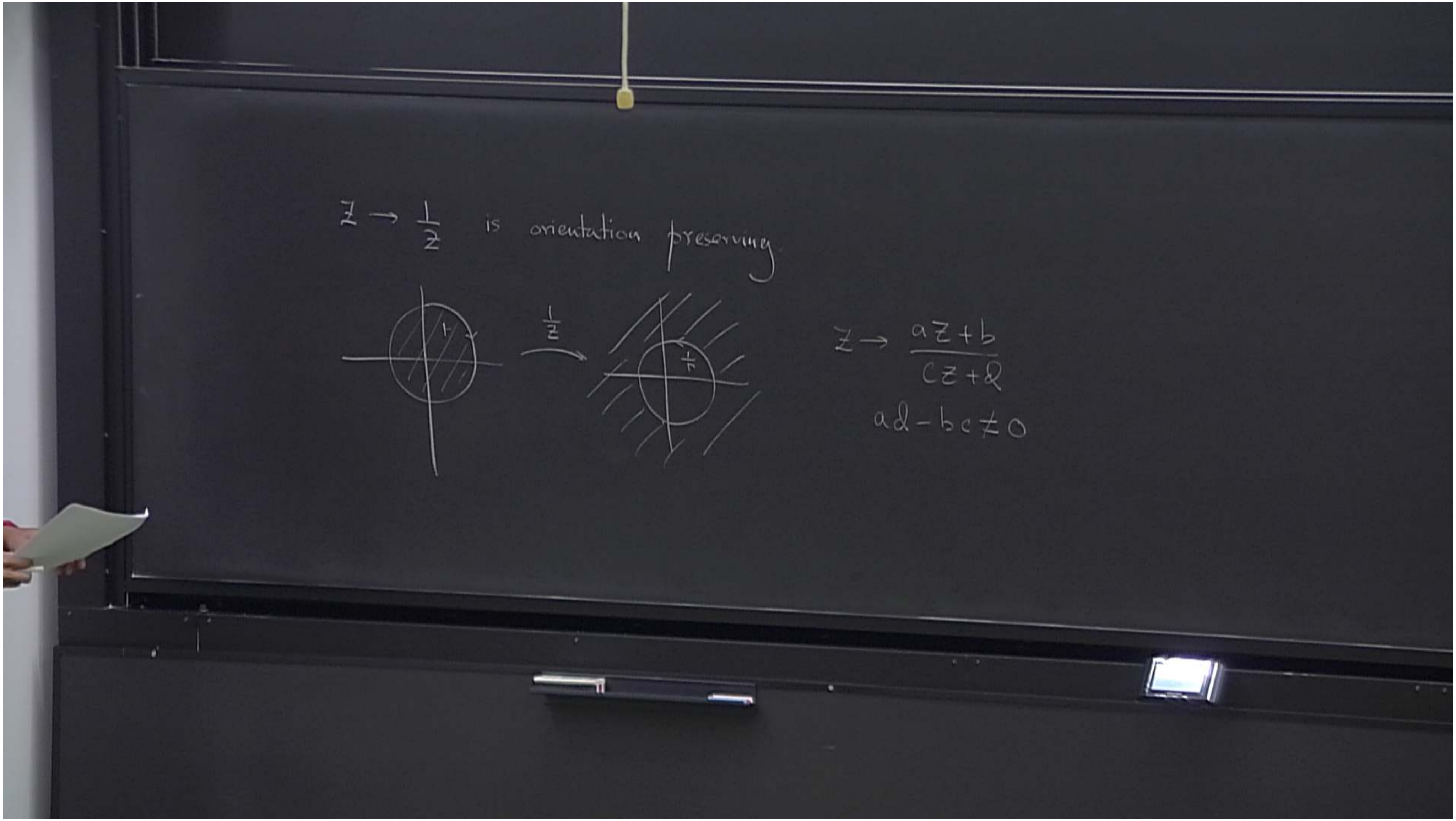
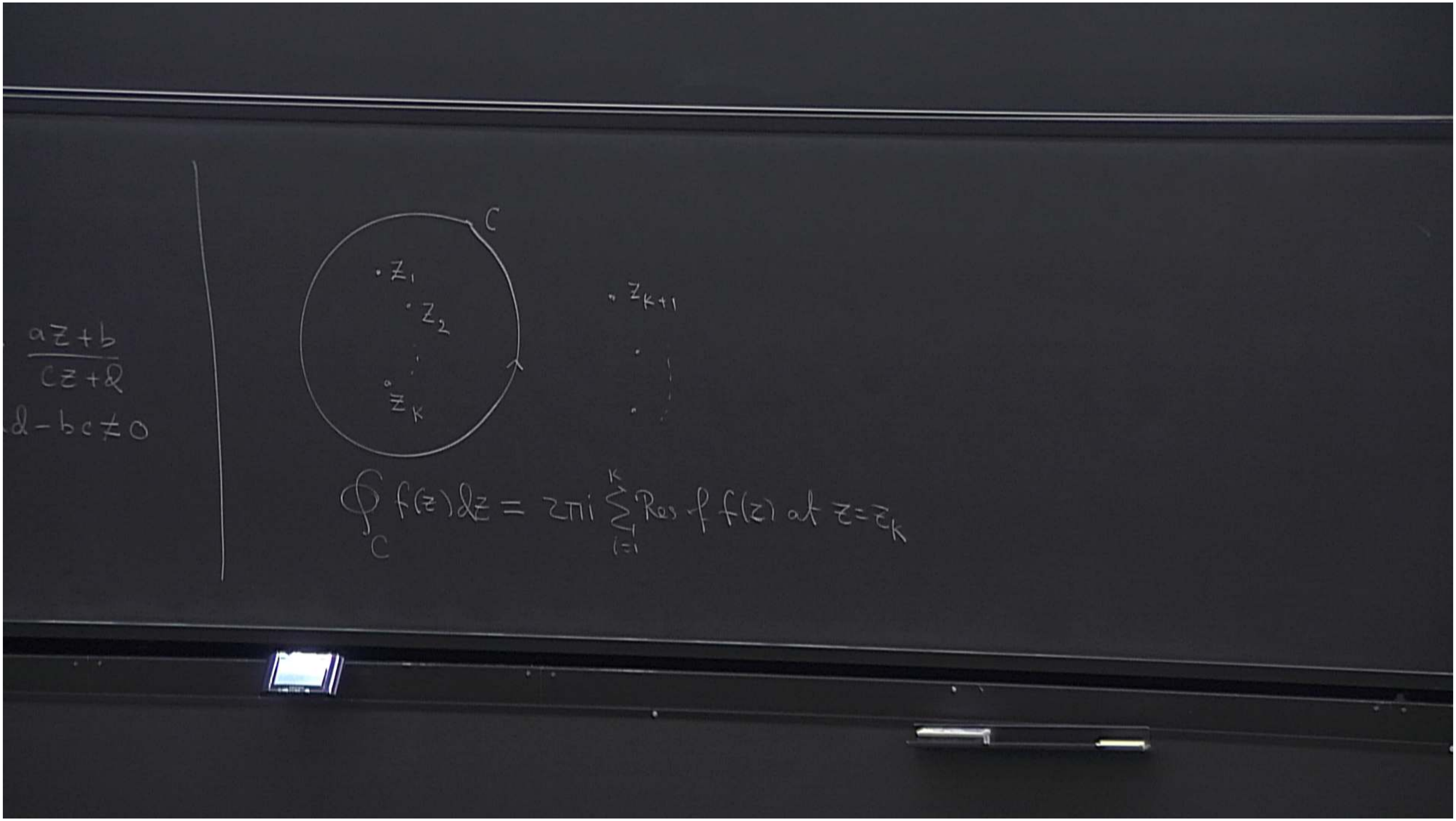
Title: Front End - Complex Analysis - 4

Date: Aug 12, 2016 09:00 AM

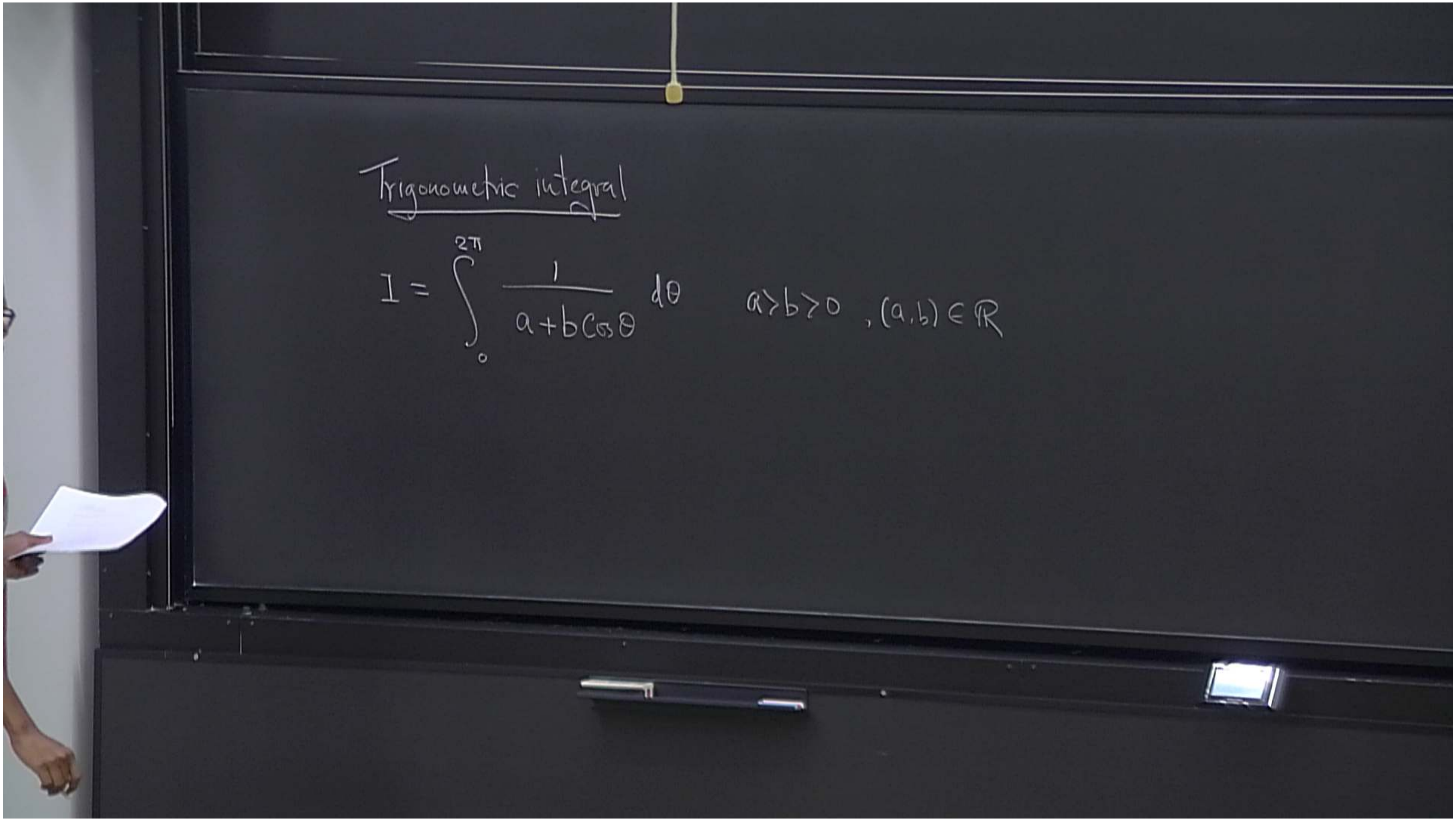
URL: http://pirsa.org/16080060

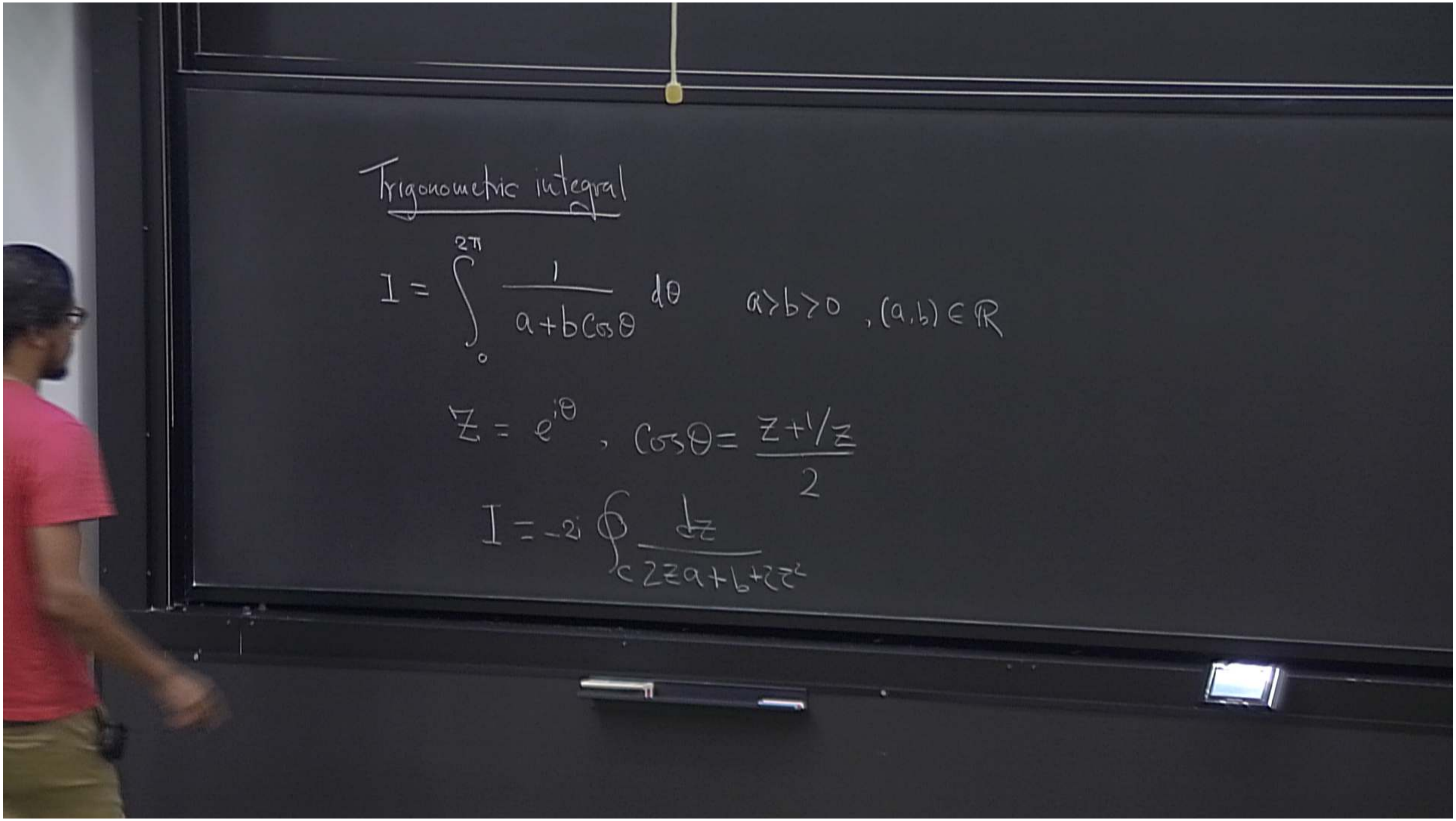
Abstract:


z → 1/z is orientation preserving.
1/z
r
1/r
z → (az+b)/(cz+d)
ad − bc ≠ 0

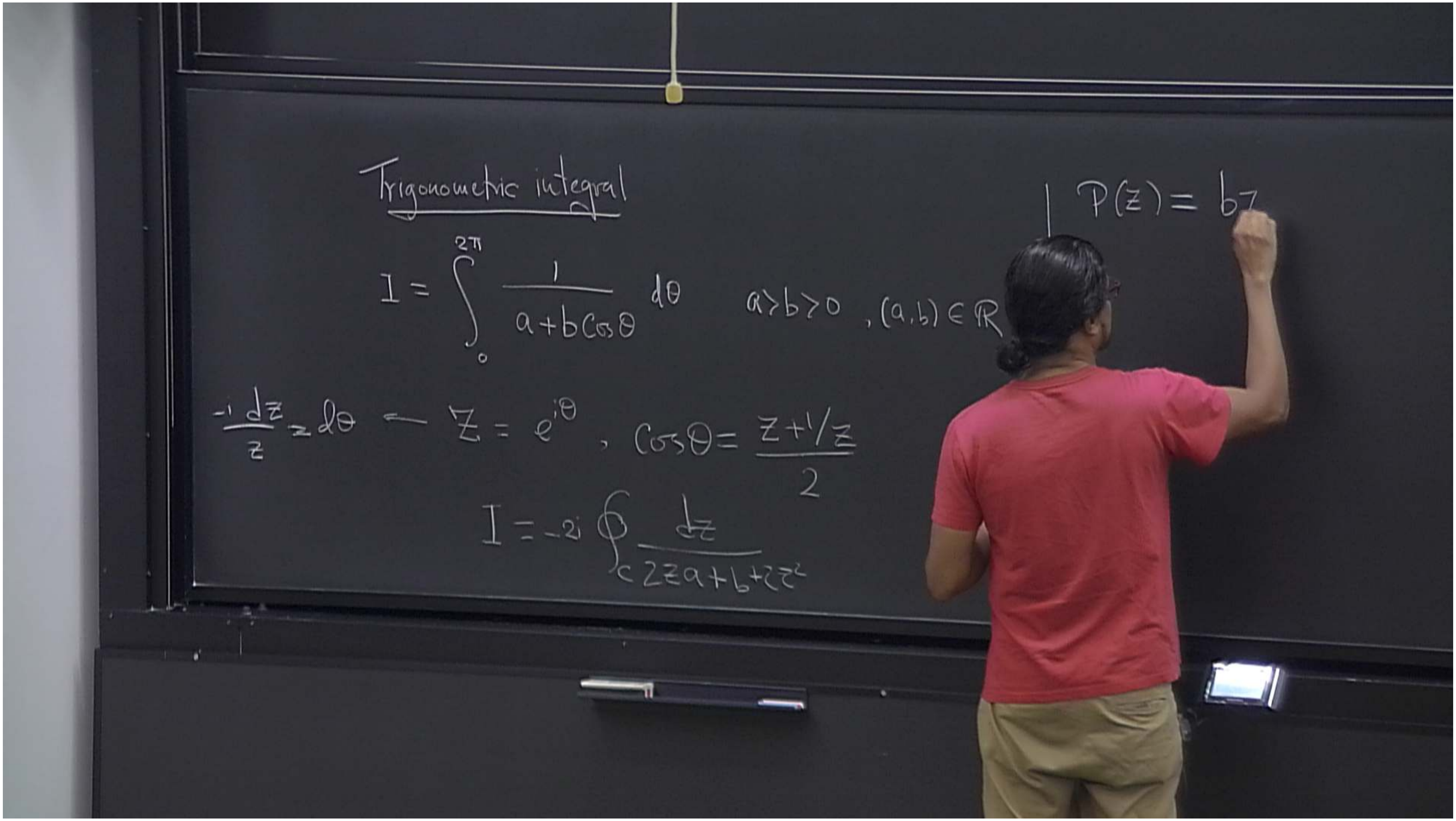


$\frac{aZ+b}{cZ+d}$
$ad-bc\neq 0$
C
z_1
z_2
z_K
z_{K+1}
$\oint_C f(z)\,dz = 2\pi i \sum_{i=1}^{K} \mathrm{Res}$ of $f(z)$ at $z=z_k$


Trigonometric integral
$I = \int_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta$
$a > b > 0\ ,\ (a, b) \in \mathbb{R}$



Trigonometric integral
$I = \int_0^{2\pi} \frac{1}{a+b\cos\theta}\, d\theta \qquad a>b>0,\ (a,b) \in \mathbb{R}$
$z = e^{i\theta}, \quad \cos\theta = \frac{z+1/z}{2}$
$I = -2i \oint_c \frac{dz}{2za+b+bz^2}$


Trigonometric integral
$I = \int_0^{2\pi} \frac{1}{a+b\cos\theta}\, d\theta$
$a>b>0$, $(a,b) \in \mathbb{R}$
$P(z) = bz$
$\frac{-i\, dz}{z} = d\theta \leftarrow z = e^{i\theta}$, $\cos\theta = \frac{z+1/z}{2}$
$I = -2i \oint_C \frac{dz}{2za+b+bz^2}$

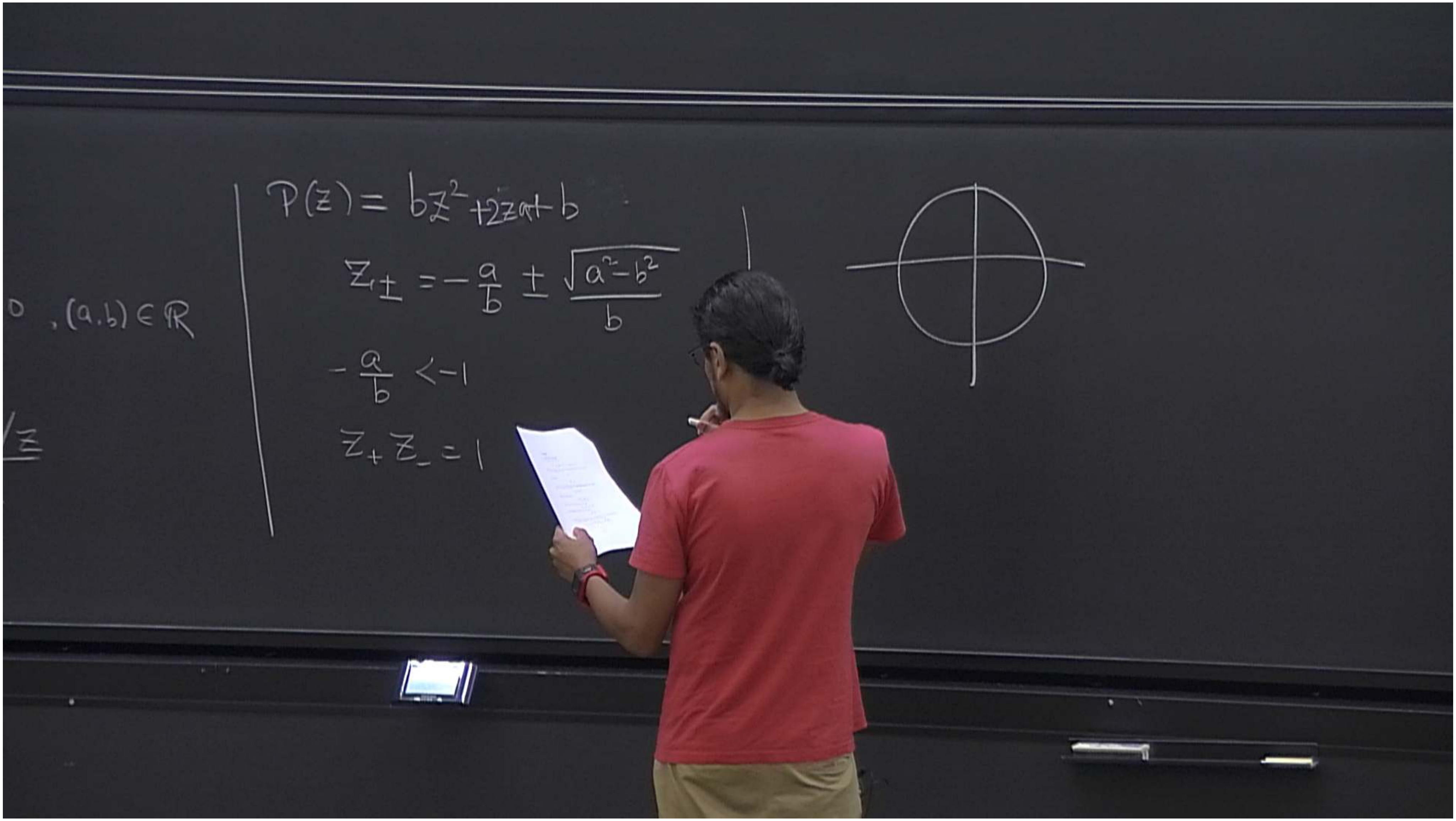
$0 \quad , (a,b) \in \mathbb{R}$

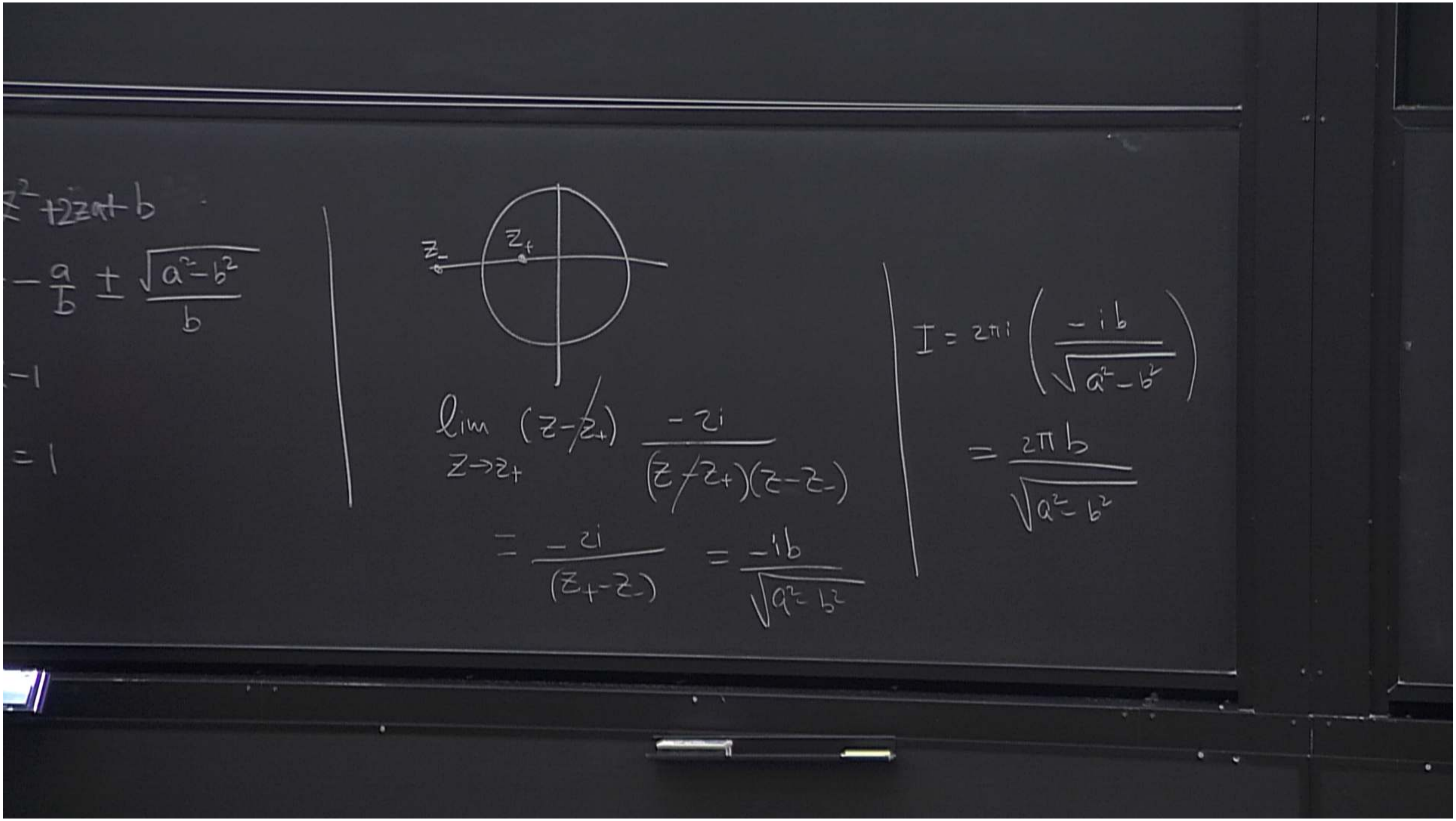
$\sqrt{z}$

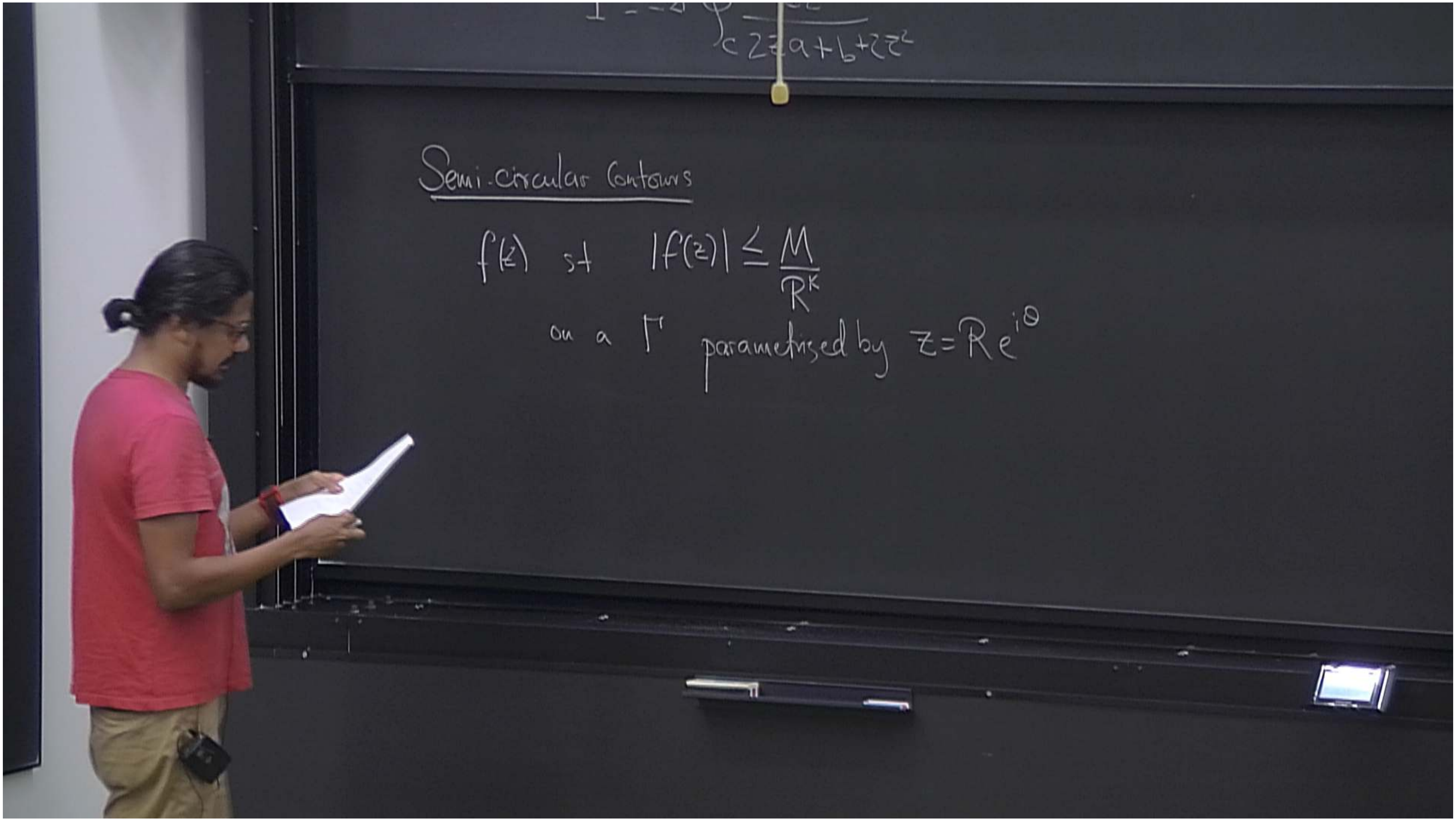
$$P(z) = bz^2 + 2za + b$$

$$z_{\pm} = -\frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}$$

$$-\frac{a}{b} < -1$$


P(z) = bz² + 2za + b
z± = − a/b ± √(a² − b²)/b
− a/b < −1
z₊ z₋ = 1
0 , (a.b) ∈ ℝ


z^2 + 2za + b
-a/b ± √(a^2 - b^2)/b
-1
= 1
z_-
z_+
lim_{z→z_+} (z - z_+) · -2i / ((z - z_+)(z - z_-))
= -2i / (z_+ - z_-) = -ib / √(a^2 - b^2)
I = 2πi (-ib / √(a^2 - b^2))
= 2πb / √(a^2 - b^2)


Semi-circular Contours
$f(z)$ s.t. $|f(z)| \le \frac{M}{R^k}$
on a Γ parametrised by $z = Re^{i\theta}$

$$\frac{}{c\ 2z\ a+b+2z^2}$$

Semi-circular Contours

$f(z)$ s.t. $|f(z)| \leq \frac{M}{R^k}$

on a Γ parametrised by $z = Re^{i\theta}$, where $0 \leq \theta \leq \pi$ or $\pi \leq \theta \leq 2\pi$

and $k > 1$ & M are real & positive

$$\underbrace{\left| \int_\Gamma f(z)\, dz \right|}_{\to 0 \text{ as } R \to \infty} \leq \frac{M\pi}{R^{k-1}}$$

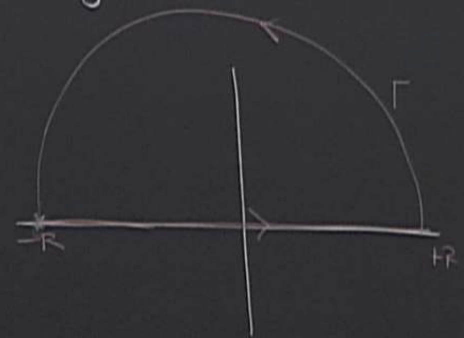
$(z_+ - z_-)$ $\sqrt{a^2 - b^2}$

or $\pi \le \theta \le 2\pi$

$\le \frac{M\pi}{R^{K-1}}$

$\to 0$ as $R \to \infty$

$$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$$



$$I = \frac{1}{2}\oint \frac{1}{(a^2+z^2)(b^2+z^2)}\,dz$$

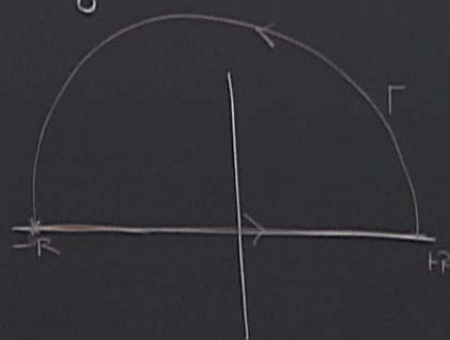
$$\frac{\quad}{(z_+ - z_-)} \qquad \frac{\quad}{\sqrt{a^2 - b^2}}$$

$\ldots$ or $\pi \leq \theta \leq 2\pi$

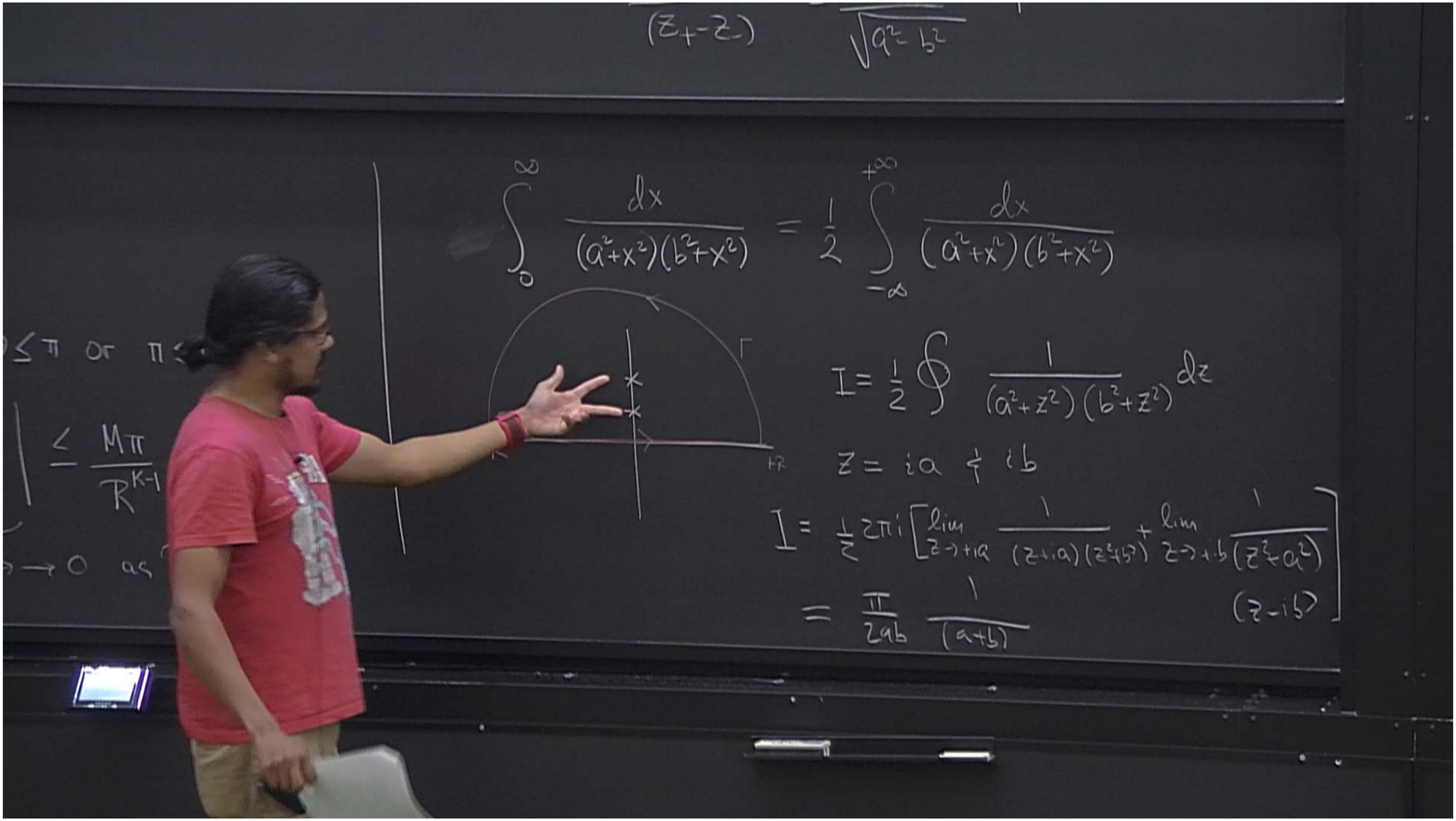
$$\leq \frac{M\pi}{R^{K-1}}$$

$\to 0$ as $R \to \infty$

$$\int_0^{\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$$



$$I = \frac{1}{2} \oint \frac{1}{(a^2+z^2)(b^2+z^2)} dz$$


$(z_+ - z_-)$
$\sqrt{a^2 - b^2}$
$\leq \pi$ or $\pi \leq$
$\leq \frac{M\pi}{R^{K-1}}$
$\to 0$ as
$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$
Γ
$-R$
$+R$
$I = \frac{1}{2} \oint \frac{1}{(a^2+z^2)(b^2+z^2)} dz$
$z = ia$ & ib
$I = \frac{1}{2} 2\pi i \left[\lim_{z \to +ia} \frac{1}{(z+ia)(z^2+b^2)} + \lim_{z \to +ib} \frac{1}{(z^2+a^2)(z-ib)} \right]$
$= \frac{\pi}{2ab} \frac{1}{(a+b)}$

Jordan's Lemma

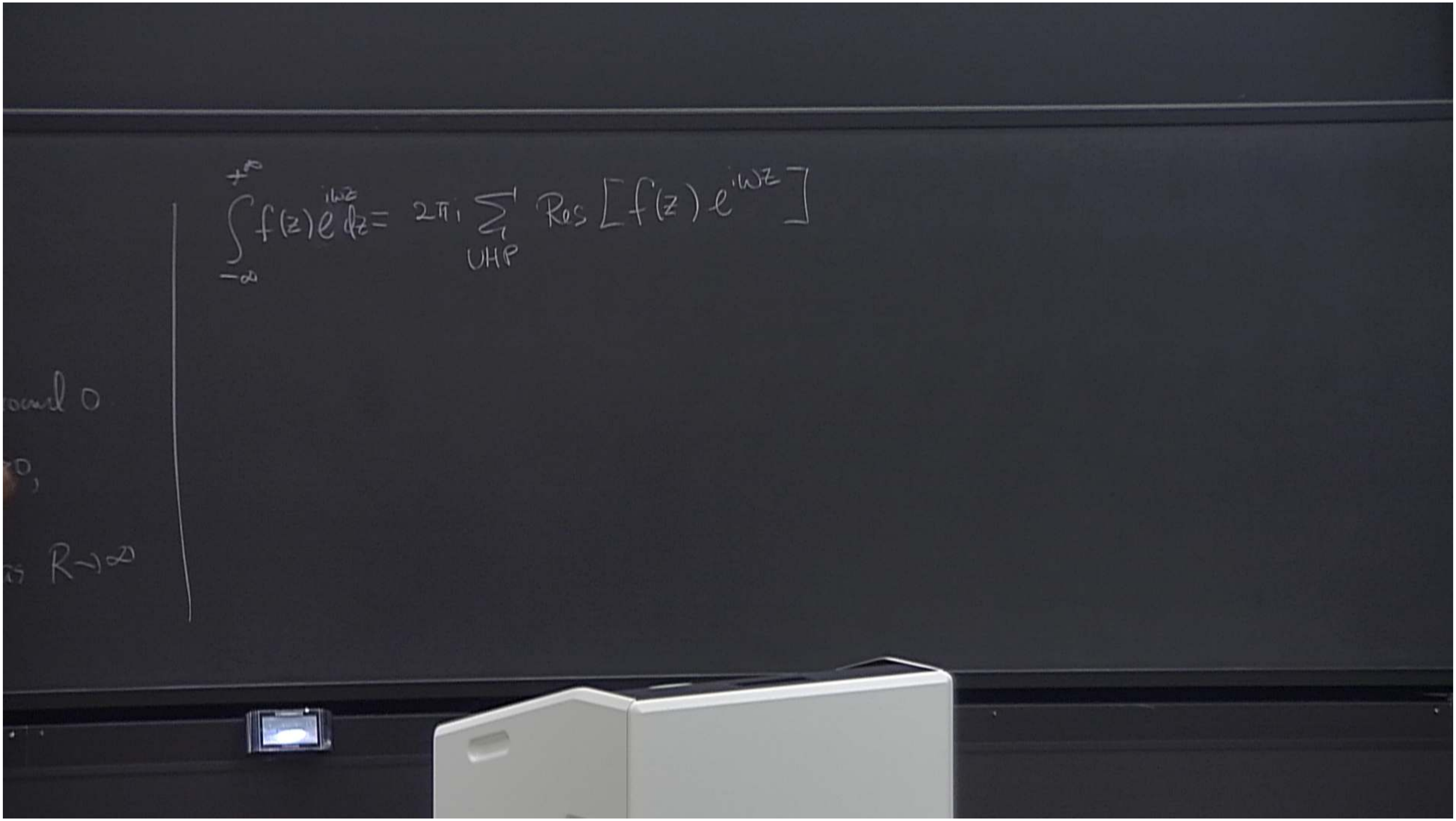
$$\int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx$$

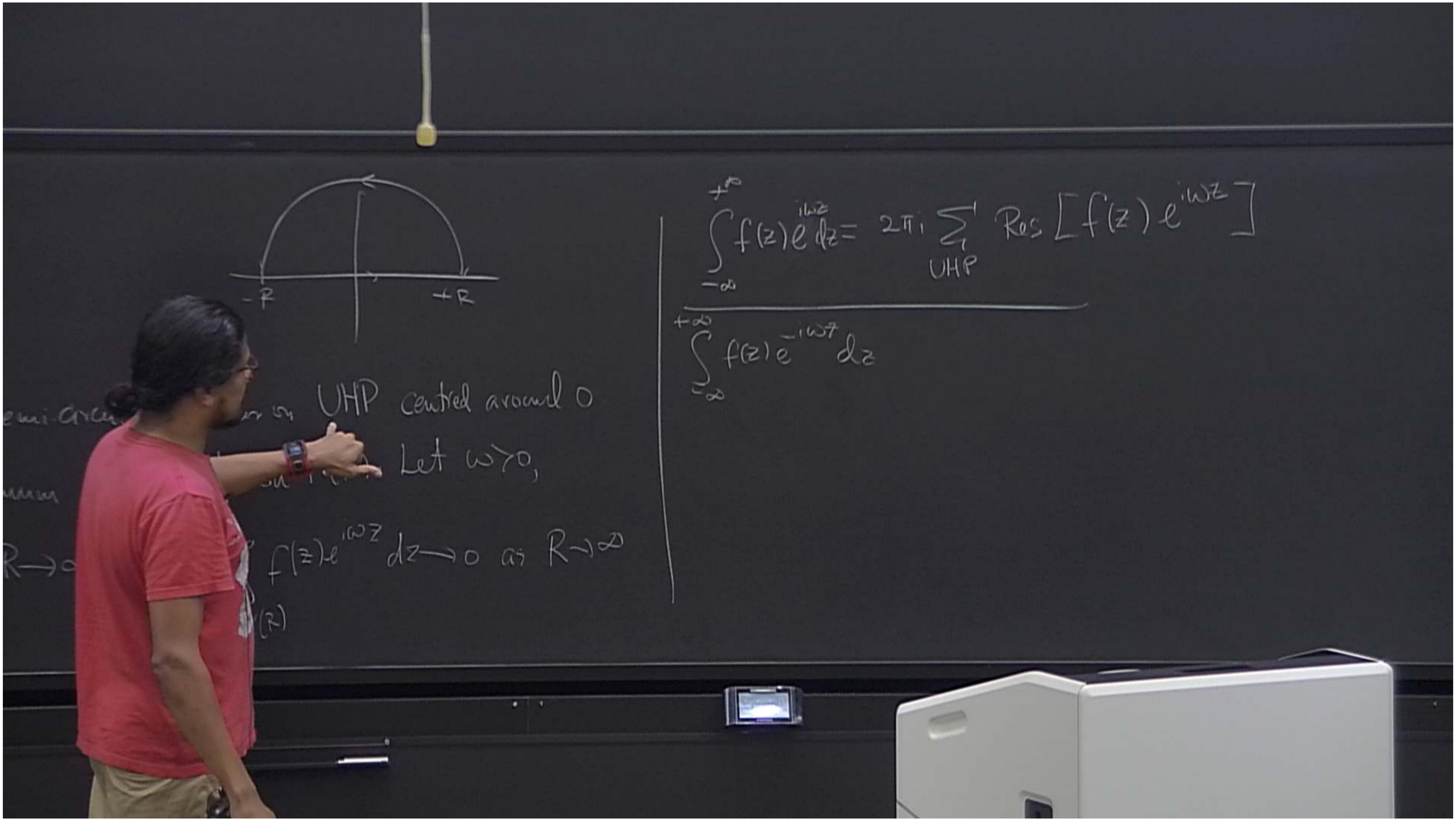
Suppose $\Gamma(R)$ is a semi-circular contour in UHP centred around 0

Let M_R be the maximum of $|f(z)|$ on $\Gamma(R)$. Let $\omega > 0$,

and $M_R \to 0$ as $R \to \infty$, then $\int_{\Gamma(R)} f(z) e^{i\omega z}\, dz \to 0$ as $R \to \infty$

$$\int_{-\infty}^{+\infty} f(z)\, dz$$


$\int_{-\infty}^{+\infty} f(z) e^{i\omega z} dz = 2\pi i \sum_{UHP}' \mathrm{Res}\left[f(z) e^{i\omega z} \right]$
$R \to \infty$


-R
+R
UHP centred around 0
Let $\omega > 0$,
$f(z)e^{i\omega z}dz \to 0$ as $R \to \infty$
$\int_{-\infty}^{+\infty} f(z)e^{i\omega z}dz = 2\pi i \sum_{UHP} \mathrm{Res}\left[f(z)e^{i\omega z}\right]$
$\int_{-\infty}^{+\infty} f(z)e^{-i\omega z}dz$

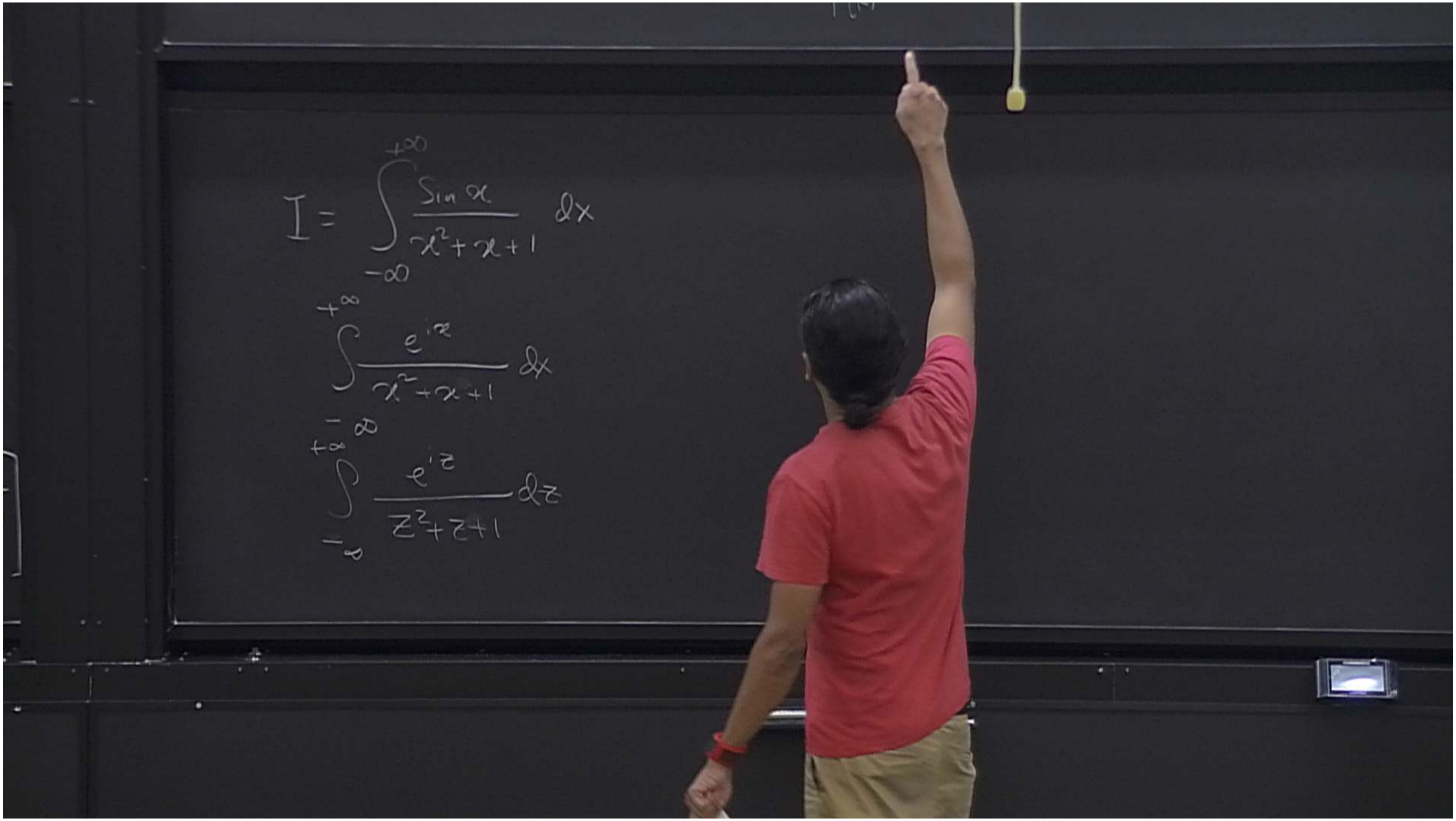
around 0

>0,

as $R \to \infty$

$$\int_{-\infty}^{+\infty} f(z) e^{i\omega z} dz = 2\pi i \sum_{UHP} \text{Res}\left[f(z) e^{i\omega z} \right]$$

$$\int_{-\infty}^{+\infty} f(z) e^{-i\omega z} dz = -2\pi i \sum_{LHP} \text{Res}\left[f(z) e^{-i\omega z} \right], \quad \omega > 0$$


$I = \int_{-\infty}^{+\infty} \frac{\sin x}{x^2+x+1}\, dx$
$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+x+1}\, dx$
$\int_{-\infty}^{+\infty} \frac{e^{iz}}{z^2+z+1}\, dz$

$f'(R)$

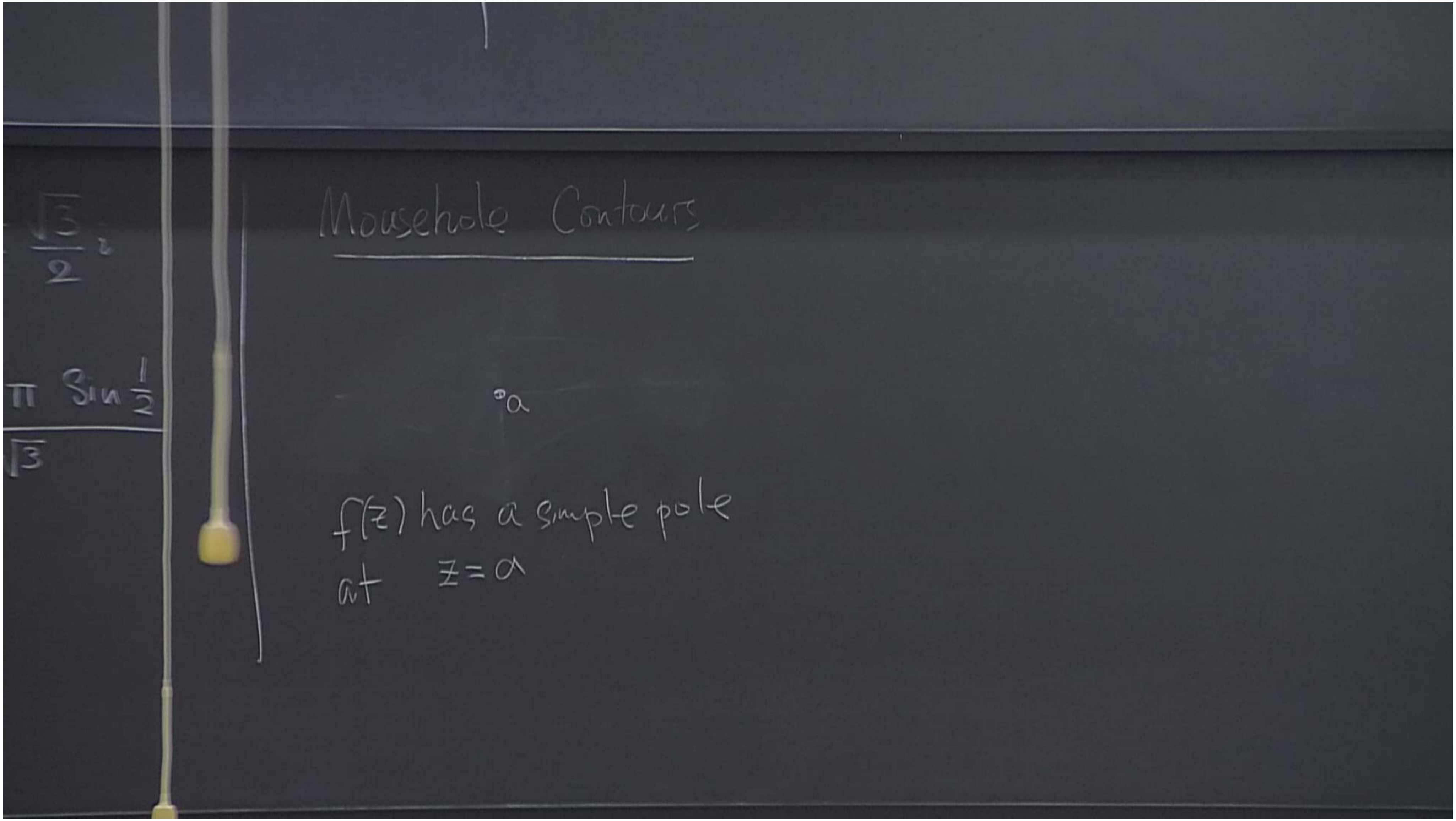
$$I = \int_{-\infty}^{+\infty} \frac{\sin x}{x^2 + x + 1}\, dx$$

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2 + x + 1}\, dx$$

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2 + z + 1}\, dz$$

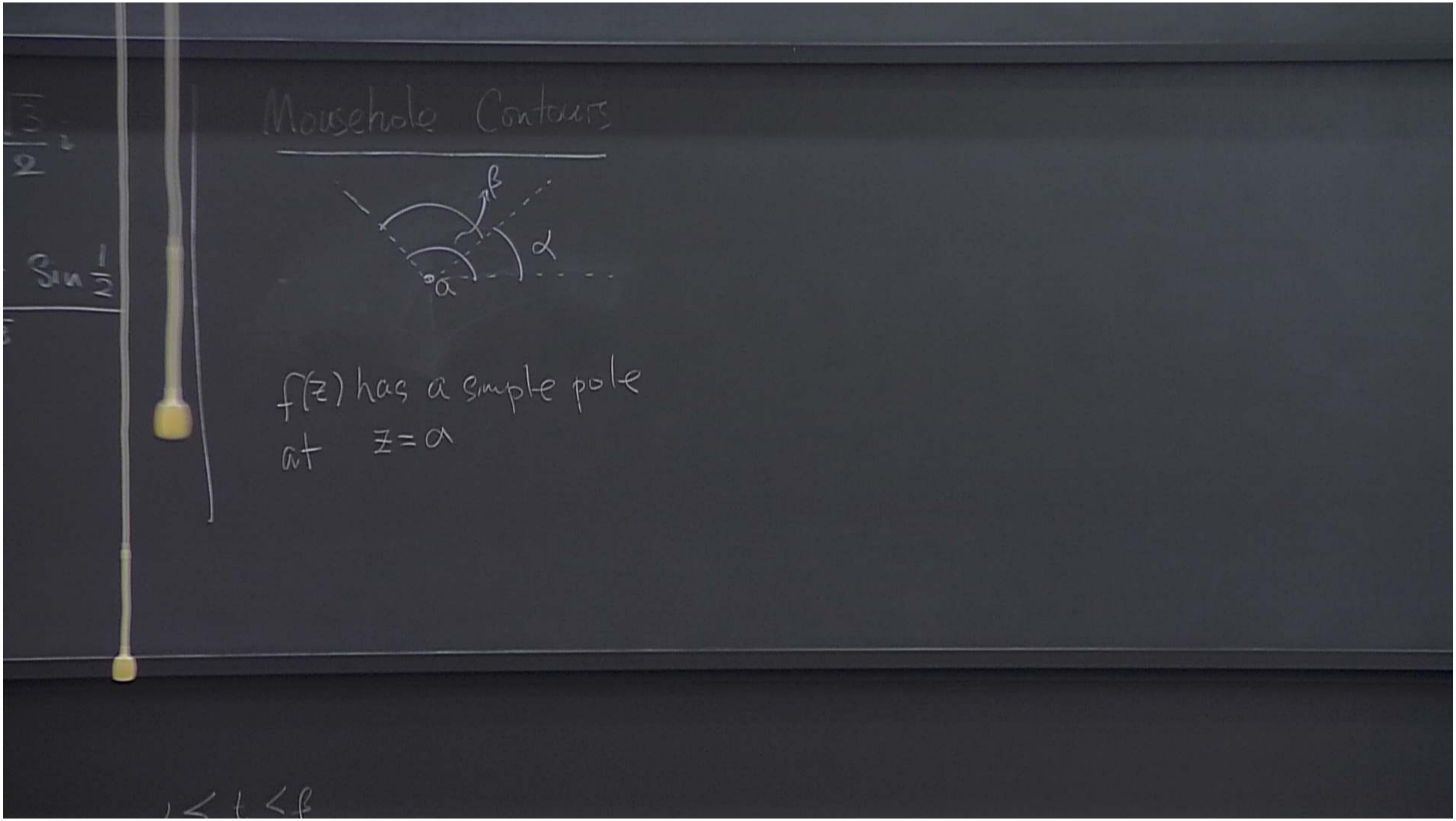
$$z_\pm = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

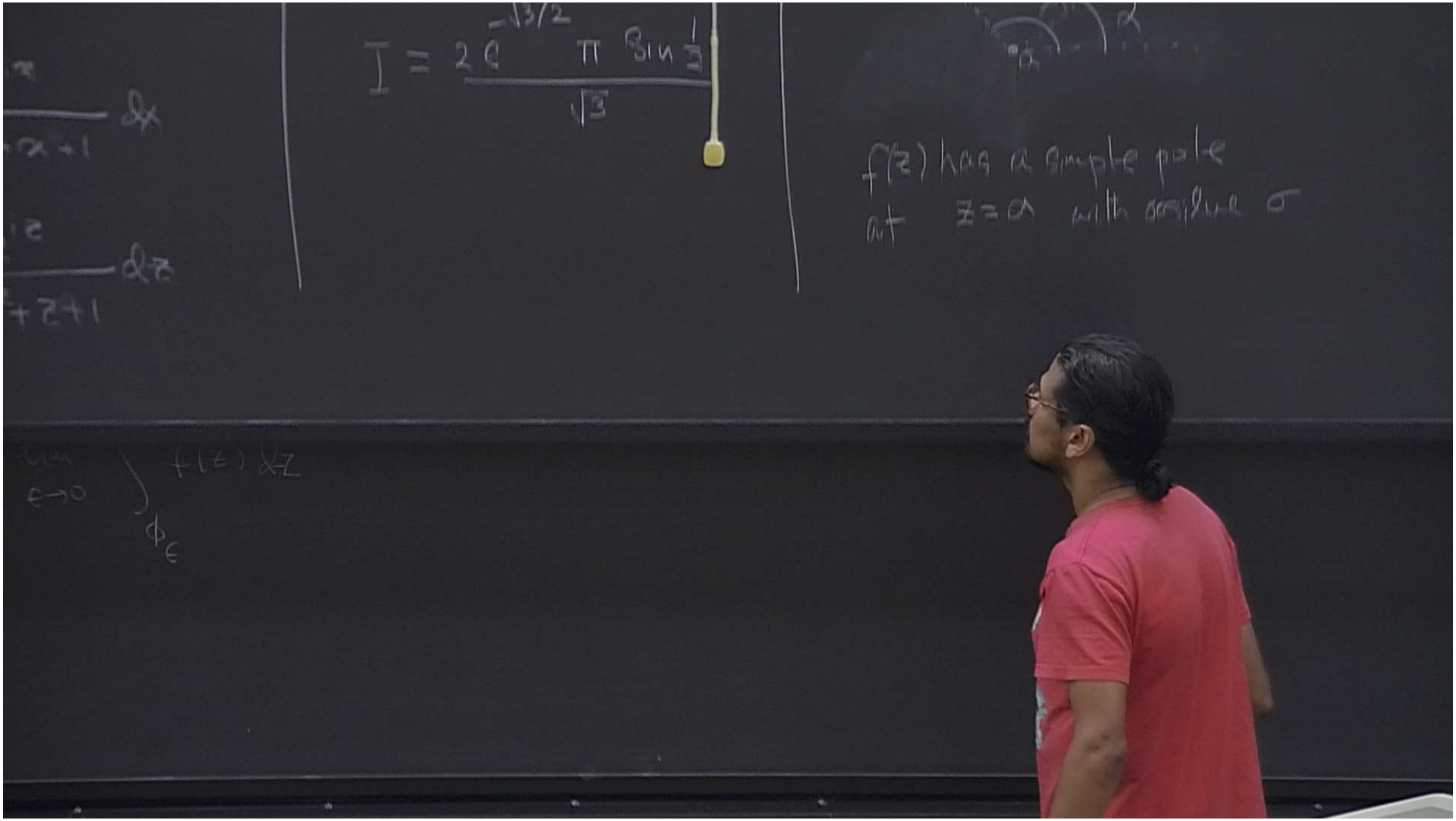
$$I = \frac{2 e^{-\sqrt{3}/2} \pi \sin \frac{1}{2}}{\sqrt{3}}$$

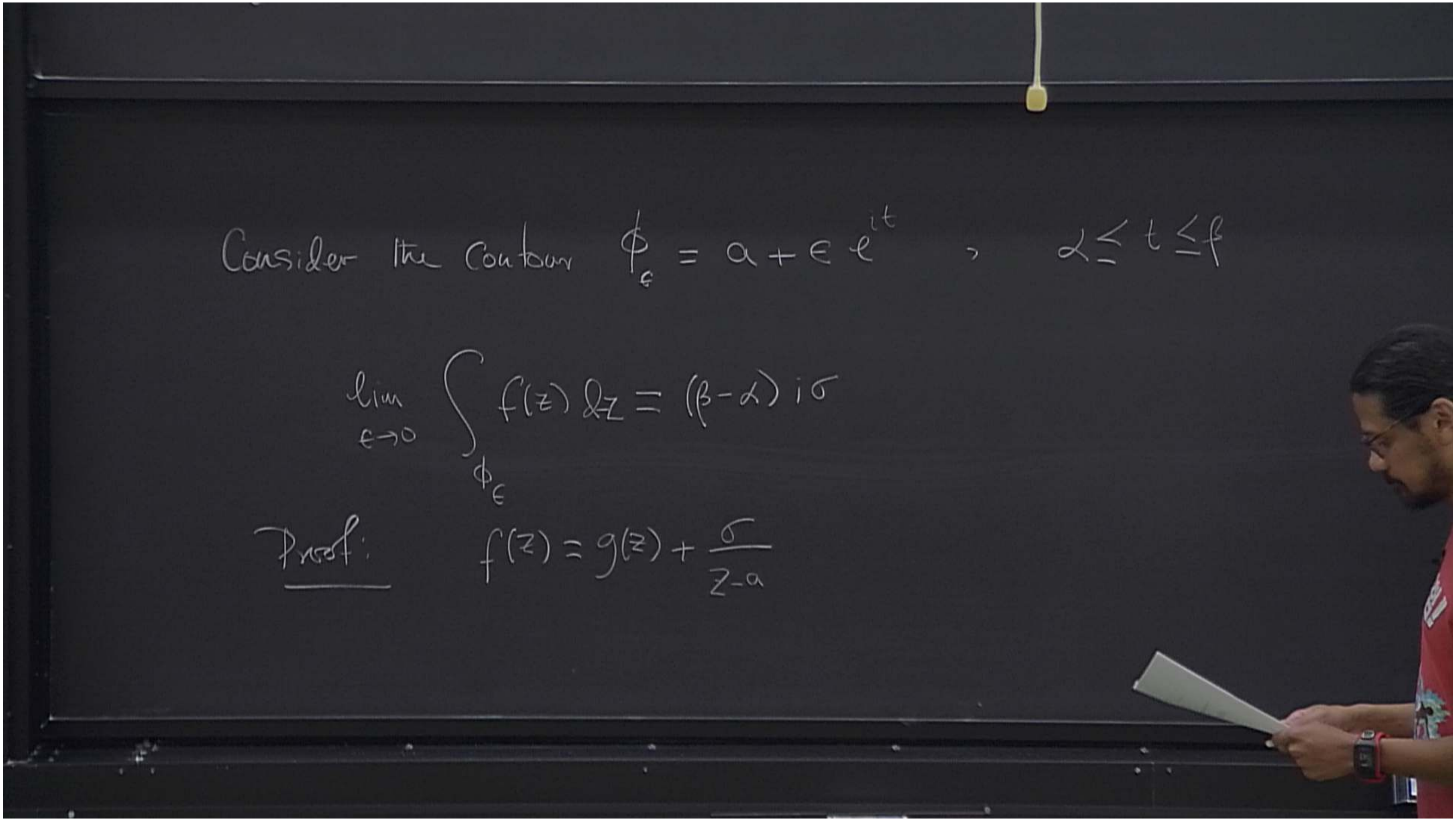

Mousehole Contours
a
f(z) has a simple pole
at z = a

$-\infty \quad z^2+z+1$

Consider the contour $\phi_\epsilon = a + \epsilon e^{it}$, $\alpha \leq t \leq \beta$

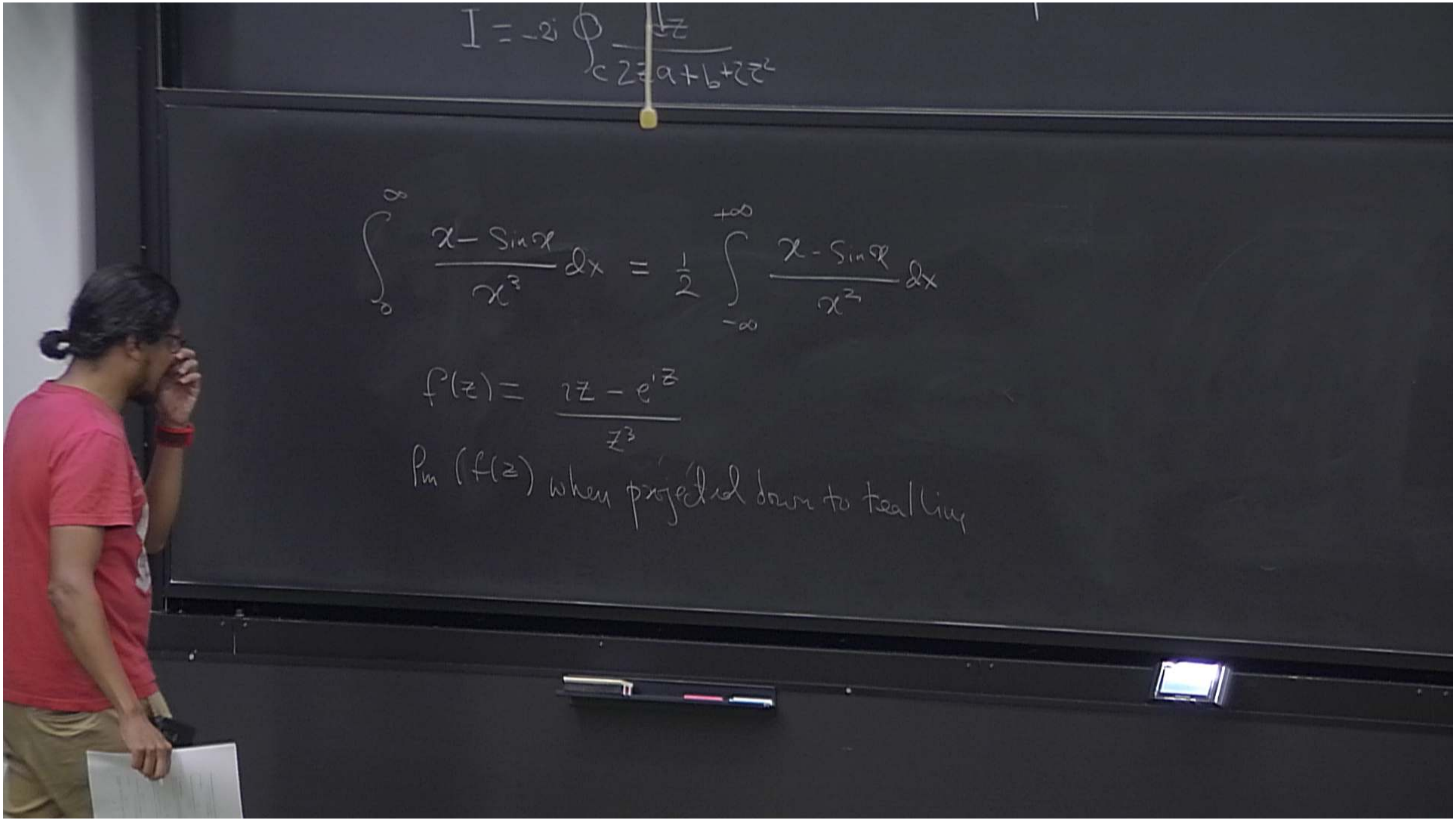

Mousehole Contours
β
α
a
f(z) has a simple pole
at z = a


f(z) has a simple pole
at z = α with residue σ
f(z) dz


Consider the contour $\phi_\epsilon = a + \epsilon e^{it}$, $\alpha \leq t \leq \beta$
$\lim_{\epsilon \to 0} \int_{\phi_\epsilon} f(z)\, dz = (\beta - \alpha) i\sigma$
Proof: $f(z) = g(z) + \frac{\sigma}{z-a}$

$$I = -2i \oint_C \frac{dz}{2za + b + bz^2}$$

$$\int_0^\infty \frac{x - \sin x}{x^3}\, dx$$


$I = -2i \oint_c \frac{dz}{z^2 a + b + z z^2}$
$\int_0^\infty \frac{x - \sin x}{x^3} dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x - \sin x}{x^3} dx$
$f(z) = \frac{iz - e^{iz}}{z^3}$
Im (f(z)) when projected down to real line

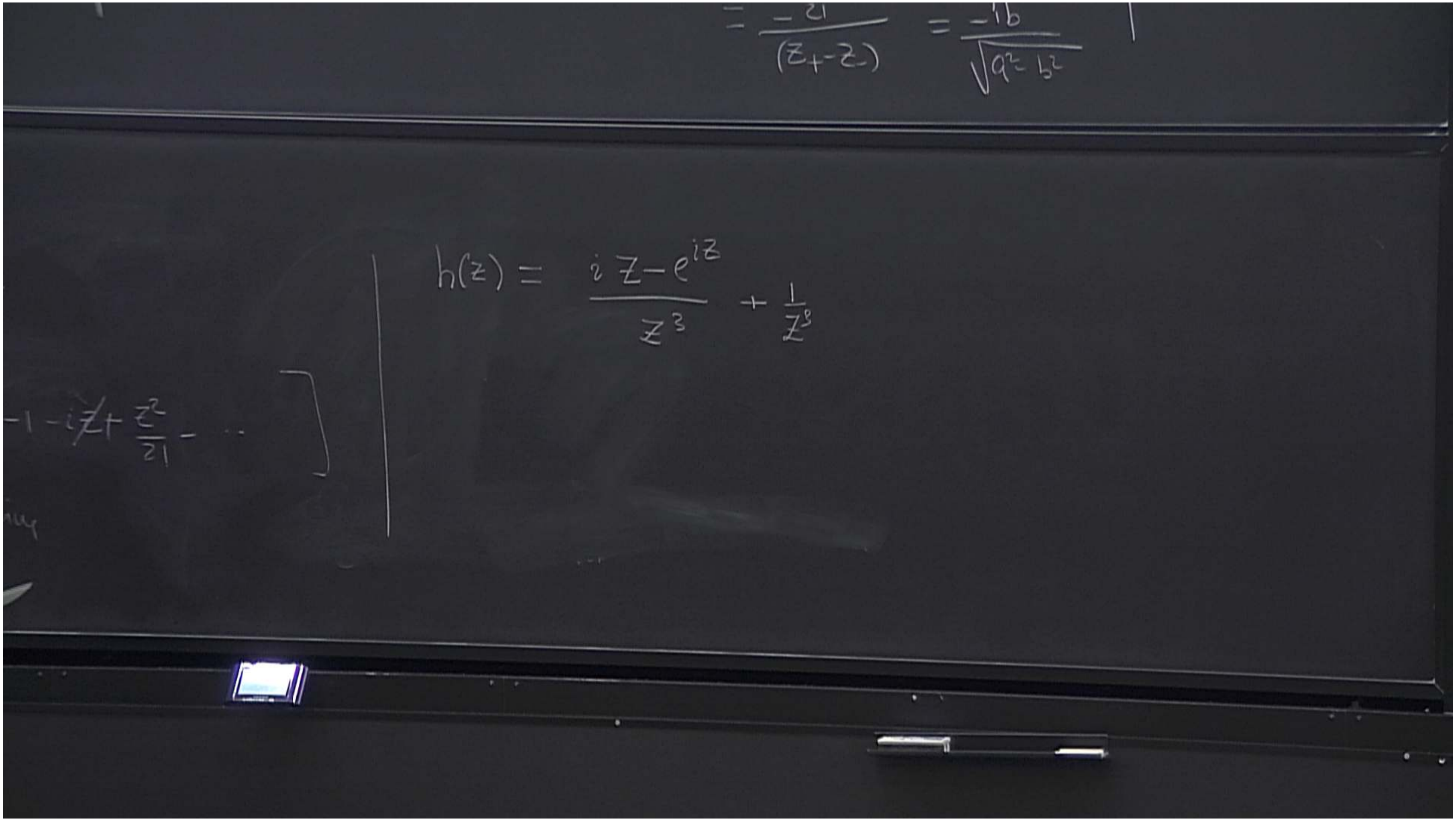
$$I = -2i \oint_c \frac{dz}{2za+b+2z^2}$$

$x - x + \frac{x^3}{3!\,x^3}$

$$\int_0^\infty \frac{x-\sin x}{x^3}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x-\sin x}{x^3}\,dx$$

$$f(z) = \frac{iz - e^{iz}}{z^3} = \frac{1}{z^3}\left[\cancel{iz} - 1 - \cancel{iz} + \frac{z^2}{2!} - \dots\right]$$

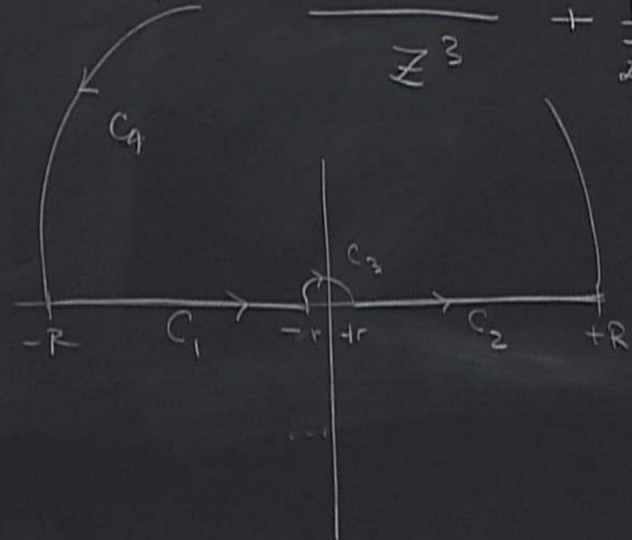
Im $(f(z))$ when projected down to real line



$= \frac{-2i}{(z_+ - z_-)} = \frac{-ib}{\sqrt{a^2-b^2}}$
$h(z) = \frac{iz - e^{iz}}{z^3} + \frac{1}{z^3}$
$-1 - i\cancel{z} + \frac{z^2}{2!} - \cdots$

$(z_+ - z_-)$ $\sqrt{a^2 - b^2}$

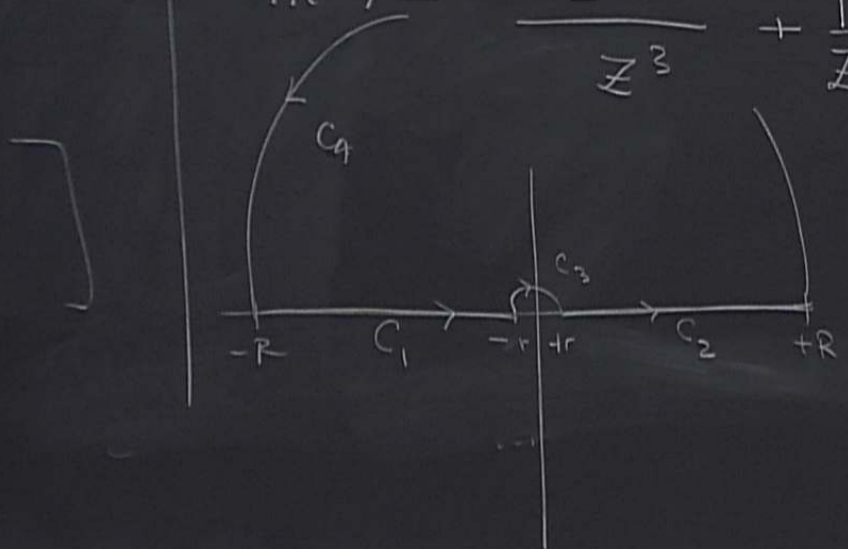
$$h(z) = \frac{iz - e^{iz}}{z^3} + \frac{1}{z^3}$$



$$\int_\Gamma h(z)\,dz$$

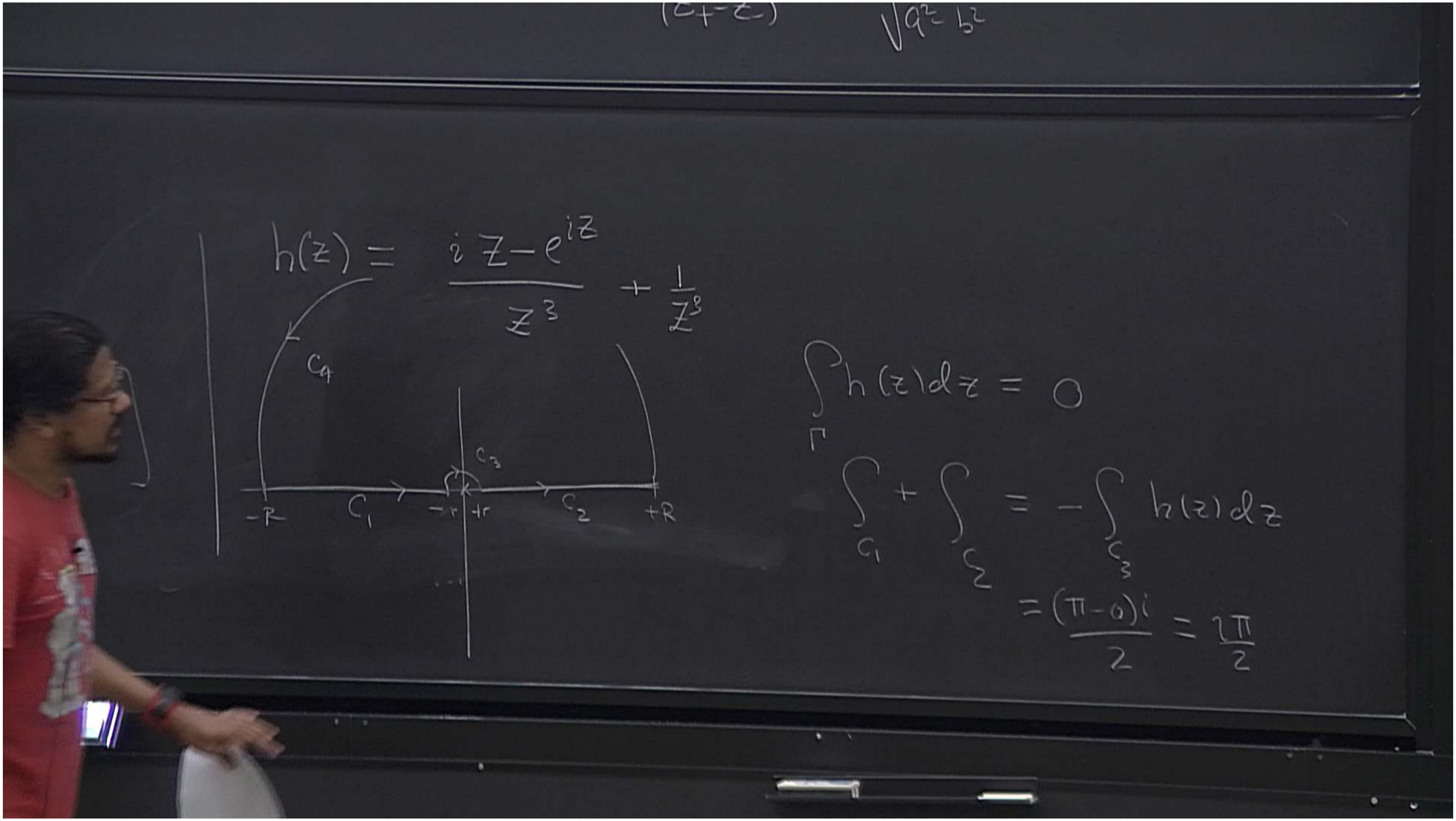
$(z_+ - z_-)$ $\sqrt{a^2 - b^2}$

$$h(z) = \frac{iz - e^{iz}}{z^3} + \frac{1}{z^3}$$

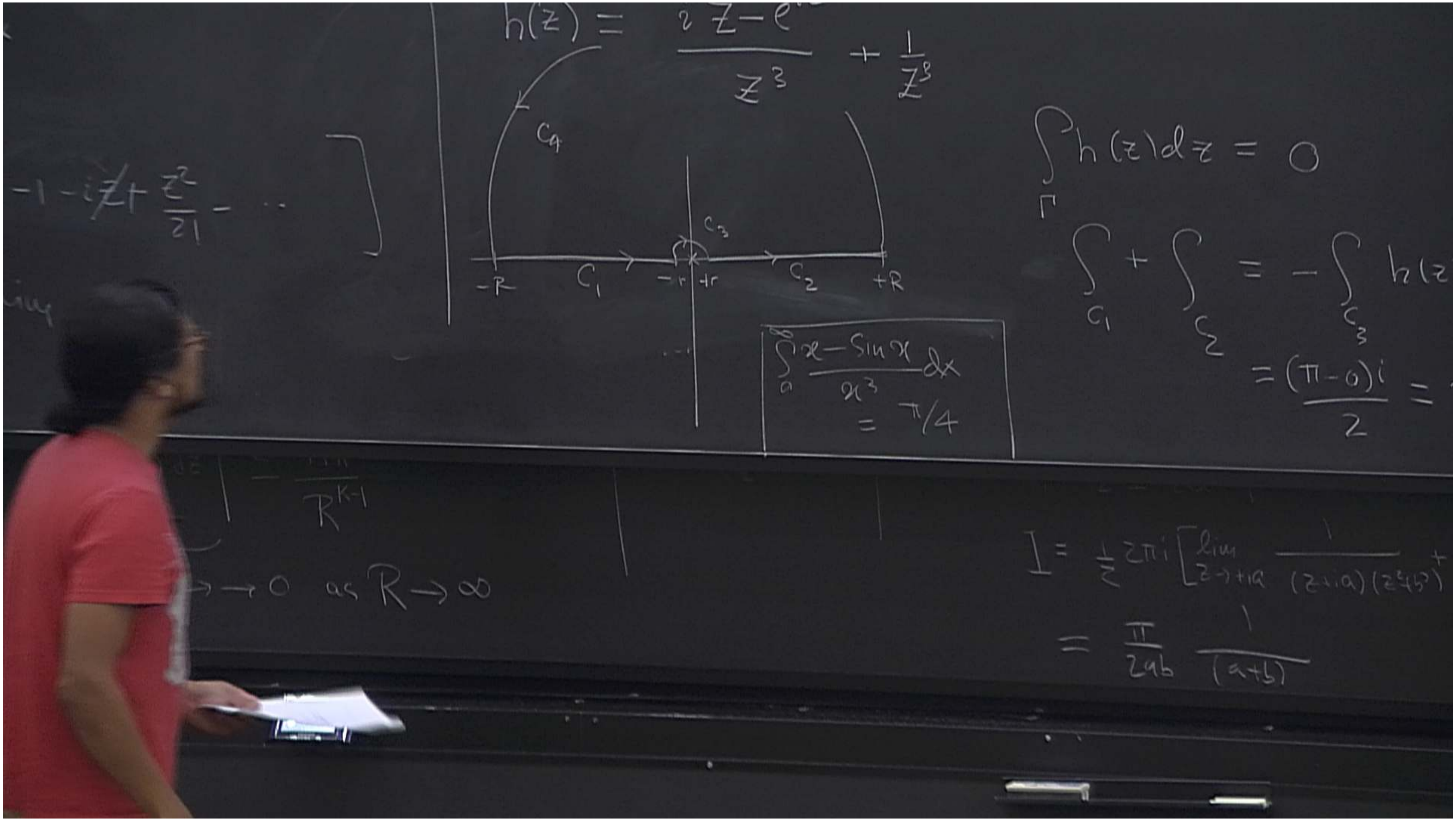


$$\int_\Gamma h(z)\,dz$$

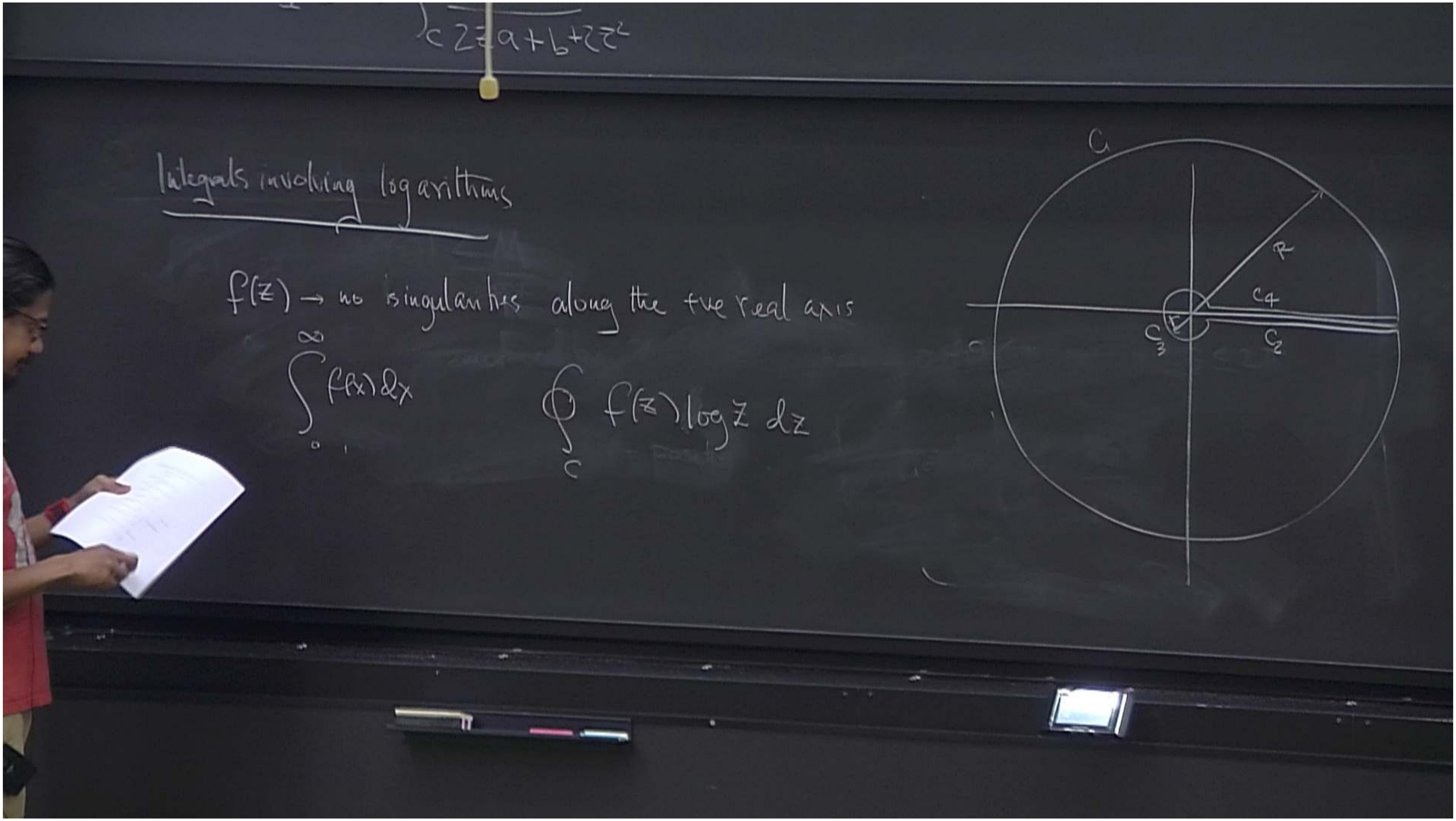
$$= \int_{C_1} + \int_{C_2} -$$

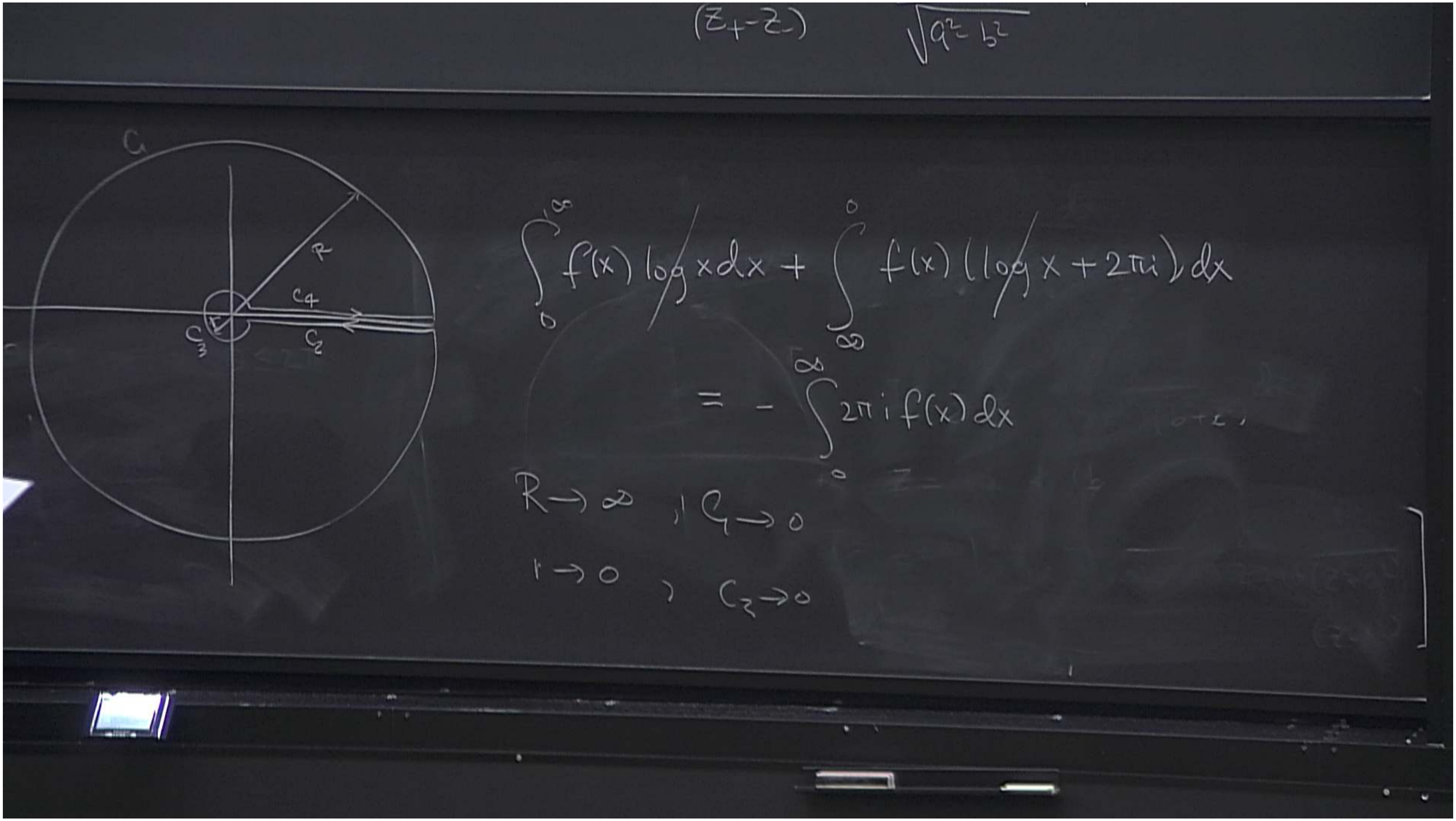


$h(z) = \frac{iz - e^{iz}}{z^3} + \frac{1}{z^3}$
C_4
C_3
$-R$ C_1 $-r$ $+r$ C_2 $+R$
$\int_\Gamma h(z)\,dz = 0$
$\int_{C_1} + \int_{C_2} = -\int_{C_3} h(z)\,dz$
$= \frac{(\pi - 0)i}{2} = \frac{i\pi}{2}$

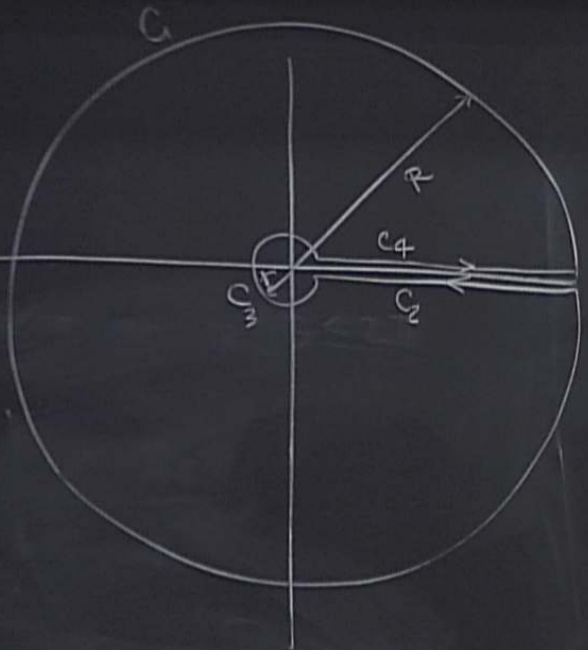


h(z) =
+ 1/z^3
C4
C3
C1
C2
-R
-r
+r
+R
$\int_\Gamma h(z)\,dz = 0$
$\int_{C_1} + \int_{C_2} = -\int_{C_3} h(z$
$= \frac{(\pi - 0)i}{2}$
$\int_0^\infty \frac{x - \sin x}{x^3}\,dx = \pi/4$
$\to 0$ as $R \to \infty$
R^{k-1}
$= \frac{\pi}{2ab}\frac{1}{(a+b)}$


Integrals involving logarithms
f(z) → no singularities along the +ve real axis
$\int_0^\infty f(x)\,dx$
$\oint_C f(z)\log z\,dz$
C
R
C4
C2
C3
ε


$(z_+ - z_-)$
$\sqrt{a^2 - b^2}$
C_1
R
C_4
r
C_3
C_2
$\int_0^\infty f(x) \cancel{\log x}\, dx + \int_\infty^0 f(x)(\cancel{\log x} + 2\pi i)\, dx$
$= - \int_0^\infty 2\pi i f(x)\, dx$
$R \to \infty \; , \; C_1 \to 0$
$r \to 0 \; , \; C_3 \to 0$

$(z_+ - z_-)$ $\sqrt{a^2 - b^2}$



$$\int_0^\infty f(x)\,\cancel{\log}\,x\,dx + \int_\infty^0 f(x)\,(\cancel{\log}\,x + 2\pi i)\,dx$$

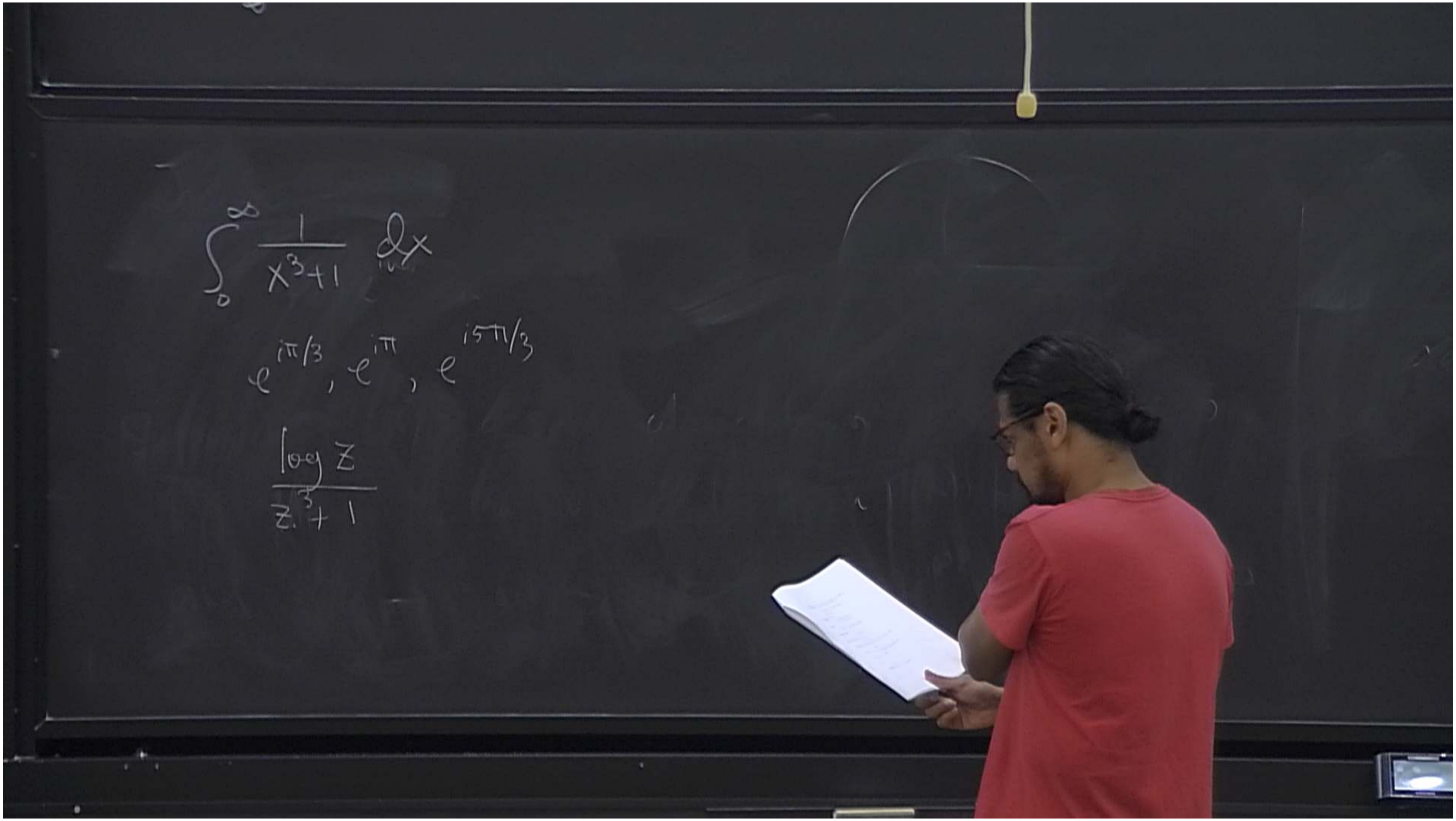
$$= -\int_0^\infty 2\pi i\, f(x)\,dx$$

$R \to \infty$, $C_1 \to 0$

$r \to 0$, $C_3 \to 0$

$$\int_0^\infty f(x)\,dx = -\sum \text{Res of } f(z)\log z$$

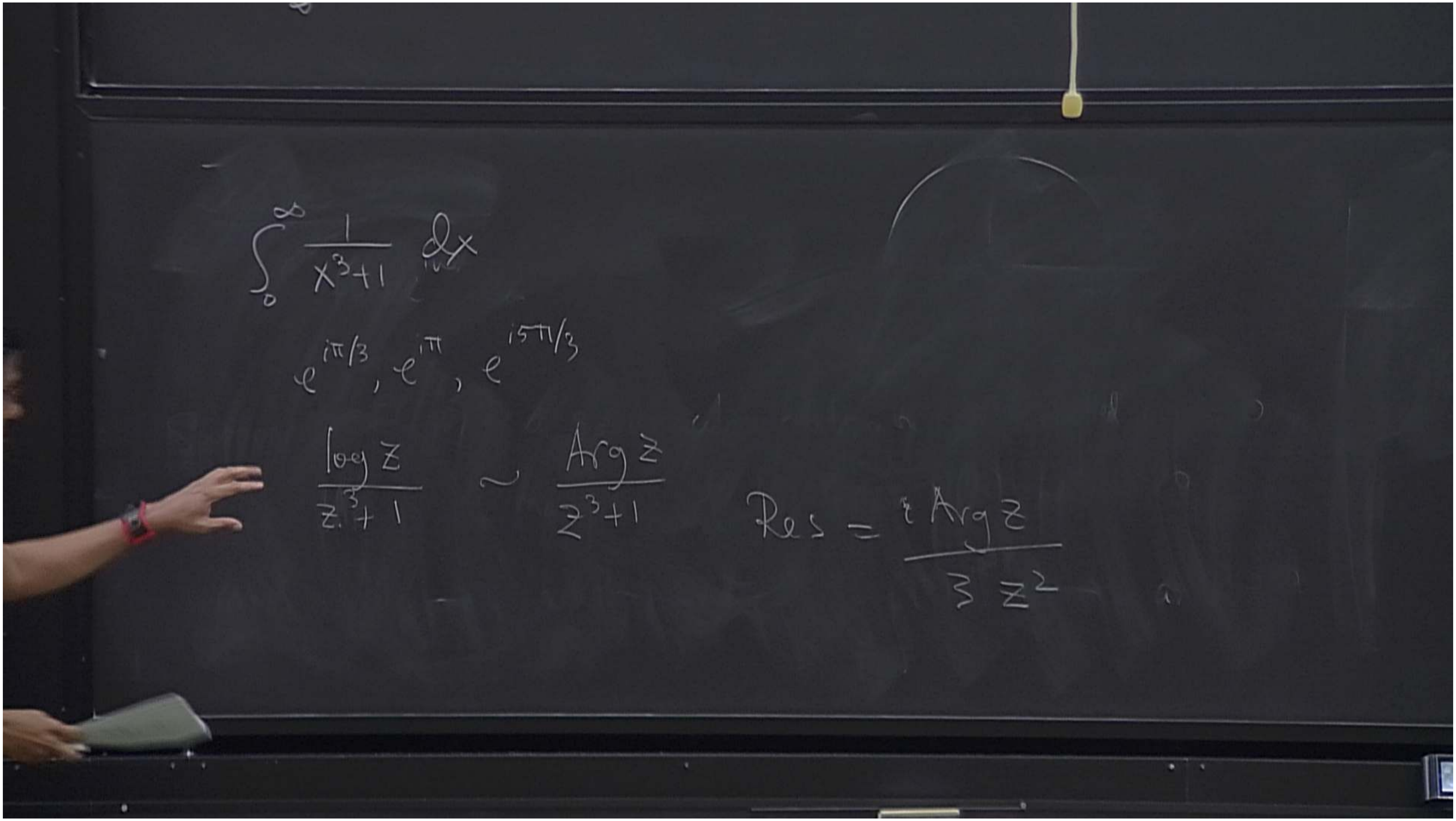
inside Γ

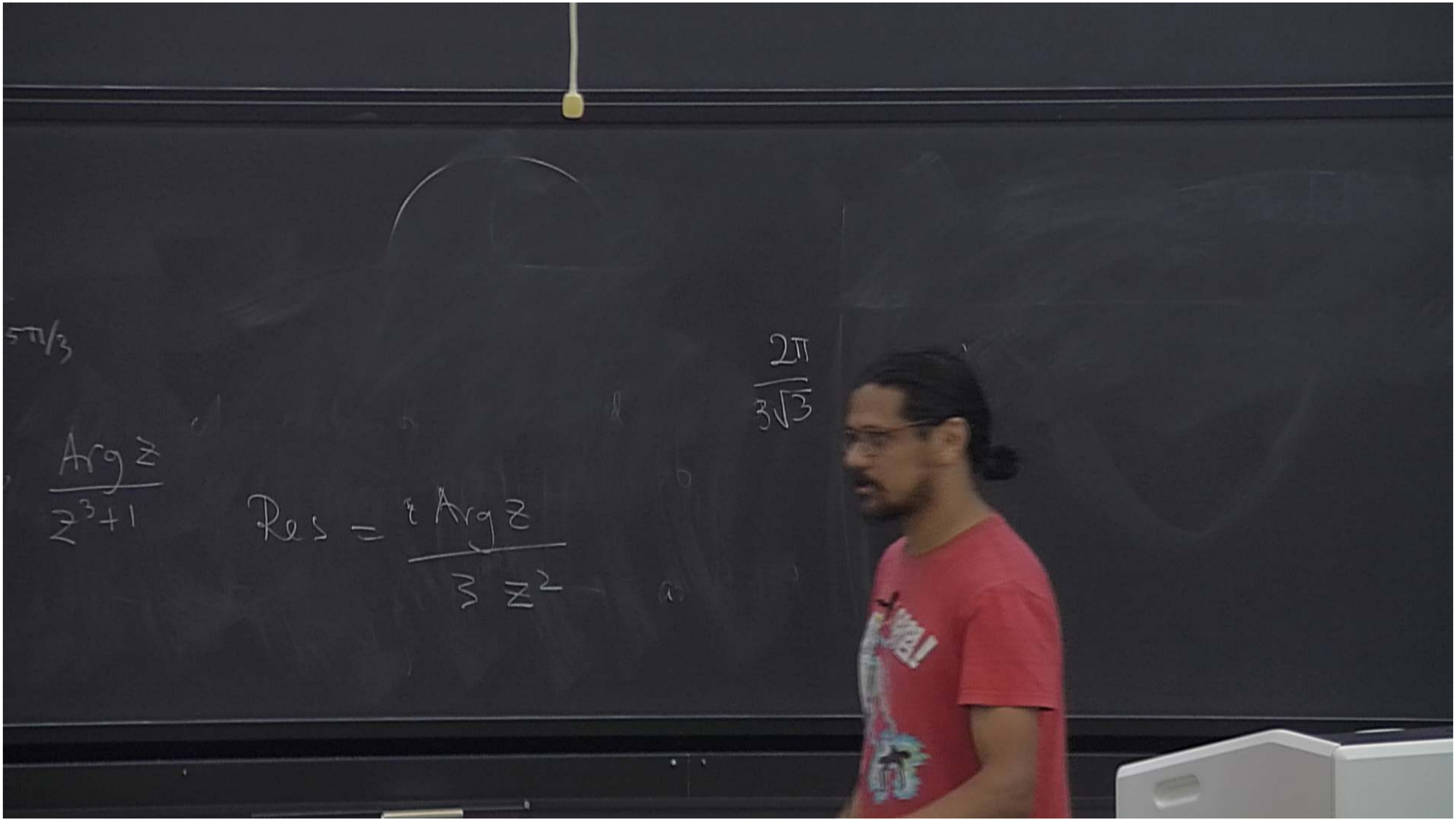

$\int_0^\infty \frac{1}{x^3+1}\,dx$
$e^{i\pi/3},\ e^{i\pi},\ e^{i5\pi/3}$
$\frac{\log z}{z^3+1}$

$$\int_0^\infty \frac{1}{x^3+1}\,dx$$

$$e^{i\pi/3},\ e^{i\pi},\ e^{i5\pi/3}$$

$$\frac{\log z}{z^3+1} \sim \frac{\operatorname{Arg} z}{z^3+1}$$


$\int_0^\infty \frac{1}{x^3+1}\,dx$
$e^{i\pi/3}, e^{i\pi}, e^{i5\pi/3}$
$\frac{\log z}{z^3+1} \sim \frac{\operatorname{Arg} z}{z^3+1}$
$\mathrm{Res} = \frac{i \operatorname{Arg} z}{3z^2}$


5π/3
Arg z / z^3+1
Res = i Arg z / 3 z^2
2π / 3√3