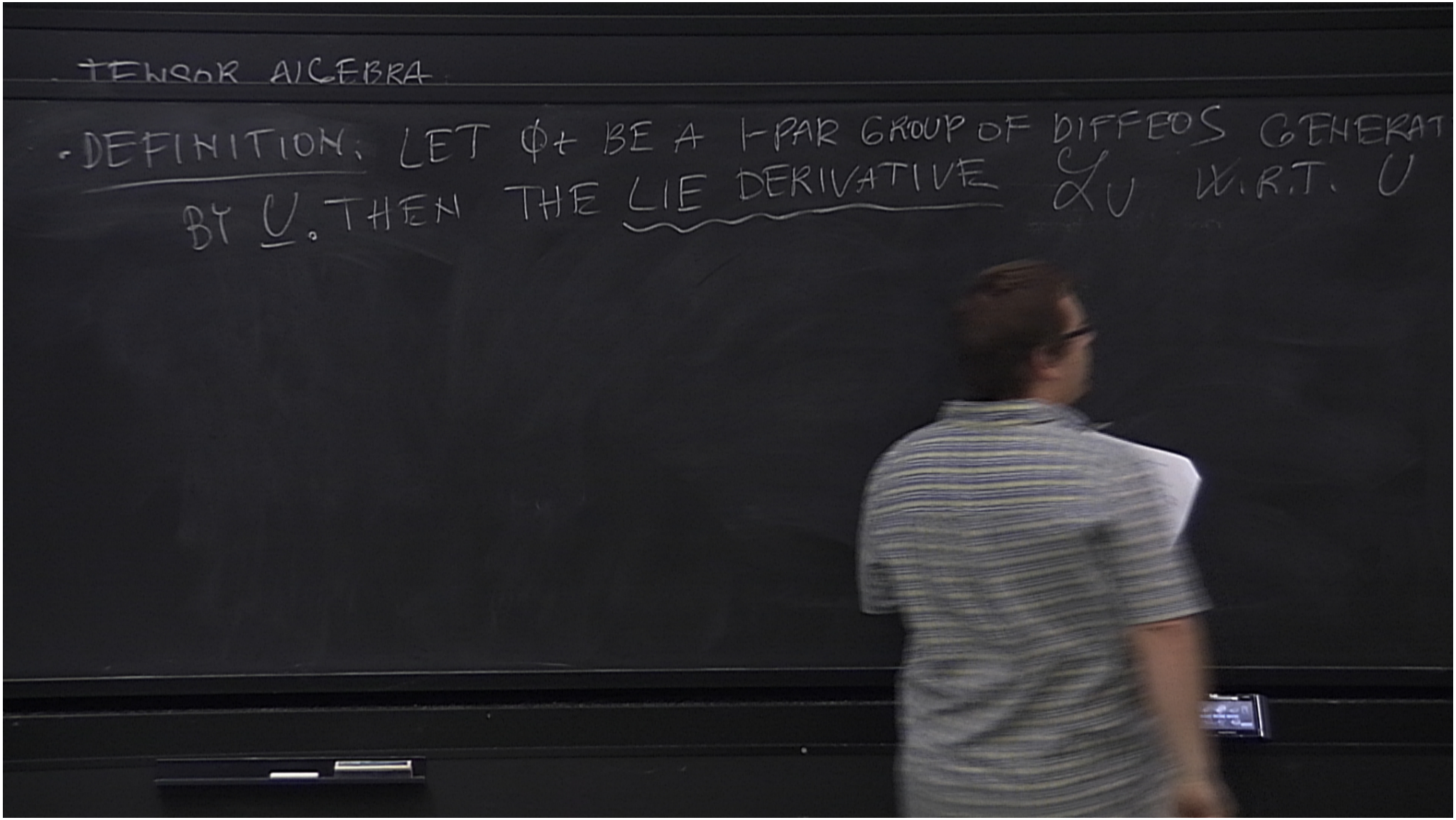
ALL YOU CAN WISH FOR

DIFFEOMORPHISM: ϕ:


$\phi_{-t} = \phi_t^{-1}$
DIFFEOMORPHISM: $\phi: M \to \tilde{M}$, 1-1, ONTO, ϕ^{-1} SMOOTH,
ONE CAN DEFINE "OVERLOADED OPERATION"
ϕ^*: TENSOR ON $M \to$ TENSOR ON $\tilde{M}$


$\phi_t = \phi_t^{-1}$
DIFFEOMORPHISM: $\phi: M \to \tilde{M}$, 1-1, ONTO, ϕ^{-1} SMOOTH,
OH CAN DEFINE "OVERLOADED OPERATION"
ϕ^*: TENSOR ON $M \to$ TENSOR ON $\tilde{M}$

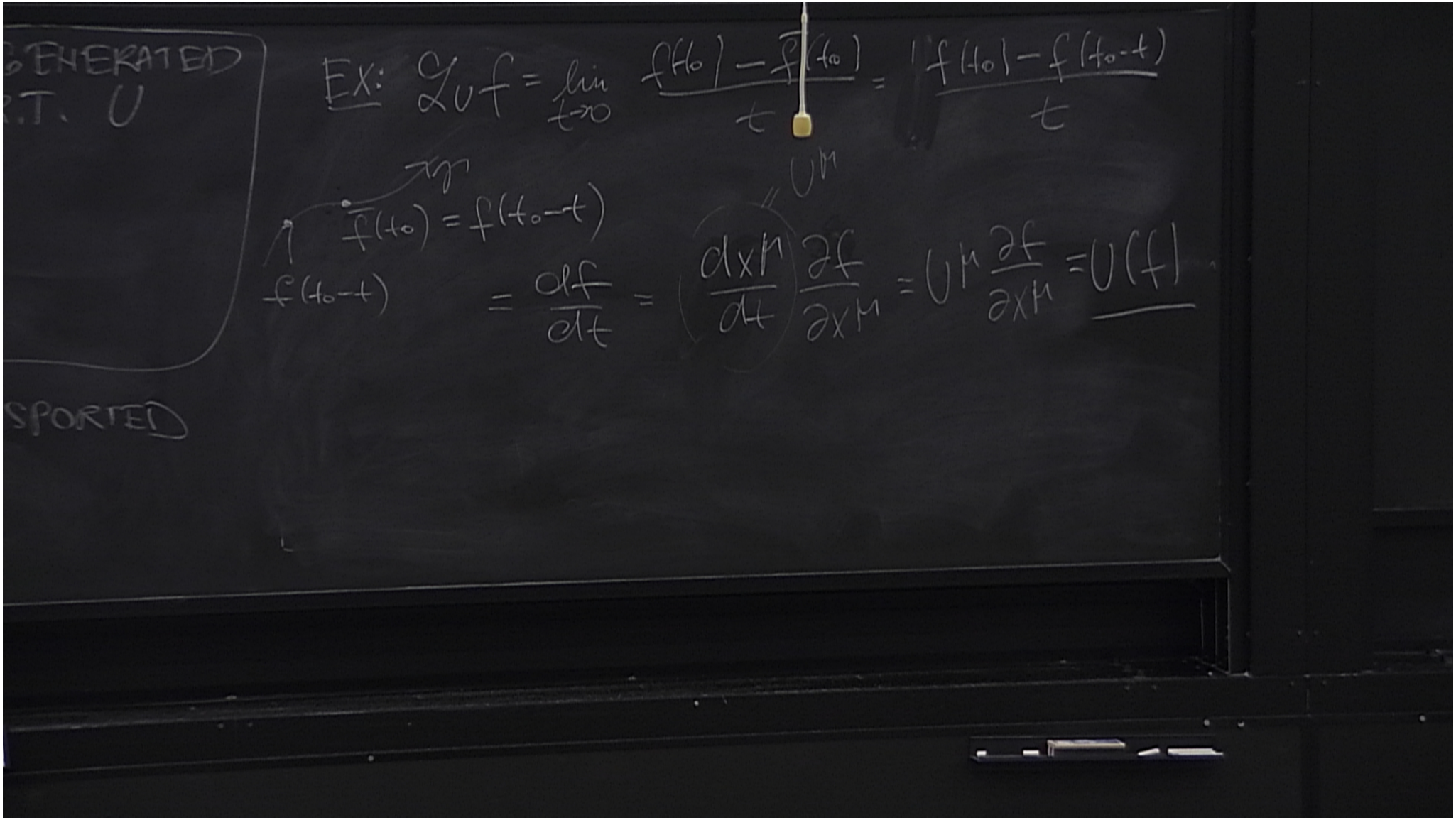

TENSOR ALGEBRA
• DEFINITION: LET ϕ_t BE A 1-PAR GROUP OF DIFFEOS

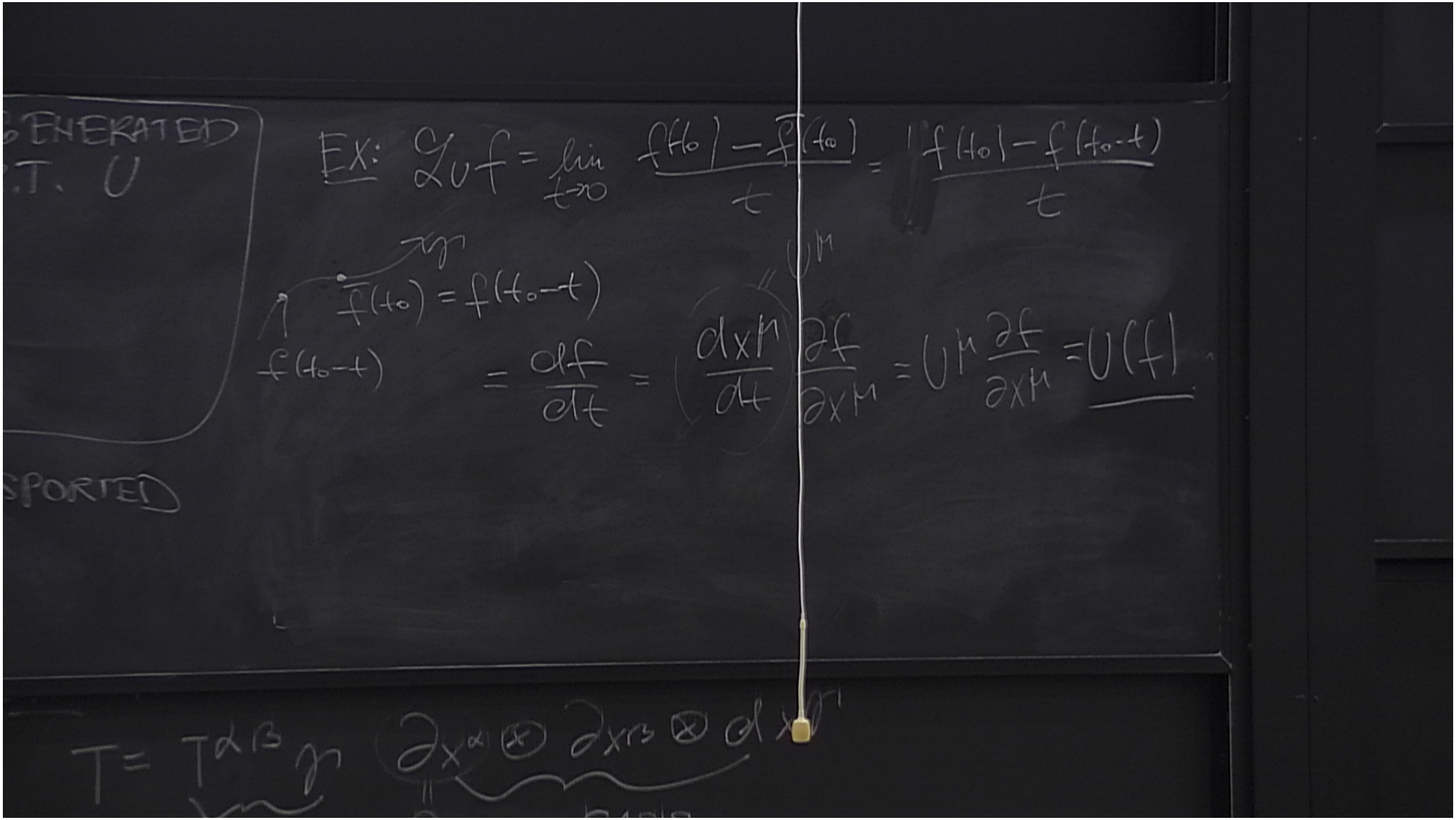

TENSOR ALGEBRA
• DEFINITION: LET ϕ_t BE A 1-PAR GROUP OF DIFFEOS GENERAT
BY $\underline{U}$. THEN THE LIE DERIVATIVE $\mathcal{L}_U$ W.R.T. U

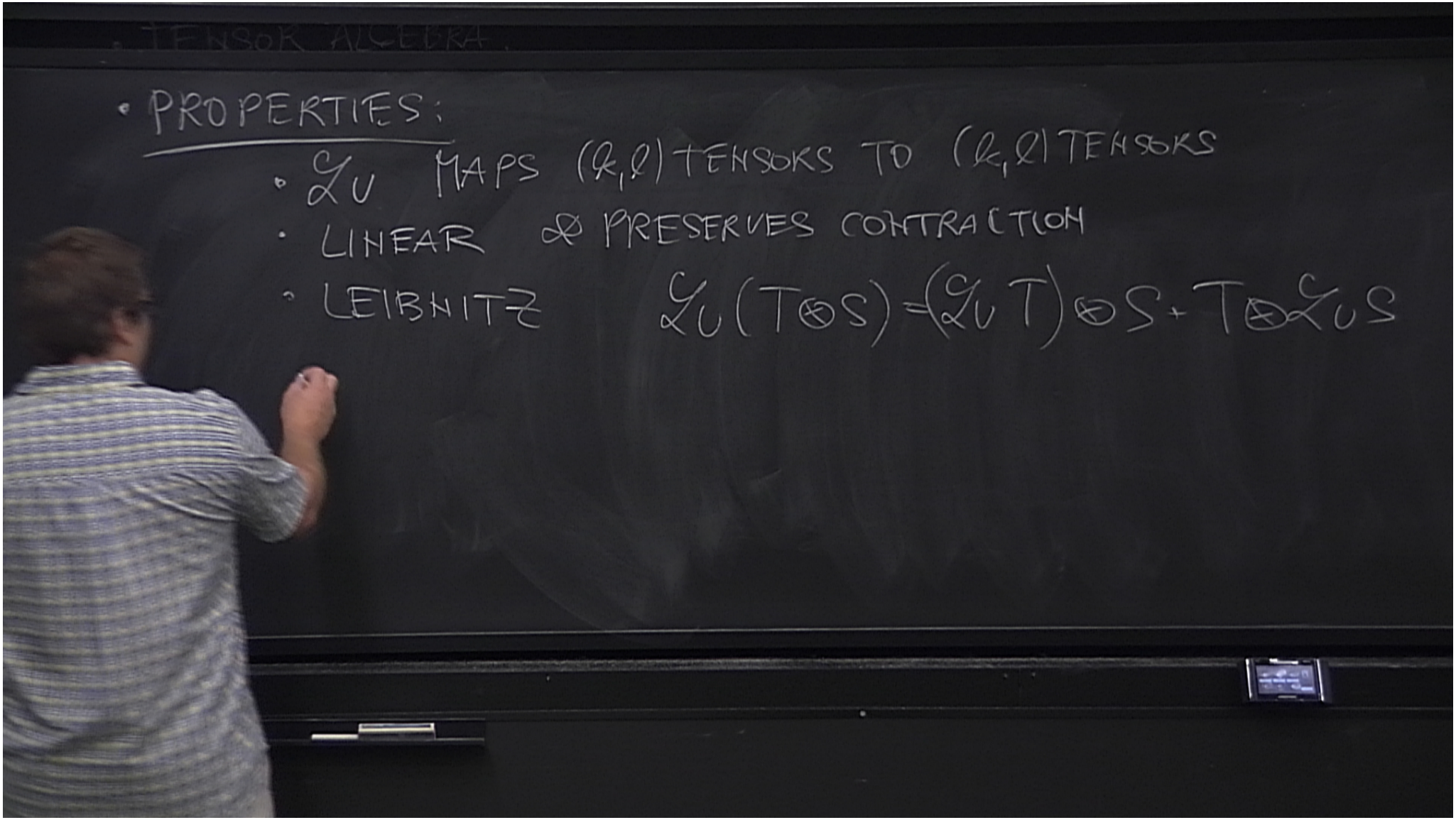
TENSOR ALGEBRA

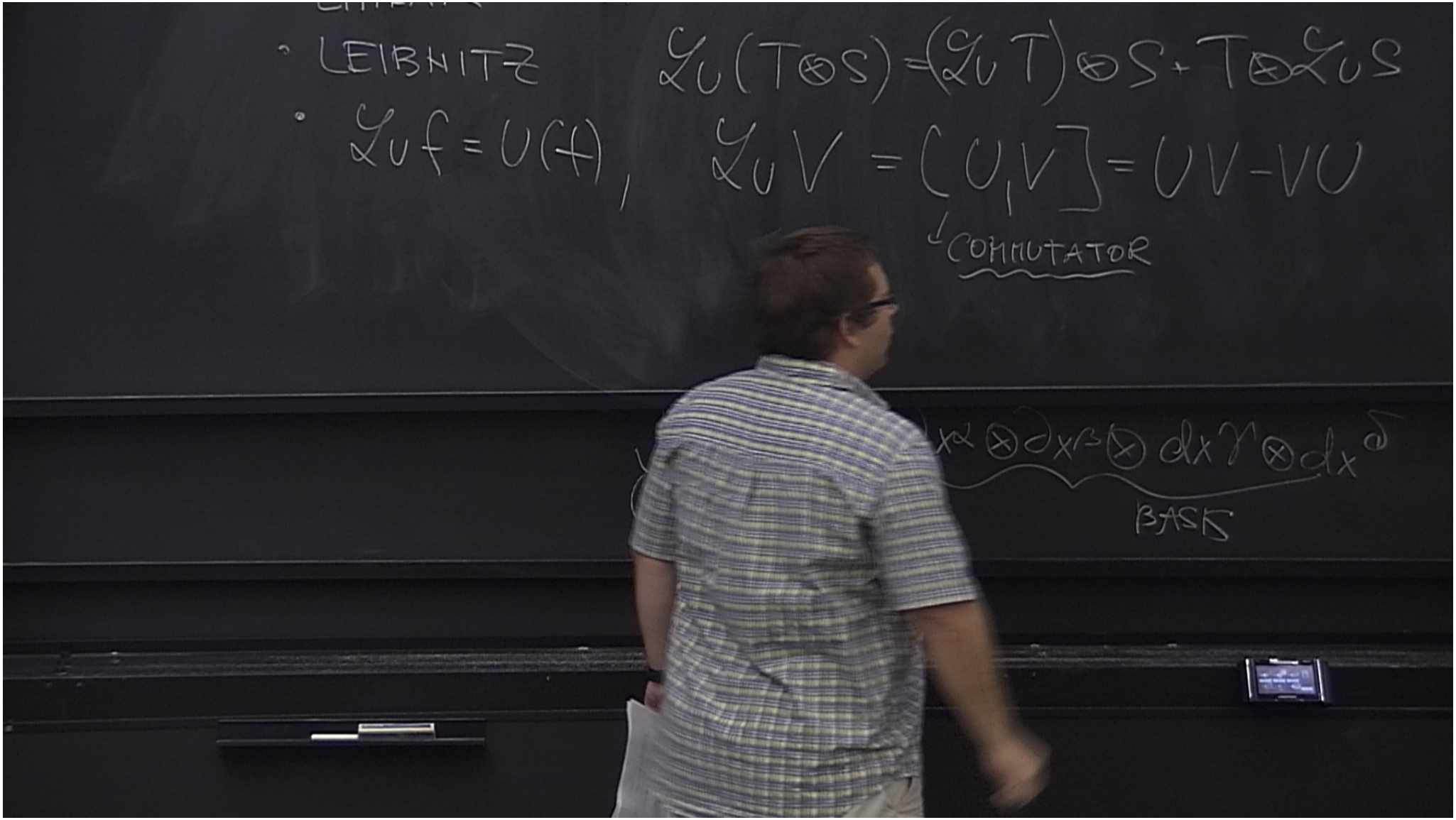
• DEFINITION: LET ϕ_t BE THE 1-PAR GROUP OF DIFFEOS GENERATED BY $\underline{U}$. THEN THE LIE DERIVATIVE $\mathcal{L}_U$ W.R.T. U IS

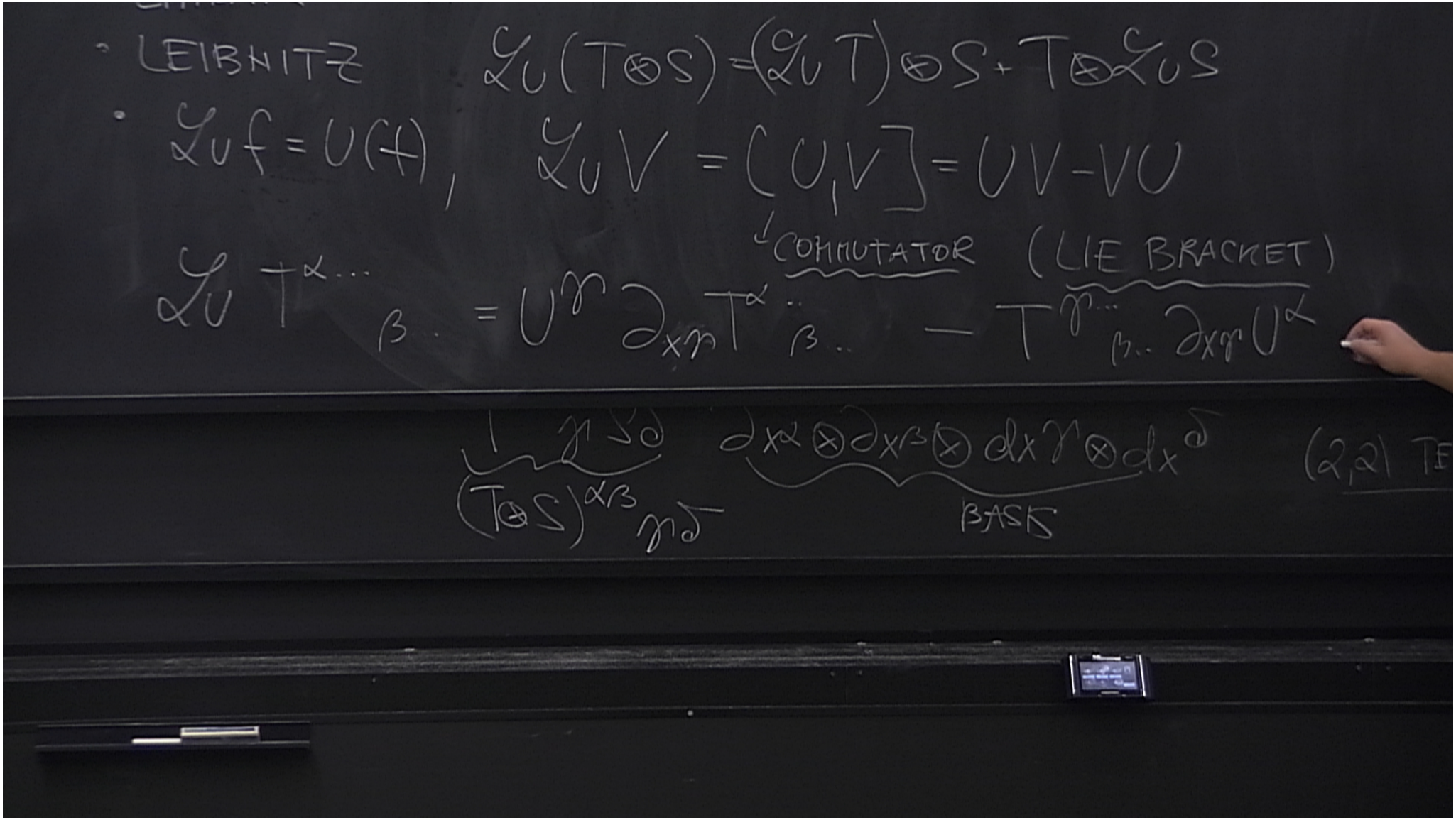
$$\mathcal{L}_U T|_p = \lim_{t \to 0} \frac{T|_p - \phi_t^* T|_p}{t}$$

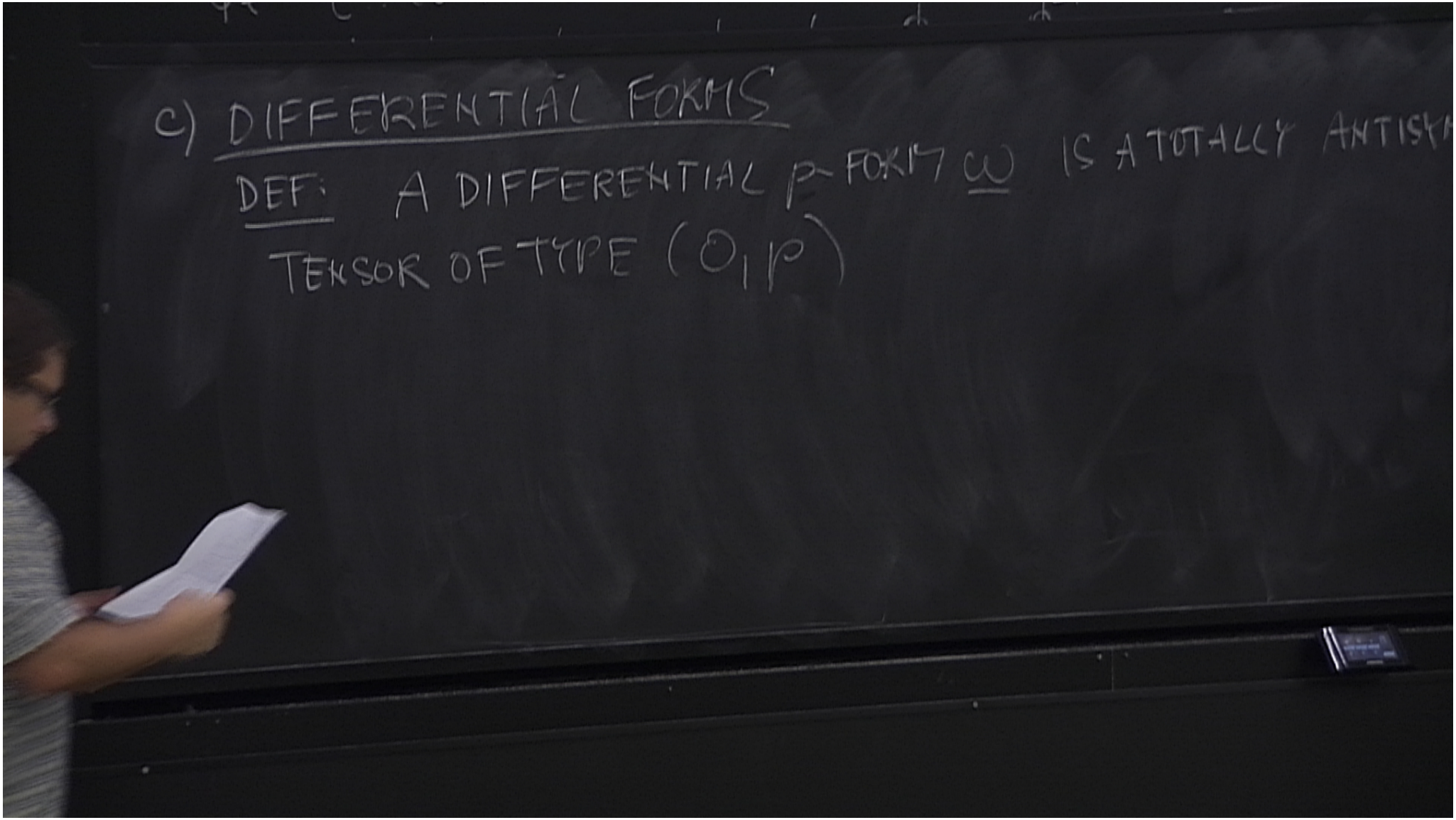

ENERATED
R.T. U
SPORTED
Ex: $\mathcal{L}_U f = \lim_{t\to 0} \frac{f(t_0) - \bar{f}(t_0)}{t} = \frac{f(t_0) - f(t_0 - t)}{t}$
$\bar{f}(t_0) = f(t_0 - t)$
$f(t_0 - t)$
$= \frac{df}{dt} = \left(\frac{dx^\mu}{dt}\right)\frac{\partial f}{\partial x^\mu} = U^\mu \frac{\partial f}{\partial x^\mu} = U(f)$

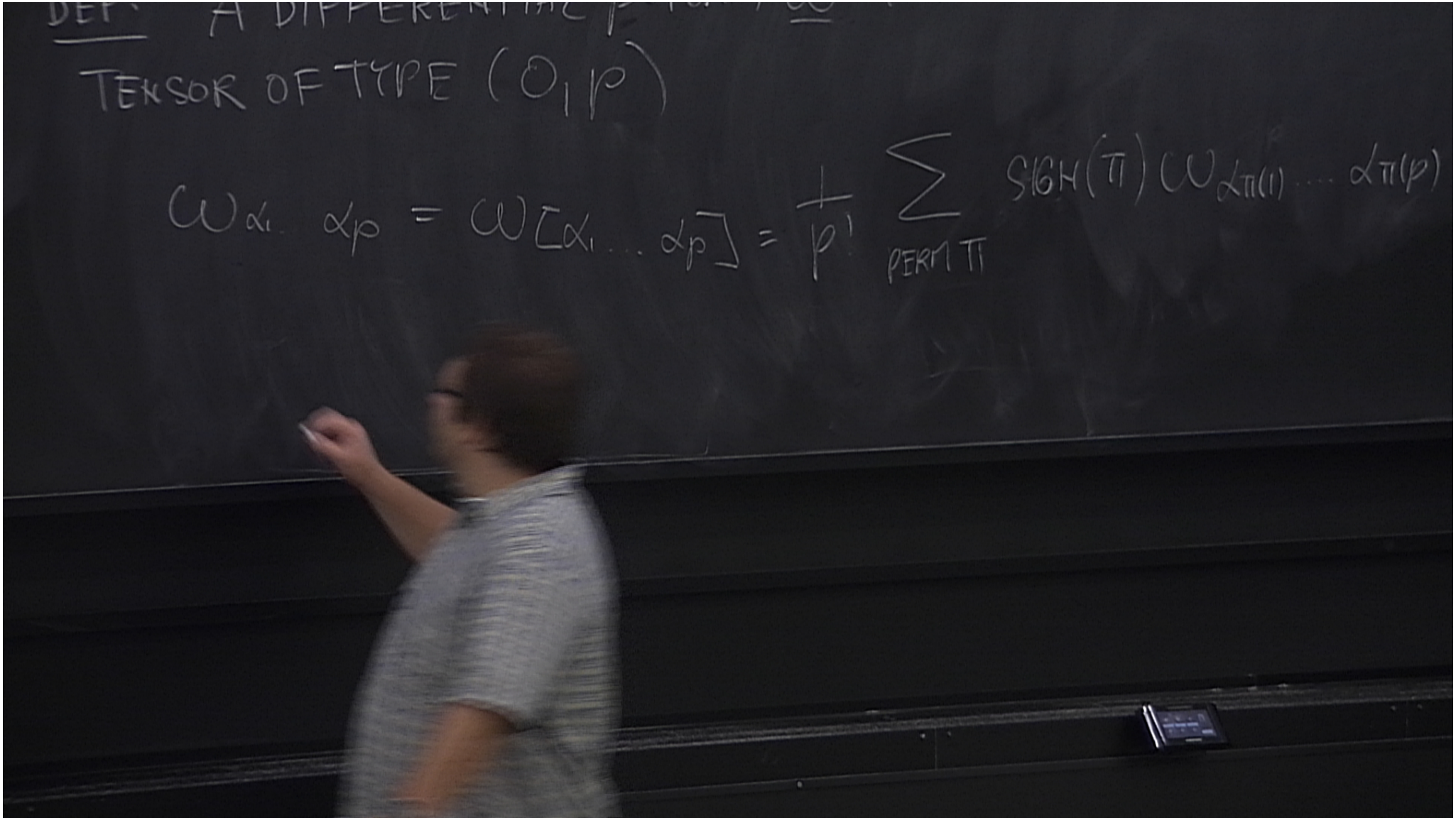

ENERATED
.T. U
SPORTED
EX: $\mathcal{L}_U f = \lim_{t\to 0} \frac{f(t_0) - \bar{f}(t_0)}{t} = \frac{f(t_0) - f(t_0 - t)}{t}$
$\bar{f}(t_0) = f(t_0 - t)$
$f(t_0 - t)$
$= \frac{df}{dt} = \frac{dx^\mu}{dt}\frac{\partial f}{\partial x^\mu} = U^\mu \frac{\partial f}{\partial x^\mu} = U(f)$
$T = T^{\alpha\beta}$
$\frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta} \otimes dx^\gamma$


TENSOR ALGEBRA
• PROPERTIES:
• $\mathcal{L}_U$ MAPS (k,ℓ) TENSORS TO (k,ℓ) TENSORS
• LINEAR & PRESERVES CONTRACTION
• LEIBNITZ $\mathcal{L}_U(T\otimes S)=(\mathcal{L}_U T)\otimes S + T\otimes\mathcal{L}_U S$


LEIBNITZ
$\nabla_U(T\otimes S)=(\nabla_U T)\otimes S + T\otimes\nabla_U S$
$\nabla_U f = U(f)$,
$\nabla_U V = [U,V] = UV - VU$
COMMUTATOR
$dx^\alpha\otimes dx^\beta\otimes dx^\gamma\otimes dx^\delta$
BASIS


LEIBNITZ
$\mathcal{L}_U(T\otimes S)=(\mathcal{L}_U T)\otimes S + T\otimes \mathcal{L}_U S$
$\mathcal{L}_U f = U(f)$, $\mathcal{L}_U V = [U,V] = UV - VU$
COMMUTATOR (LIE BRACKET)
$\mathcal{L}_U T^{\alpha\ldots}{}_{\beta\ldots} = U^\gamma \partial_{x^\gamma} T^{\alpha\ldots}{}_{\beta\ldots} - T^{\gamma\ldots}{}_{\beta\ldots}\partial_{x^\gamma} U^\alpha$
$(T\otimes S)^{\alpha\beta}{}_{\gamma\delta}$
$\partial_{x^\alpha}\otimes\partial_{x^\beta}\otimes dx^\gamma\otimes dx^\delta$
BASIS
(2,2) TE


c) DIFFERENTIAL FORMS
DEF: A DIFFERENTIAL p-FORM ω IS A TOTALLY ANTISY
TENSOR OF TYPE (0,p)

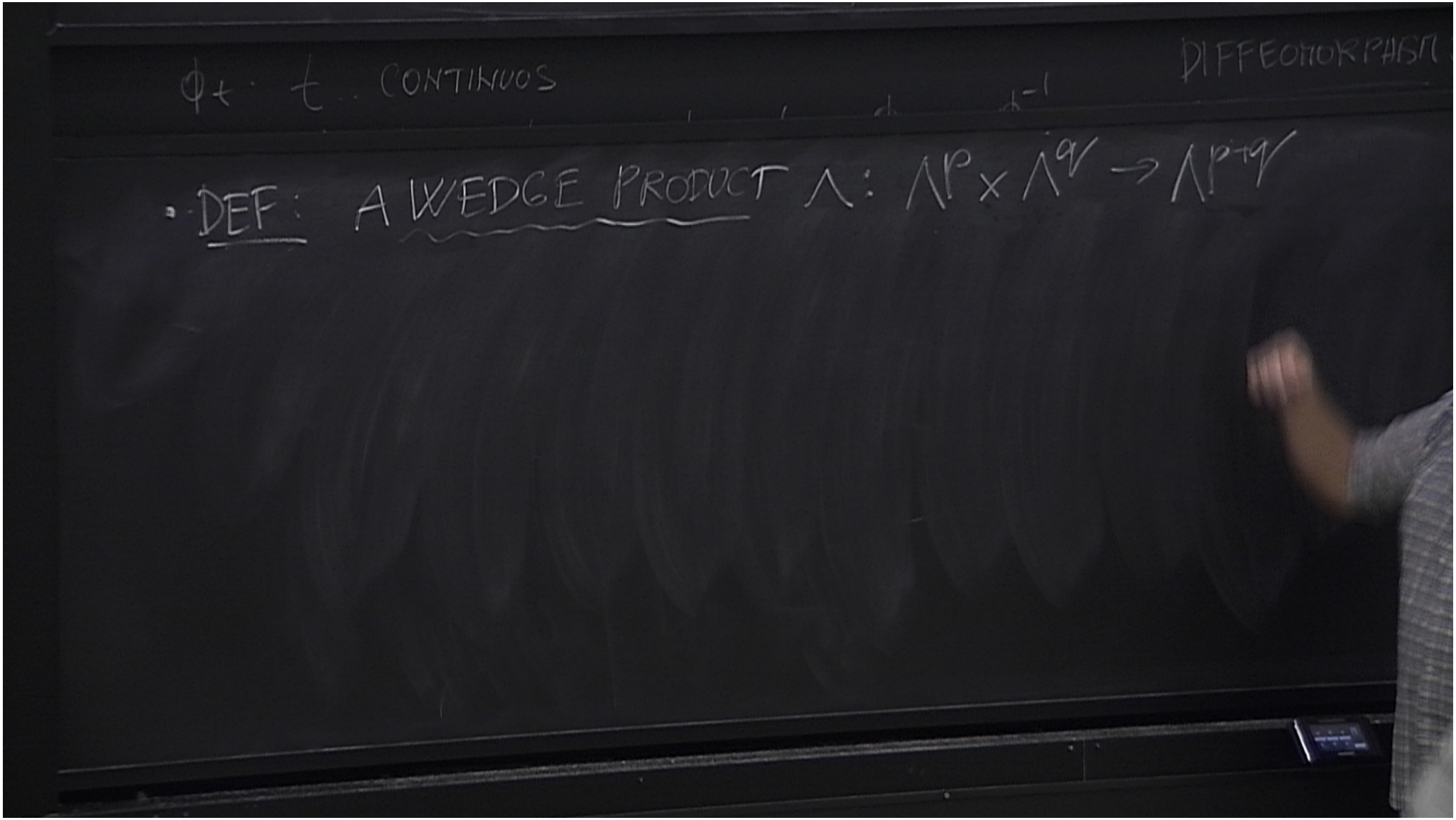

DEF: A DIFFERENTIAL p-FORM ω
TENSOR OF TYPE (0,p)
ω α₁ ... αₚ = ω[α₁ ... αₚ] = 1/p! Σ_PERM π SIGN(π) ω α_π(1) ... α_π(p)

DIFFERENTIAL FORMS

DEF: A DIFFERENTIAL p-FORM ω IS A TOTALLY ANTISYMMETRIC TENSOR OF TYPE $(0,p)$

$$\omega_{\alpha_1 \ldots \alpha_p} = \omega[\alpha_1 \ldots \alpha_p] = \frac{1}{p!} \sum_{\text{PERM } \pi} \text{SIGN}(\pi)\, \omega_{\alpha_{\pi(1)} \ldots \alpha_{\pi(p)}}$$

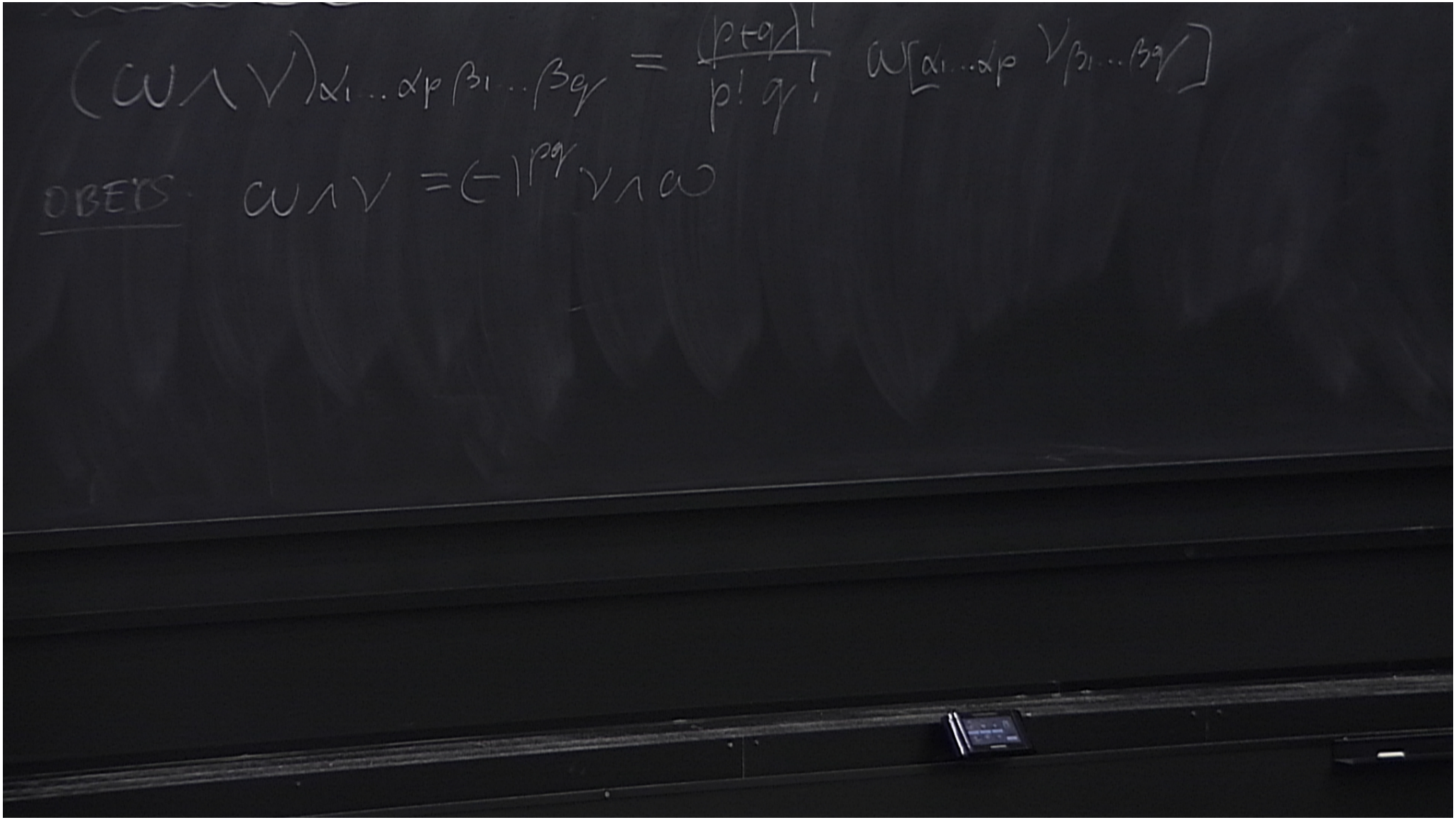
$$\omega[\alpha\beta] = \frac{1}{2}\left(\omega_{\alpha\beta} - \omega_{\beta\alpha}\right)$$

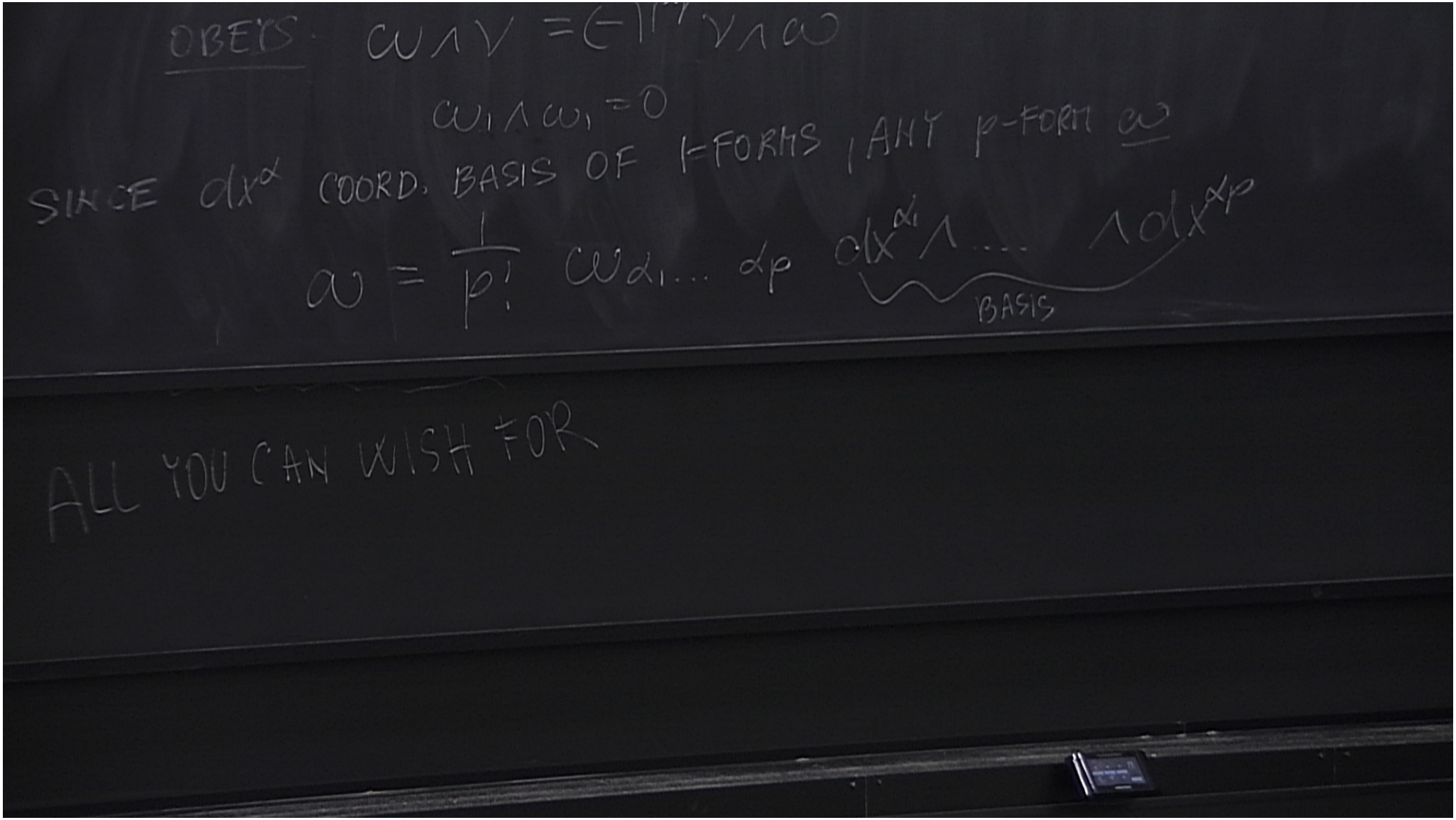

ϕ_t · t . CONTINUOS
DIFFEOMORPHISM
d
$d-1$
• DEF: A WEDGE PRODUCT $\wedge : \Lambda^p \times \Lambda^q \to \Lambda^{p+q}$

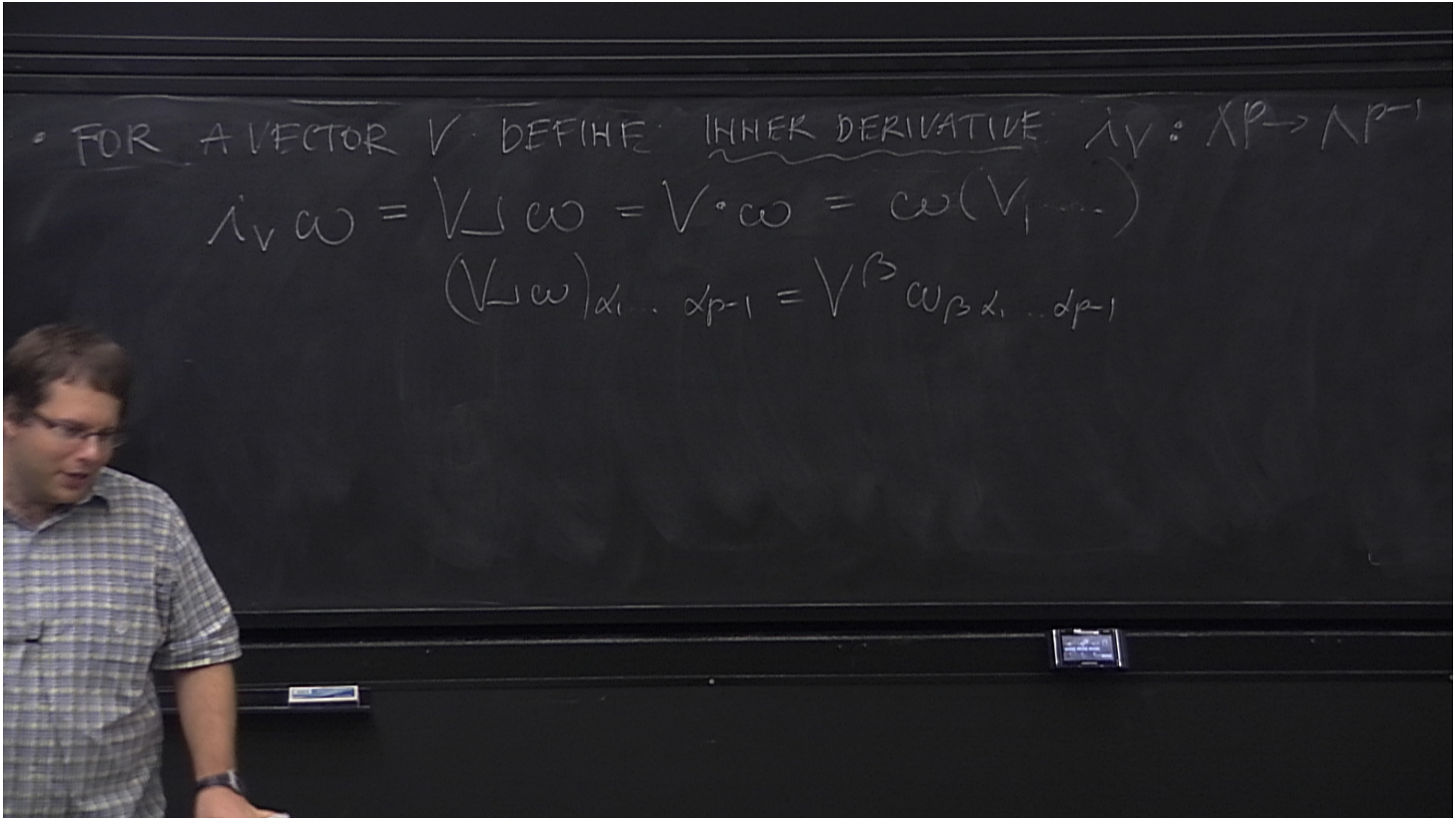
ϕ_t : t .. CONTINUOS DIFFEOMORPHISM

- DEF: A WEDGE PRODUCT $\wedge : \Lambda^p \times \Lambda^q \to \Lambda^{p+q}$

$$(\omega \wedge \nu)_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q} =$$


$(\omega \wedge \nu)_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q} = \frac{(p+q)!}{p!\,q!}\, \omega_{[\alpha_1 \ldots \alpha_p} \nu_{\beta_1 \ldots \beta_q]}$
OBEYS: $\omega \wedge \nu = (-1)^{pq}\, \nu \wedge \omega$


OBEYS $\omega \wedge \nu = (-1)^{pq} \nu \wedge \omega$
$\omega_1 \wedge \omega_1 = 0$
SINCE dx^α COORD. BASIS OF 1-FORMS, ANY p-FORM ω
$\omega = \frac{1}{p!} \omega_{\alpha_1 \ldots \alpha_p} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_p}$
BASIS
ALL YOU CAN WISH FOR


• FOR A VECTOR V DEFINE INNER DERIVATIVE $i_V : \Lambda^p \to \Lambda^{p-1}$
$i_V \omega = V \lrcorner \omega = V \cdot \omega = \omega(V, \ldots)$
$(V \lrcorner \omega)_{\alpha_1 \ldots \alpha_{p-1}} = V^\beta \omega_{\beta \alpha_1 \ldots \alpha_{p-1}}$

- FOR A VECTOR V DEFINE: INNER DERIVATIVE $\imath_V : \Lambda^p \to \Lambda^{p-1}$

$$\imath_V \omega = V \lrcorner \omega = V \cdot \omega = \omega(V, \dots)$$

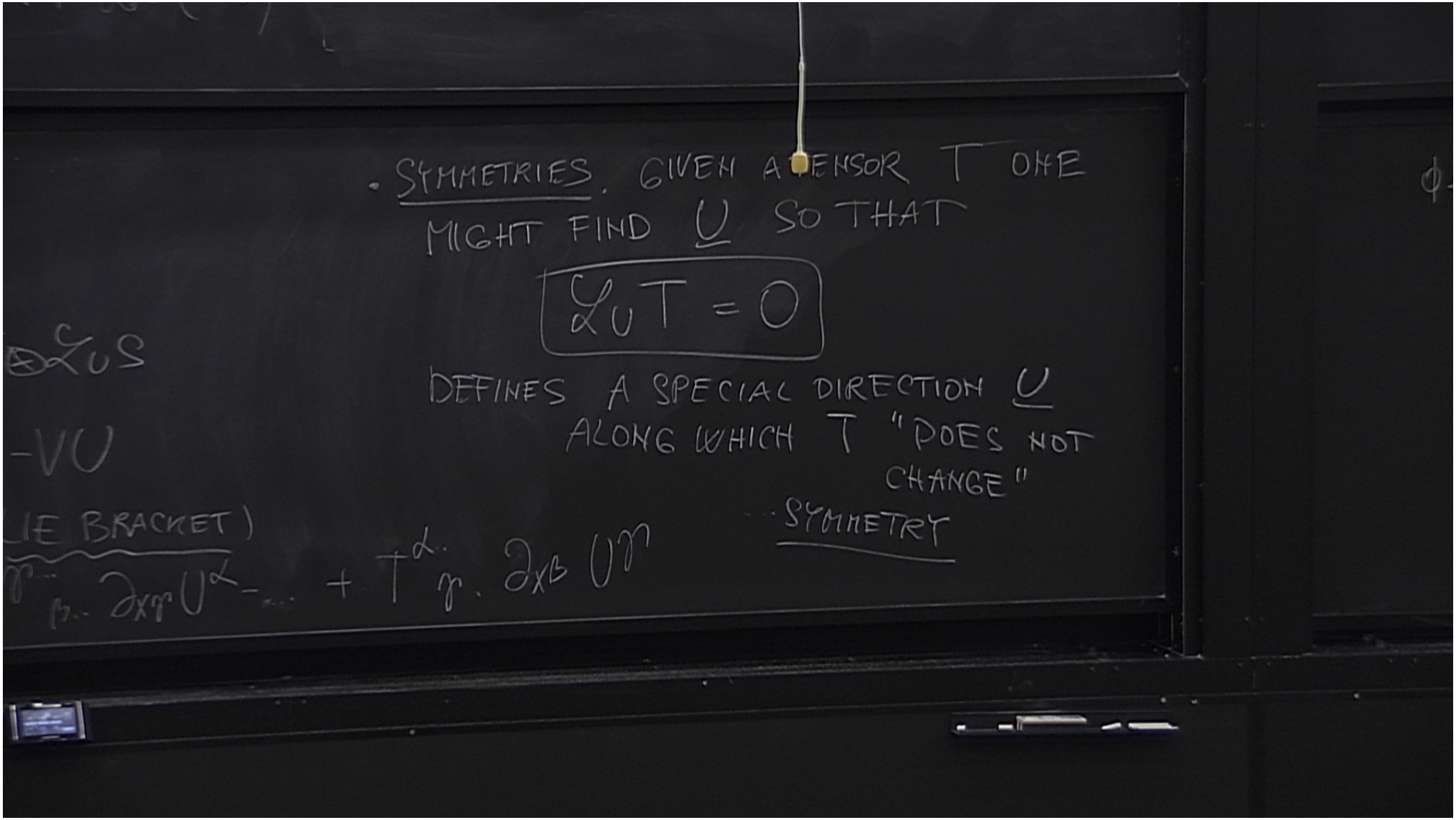
$$(V \lrcorner \omega)_{\alpha_1 \dots \alpha_{p-1}} = V^\beta \omega_{\beta \alpha_1 \dots \alpha_{p-1}}$$

PROP:

i) $\imath_V$ .. LINEAR, LINEAR IN V

ii) LEIBNITZ (GRADED): FOR $\omega \in \Lambda^p$

$$\imath_V(\omega \wedge \nu) = (\imath_V \omega) \wedge \nu + (-1)^p \omega \wedge (\imath_V \nu)$$


· SYMMETRIES. GIVEN A TENSOR T ONE
MIGHT FIND U SO THAT
$\mathcal{L}_U T = 0$
DEFINES A SPECIAL DIRECTION U
ALONG WHICH T "DOES NOT
CHANGE"
SYMMETRY
$\mathcal{L}_U S$
$-VU$
(E BRACKET)
$+ T^{\alpha\ldots}_{\gamma\ldots} \partial_{x^\beta} U^\gamma$

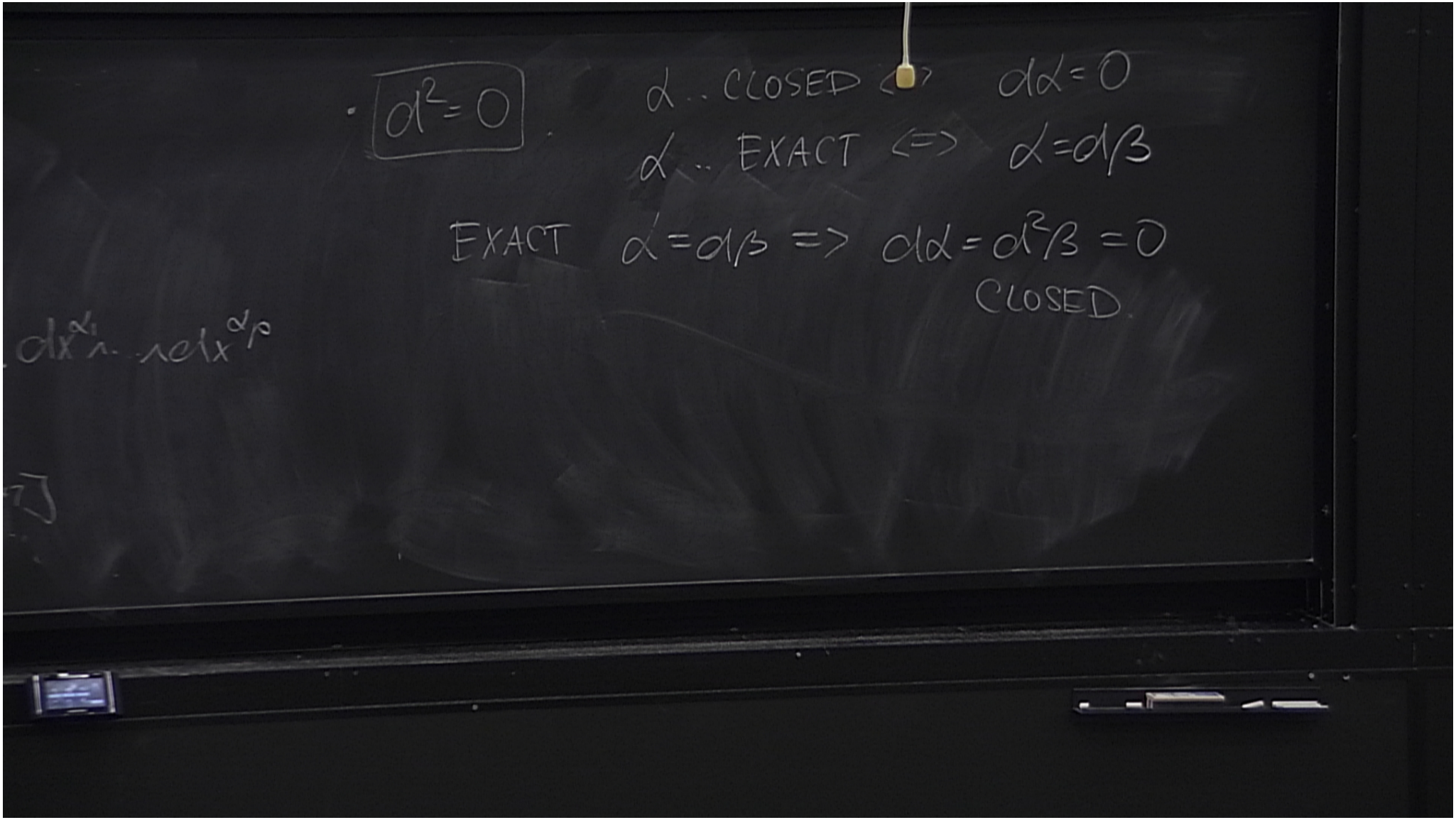
DEF: EXTERIOR DERIVATIVE $d: \Lambda^p \to \Lambda^{p+1}$

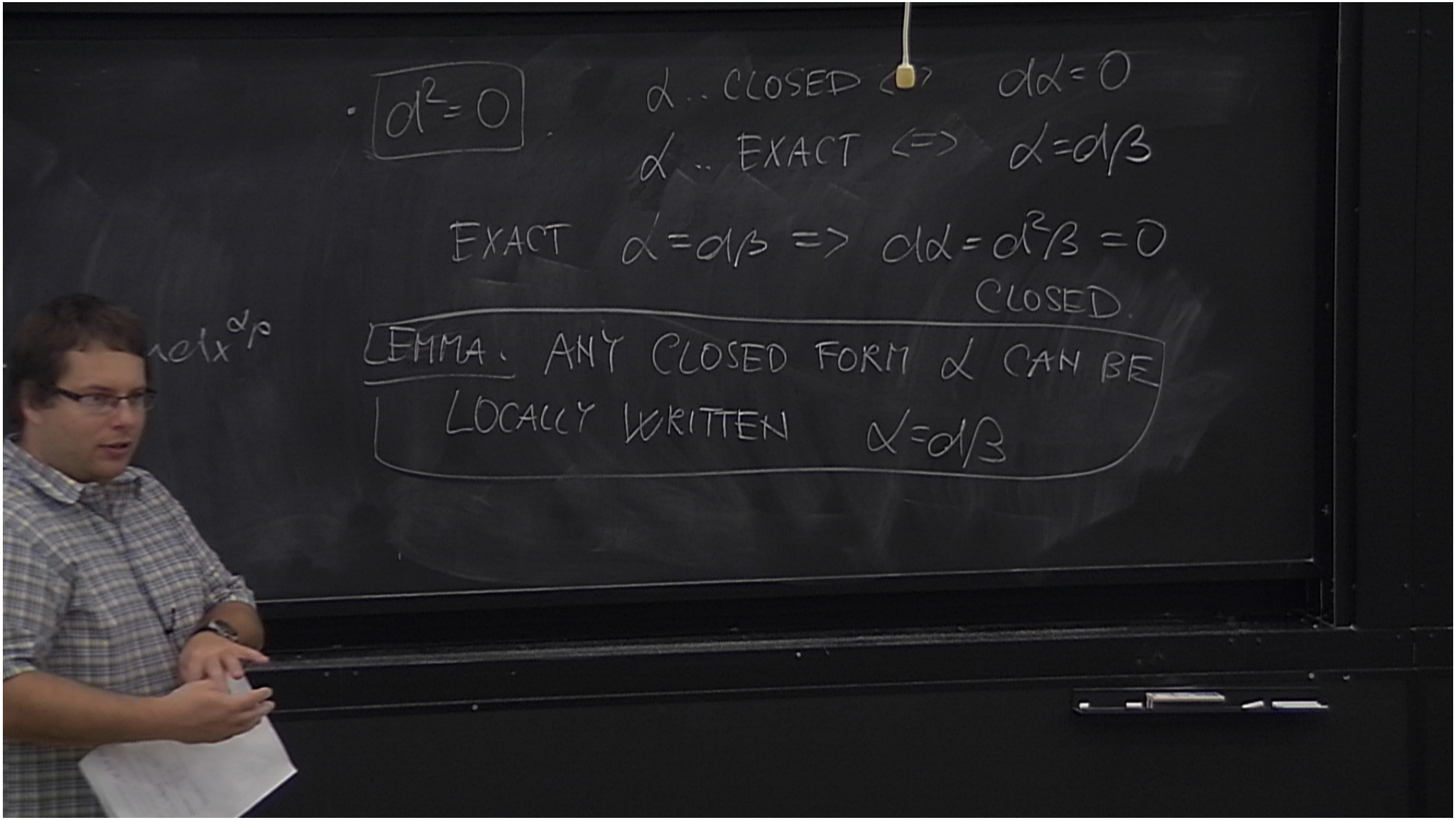
i) ON A FUNCTION: $d: f \to df = \frac{\partial f}{\partial x^\alpha} dx^\alpha$

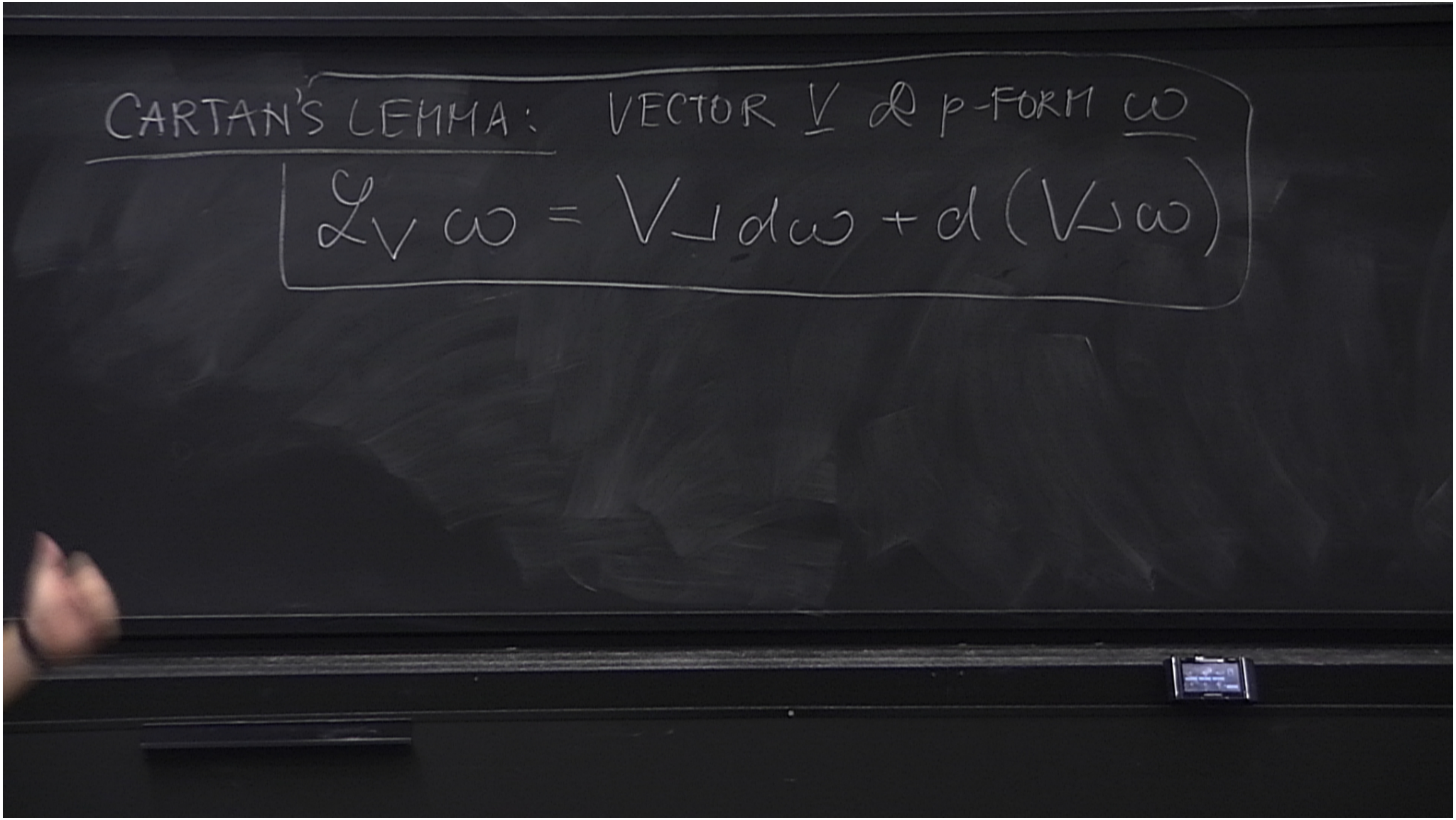
ii) ON A p-FORM ω.

$$d\omega \to d\omega = \frac{1}{p!} d(\omega_{\alpha_1 \ldots \alpha_p}) \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_p}$$

$$(d\omega)_{\alpha_1 \ldots \alpha_{p+1}} = (p+1)\, \partial_{[\alpha_1} \omega_{\alpha_2 \ldots \alpha_{p+1}]}$$


$d^2 = 0$
α .. CLOSED $\Leftrightarrow$ $d\alpha = 0$
α .. EXACT $\Leftrightarrow$ $\alpha = d\beta$
EXACT $\alpha = d\beta \Rightarrow d\alpha = d^2\beta = 0$
CLOSED
$dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_p}$


$d^2 = 0$
α .. CLOSED $\Leftrightarrow$ $d\alpha = 0$
α .. EXACT $\Leftrightarrow$ $\alpha = d\beta$
EXACT $\alpha = d\beta \Rightarrow d\alpha = d^2\beta = 0$
CLOSED
LEMMA. ANY CLOSED FORM α CAN BE LOCALLY WRITTEN $\alpha = d\beta$


CARTAN'S LEMMA: VECTOR V & p-FORM ω
$\mathcal{L}_V \omega = V \lrcorner d\omega + d(V \lrcorner \omega)$