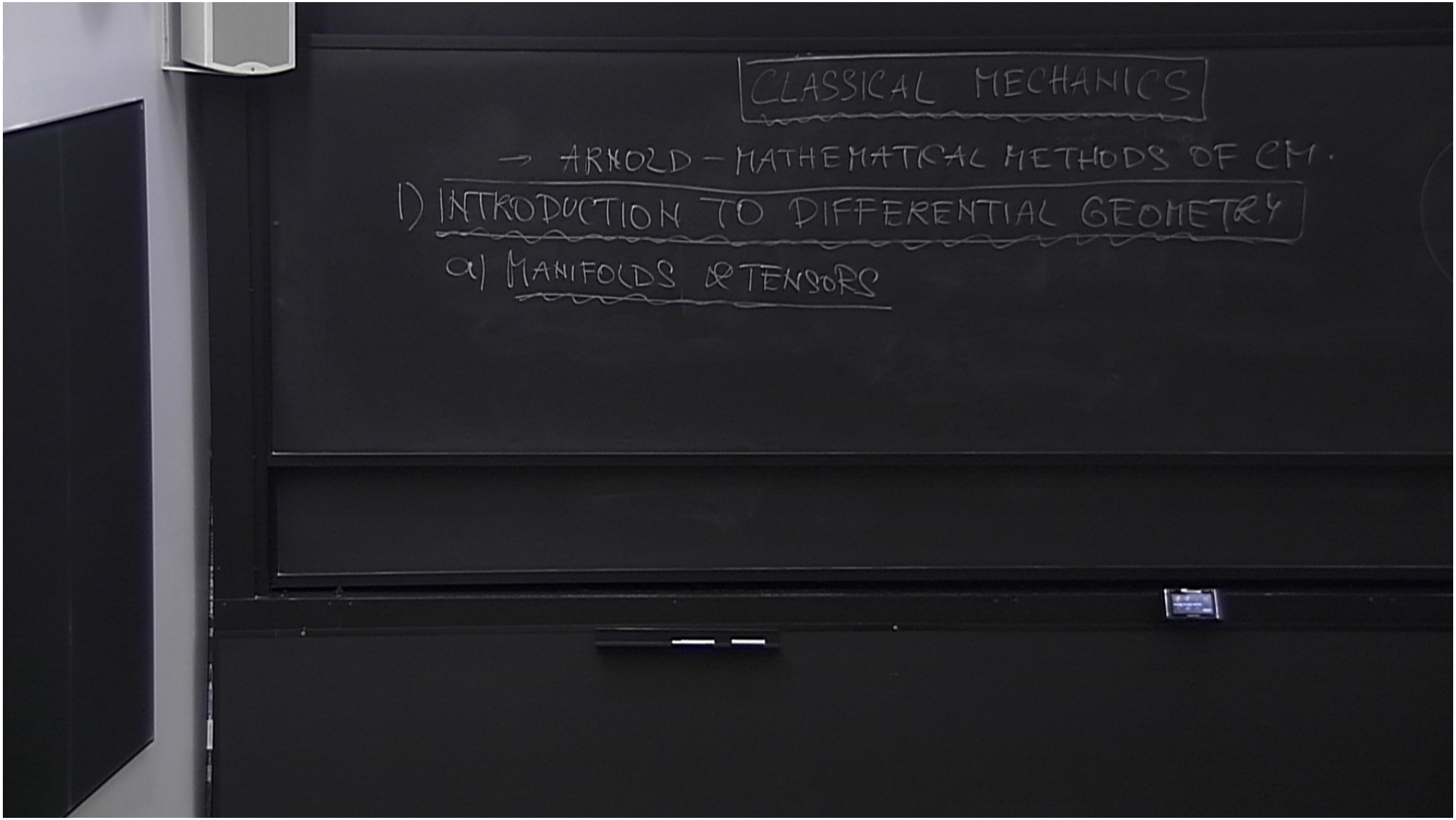
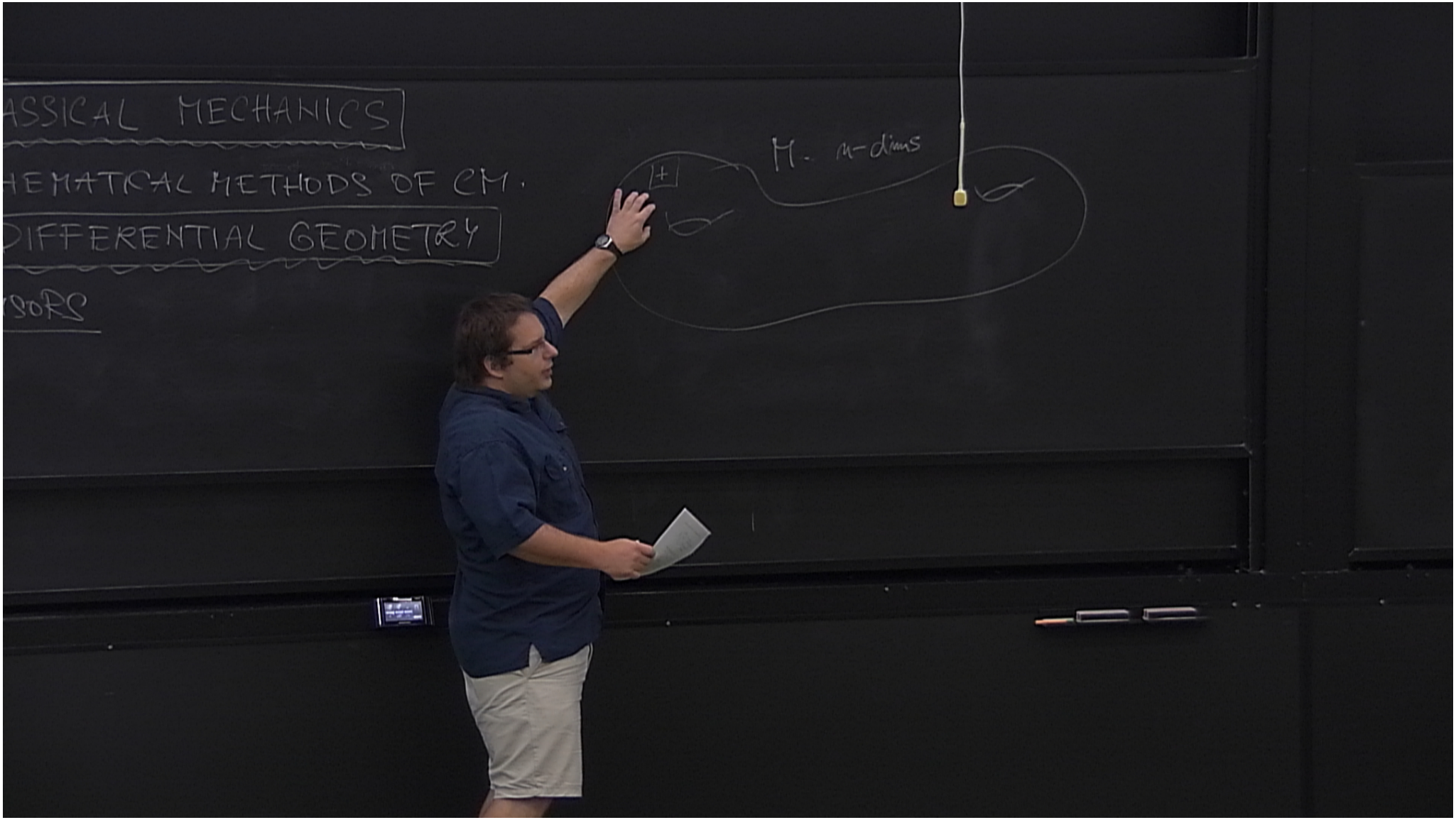
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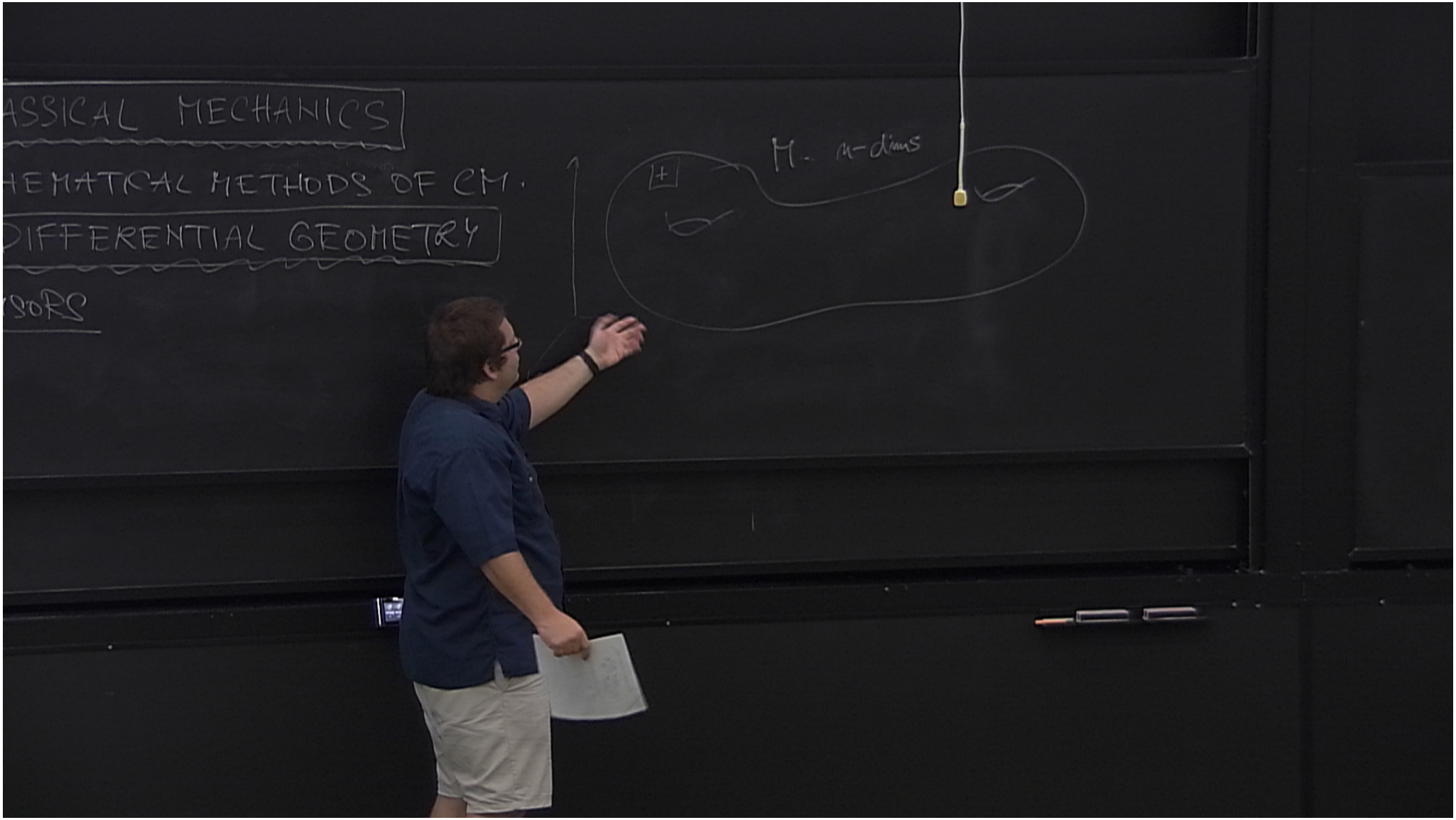
Date: Aug 09, 2016 10:30 AM

URL: http://pirsa.org/16080061

Abstract:


CLASSICAL MECHANICS
→ ARNOLD - MATHEMATICAL METHODS OF CM.
1) INTRODUCTION TO DIFFERENTIAL GEOMETRY
a) MANIFOLDS & TENSORS


ASSICAL MECHANICS
HEMATICAL METHODS OF CM.
DIFFERENTIAL GEOMETRY
M. n-dims

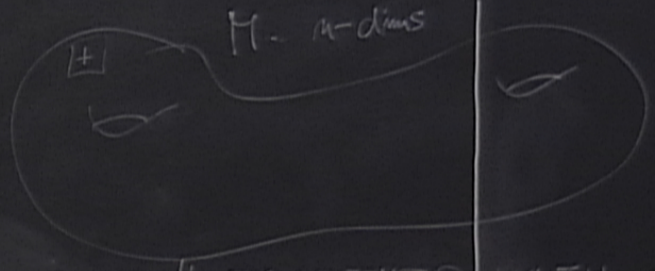

ASSICAL MECHANICS
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DIFFERENTIAL GEOMETRY
SORS
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CLASSICAL MECHANICS

→ ARNOLD - MATHEMATICAL METHODS OF CM.

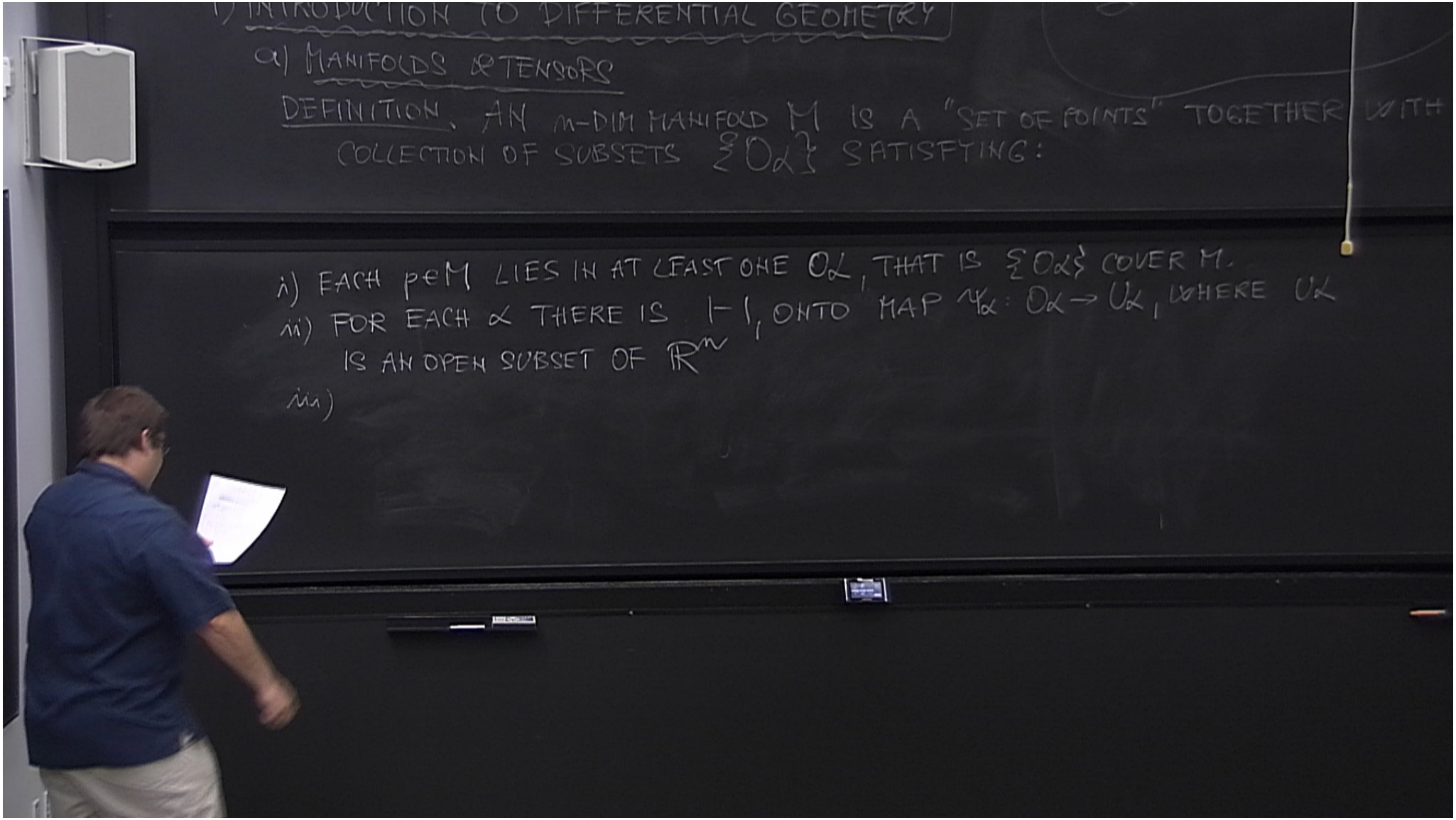
1) INTRODUCTION TO DIFFERENTIAL GEOMETRY

a) MANIFOLDS & TENSORS



DEFINITION. AN n-DIM MANIFOLD M IS A "SET OF POINTS" TOGETHER WITH A COLLECTION OF SUBSETS $\{O_\alpha\}$ SATISFYING:

i) EACH $p \in M$ LIES IN


1) INTRODUCTION TO DIFFERENTIAL GEOMETRY
a) MANIFOLDS & TENSORS
DEFINITION: AN n-DIM MANIFOLD M IS A "SET OF POINTS" TOGETHER WITH
COLLECTION OF SUBSETS $\{O_\alpha\}$ SATISFYING:
i) EACH $p \in M$ LIES IN AT LEAST ONE O_α, THAT IS $\{O_\alpha\}$ COVER M.
ii) FOR EACH α THERE IS 1-1, ONTO MAP $\psi_\alpha : O_\alpha \to U_\alpha$, WHERE U_α
IS AN OPEN SUBSET OF $\mathbb{R}^n$
iii)

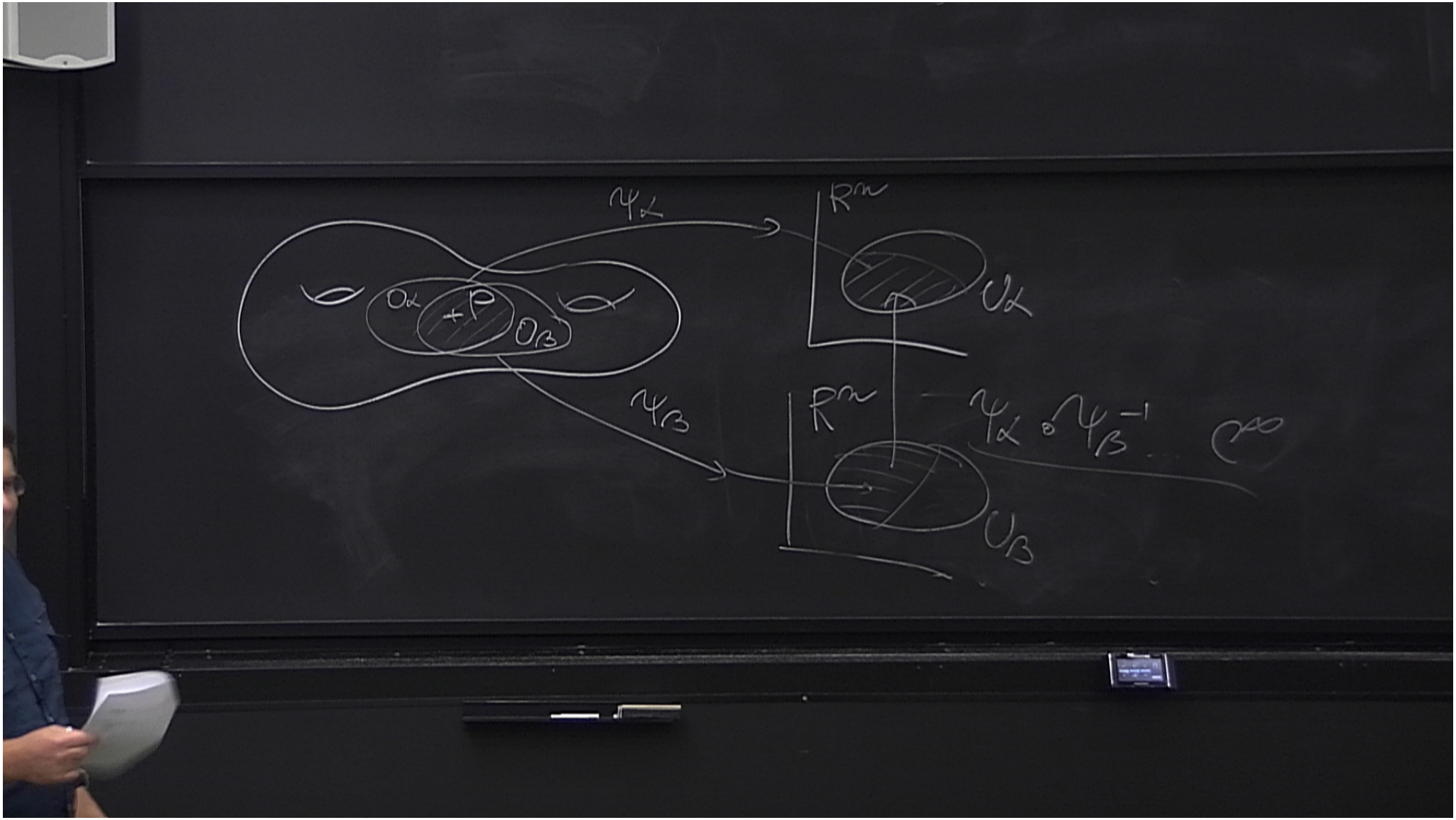
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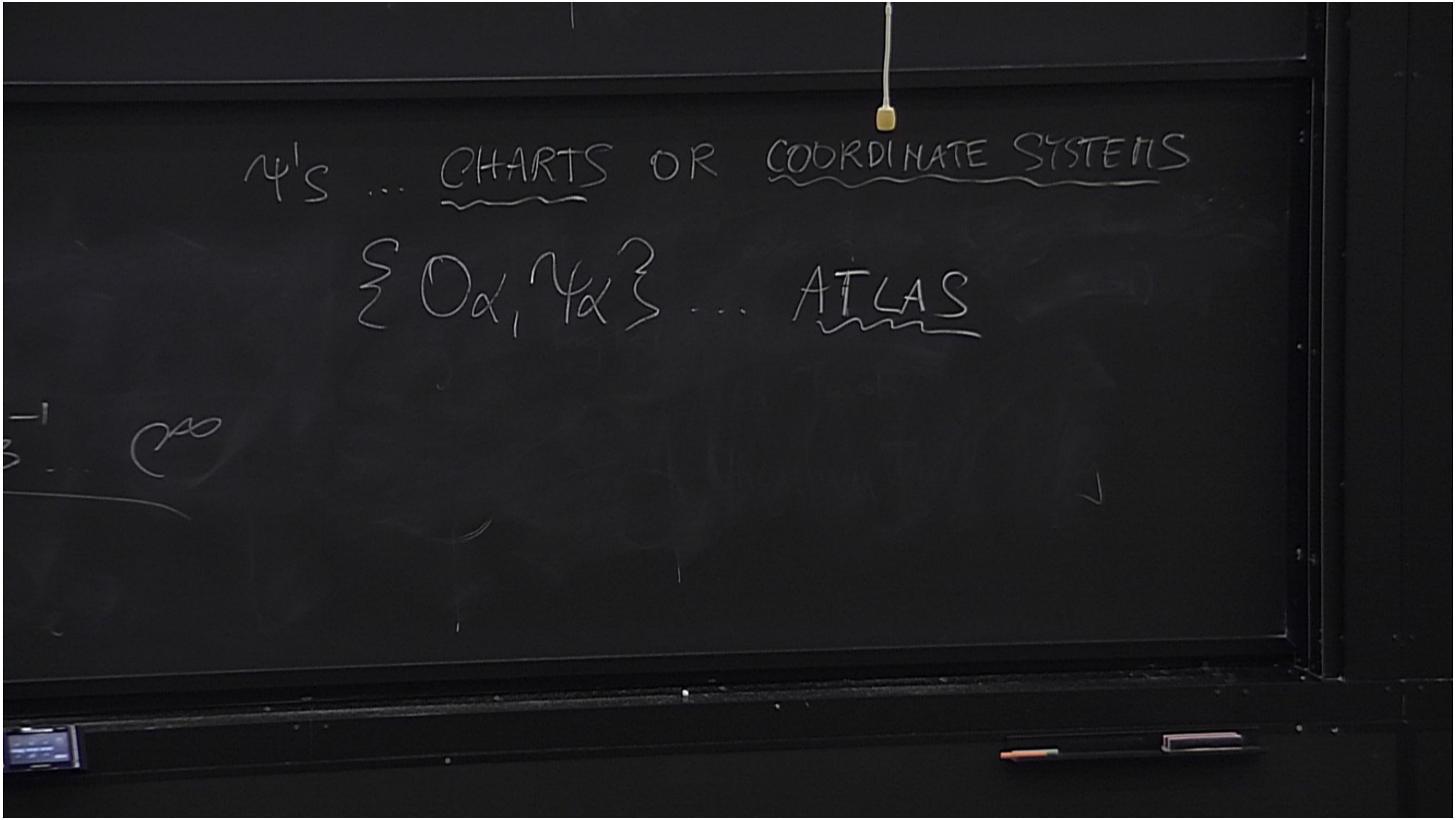
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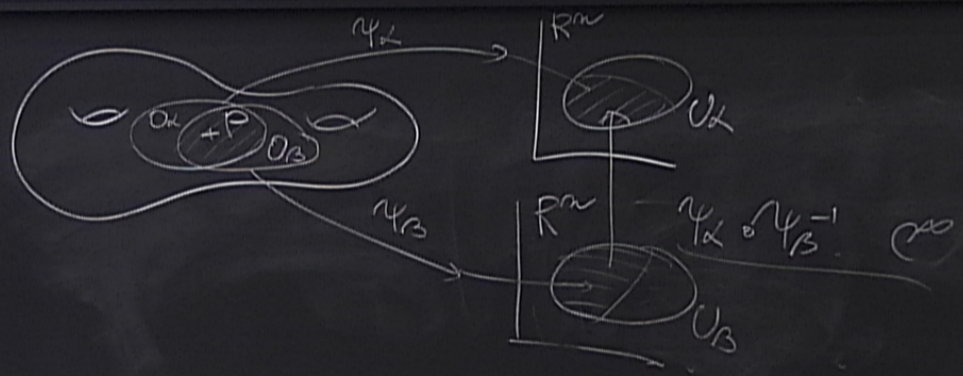
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ψ_α
R^m
U_α
O_α
p
O_β
ψ_β
R^m
$\psi_\alpha \circ \psi_\beta^{-1}$
C^∞
U_β


ψ's ... CHARTS OR COORDINATE SYSTEMS
$\{O_\alpha, \psi_\alpha\}$ ... ATLAS

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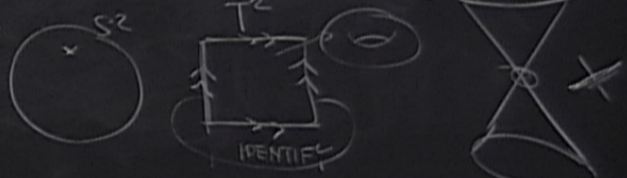
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ψ's ... CHARTS OR COORDINATE SYSTEMS

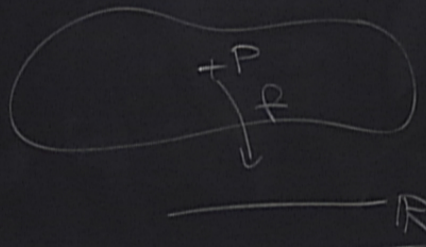
$\{O_\alpha, \psi_\alpha\}$ ... ATLAS

EX:



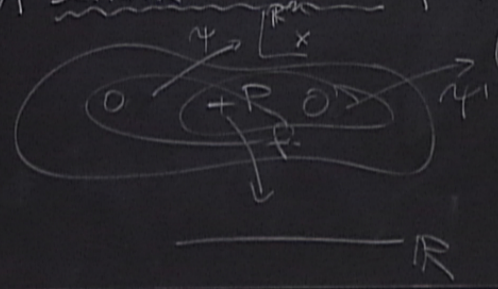
NOTE: IN GR NEED MANIFOLDS. OTHERWISE POINT PARTICLES HAVE PROBLEMS.
IN STRING THEORY. CAN USE ORBIFOLD. ($X \sim -X$ ... $R/\mathbb{Z}_2$)

- A SCALAR FUNCTION f IS A MAP $f: M \to \mathbb{R}$



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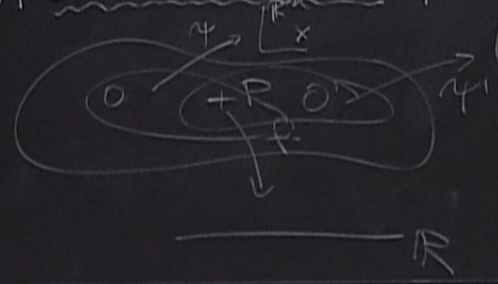


$$f(p) = f(x(p)) = f'(x'(p))$$

$$\boxed{f'(x') = f(x)}$$

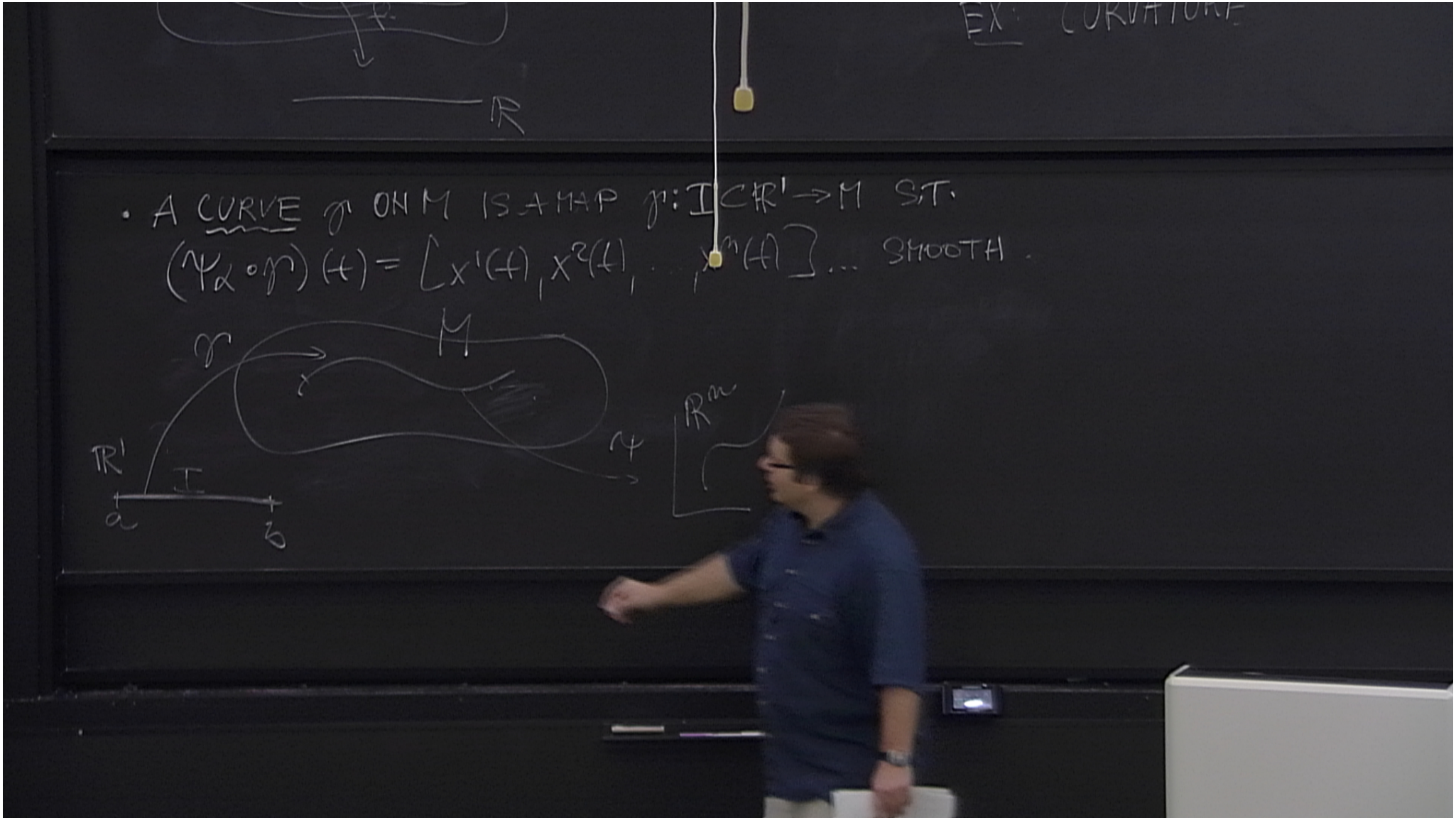
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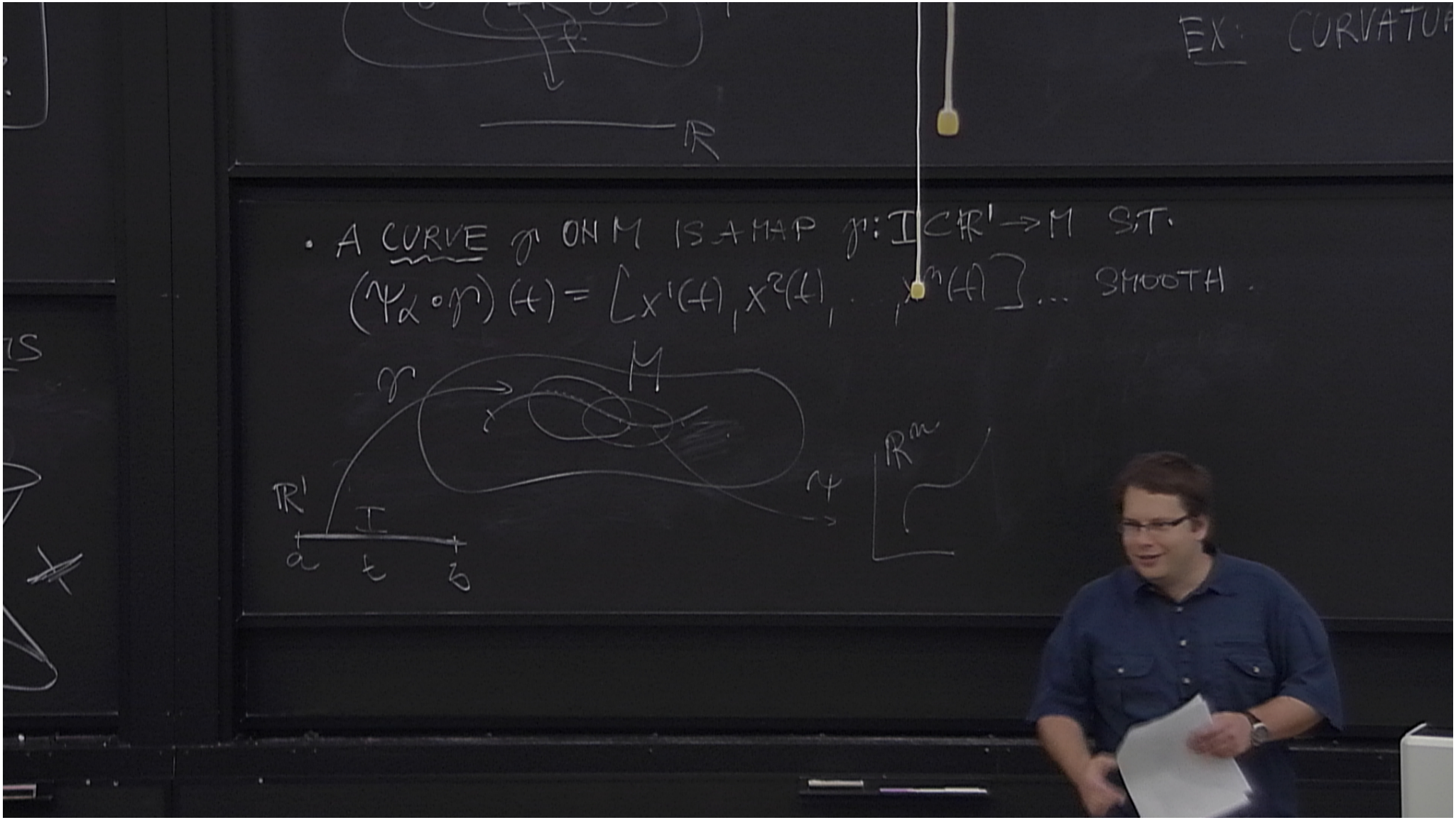
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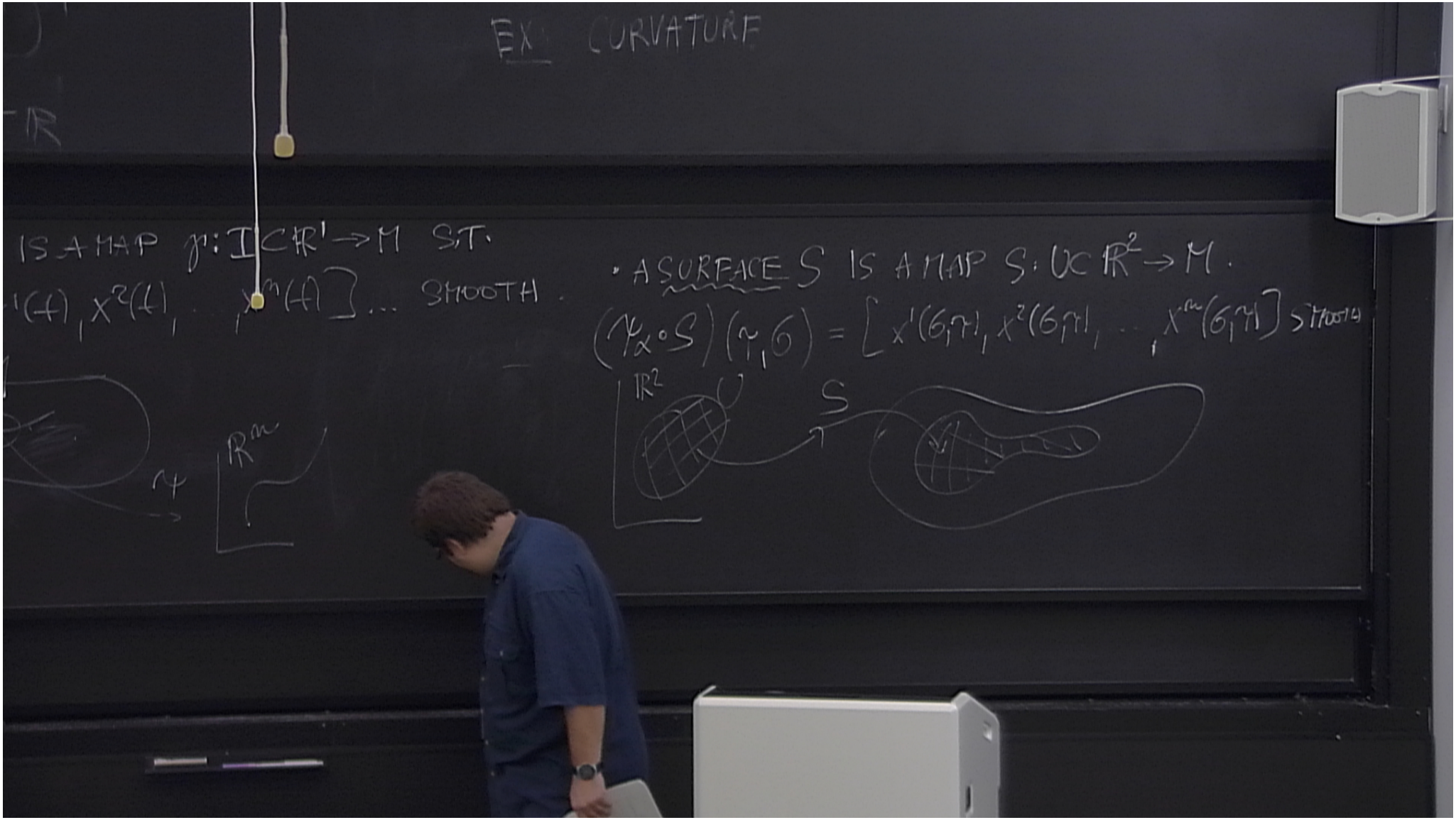


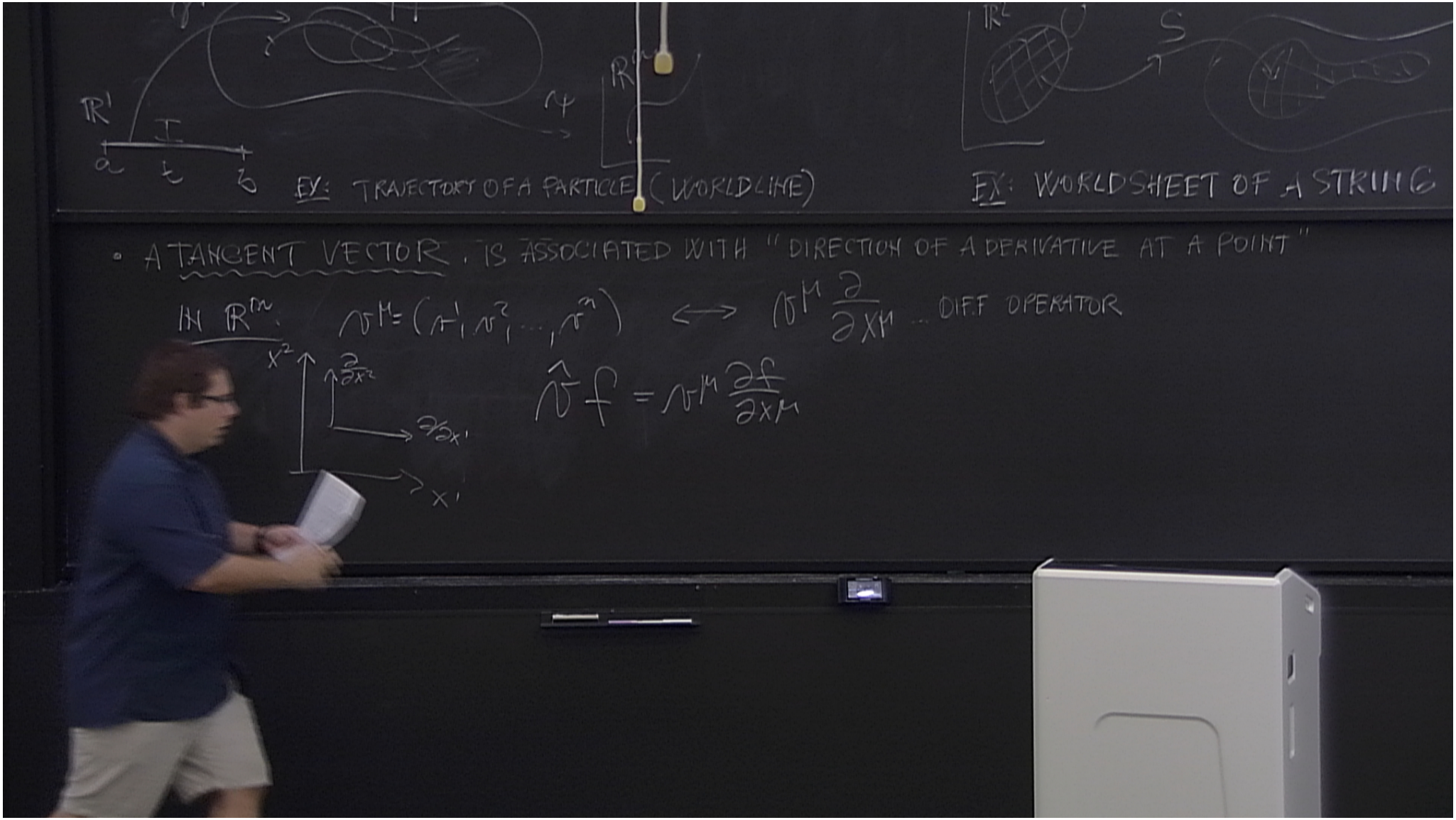
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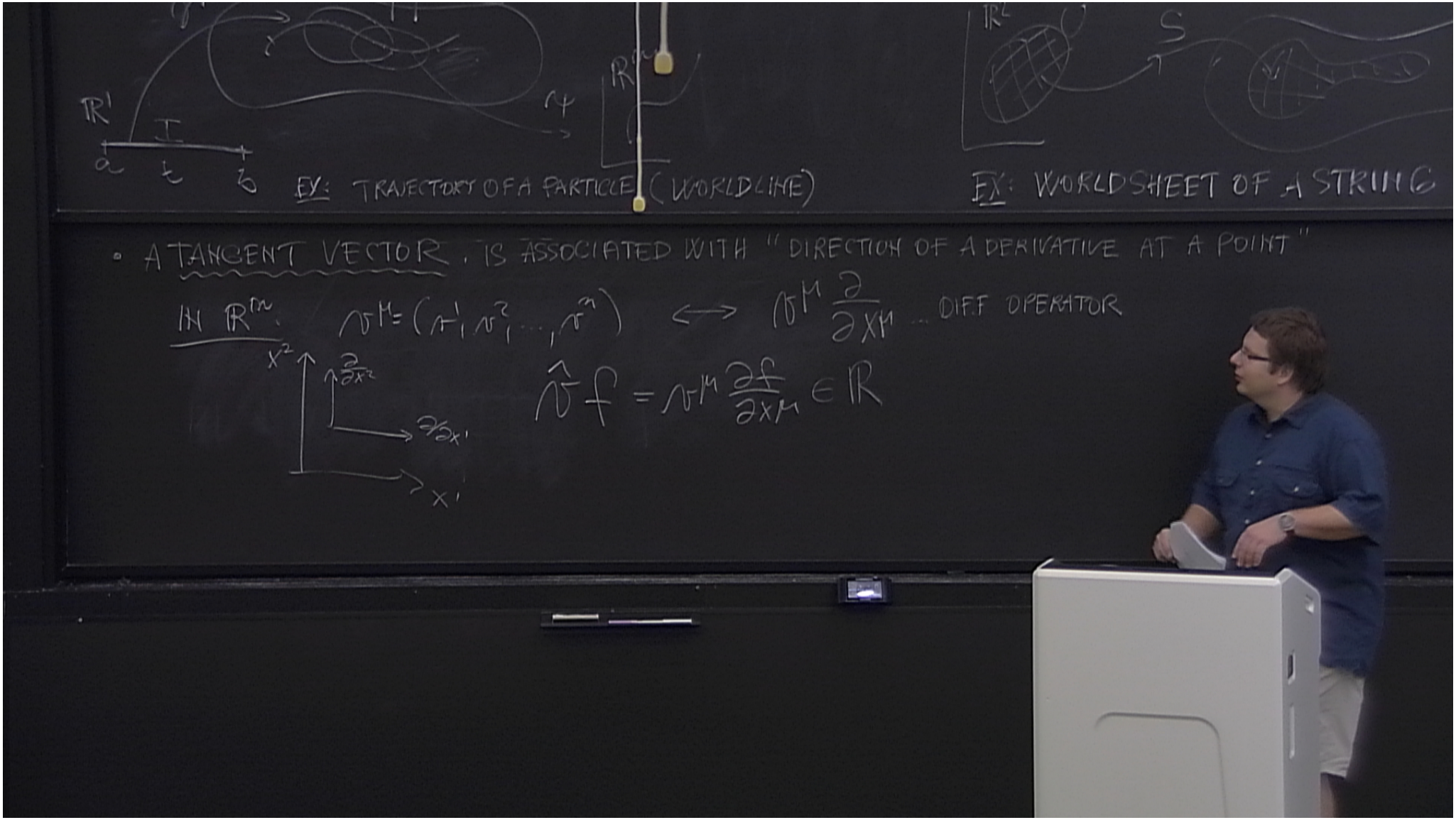
EX: CURVATURE


R
EX: CURVATURE
• A CURVE γ ON M IS A MAP γ: I ⊂ R' → M S.T.
(ψα ∘ γ)(t) = [x'(t), x²(t), ..., xⁿ(t)] ... SMOOTH.
γ
M
R'
I
a
b
ψα
Rⁿ

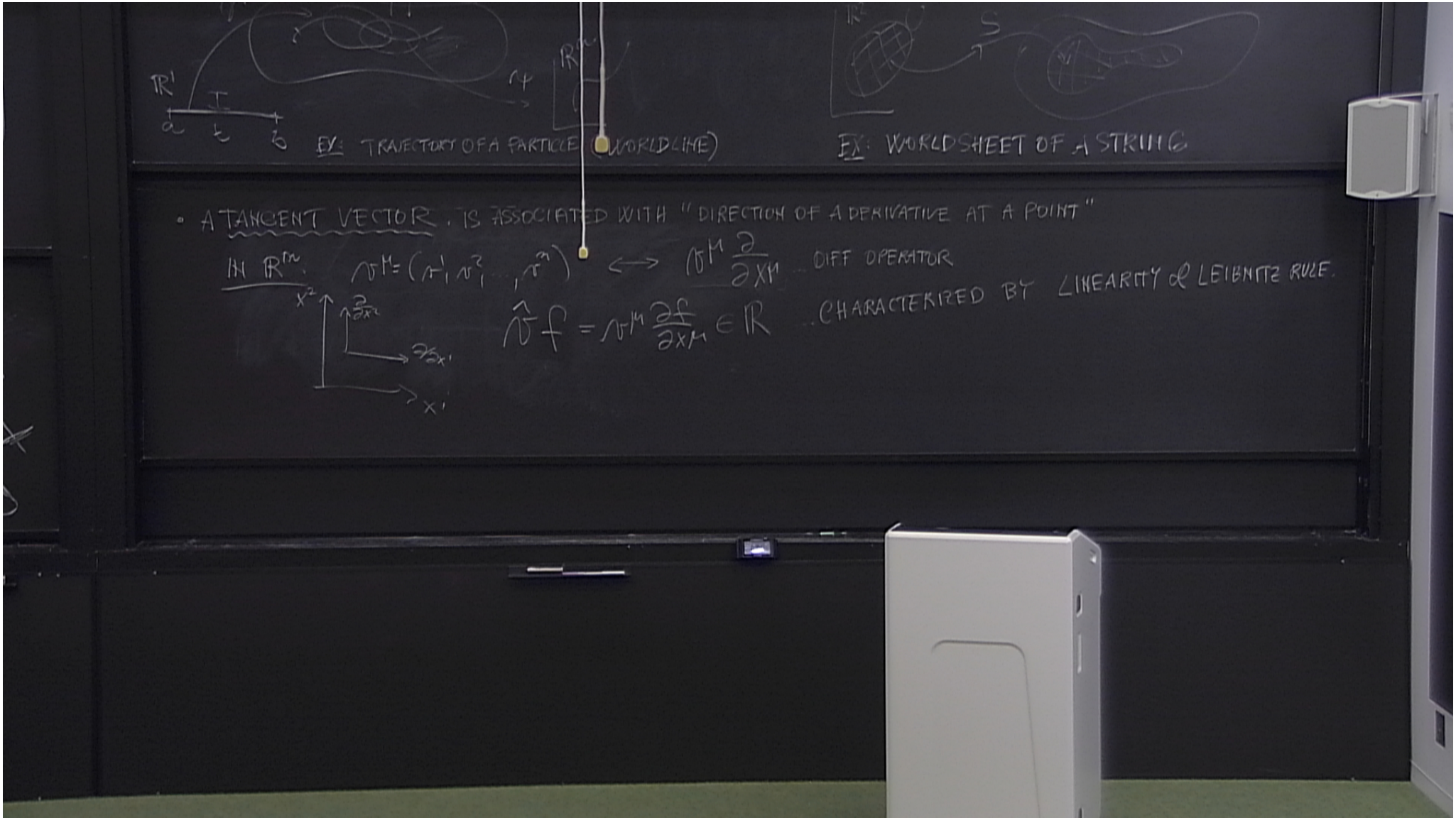

EX: CURVATU
R
• A CURVE γ ON M IS A MAP γ: I ⊂ R' → M S.T.
(ψα ∘ γ)(t) = [x'(t), x²(t), ... , xⁿ(t)] ... SMOOTH.
γ
M
R'
I
a
t
b
Rⁿ


EX: CURVATURE
IS A MAP $\gamma: I \subset \mathbb{R}^1 \to M$ S.T.
$x^1(t), x^2(t), \ldots, x^m(t)]$ ... SMOOTH.
• A SURFACE S IS A MAP $S: U \subset \mathbb{R}^2 \to M$.
$(\psi_\alpha \circ S)(\tau, \sigma) = [x^1(\sigma,\tau), x^2(\sigma,\tau), \ldots, x^m(\sigma,\tau)]$ SMOOTH
$\mathbb{R}^2$
S
$\mathbb{R}^m$


$\mathbb{R}^1$
I
a
b
$\mathbb{R}^N$
$\mathbb{R}^2$
S
EX: TRAJECTORY OF A PARTICLE (WORLDLINE)
EX: WORLDSHEET OF A STRING
• A TANGENT VECTOR IS ASSOCIATED WITH "DIRECTION OF A DERIVATIVE AT A POINT"
IN $\mathbb{R}^N$: $v^\mu = (v^1, v^2, \ldots, v^N) \longleftrightarrow v^\mu \frac{\partial}{\partial x^\mu}$ ... DIFF OPERATOR
x^2
$\frac{\partial}{\partial x^2}$
$\partial/\partial x^1$
x^1
$\hat{v} f = v^\mu \frac{\partial f}{\partial x^\mu}$


$\mathbb{R}^1$
I
a
t
b
γ
$\mathbb{R}^n$
EX: TRAJECTORY OF A PARTICLE (WORLDLINE)
$\mathbb{R}^2$
S
EX: WORLDSHEET OF A STRING
• A TANGENT VECTOR IS ASSOCIATED WITH "DIRECTION OF A DERIVATIVE AT A POINT"
IN $\mathbb{R}^n$: $v^\mu = (v^1, v^2, \ldots, v^n) \longleftrightarrow v^\mu \frac{\partial}{\partial x^\mu}$ ... DIFF OPERATOR
x^2
$\frac{\partial}{\partial x^2}$
$\partial/\partial x^1$
x^1
$\hat{v} f = v^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R}$




EX: TRAJECTORY OF A PARTICLE (WORLDLINE)
EX: WORLDSHEET OF A STRING
• A TANGENT VECTOR IS ASSOCIATED WITH "DIRECTION OF A DERIVATIVE AT A POINT"
IN $\mathbb{R}^m$: $v^\mu = (v^1, v^2, \ldots, v^m) \longleftrightarrow v^\mu \frac{\partial}{\partial x^\mu}$ ... DIFF OPERATOR
$\hat{v} f = v^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R}$ ... CHARACTERIZED BY LINEARITY & LEIBNITZ RULE.

iii) IF ANY TWO SETS O_α AND O_β OVERLAP, $O_\alpha \cap O_\beta \neq 0$, THE MAP $\psi_\beta \circ \psi_\alpha^{-1}$ IS C^∞.

DEFINITION: LET $\mathcal{F}$ BE A COLLECTION OF C^∞ FUNCTIONS. A TANGENT VECTOR V AT POINT $p \in M$ IS A MAP $V: \mathcal{F} \to \mathbb{R}$ THAT IS

i) LINEAR $V(af+bg) = aV(f)+bV(g)$, $a,b \in \mathbb{R}$

ii) OBEYS LEIBNITZ: $V(fg) = V(f)\,g(p) + V(g)\,f(p)$

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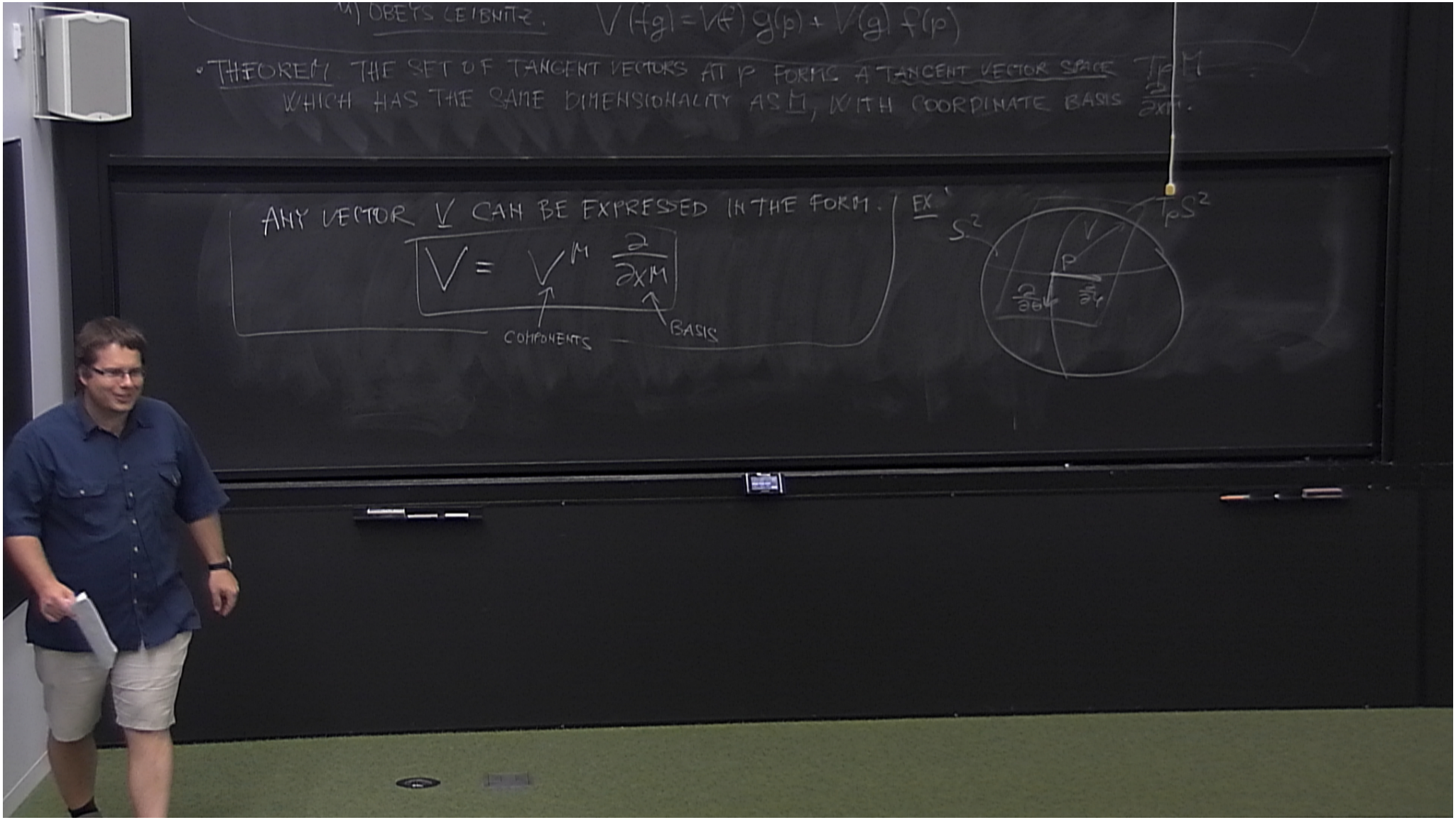
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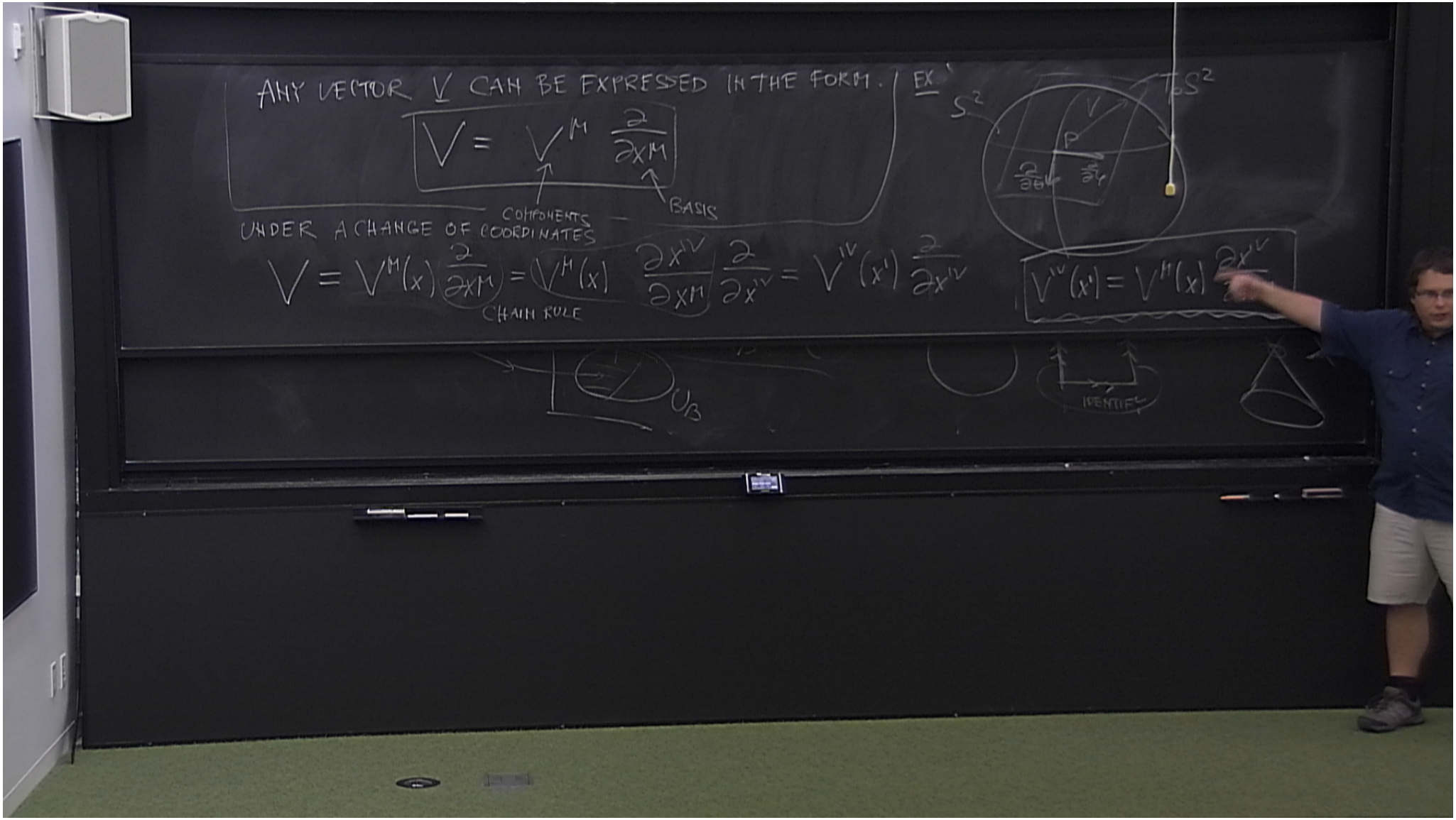
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- **THEOREM:** THE SET OF TANGENT VECTORS AT p FORMS A TANGENT VECTOR SPACE T_pM WHICH HAS THE SAME DIMENSIONALITY AS M, WITH COORDINATE BASIS $\frac{\partial}{\partial x^\mu}$.

- , THAT IS $\{O_\alpha\}$ COVER M.
- MAP $\psi_\alpha : O_\alpha \to U_\alpha$, WHERE U_α

AP, $O_\alpha \cap O_\beta \neq \emptyset$, THE MAP $\psi_\beta \circ \psi_\alpha^{-1}$ IS C^∞.


OBEYS LEIBNITZ. $V(fg) = V(f)\,g(p) + V(g)\,f(p)$
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ANY VECTOR V CAN BE EXPRESSED IN THE FORM.
$V = V^\mu \frac{\partial}{\partial x^\mu}$
COMPONENTS
BASIS
EX
S^2
T_pS^2
V
p


ANY VECTOR V CAN BE EXPRESSED IN THE FORM.
$V = V^M \frac{\partial}{\partial x^M}$
COMPONENTS
BASIS
UNDER A CHANGE OF COORDINATES
$V = V^M(x) \frac{\partial}{\partial x^M} = V'^M(x) \frac{\partial x'^N}{\partial x^M} \frac{\partial}{\partial x'^N} = V'^N(x') \frac{\partial}{\partial x'^N}$
CHAIN RULE
$V'^N(x') = V'^M(x) \frac{\partial x'^N}{\partial x^M}$
EX
S^2
$T_p S^2$
V
P
$\frac{\partial}{\partial \theta}$
$\frac{\partial}{\partial \varphi}$
U_B
IDENTIFY

$V = V^\mu \frac{\partial}{\partial x^\mu}$

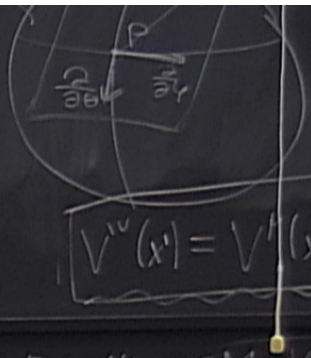
COMPONENTS — BASIS

UNDER A CHANGE OF COORDINATES

$$V = V^\mu(x) \frac{\partial}{\partial x^\mu} = V^\mu(x) \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} = V'^\nu(x') \frac{\partial}{\partial x'^\nu}$$

INVARIANT OBJECT — CHAIN RULE

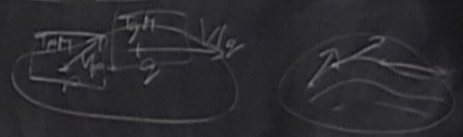
$$V'^\nu(x') = V^\mu(x) \frac{\partial x'^\nu}{\partial x^\mu}$$

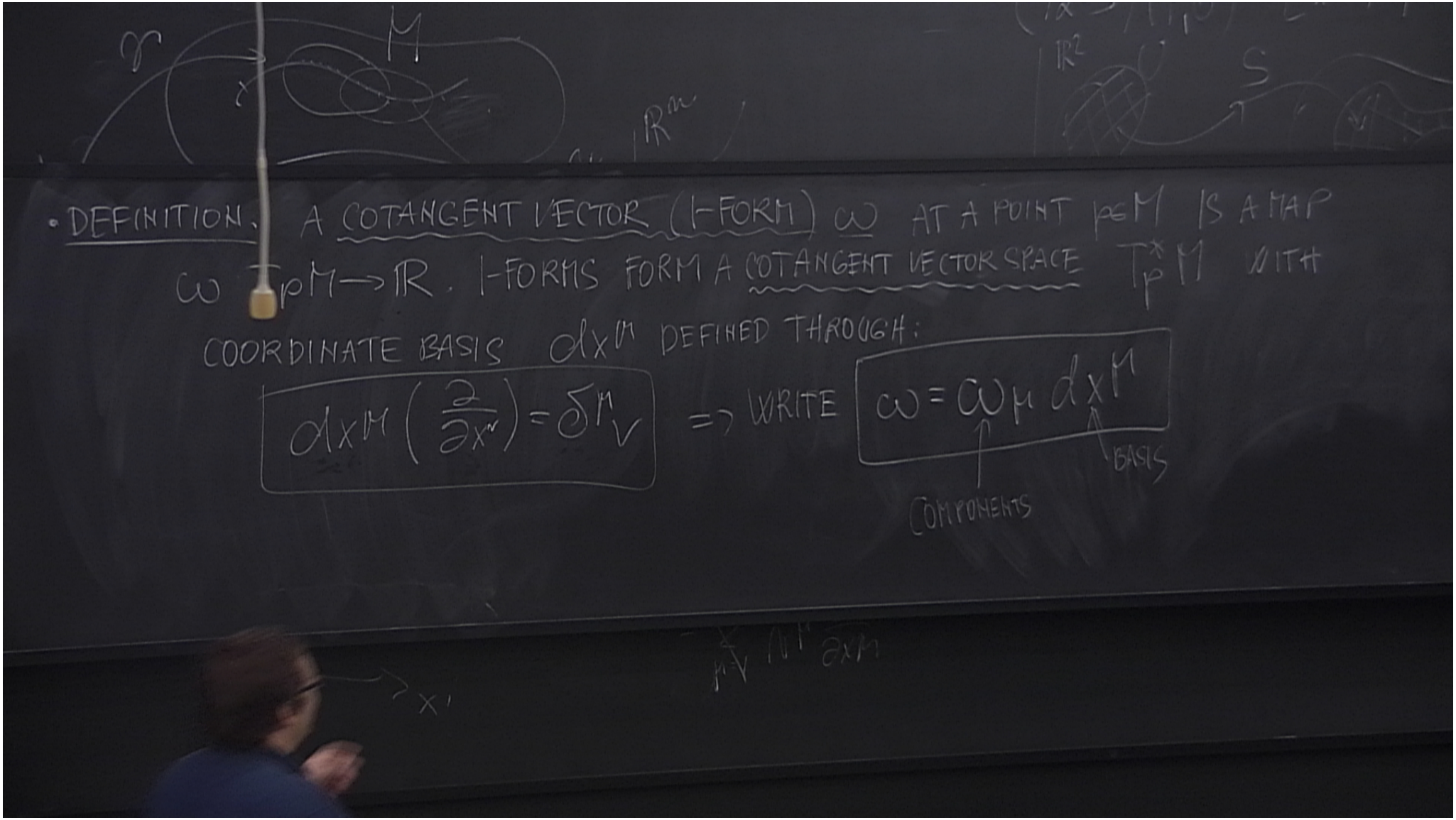


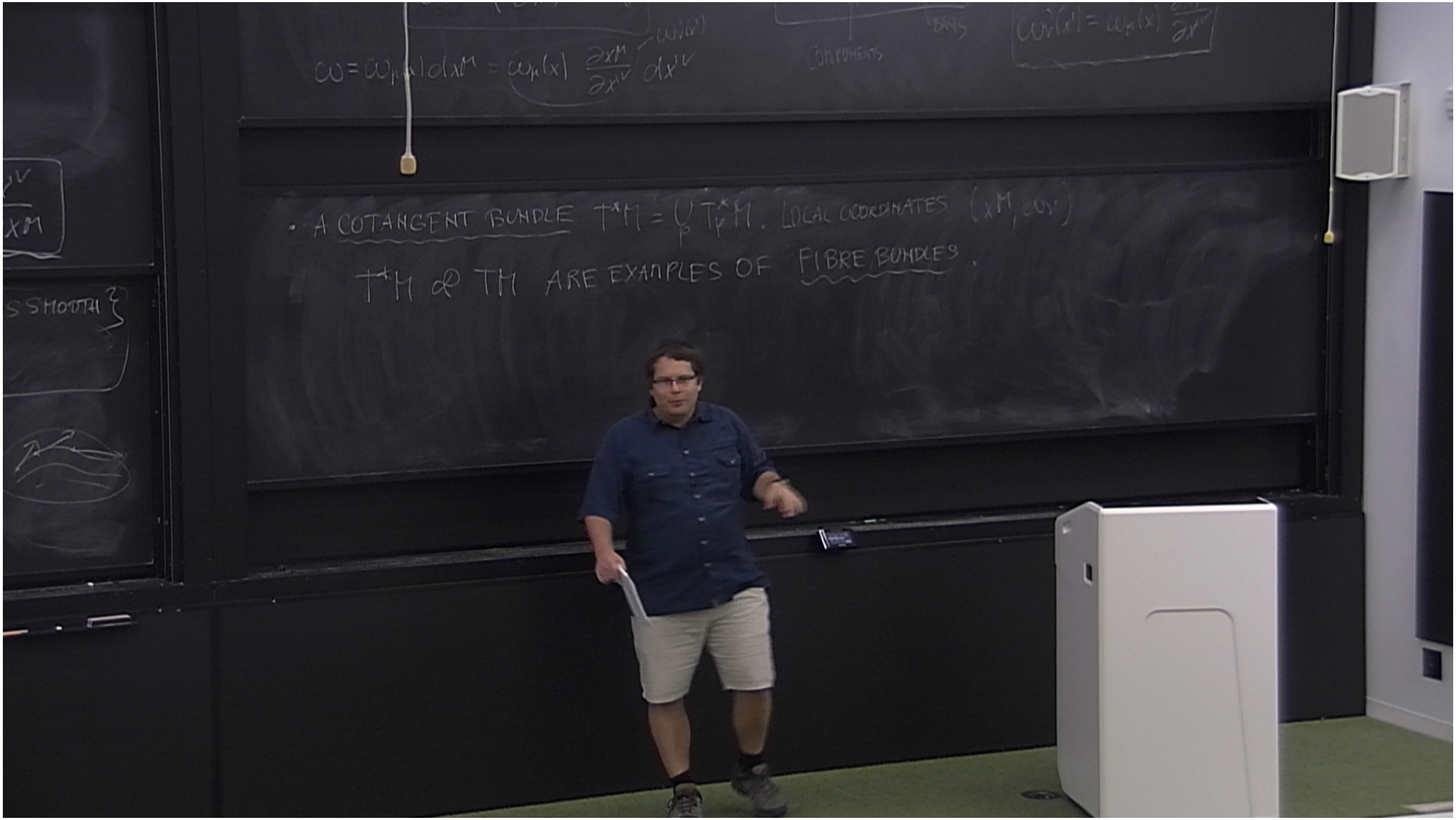
DEFINITION:

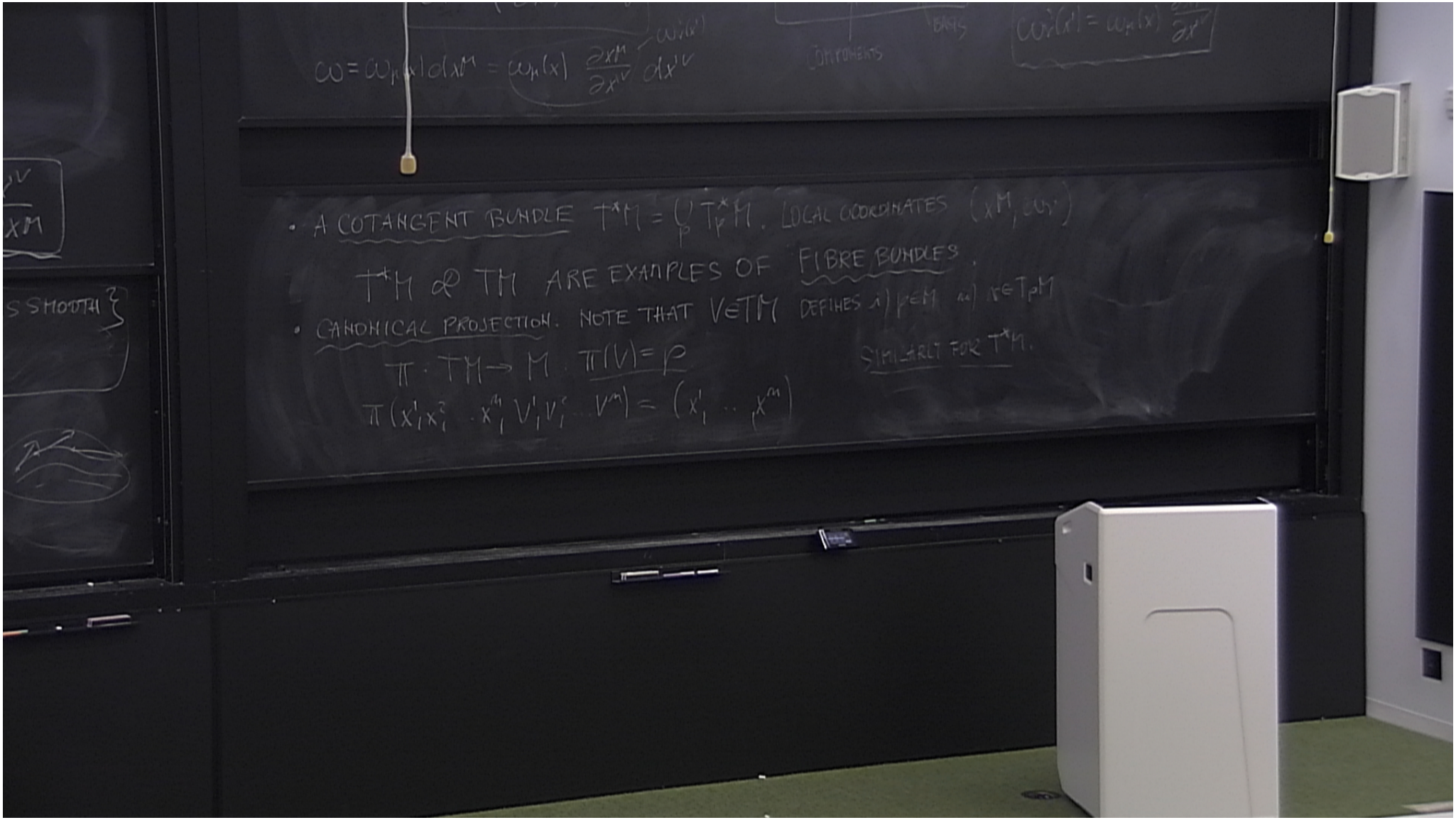
- A TANGENT VECTOR V IS DEFINED AS $\{ V|_p \in T_pM$ FOR ALL $p \in M$: $V(f)$ IS SMOOTH $\}$
- A TANGENT BUNDLE $TM = \bigcup_p T_pM$

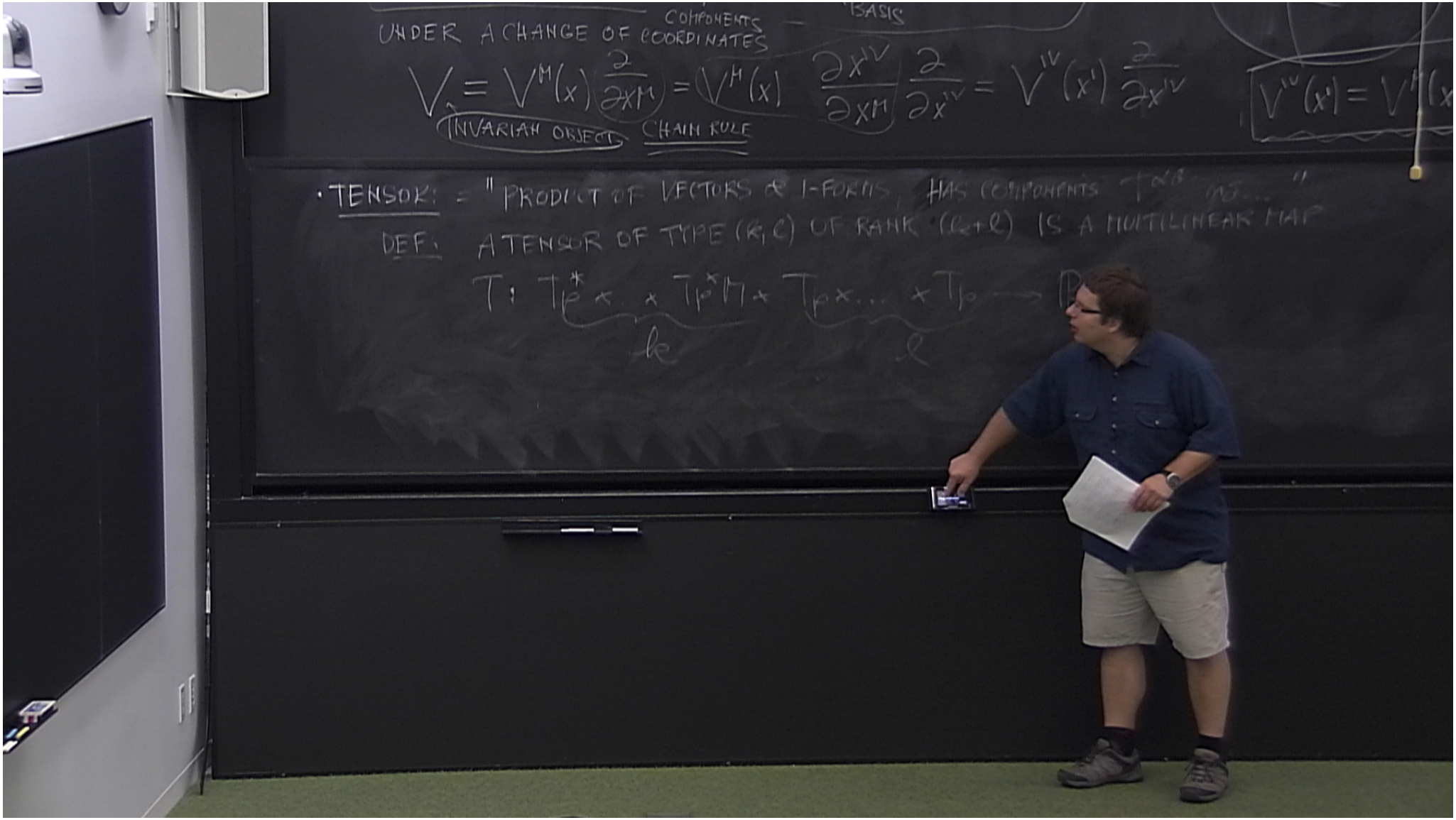
NOTE THAT TM HAS "LOCAL COORDINATES" (x^μ, V^ν)




M
$\mathbb{R}^n$
$\mathbb{R}^2$
S
• DEFINITION. A COTANGENT VECTOR (1-FORM) ω AT A POINT $p \in M$ IS A MAP
$\omega : T_pM \to \mathbb{R}$. 1-FORMS FORM A COTANGENT VECTOR SPACE T_p^*M WITH
COORDINATE BASIS dx^μ DEFINED THROUGH:
$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu$
=> WRITE $\omega = \omega_\mu dx^\mu$
COMPONENTS
BASIS


$\omega = \omega_\mu(x)\, dx^\mu = \omega_\mu(x) \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$
COMPONENTS
BASIS
$\omega_\nu(x') = \omega_\mu(x) \frac{\partial x^\mu}{\partial x'^\nu}$
• A COTANGENT BUNDLE $T^*M = \bigcup_p T^*_p M$. LOCAL COORDINATES (x^μ, ω_ν)
T^*M & TM ARE EXAMPLES OF FIBRE BUNDLES.
S SMOOTH


$\omega = \omega_\mu(x) dx^\mu = \omega_\mu(x) \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$
COMPONENTS
BASIS
$\omega_\nu(x') = \omega_\mu(x) \frac{\partial x^\mu}{\partial x'^\nu}$
• A COTANGENT BUNDLE $T^*M = \bigcup_p T_p^*M$. LOCAL COORDINATES (x^μ, ω_ν)
T^*M & TM ARE EXAMPLES OF FIBRE BUNDLES.
• CANONICAL PROJECTION. NOTE THAT $V \in TM$ DEFINES a) $p \in M$ and b) $v \in T_pM$
$\pi : TM \to M \quad \pi(V) = p$
SIMILARLY FOR T^*M.
$\pi(x^1, x^2, \ldots x^n, V^1, V^2, \ldots V^n) = (x^1, \ldots x^n)$
S SMOOTH


UNDER A CHANGE OF COORDINATES
COMPONENTS
BASIS
INVARIAN OBJECT
CHAIN RULE
TENSOR: = "PRODUCT OF VECTORS & 1-FORMS, HAS COMPONENTS
DEF: A TENSOR OF TYPE (k, ℓ) OF RANK (k+ℓ) IS A MULTILINEAR MAP