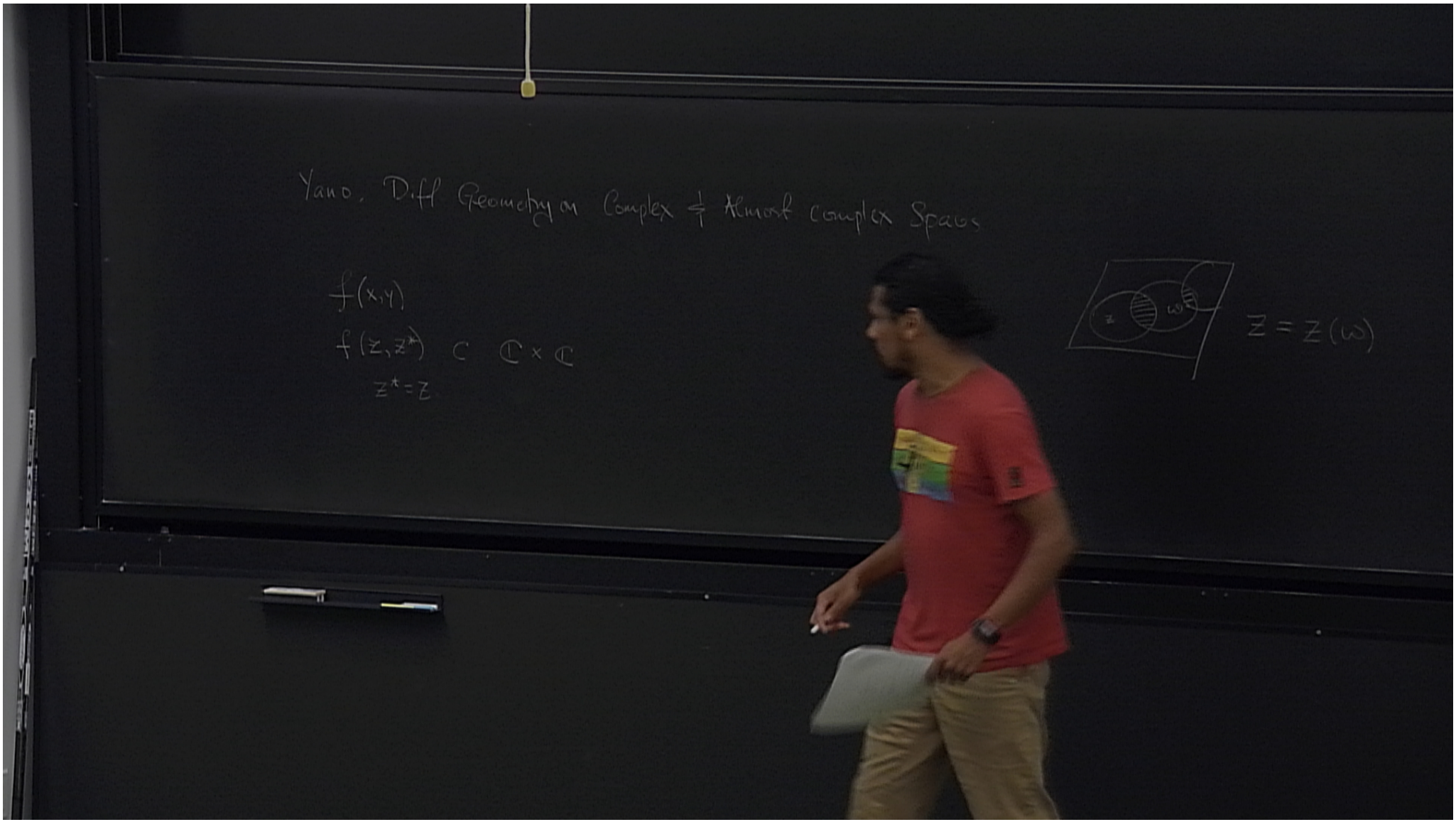
Title: Front End - Complex Analysis - 3

Date: Aug 11, 2016 09:00 AM

URL: http://pirsa.org/16080059

Abstract:


Yano, Diff Geometry on Complex & Almost Complex Spaces
f(x,y)
f(z,z*) ⊂ ℂ × ℂ
z* = z
z = z(ω)

$$\int_{\Gamma} f(z)\,dz = \int_a^b f(\Gamma(t))\,\Gamma'(t)\,dt$$

$$\oint_{S^1} z^n\,dz = \begin{cases} 2\pi i & \text{if } n=-1 \\ 0 & \text{otherwise} \end{cases}$$

U be an open subset of $\mathbb{C}$. Let $\Gamma : [a,b] \to \mathbb{C}$. $f : U \to \mathbb{C}$ holomorphic & continuous in U

Suppose $\exists$ $F : U \to \mathbb{C}$ s.t. $F' = f$.

$$\int_\Gamma f(z)\,dz = F(\Gamma(b)) - F(\Gamma(a))$$

$\Gamma \to$ closed loop & F is continuous then 0

$\Gamma \longrightarrow c$

Cauchy's Theorem

$\underline{A}$ be two dim vector field with continuous first derivative in some simply connected region $\mathcal{D}$.

$$\oint_{\Gamma} \underline{A} \cdot d\underline{l} = \iint_{\mathcal{D}} (\underline{\nabla} \times \underline{A}) \cdot d\underline{a}$$

$\Gamma = \partial \mathcal{D}$

$$\boxed{\partial^2 = \emptyset}$$

e-in some

$f'(z)$ exists

$$\oint_\Gamma f(z)\,dz = \oint_\Gamma \{(u+iv)\,dx + (iu-v)\,dy$$

$$= \iint_D \left(\frac{\partial(iu-v)}{\partial x} - \frac{\partial(u-iv)}{\partial y}\right) dx\,dy$$

$$= \iint_D i \underbrace{\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]}_{=2\frac{\partial}{\partial \bar{z}}} \underbrace{(u+iv)}_{f(z)}\,dx\,dy = 0$$

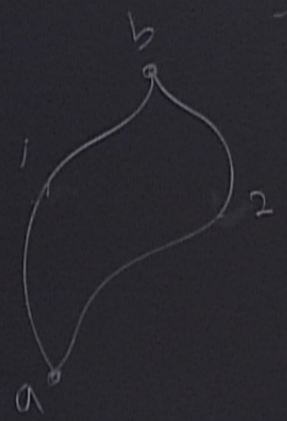
Let U be a simply connected region in $\mathbb{C}$. Let Γ be a closed loop in U

If $f(z)$ is a hol. fn on U & $f'(z)$ exists then

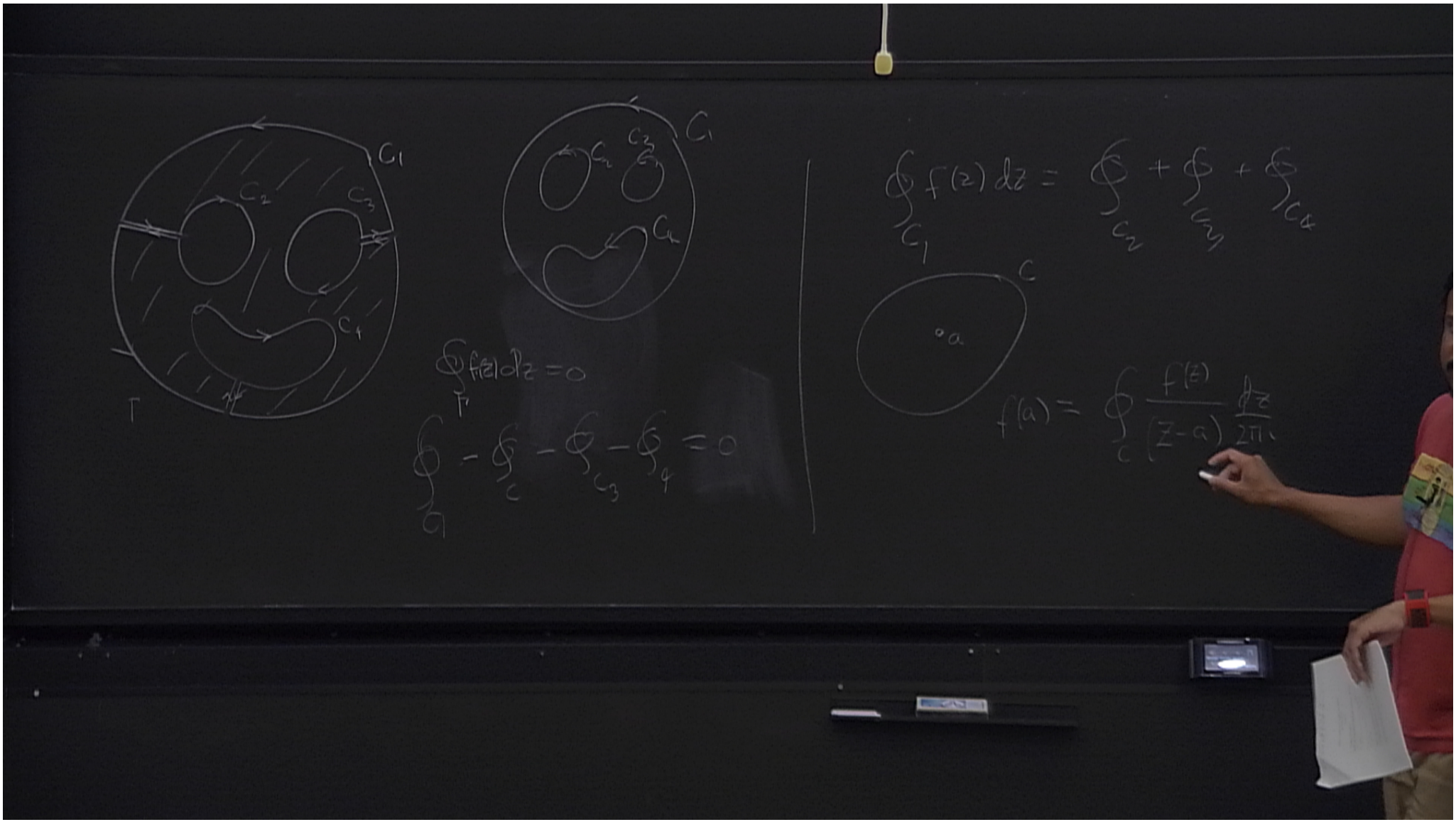
$$\oint_{\Gamma} f(z)\, dz = 0$$

Corollary.

#



$$\int_{1} f(z)\,dz = \int_{2} f(z)\,dz$$


C1
C2
C3
C4
Γ
$\oint_{\Gamma} f(z)\,dz = 0$
$\oint_{C_1} - \oint_{C} - \oint_{C_3} - \oint_{C_4} = 0$
$\oint_{C_1} f(z)\,dz = \oint_{C_2} + \oint_{C_3} + \oint_{C_4}$
C
a
$f(a) = \oint_{C} \frac{f(z)}{(z-a)} \frac{dz}{2\pi i}$

$$f(a) = \oint \frac{f(z)}{(z-a)} \frac{dz}{2\pi i}$$

$$\frac{\partial^n f(z)}{\partial z^n} := f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(t)}{(t-z)^{n+1}} dt$$

$$f(a+z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} f^{(n)}(a)$$

$$\oint_{C_1} - \oint_{C} - \oint_{C_3} - \oint_{C_4} = 0$$

$$f(a) = \oint_C \frac{f(z)}{(z-a)} \frac{dz}{2\pi i}$$

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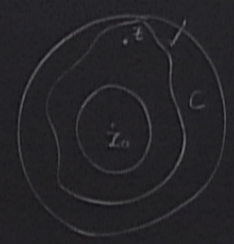
Laurent's series

$f(z)$ is holomorphic in some annular region



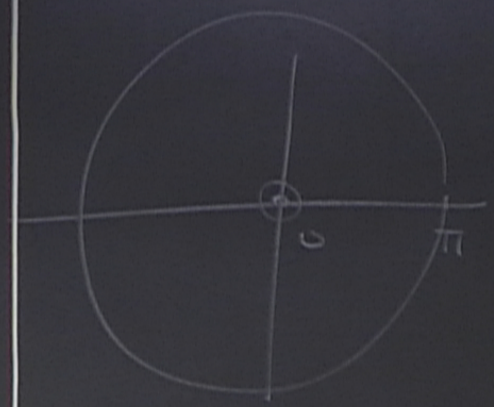
$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$$

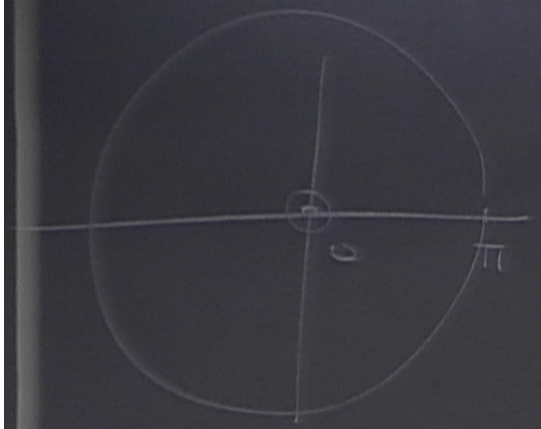
$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t-z_0)^{n+1}} dt$$

$$f(z) = \frac{1}{\sin z}$$

$$= \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!}}$$

$$= \frac{1}{z} + \frac{z}{6} + \frac{7}{360} z^3 + \cdots$$

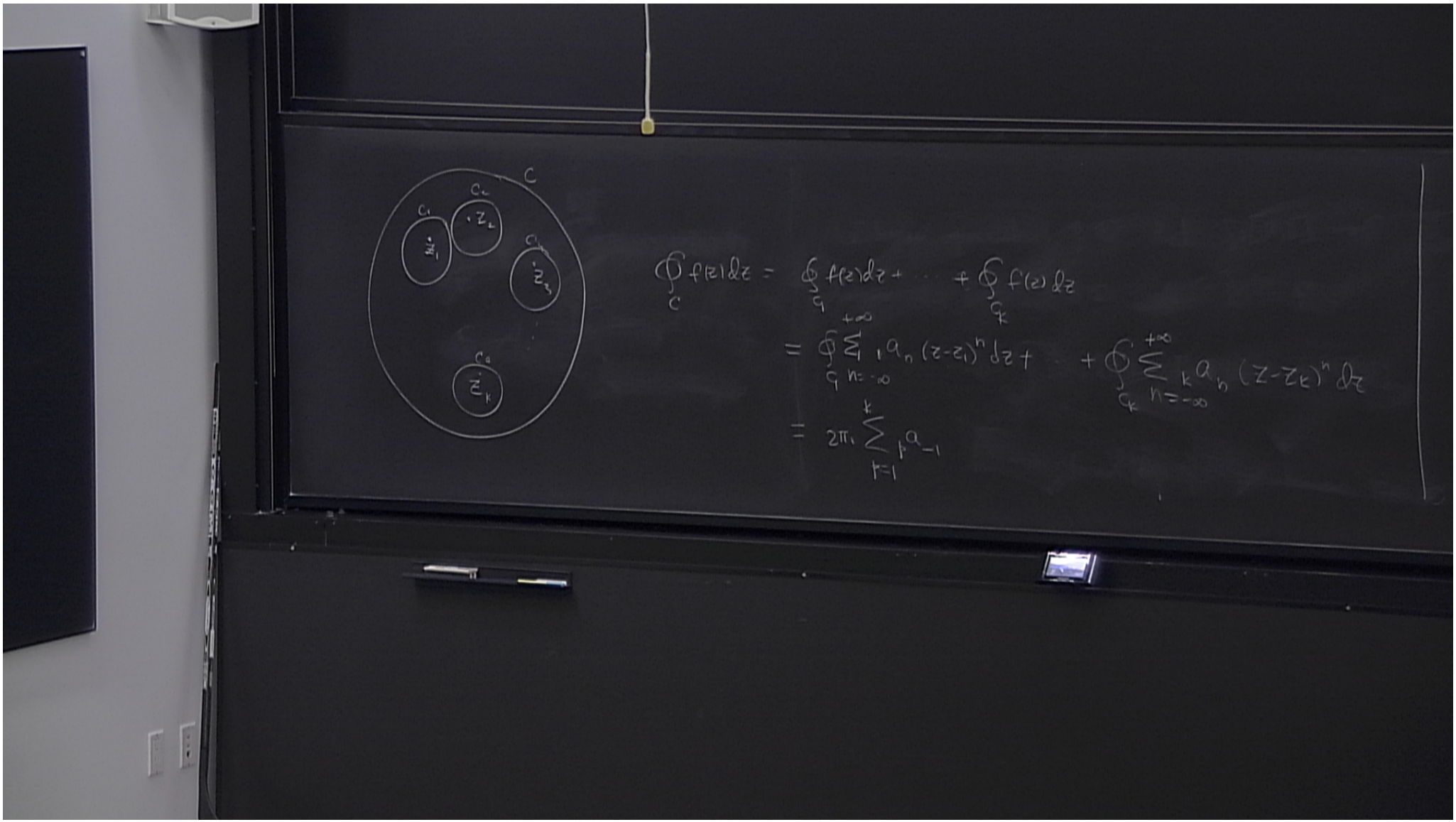




Cauchy's Residue theorem

$$\oint_C z^n \cdot dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{, otherwise} \end{cases}$$

$a_{-1} \rightsquigarrow$ Residue of $f(z)$ at $z = z_0$


C
C_1
C_2
C_3
C_k
z_1
z_2
z_3
z_k
$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz$
$= \oint_{C_1} \sum_{n=-\infty}^{+\infty} {}_1a_n (z-z_1)^n\,dz + \cdots + \oint_{C_k} \sum_{n=-\infty}^{+\infty} {}_k a_n (z-z_k)^n\,dz$
$= 2\pi i \sum_{k=1}^{k} {}_k a_{-1}$

$f'(z)$ exists

$$\oint_\Gamma f(z)\,dz = \oint_\Gamma \{(u+iv)\,dx + (iu-v)\,dy$$

$$= \iint_D \left(\frac{\partial(iu-v)}{\partial x} - \frac{\partial(u-iv)}{\partial y}\right) dx\,dy$$

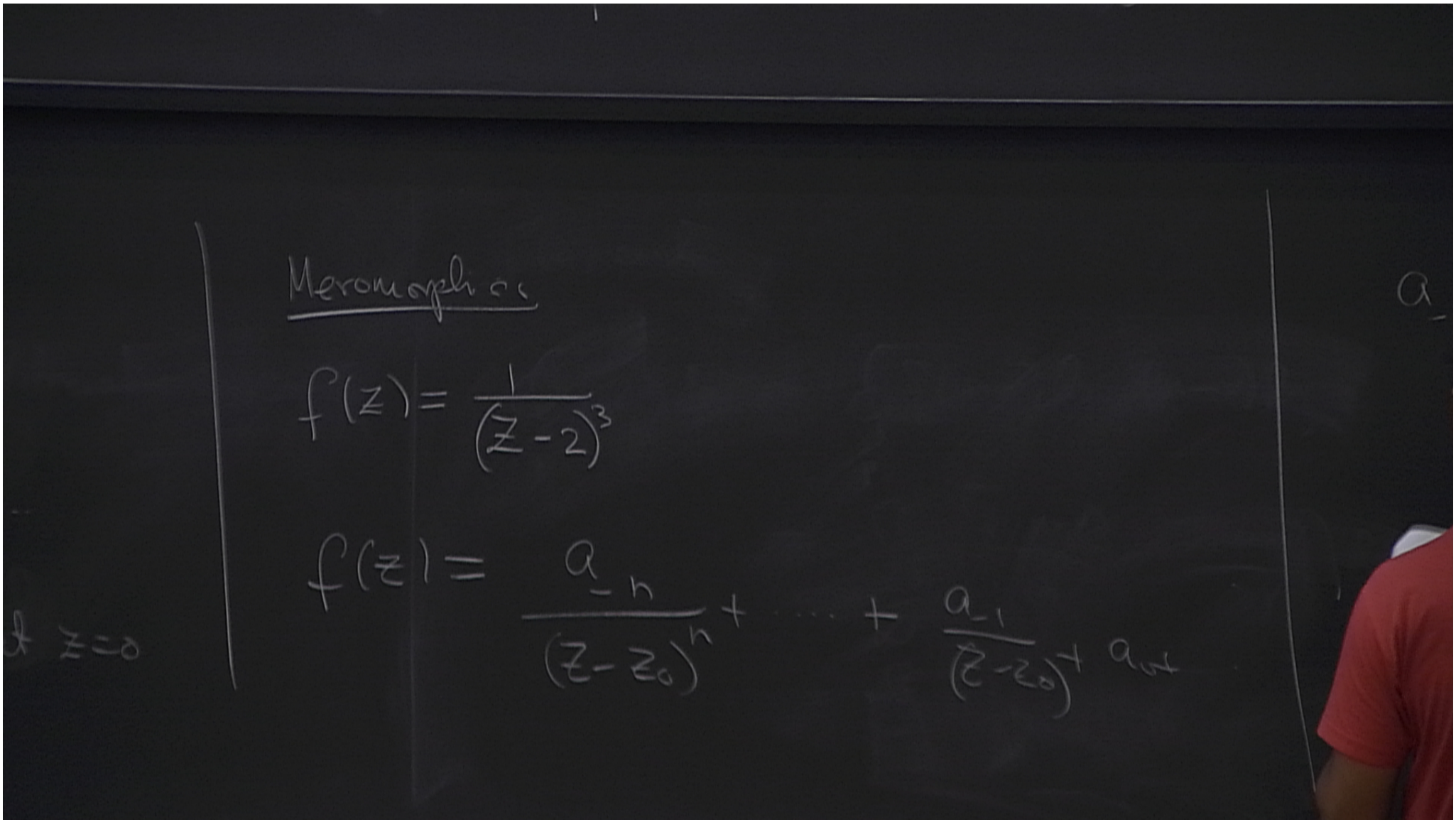
$$= \iint_D i \underbrace{\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]}_{=2\frac{\partial}{\partial \bar{z}}} \underbrace{(u+iv)}_{f(z)}\,dx\,dy = 0$$

$\underline{n=1}$ ~ Simple pole

$\underline{n\neq 1}$ ~ higher order pole

2. Essential singularity: $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots$

↳ an essential singularity at $z=0$


Meromorphics
$f(z) = \frac{1}{(z-2)^3}$
$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_{0+}$
$z=0$
a_-

$$\oint_{C_k} \sum_{n=-\infty}^{k} a_n (z-z_k)^n \, dz$$

$$2\pi i \sum_{k=1}^{k} a_{-1}$$

$$z_0 \longrightarrow \text{Pole}$$

$$(z-z_0)^n f(z) = g(z)$$

$$a_1 (z-z_0)^{-1}$$

$$z + \cdots$$

nity at $z=0$

Meromorphic

$$f(z) = \frac{1}{(z-2)^3}$$

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{(z-z_0)^1} + a_0 +$$

$$a_{-1} = \lim_{(z \to z_0)} \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial z^{n-1}} (z-z_0)^n f(z)$$

$$f(z) = \frac{1}{a^2+z^2} = \frac{1}{(z+ia)(z-ia)}$$

Res of $f(z)$ at $(\pm ia) = \frac{\pm 1}{2ia}$

$$\frac{e^{z}}{\sin z}$$

at $z=0$ $\quad 1$

$$\frac{e^{z+n\pi}}{\sin(z+n\pi)} = e^{n}$$

$$z = n\pi$$

$$\frac{e^{z}}{\sin^2 z} = \frac{\left(1+z+\frac{z^2}{2!}+\cdots\right)}{\left(z-\frac{z^3}{3!}+\cdots\right)^2} = \frac{1}{z}\left(1+z+\frac{z^2}{2!}+\cdots\right)\left(1-\frac{z^2}{3!}\right)^{-1}$$

$$= \frac{1}{z}\left(1+z+\frac{z^2}{2!}+\cdots\right)\left(1+\frac{z^2}{3!}\right)$$

$$= \frac{1}{z}\left(1+\frac{z^2}{3!}+z+\frac{z^3}{3!}+\cdots\right)$$

Res at

$$\frac{e^{z+n\pi}}{\sin(z+n\pi)} = \frac{e^{n\pi}\, e^{z}}{(-)^n \sin z}$$

Res at $z = n\pi$ is $(-)^n e^{n\pi}$

$$\left(1+z+\frac{z^2}{2!}+\cdots\right)\left(1-\frac{z^2}{3!}\right)^{-1}$$

$$\left(1+z+\frac{z^2}{2!}+\cdots\right)\left(1+\frac{z^2}{3!}\right)$$

$$\left(1+\frac{z^2}{3!}+z+\frac{z^3}{3!}+\cdots\right)$$