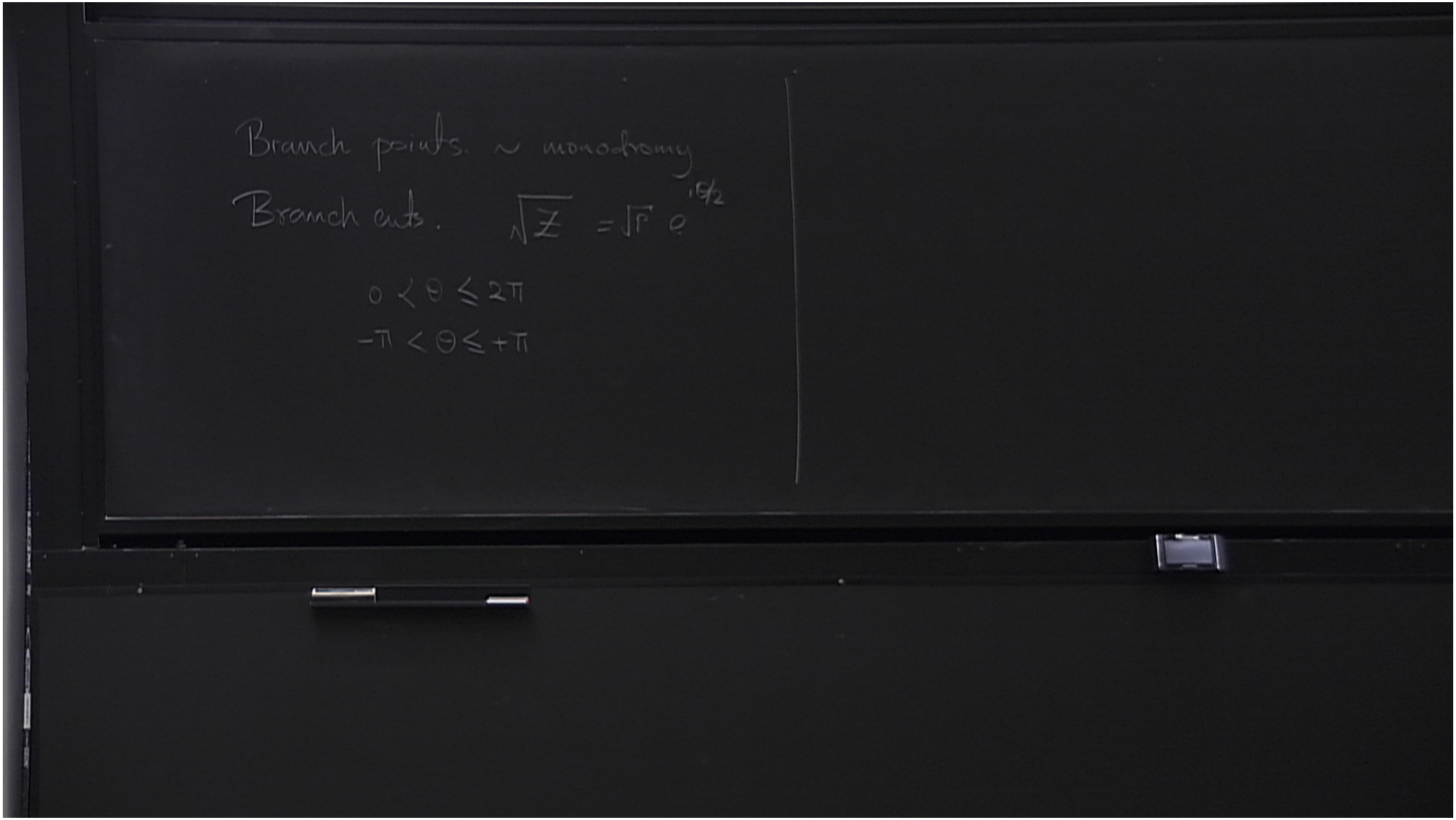
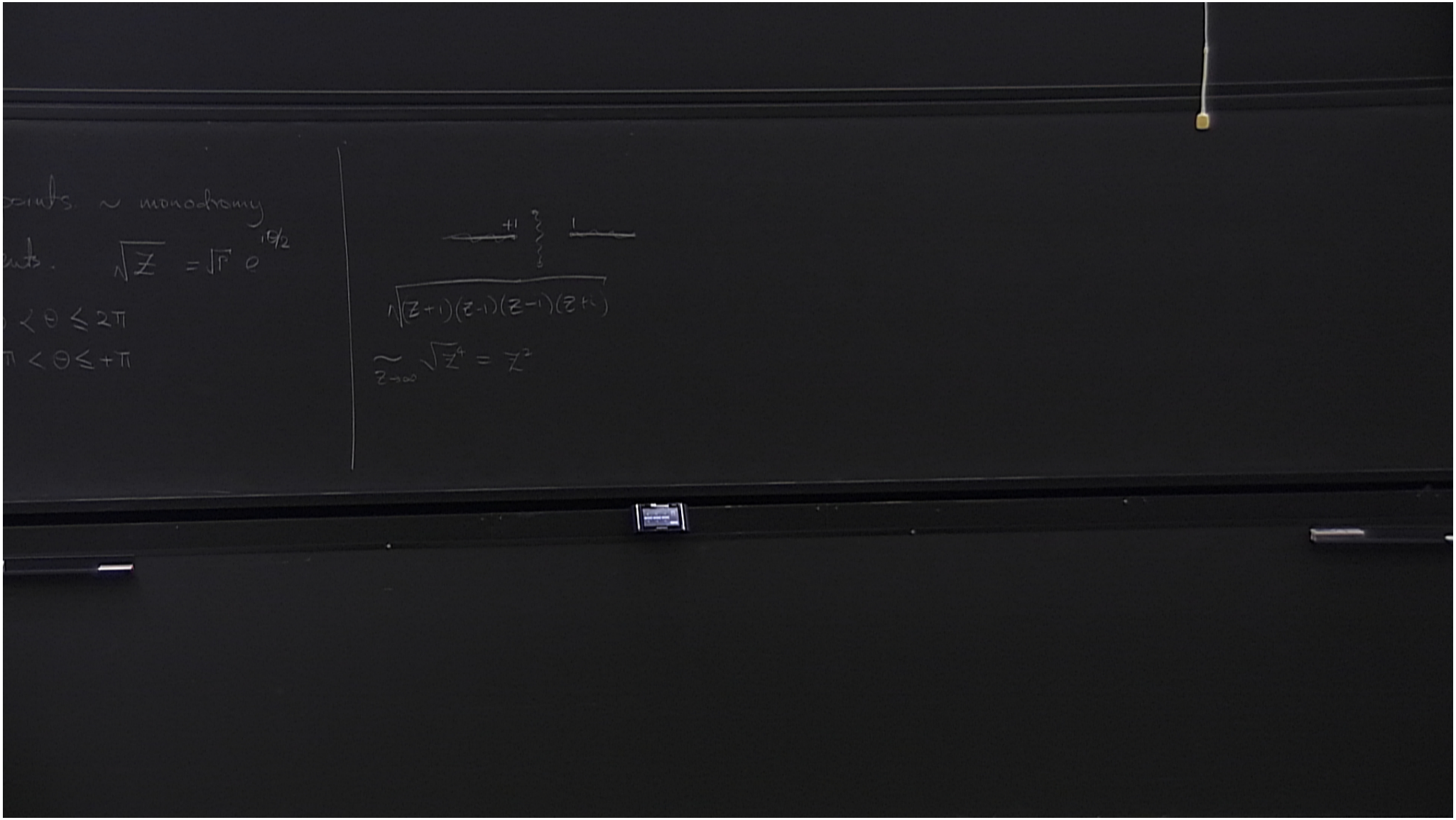
Title: Front End - Complex Analysis - 1

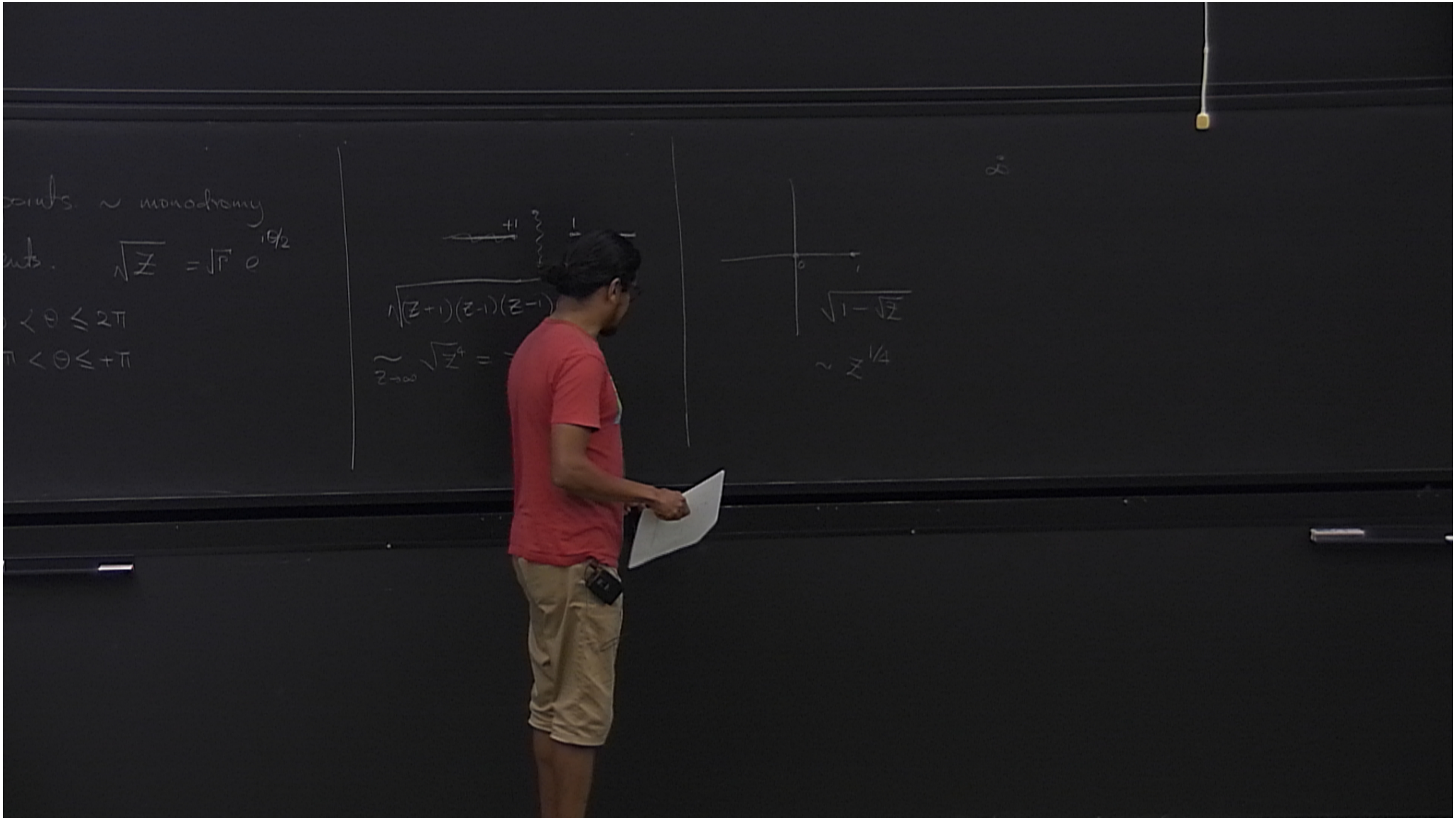
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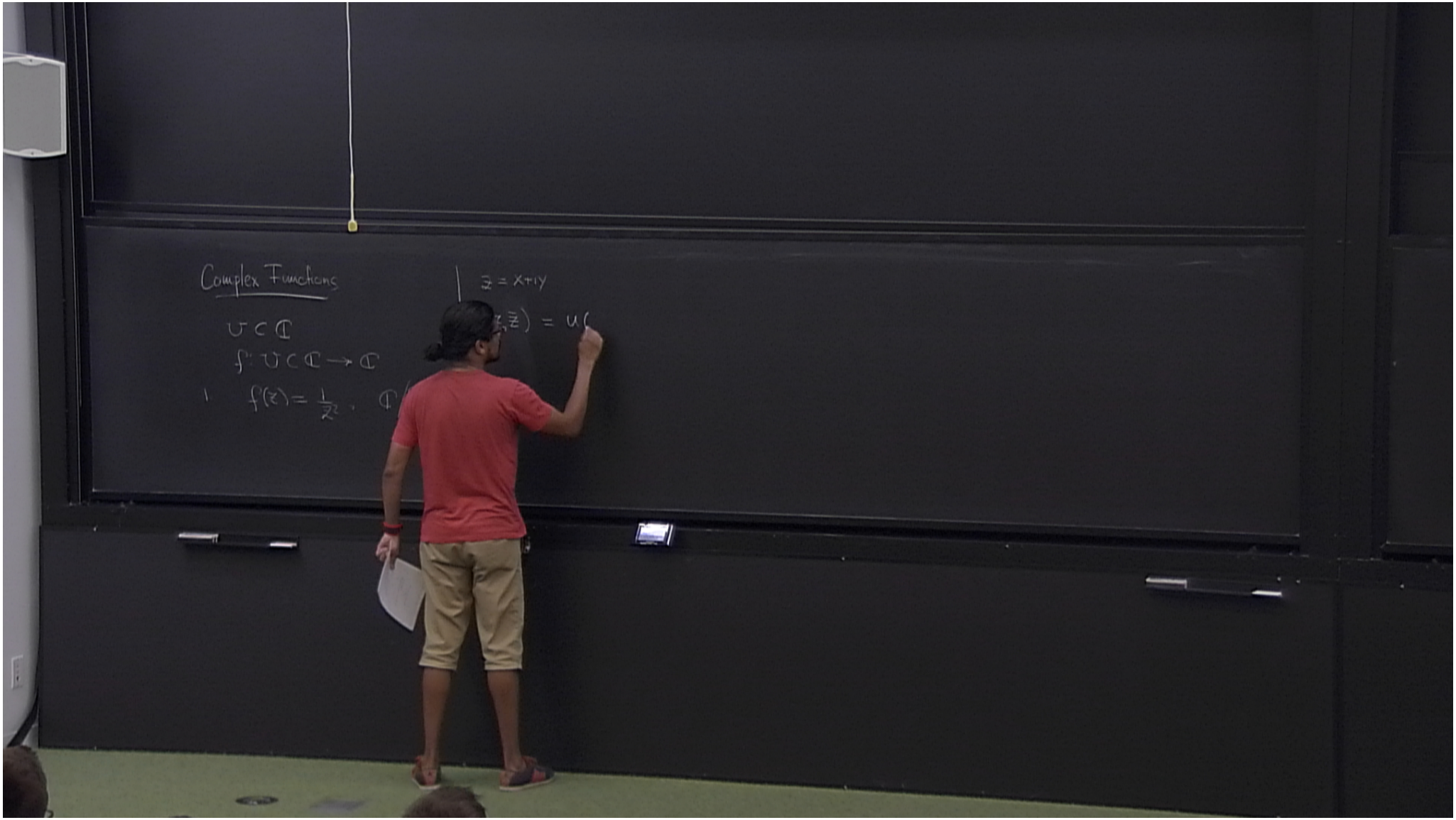
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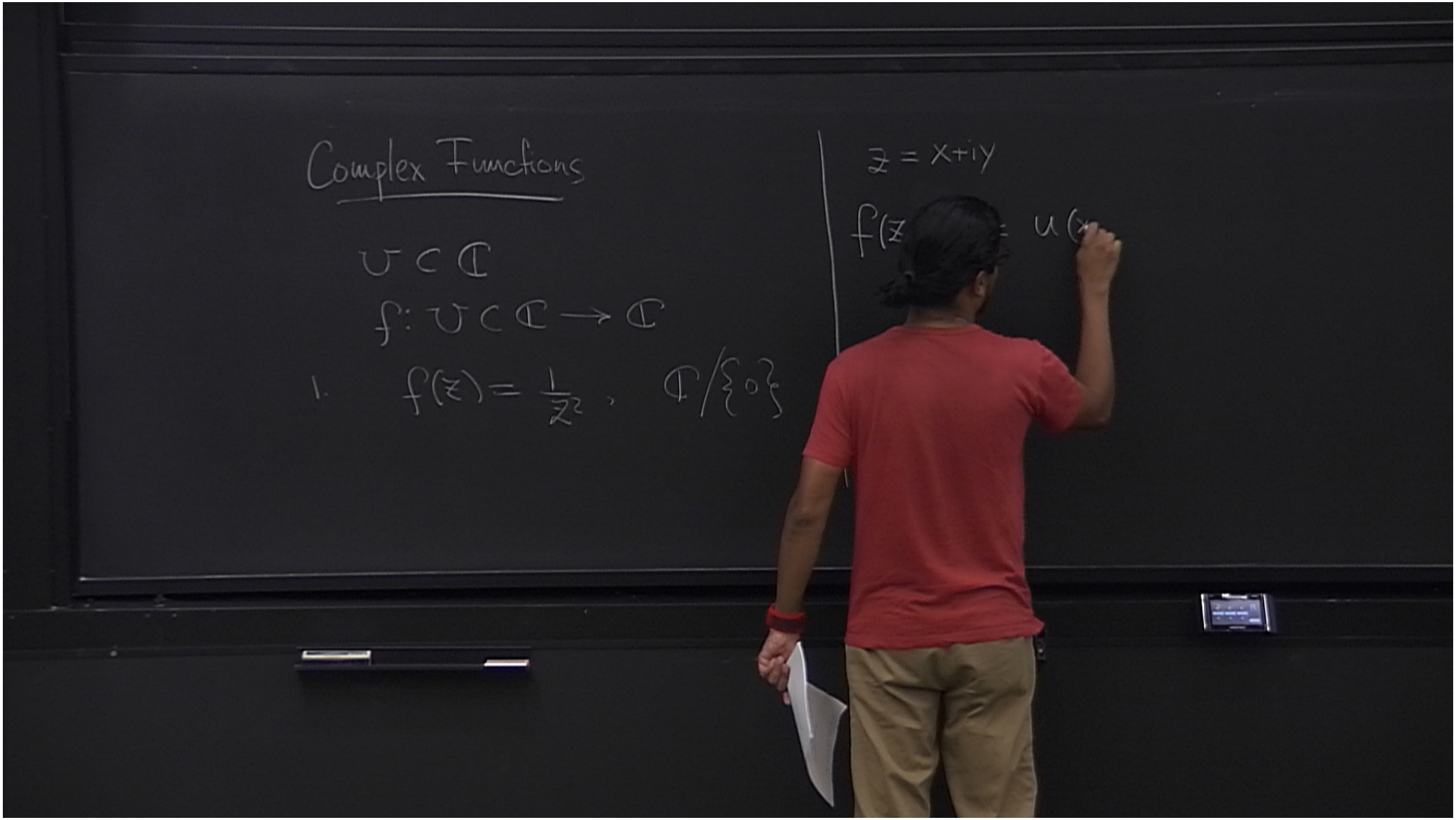
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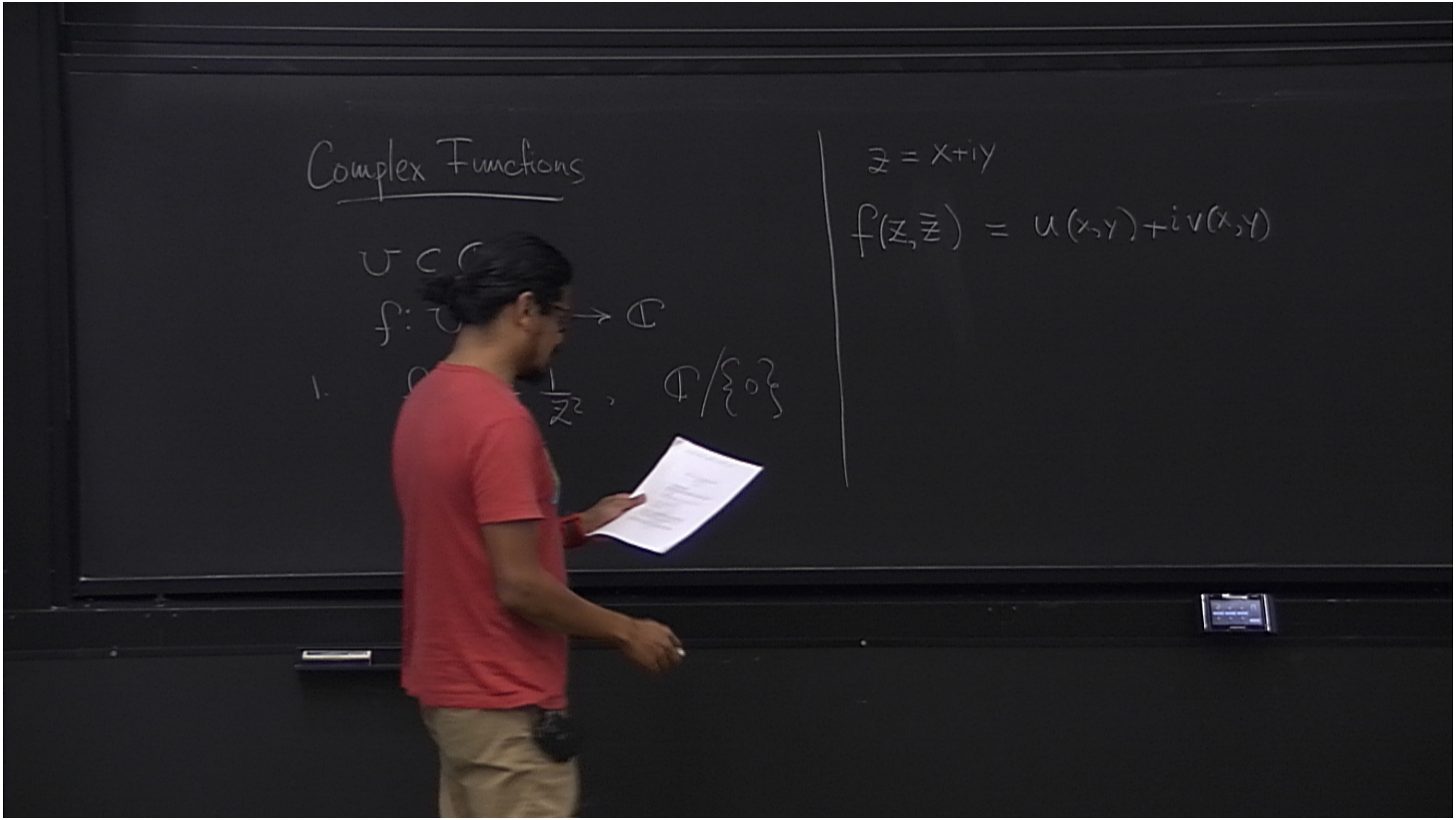

Branch points. ~ monodromy
Branch cuts. $\sqrt{z} = \sqrt{r}\, e^{i\theta/2}$
$0 < \theta \leq 2\pi$
$-\pi < \theta \leq +\pi$

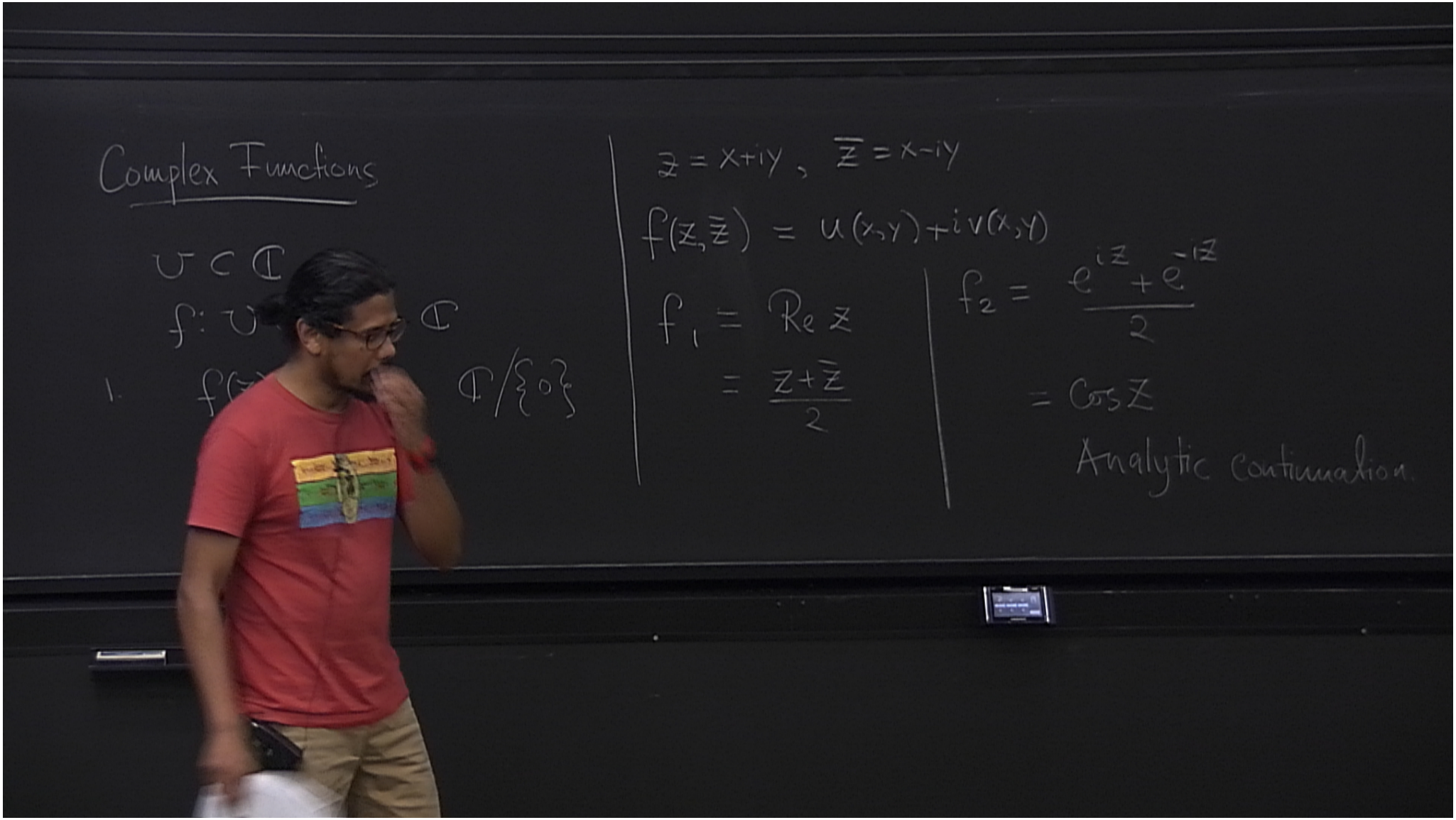

points ~ monodromy
$\sqrt{z} = \sqrt{r}\, e^{i\theta/2}$
$-\pi < \theta \leq +\pi$
$+1$
-1
$\sqrt{(z+1)(z-1)(z-1)(z+1)}$
$\underset{z\to\infty}{\sim} \sqrt{z^4} = z^2$

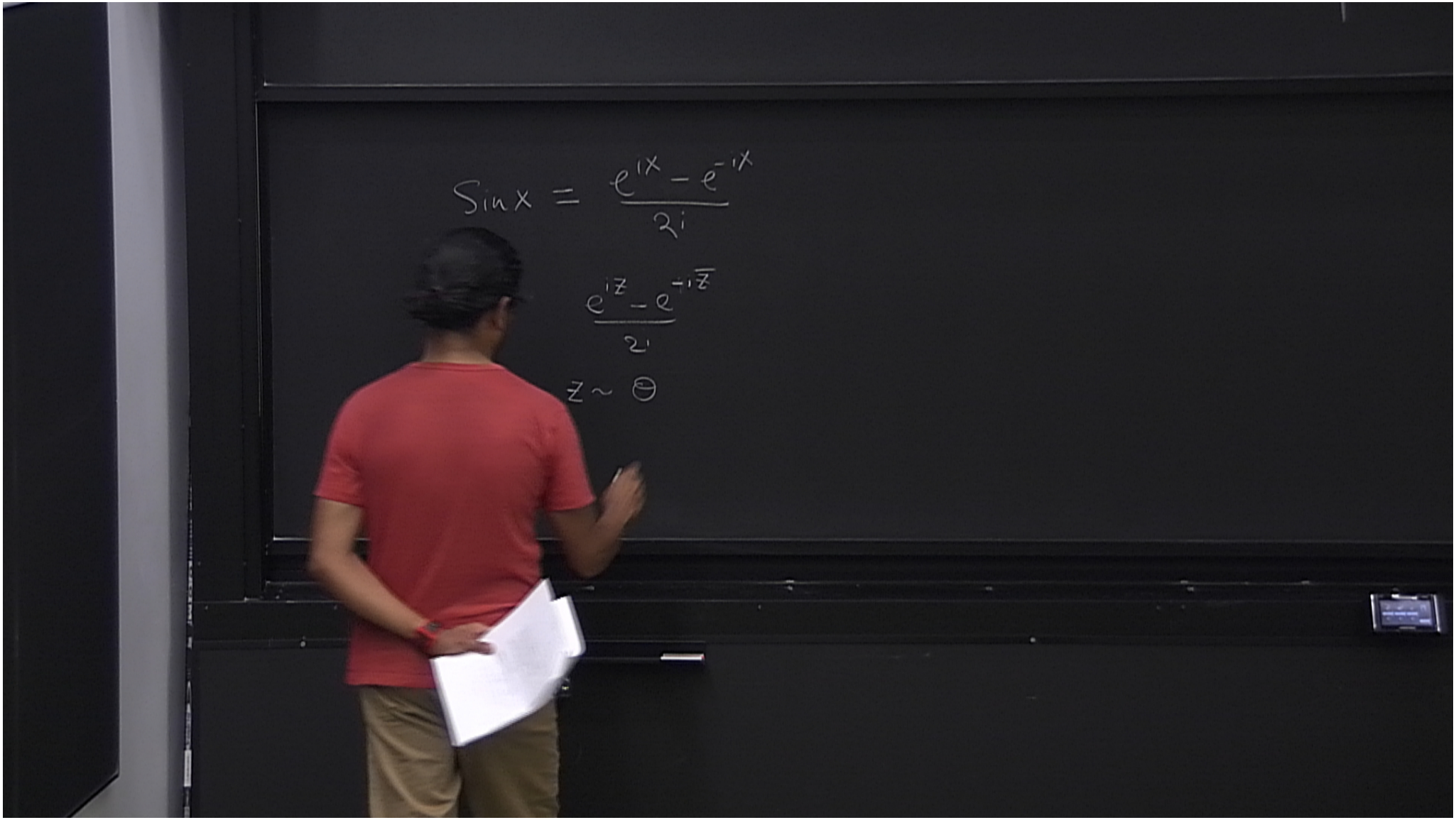

monodromy
$\sqrt{z} = \sqrt{r}\, e^{i\theta/2}$
$-\pi < \theta \leq +\pi$
$\sqrt{(z+1)(z-1)(z-1)}$
$\sqrt{z^4}$
$\sqrt{1-\sqrt{z}}$
$\sim z^{1/4}$

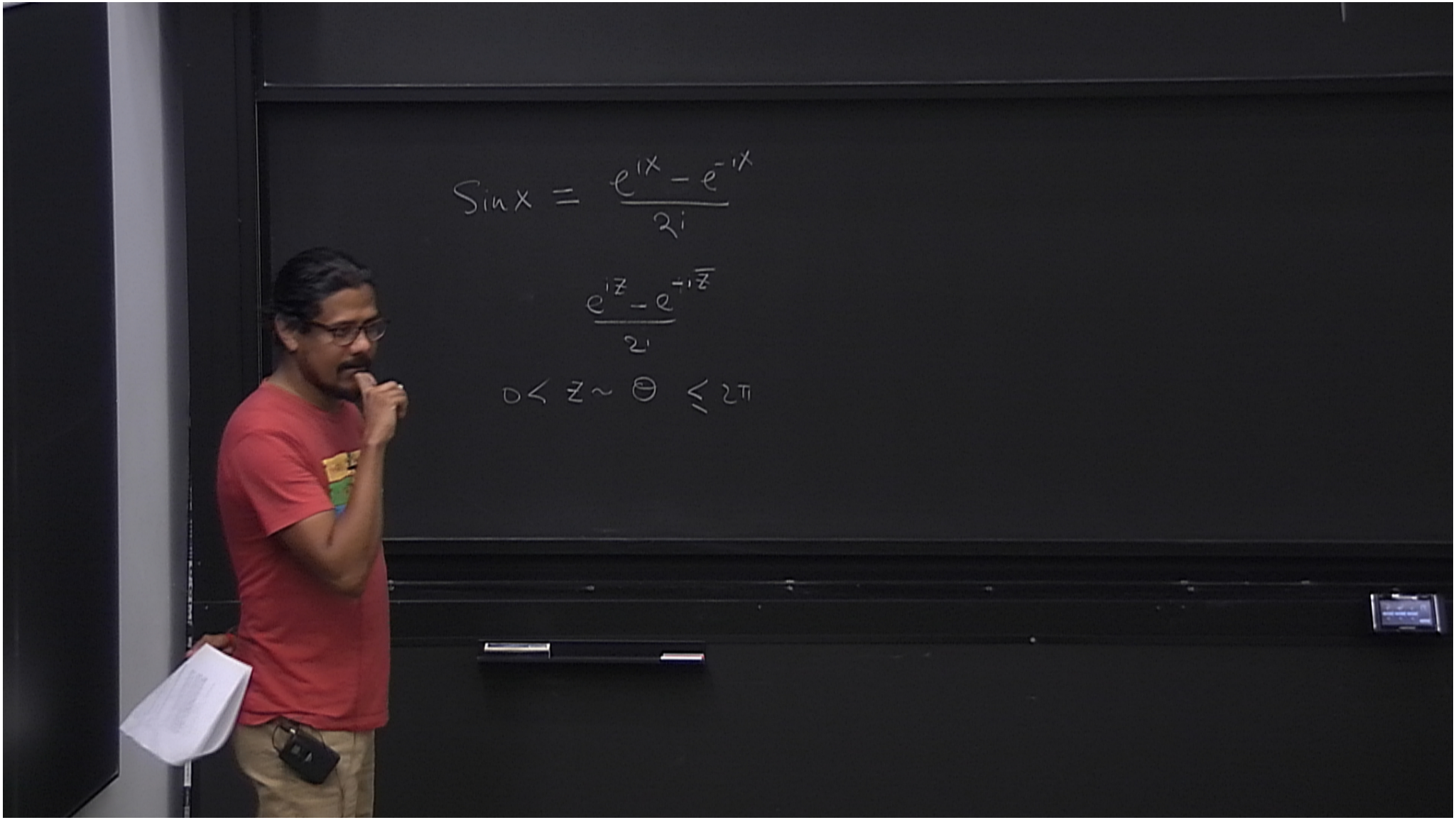

Complex Functions
$z = x+iy$
$U \subset \mathbb{C}$
$f: U \subset \mathbb{C} \to \mathbb{C}$
$f(z) = \frac{1}{z^2}$

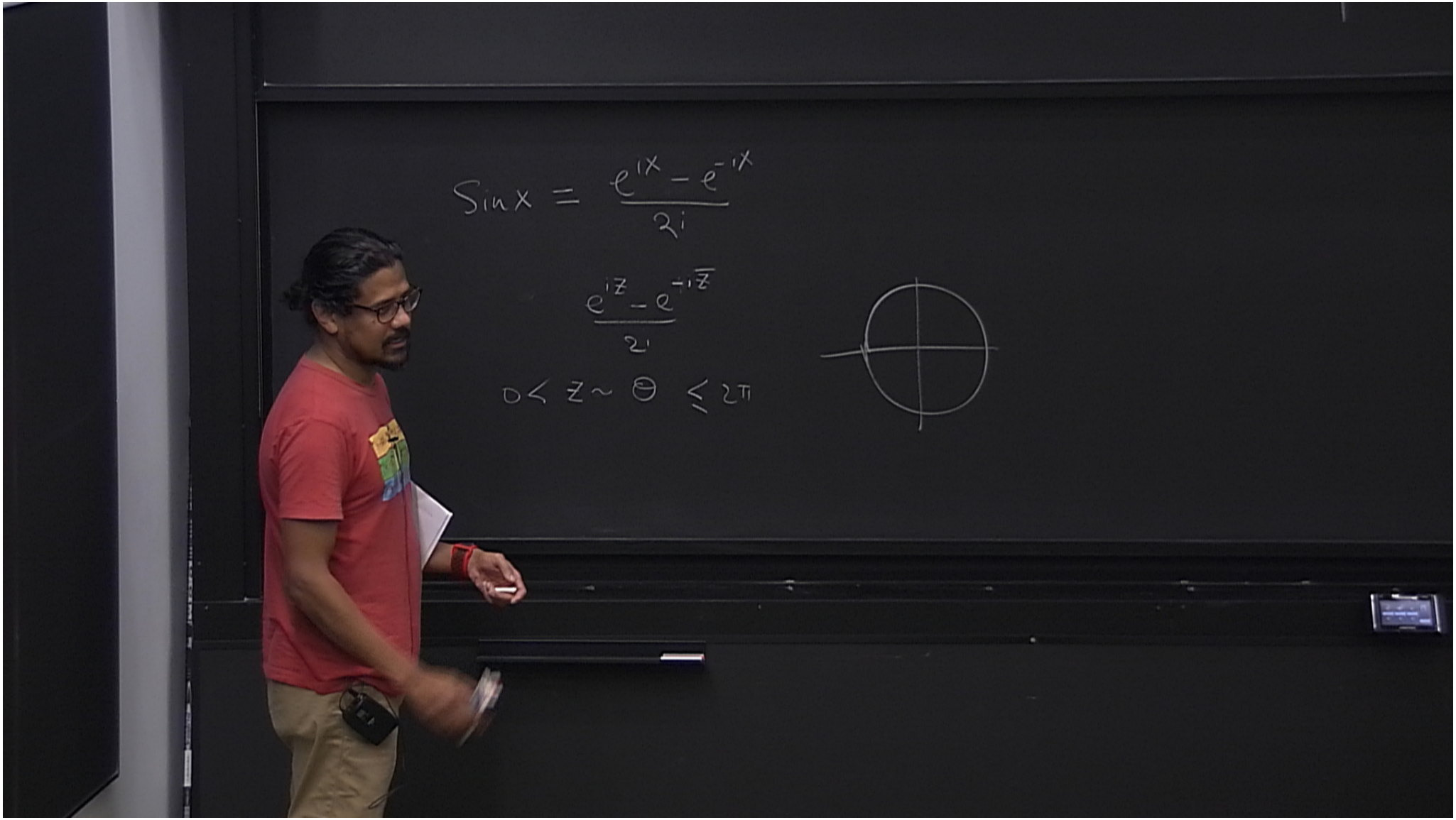

Complex Functions
$U \subset \mathbb{C}$
$f: U \subset \mathbb{C} \to \mathbb{C}$
1. $f(z) = \frac{1}{z^2}$, $\mathbb{C}/\{0\}$
$z = x+iy$

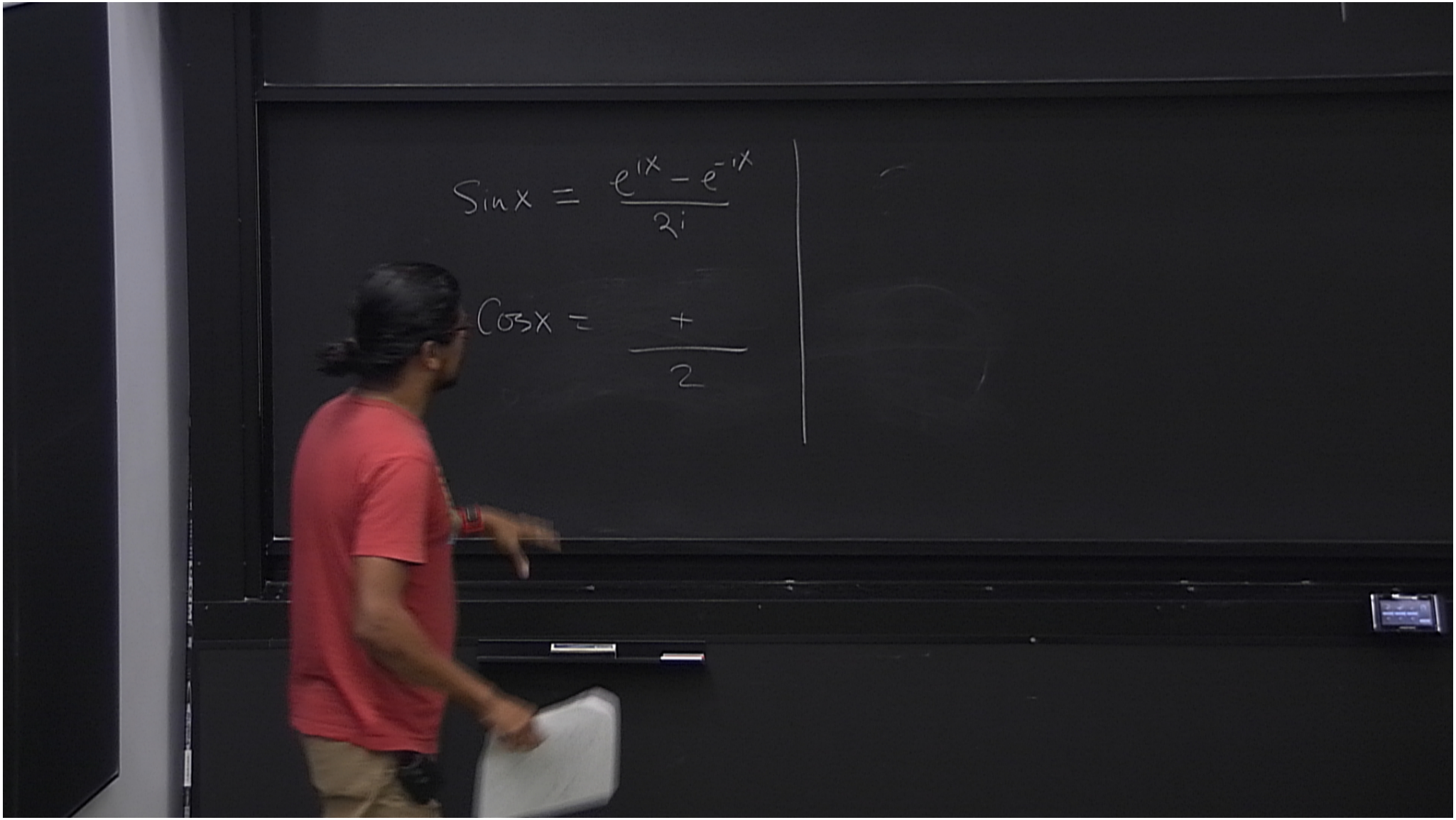

Complex Functions
$z = x+iy$
$f(z,\bar{z}) = u(x,y)+iv(x,y)$
$U \subset \mathbb{C}$
$f: U \to \mathbb{C}$
1.
$\frac{1}{z^2}$,
$\mathbb{C}/\{0\}$

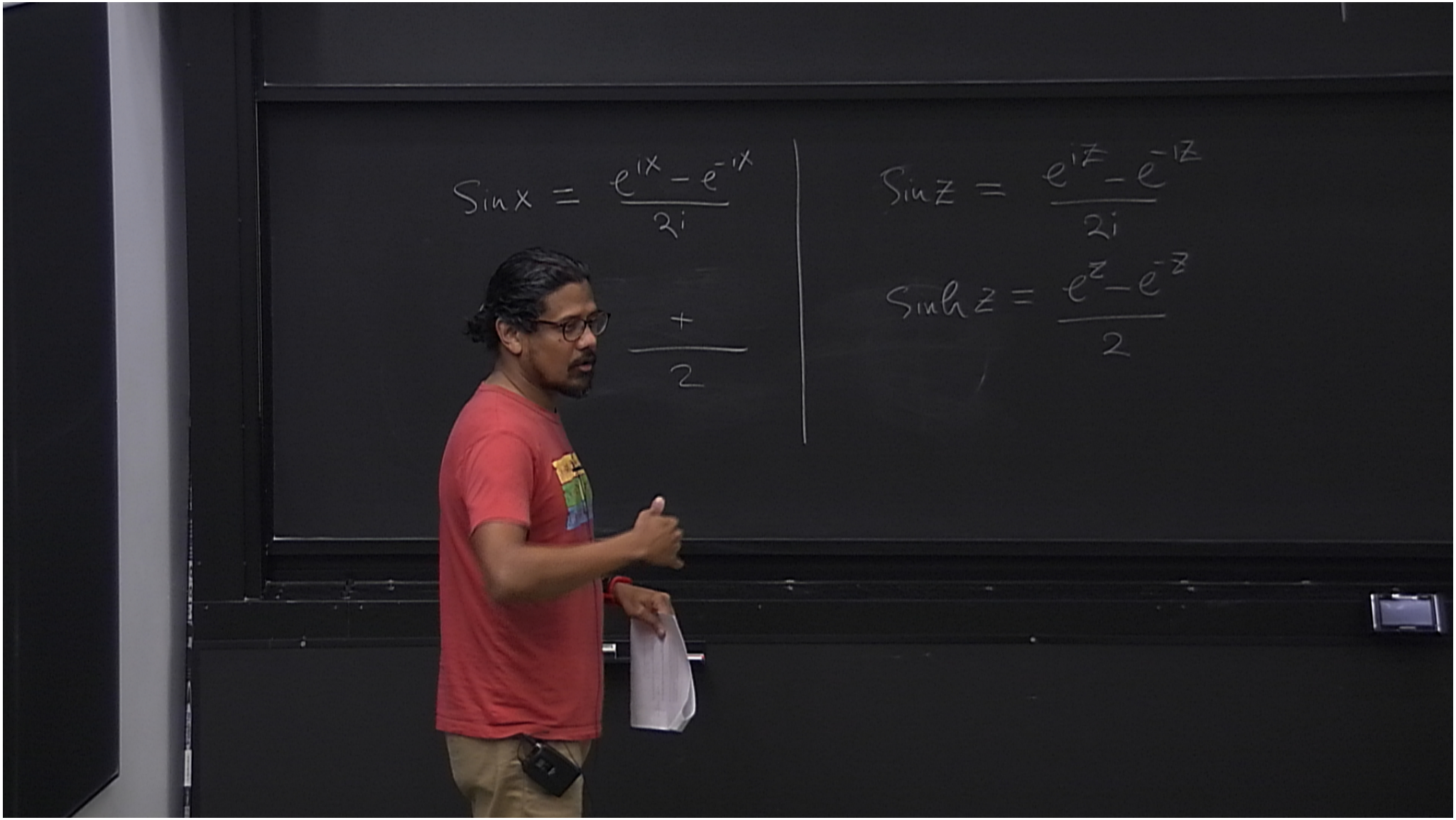

Complex Functions
$U \subset \mathbb{C}$
$f: U \to \mathbb{C}$
1. $f(\mathbb{C}) = \mathbb{C}/\{0\}$
$z = x+iy$, $\bar{z} = x-iy$
$f(z,\bar{z}) = u(x,y) + iv(x,y)$
$f_1 = \mathrm{Re}\, z = \frac{z+\bar{z}}{2}$
$f_2 = \frac{e^{iz}+e^{-iz}}{2} = \cos z$
Analytic continuation.

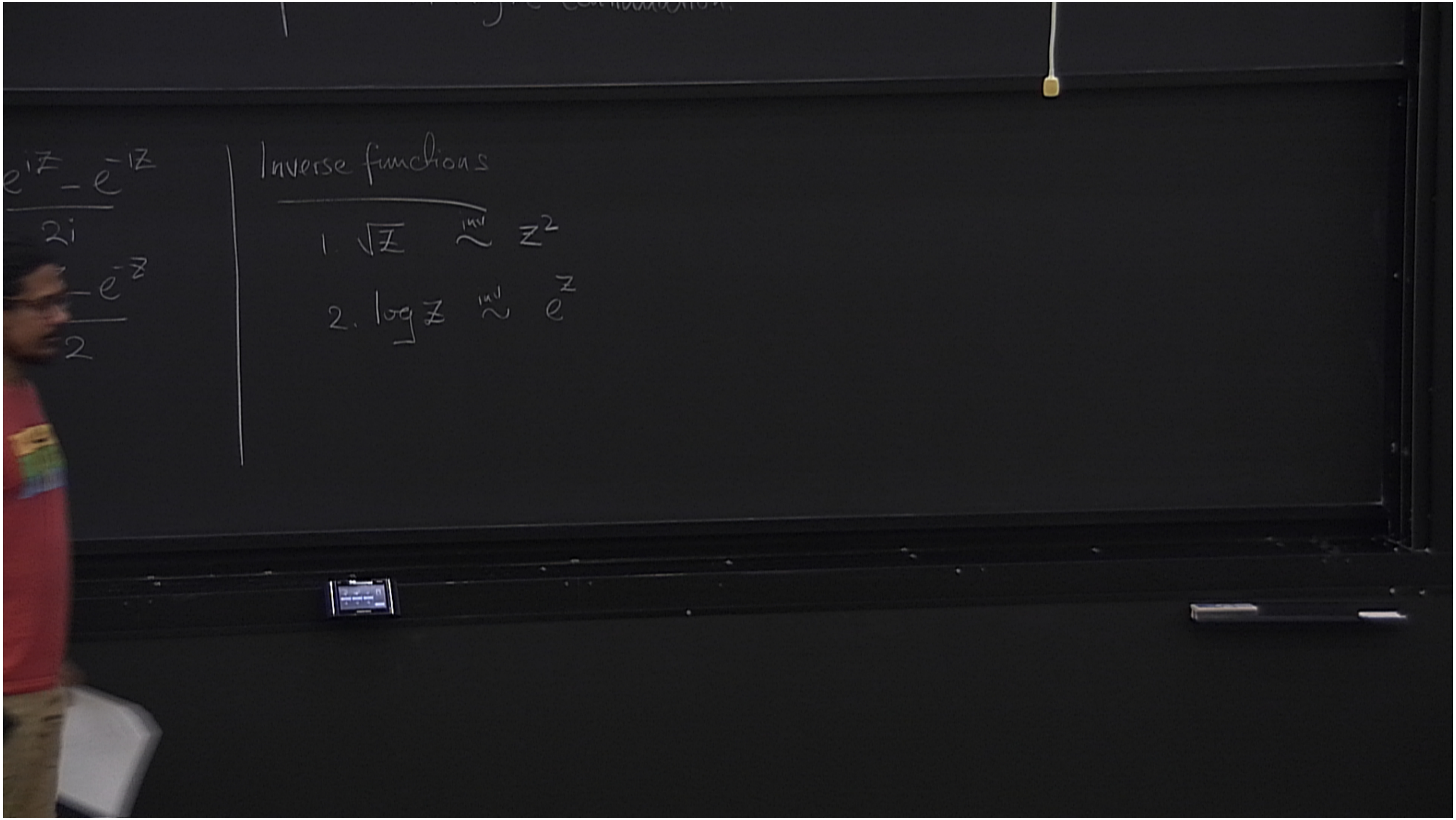

Sin x = (e^{ix} - e^{-ix}) / 2i
(e^{iz} - e^{-iz}) / 2i
z ~ Θ


$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$
$\frac{e^{iz} - e^{-iz}}{2i}$
$0 < z \sim \theta \leq 2\pi$


$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$
$\frac{e^{iz} - e^{-i\bar{z}}}{2i}$
$0 < z \sim \theta \leq 2\pi$


$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$
$\cos x = \frac{+}{2}$


$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$
$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$
$\sinh z = \frac{e^{z} - e^{-z}}{2}$
$+$
2


$\frac{e^{iz} - e^{-iz}}{2i}$
$\frac{e^{z} - e^{-z}}{2}$
Inverse functions
1. $\sqrt{z} \overset{inv}{\simeq} z^2$
2. $\log z \overset{inv}{\simeq} e^z$

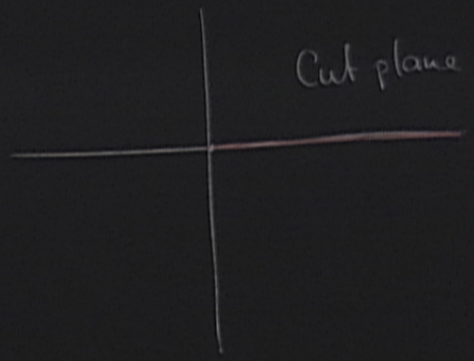
$\frac{e^{iz} - e^{-iz}}{2i}$

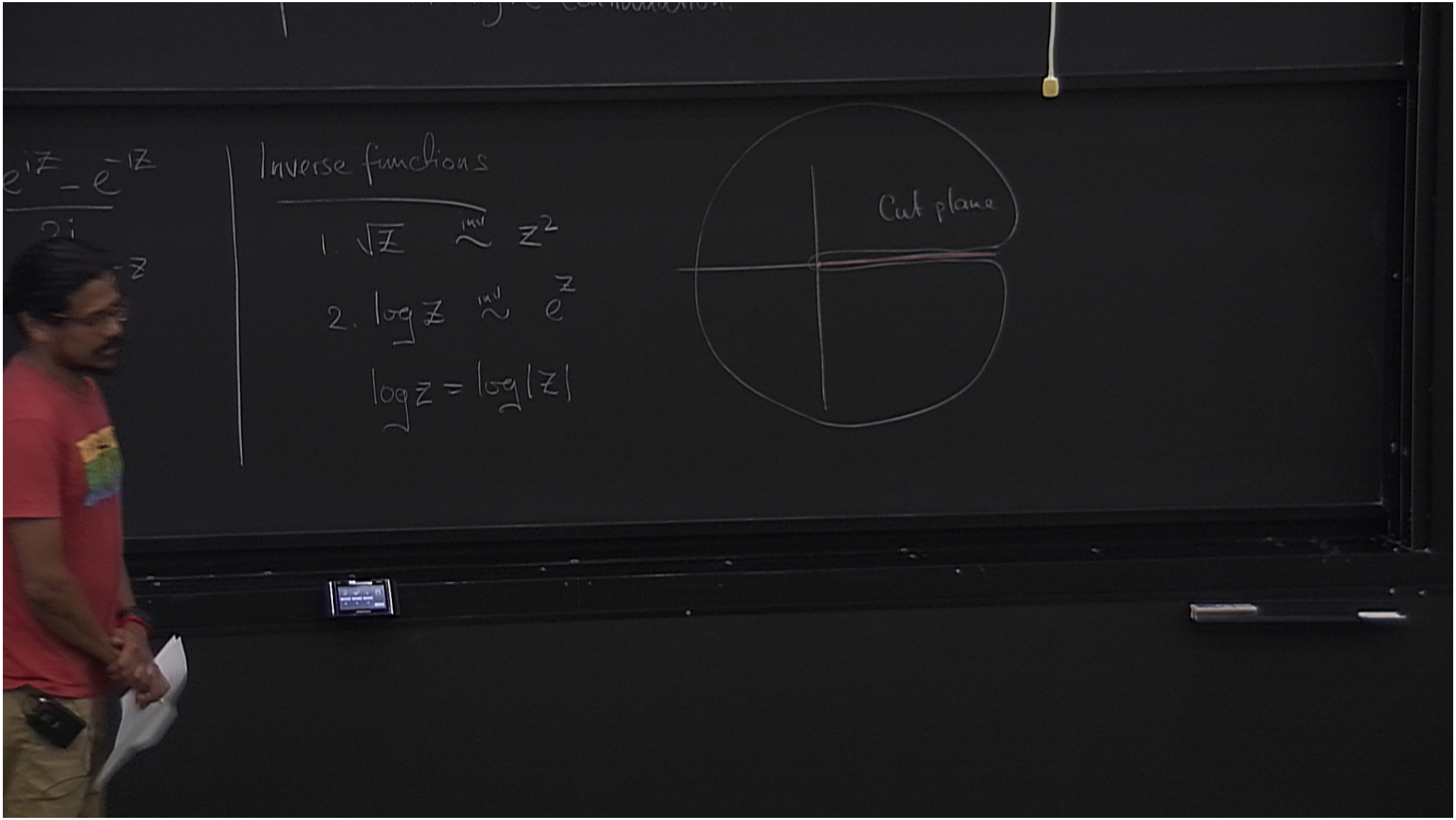
$\frac{e^{z} - e^{-z}}{2}$

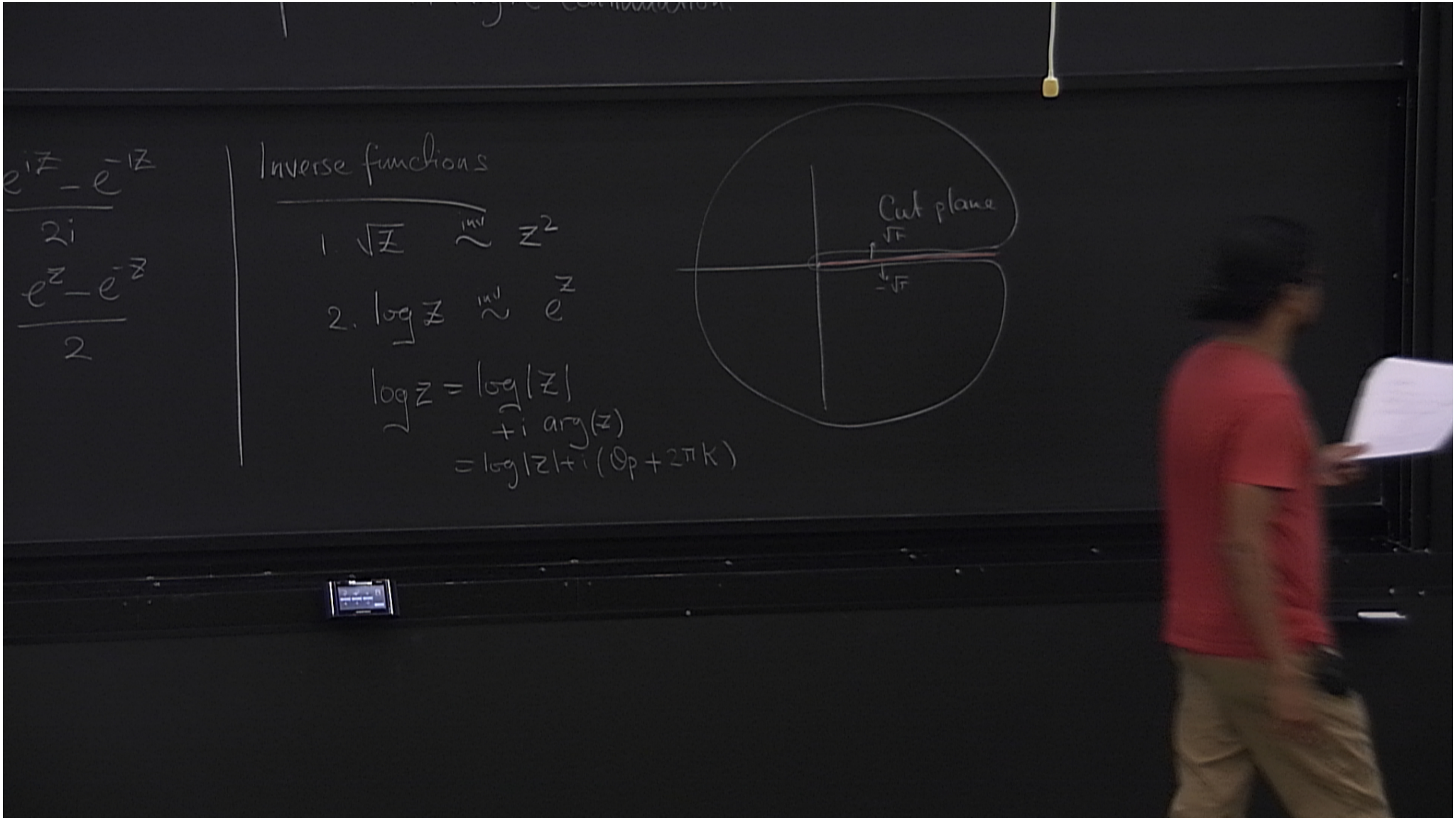
Inverse functions

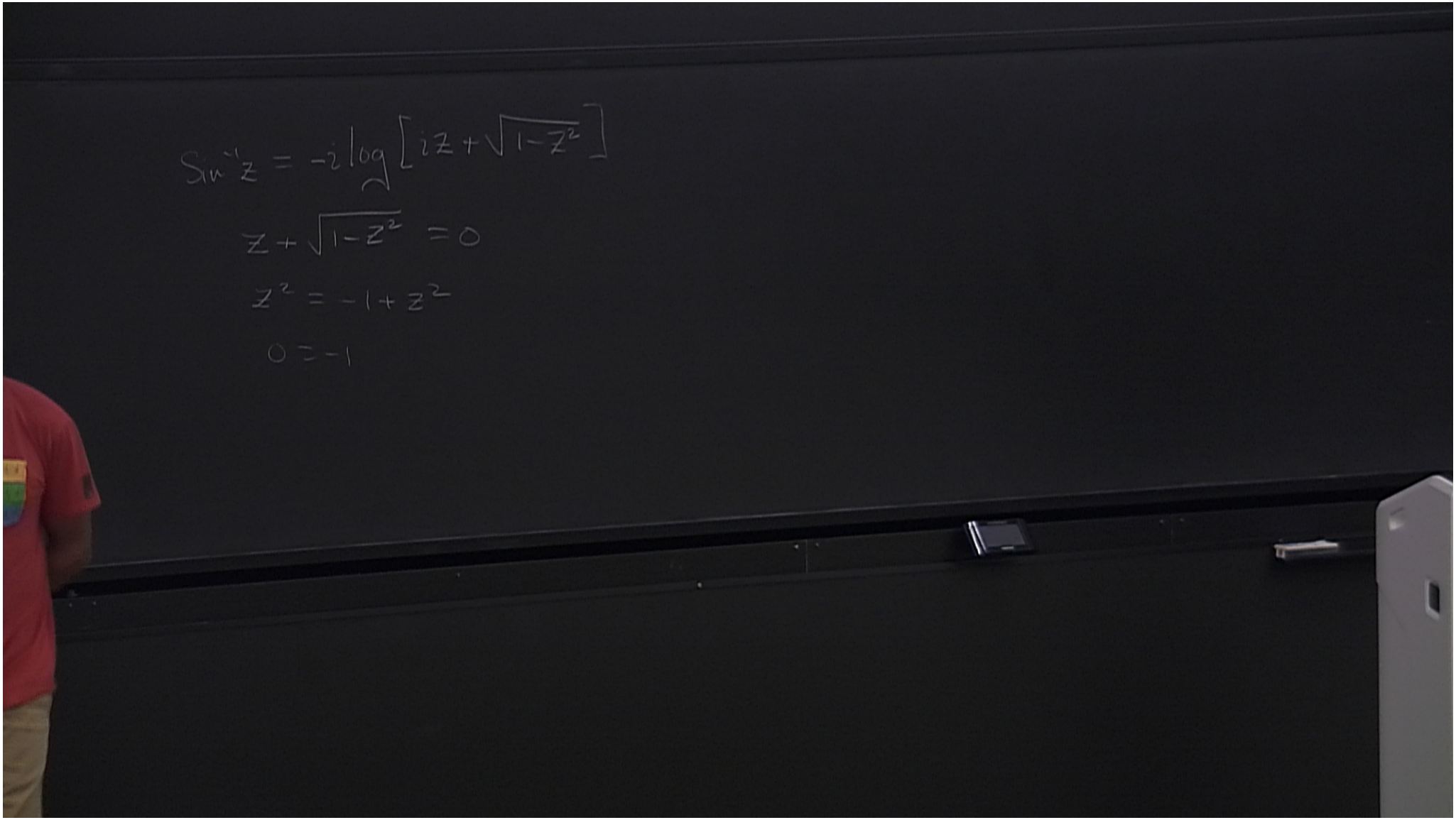
1. $\sqrt{z} \overset{inv}{\simeq} z^2$

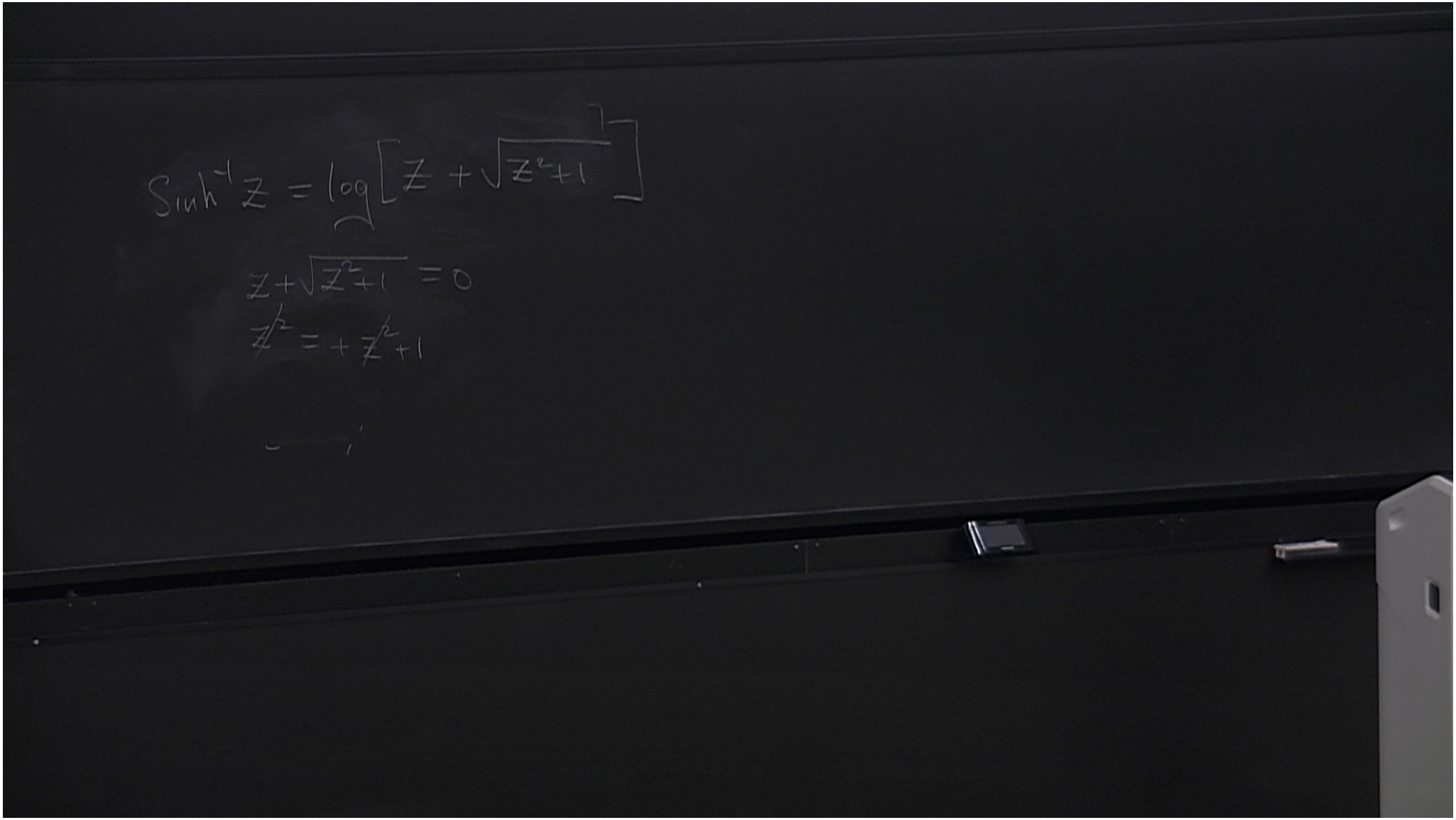
2. $\log z \overset{inv}{\simeq} e^z$

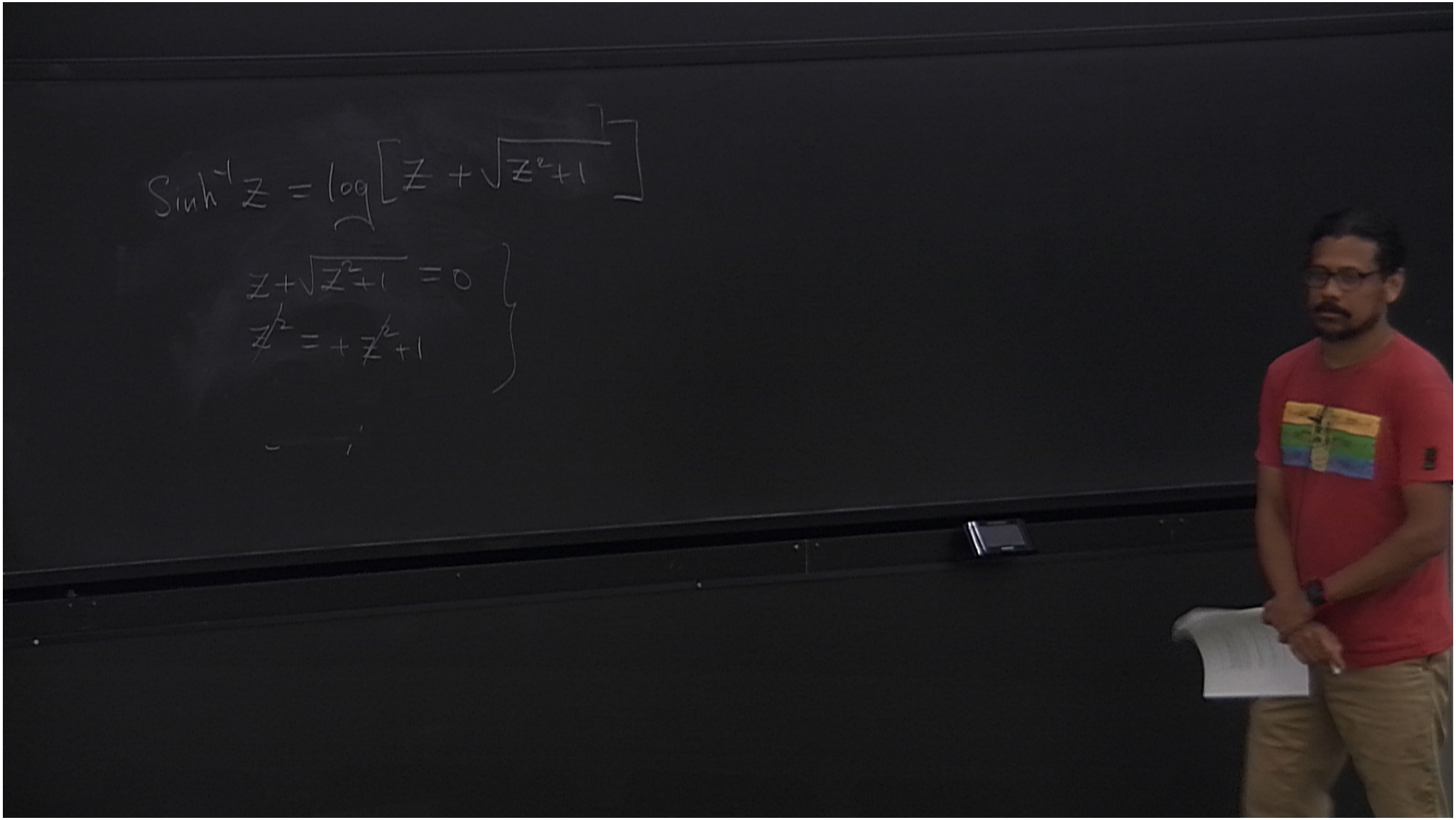


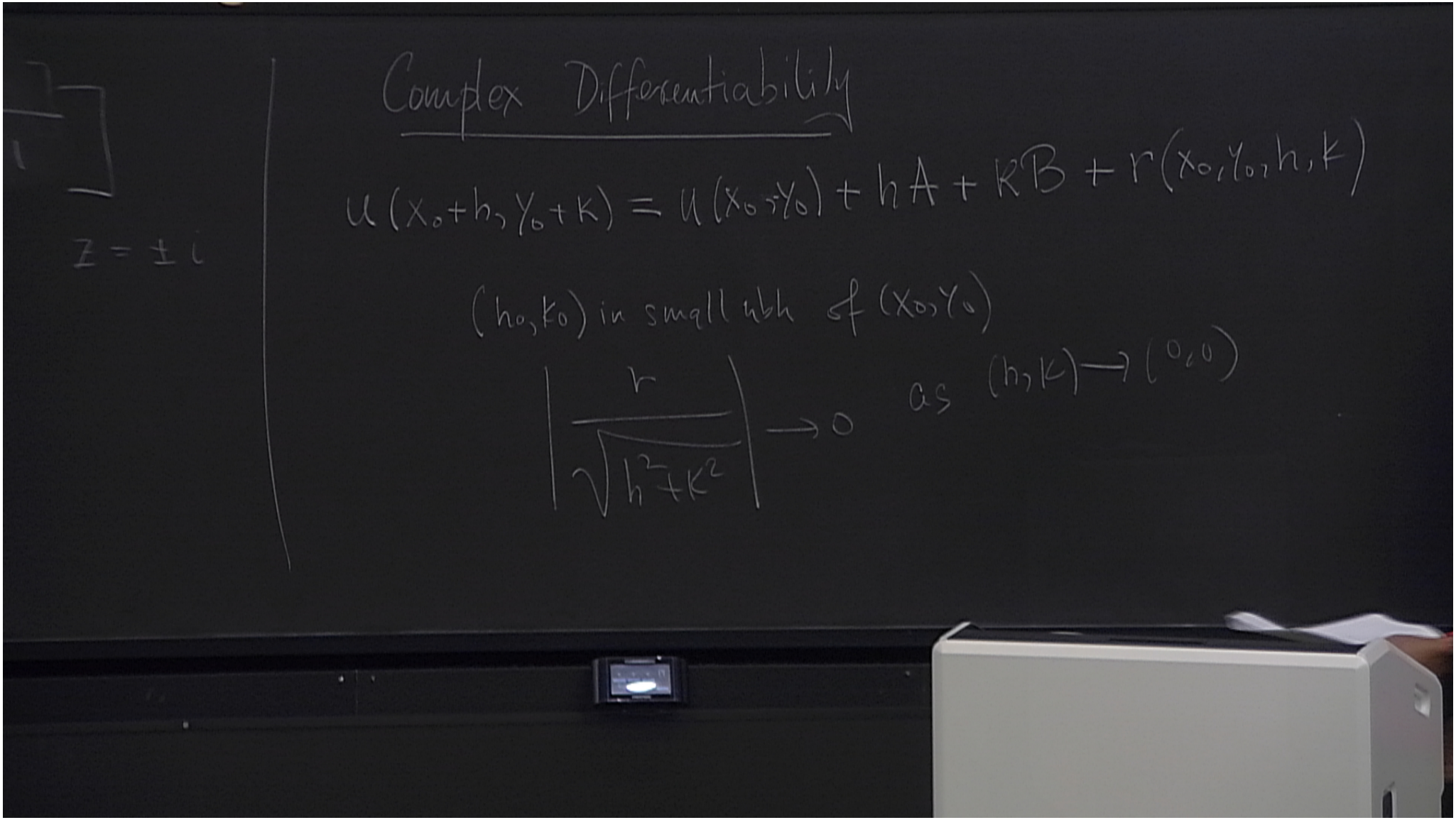

Inverse functions
1. $\sqrt{z} \overset{inv}{\sim} z^2$
2. $\log z \overset{inv}{\sim} e^z$
$\log z = \log|z|$
Cut plane

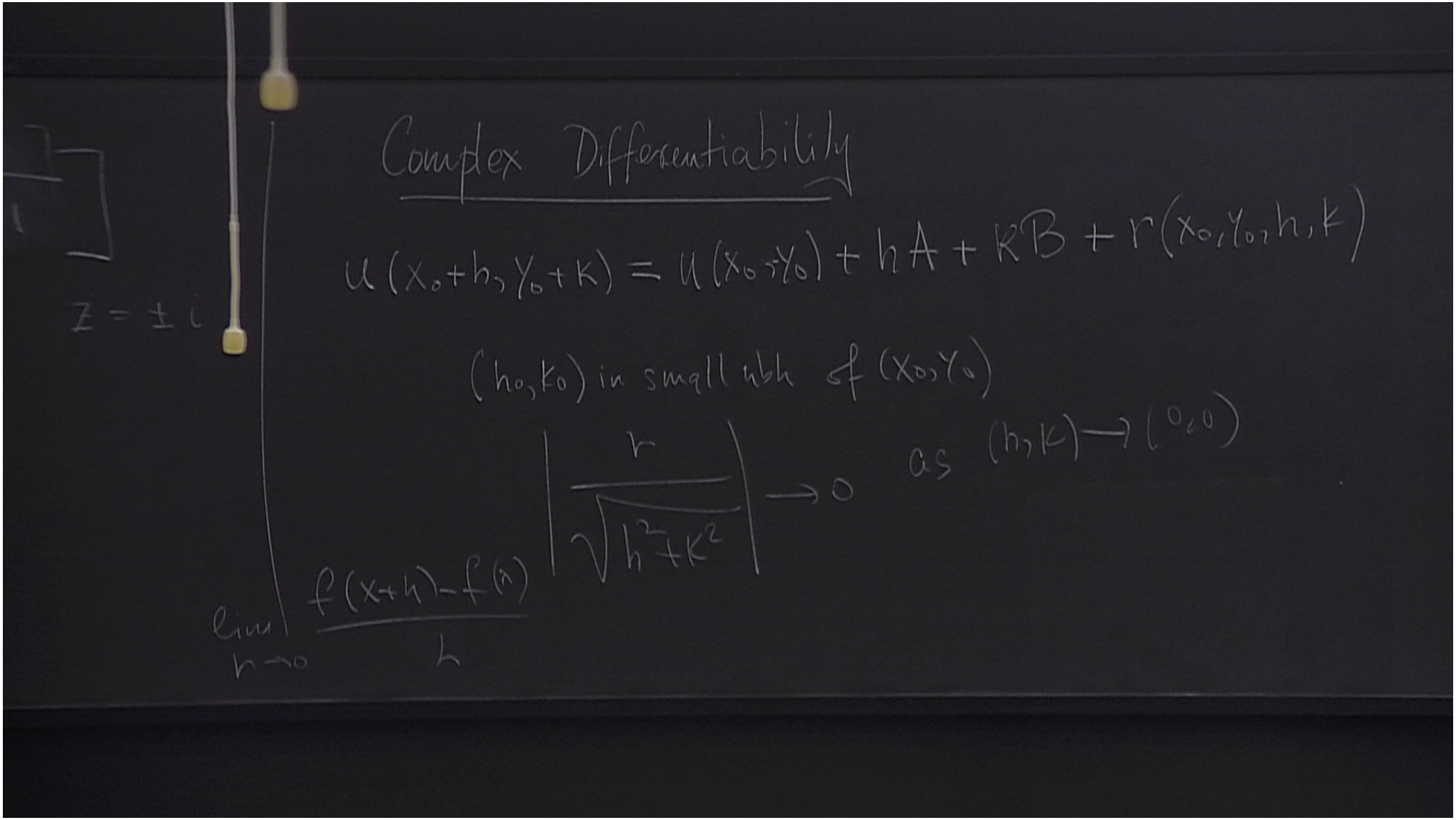

$\frac{e^{iz} - e^{-iz}}{2i}$
$\frac{e^{z} - e^{-z}}{2}$
Inverse functions
1. $\sqrt{z} \overset{inv}{\sim} z^2$
2. $\log z \overset{inv}{\sim} e^z$
$\log z = \log|z| + i \arg(z)$
$= \log|z| + i(\theta_p + 2\pi k)$
Cut plane
$\sqrt{r}$
$-\sqrt{r}$

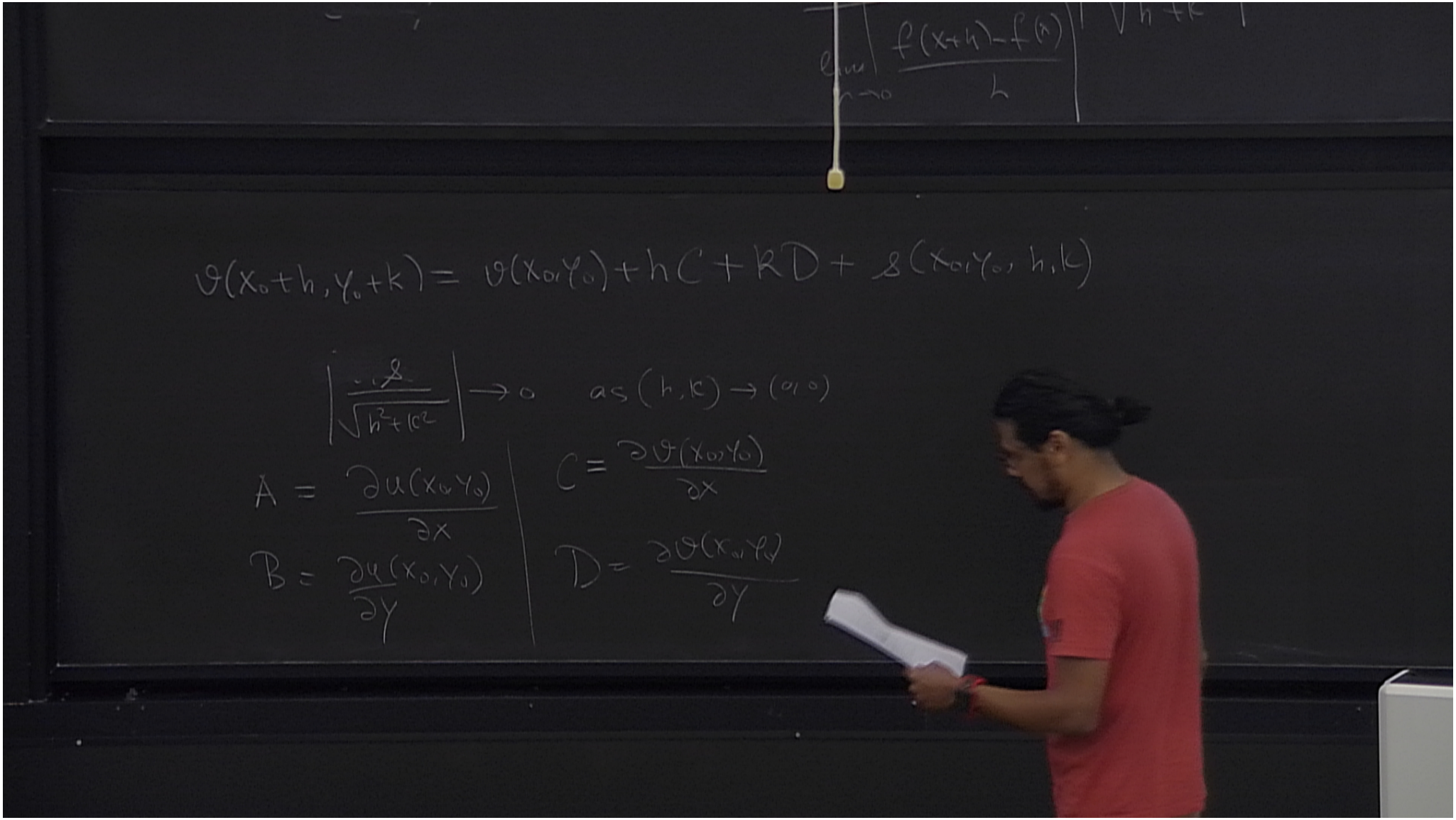

$\sin^{-1} z = -i \log\left[iz + \sqrt{1-z^2}\right]$
$z + \sqrt{1-z^2} = 0$
$z^2 = -1 + z^2$
$0 = -1$

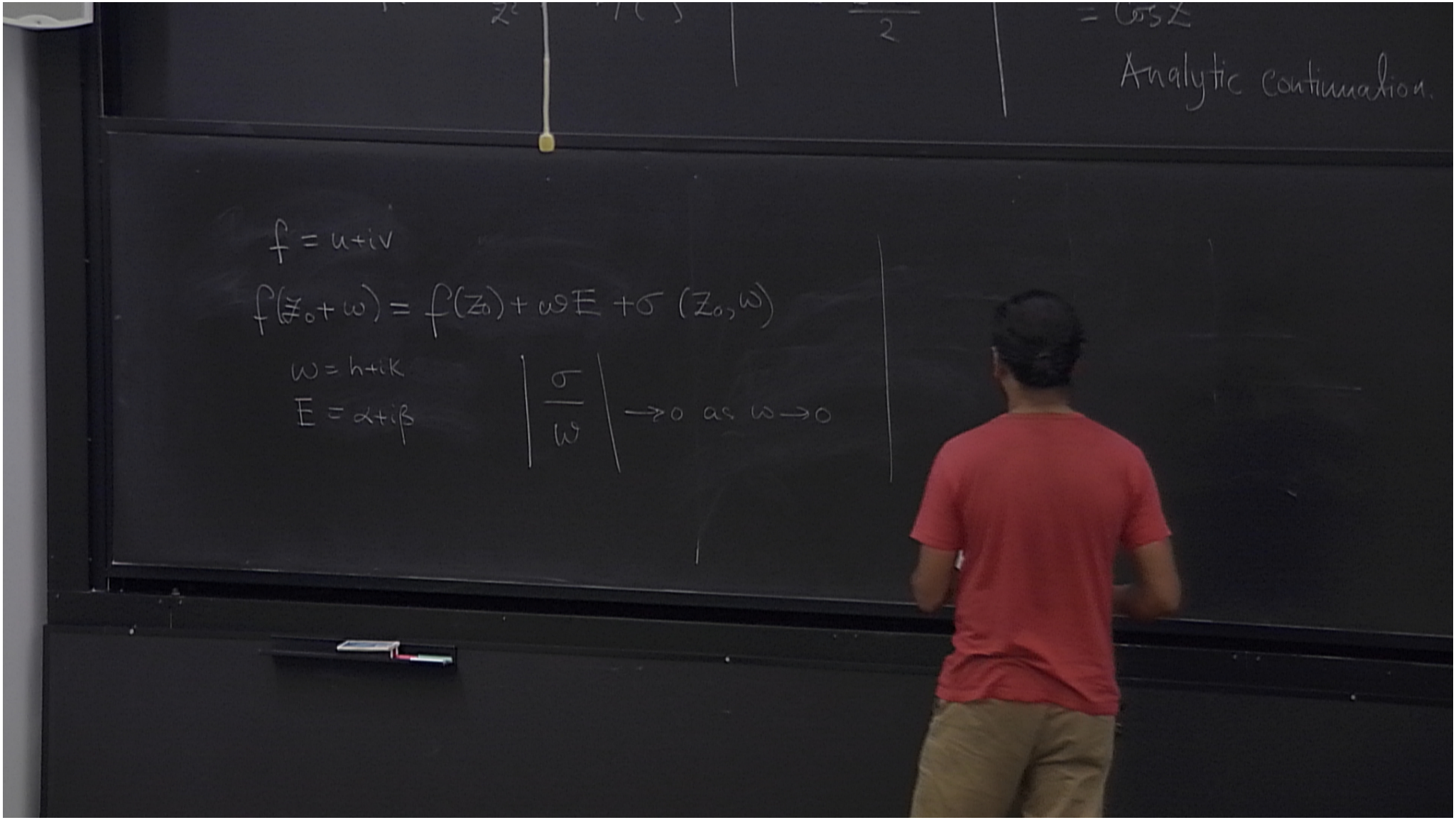

$\sinh^{-1} z = \log\left[z + \sqrt{z^2+1}\right]$
$z + \sqrt{z^2+1} = 0$
$z^2 = +z^2 + 1$

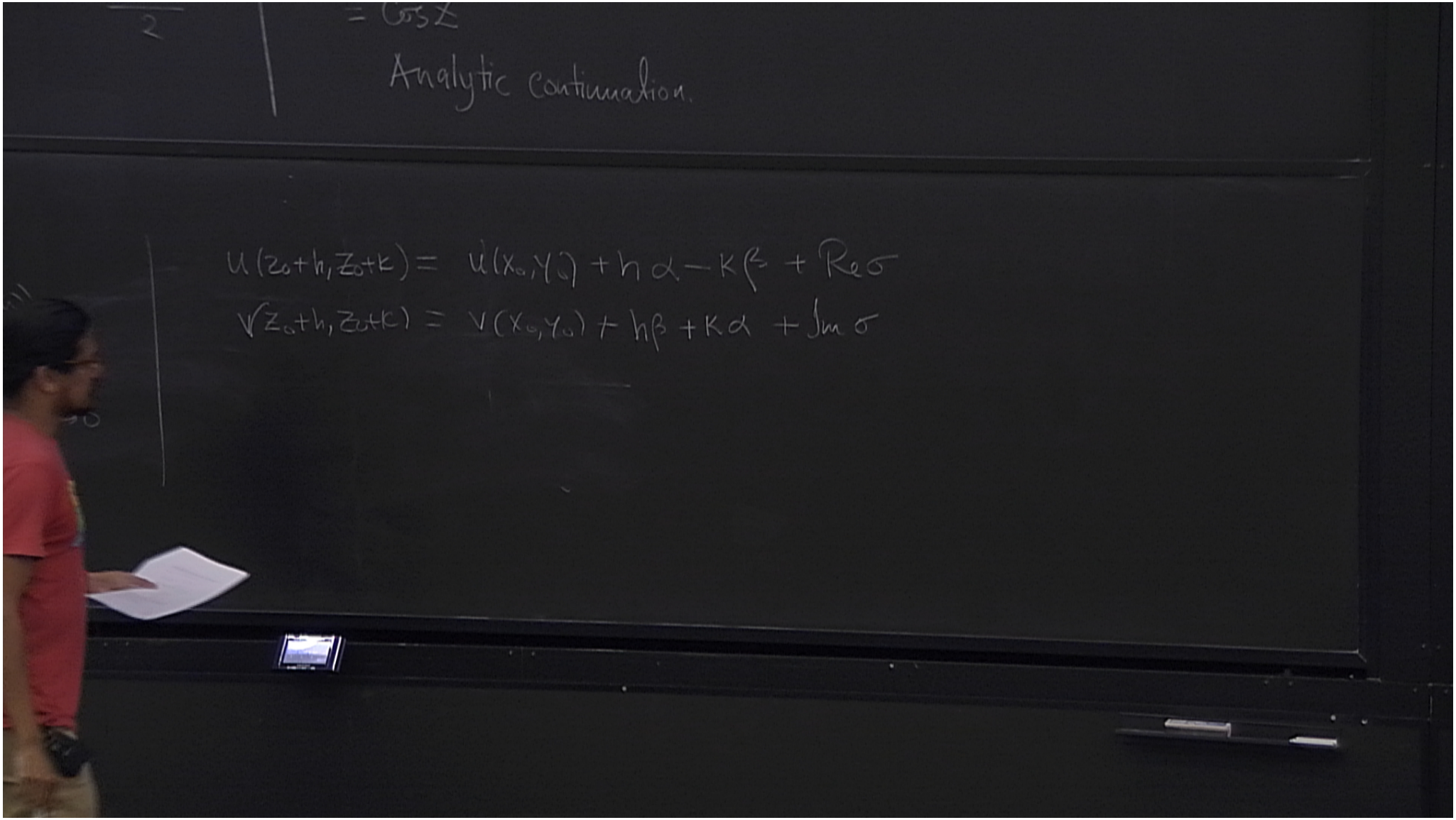

$\sinh^{-1} z = \log\left[z + \sqrt{z^2+1}\right]$
$z + \sqrt{z^2+1} = 0$
$z^2 = +z^2 + 1$

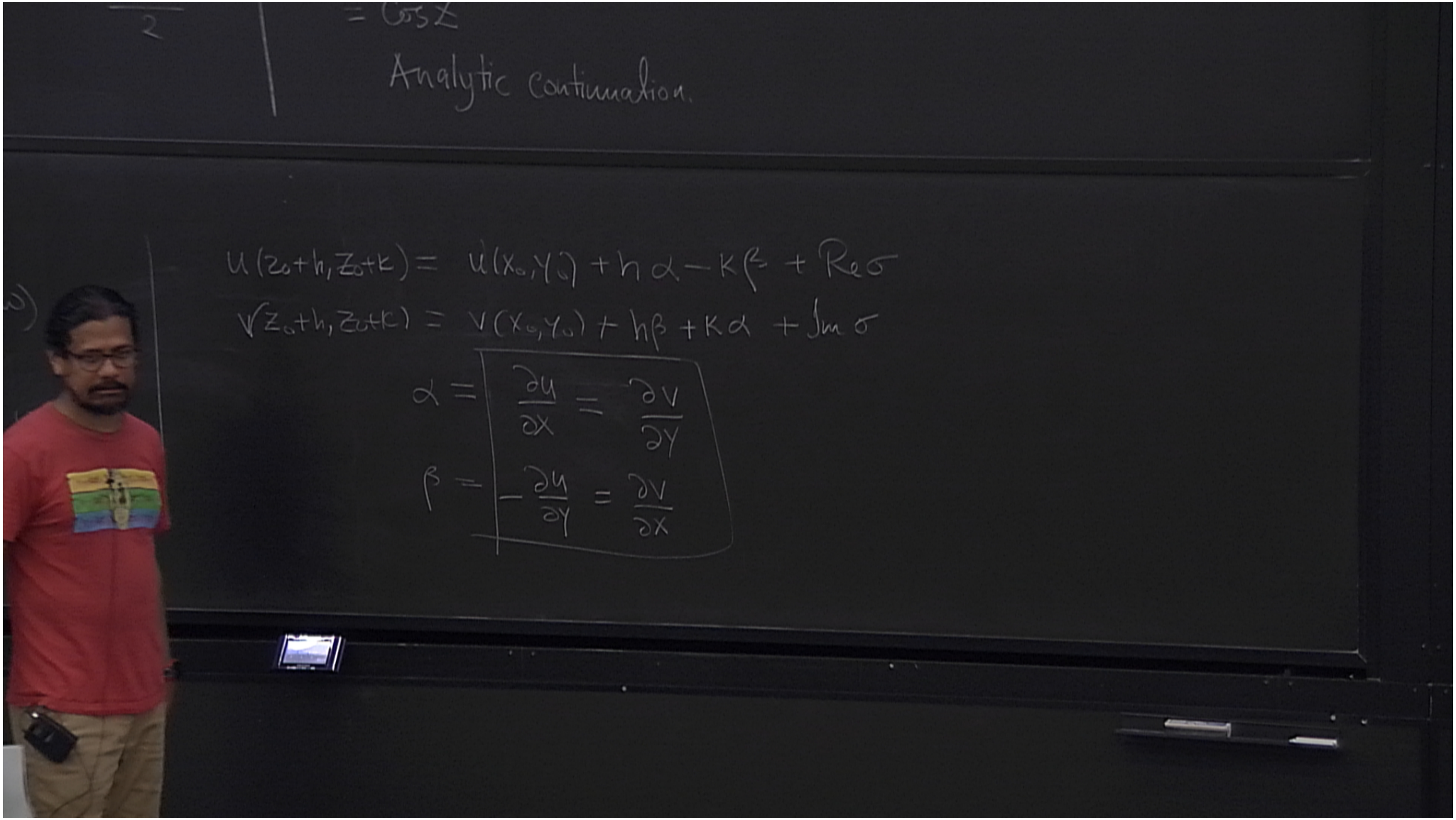

Complex Differentiability
$u(x_0+h, y_0+k) = u(x_0, y_0) + hA + kB + r(x_0, y_0, h, k)$
(h_0, k_0) in small nbh of (x_0, y_0)
$\left| \frac{r}{\sqrt{h^2+k^2}} \right| \to 0$ as $(h,k) \to (0,0)$
$z = \pm i$


Complex Differentiability
$u(x_0+h, y_0+k) = u(x_0, y_0) + hA + kB + r(x_0, y_0, h, k)$
(h_0, k_0) in small nbh of (x_0, y_0)
$\left|\frac{r}{\sqrt{h^2+k^2}}\right| \to 0$ as $(h,k) \to (0,0)$
$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$
$z = \pm i$


$\lim_{h \to 0} \left| \frac{f(x+h)-f(x)}{h} \right|$
$\vartheta(x_0+h, y_0+k) = \vartheta(x_0,y_0) + hC + kD + \varepsilon(x_0,y_0,h,k)$
$\left| \frac{\varepsilon}{\sqrt{h^2+k^2}} \right| \to 0 \quad as \ (h,k) \to (0,0)$
$A = \frac{\partial u(x_0,y_0)}{\partial x}$
$B = \frac{\partial u(x_0,y_0)}{\partial y}$
$C = \frac{\partial \vartheta(x_0,y_0)}{\partial x}$
$D = \frac{\partial \vartheta(x_0,y_0)}{\partial y}$


Analytic continuation.
$f = u + iv$
$f(z_0 + \omega) = f(z_0) + \omega E + \sigma(z_0, \omega)$
$\omega = h + ik$
$E = \alpha + i\beta$
$\left|\frac{\sigma}{\omega}\right| \to 0$ as $\omega \to 0$


Analytic continuation.
$u(z_0+h, z_0+k) = u(x_0, y_0) + h\alpha - k\beta + \mathrm{Re}\,\sigma$
$v(z_0+h, z_0+k) = v(x_0, y_0) + h\beta + k\alpha + \mathrm{Im}\,\sigma$


Analytic continuation.
$u(z_0+h, z_0+k) = u(x_0, y_0) + h\alpha - k\beta + \mathrm{Re}\,\sigma$
$v(z_0+h, z_0+k) = v(x_0, y_0) + h\beta + k\alpha + \mathrm{Im}\,\sigma$
$\alpha = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$
$\beta = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

$= G_5 \Sigma$

Analytic continuation.

$$u(z_0+h, z_0+k) = u(x_0, y_0) + h\alpha - k\beta + \mathrm{Re}\,\sigma$$

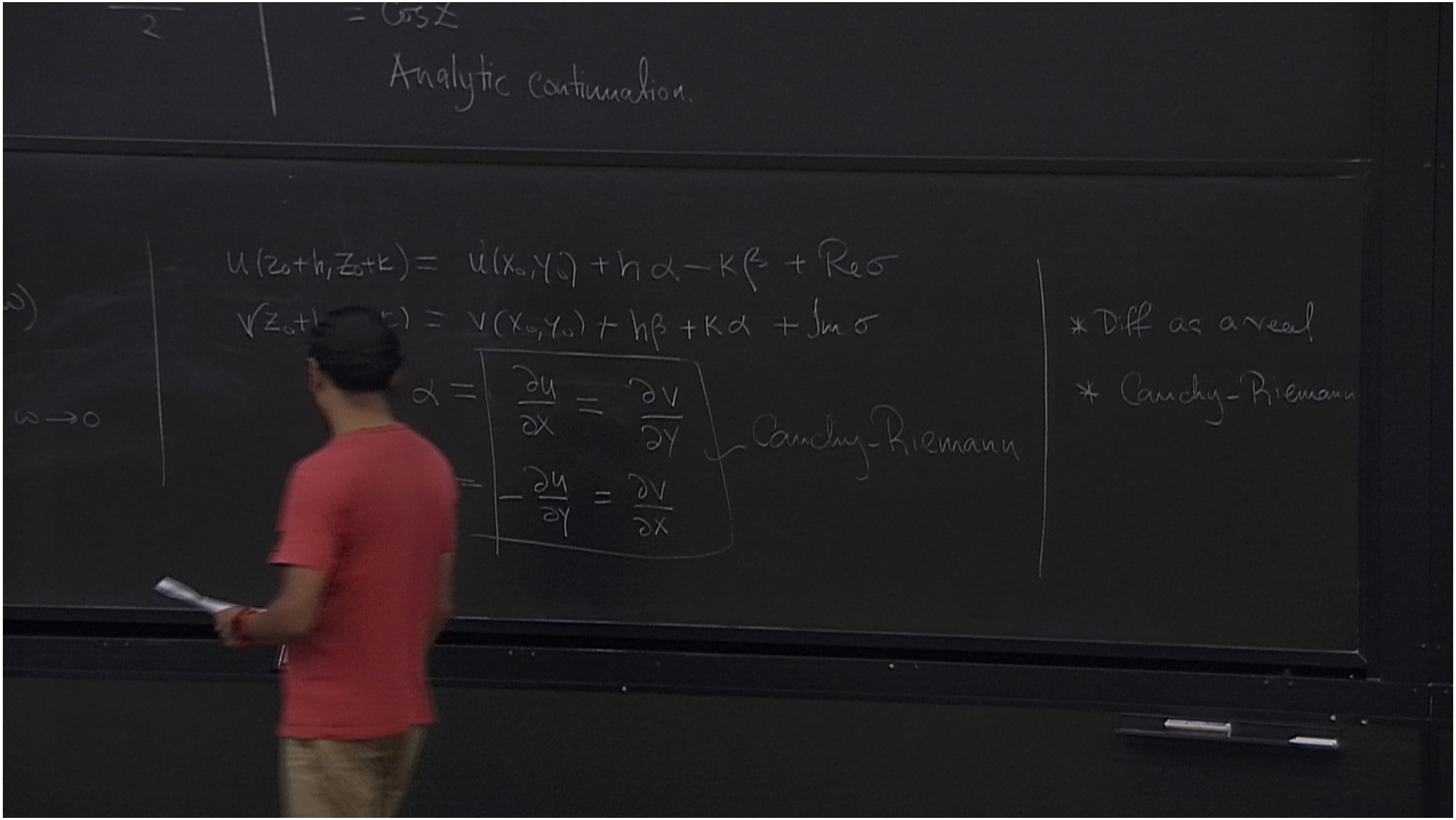
$$v(z_0+h, z_0+k) = v(x_0, y_0) + h\beta + k\alpha + \mathrm{Im}\,\sigma$$

$$\alpha = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\beta = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

Cauchy-Riemann

$\omega \to 0$


Analytic continuation.
$u(z_0+h, z_0+k) = u(x_0, y_0) + h\alpha - k\beta + \mathrm{Re}\,\sigma$
$v(z_0+h, z_0+k) = v(x_0, y_0) + h\beta + k\alpha + \mathrm{Im}\,\sigma$
$\alpha = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$
$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$
Cauchy-Riemann
* Diff as a real
* Cauchy-Riemann

Analytic continuation.

$$u(z_0+h, z_0+k) = u(x_0, y_0) + h\alpha - k\beta + \mathrm{Re}\,\sigma$$

$$v(z_0+h, z_0+k) = v(x_0, y_0) + h\beta + k\alpha + \mathrm{Im}\,\sigma$$

$$\alpha = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

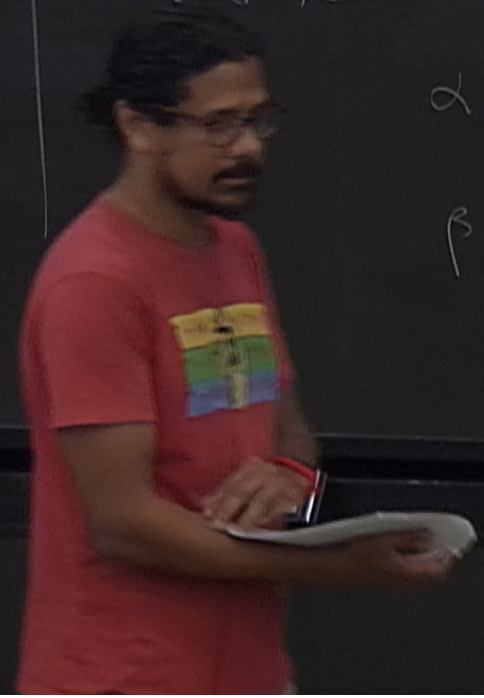
$$\beta = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

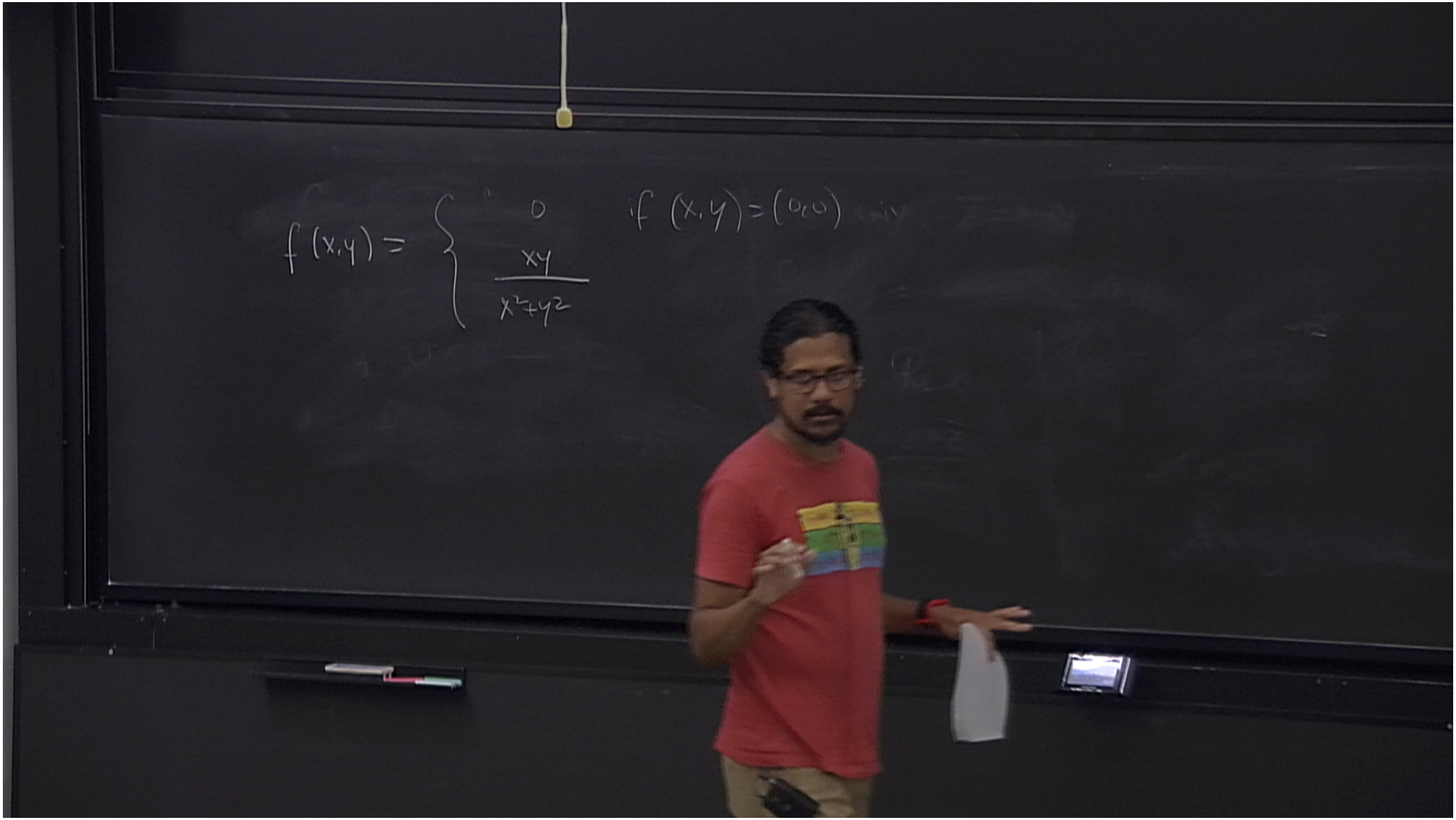
Cauchy-Riemann

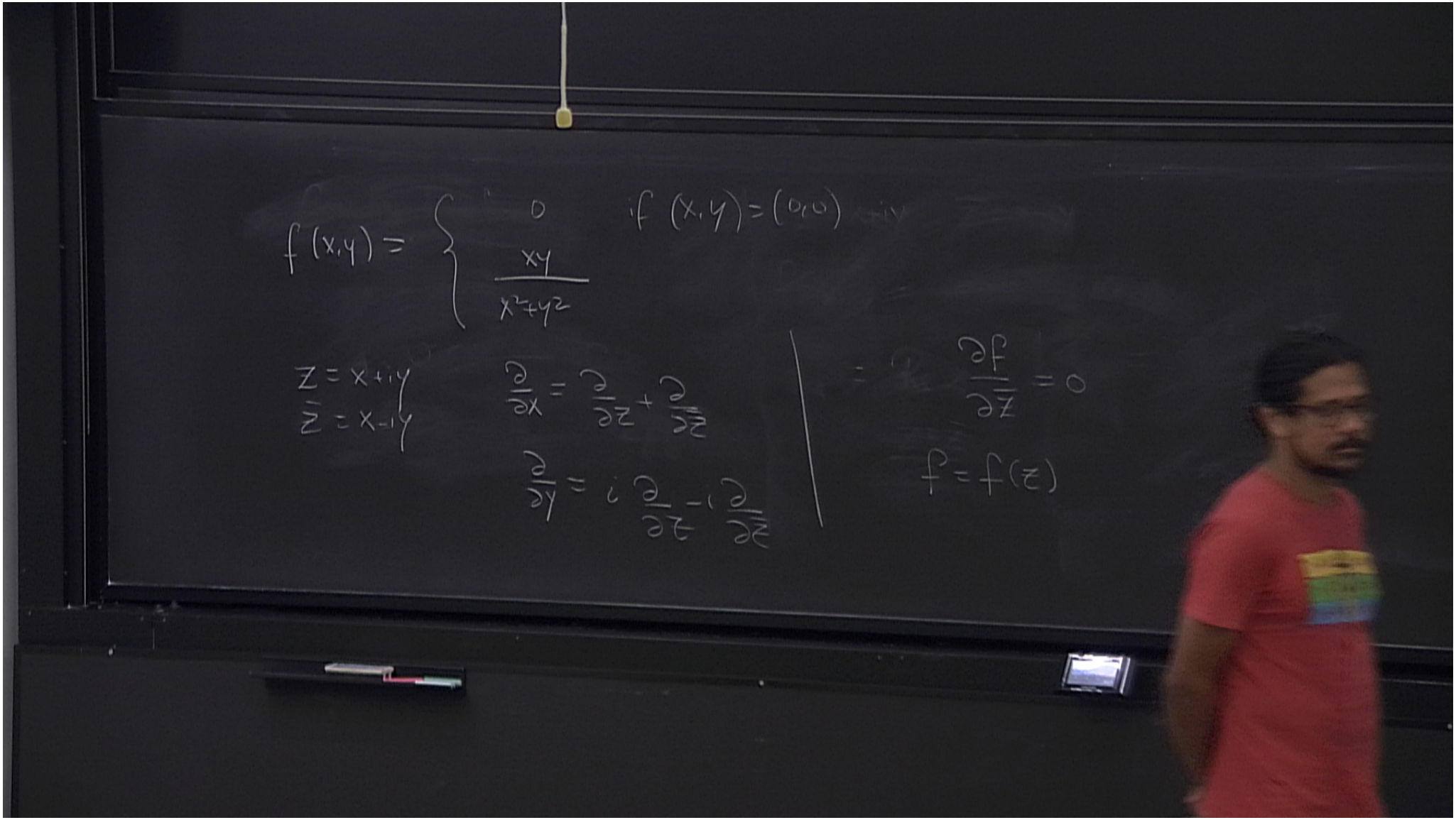
* Diff as a real
* Cauchy-Riemann

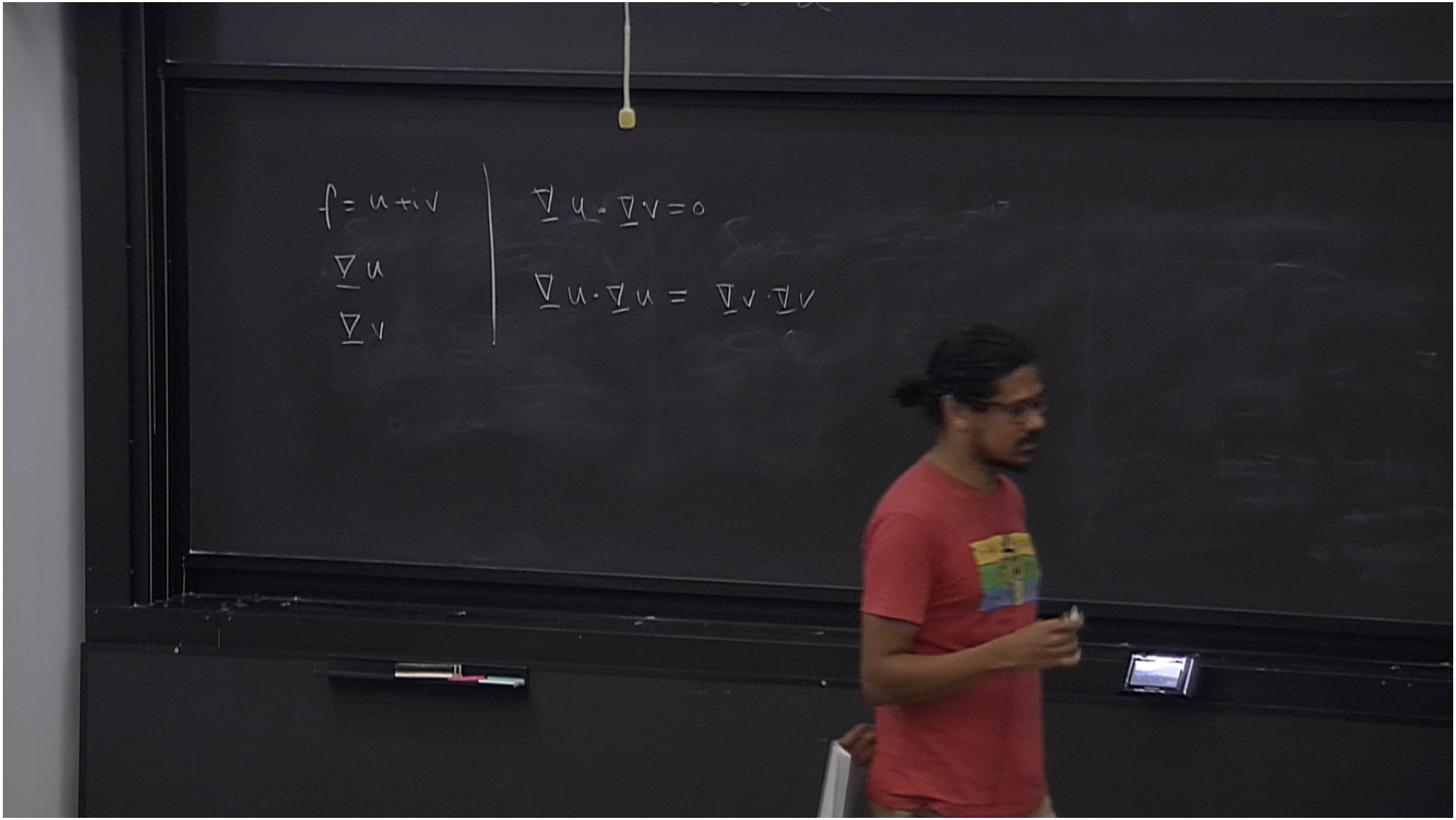
Analytic at x_0, y_0

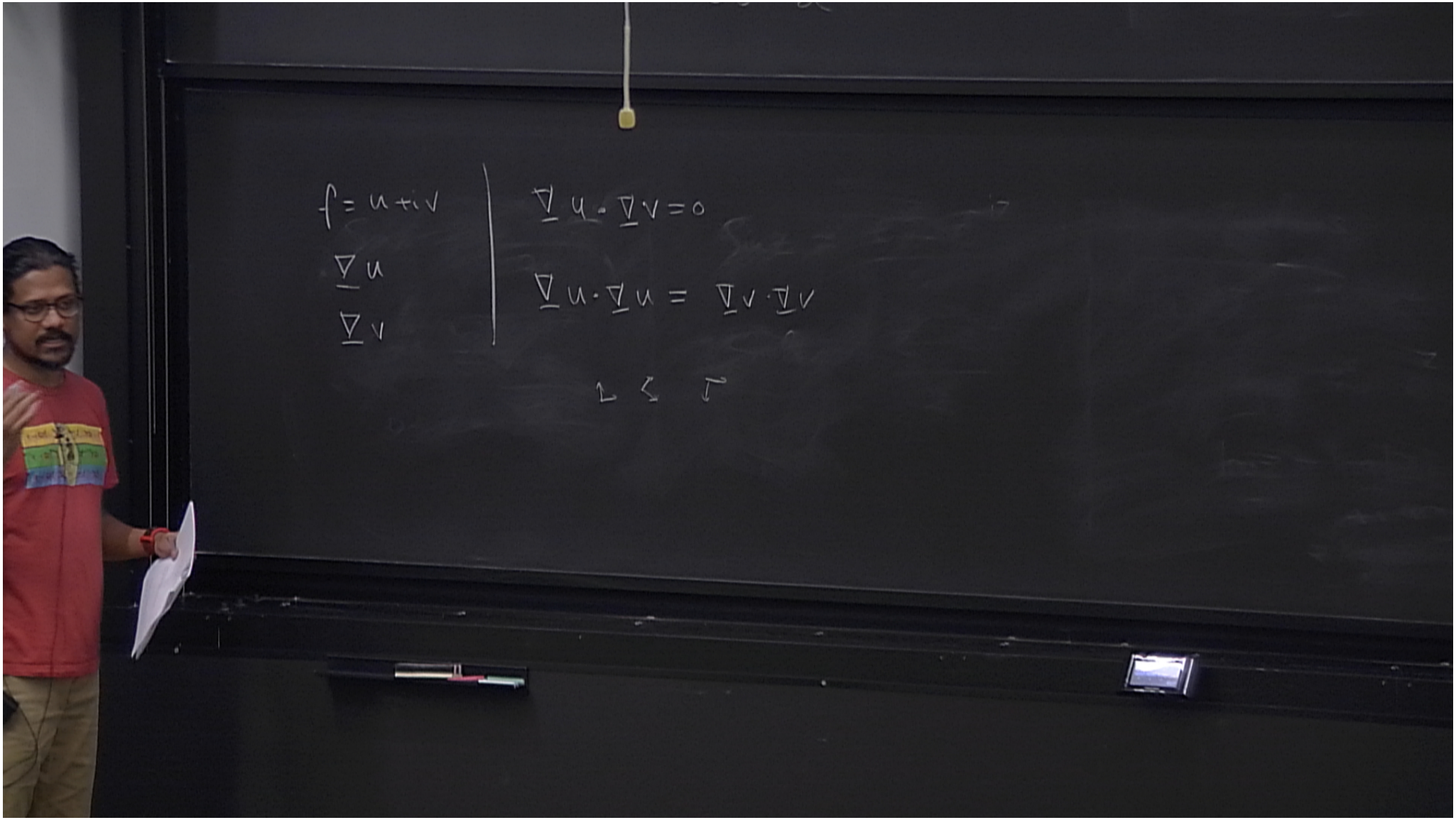
holomorphic

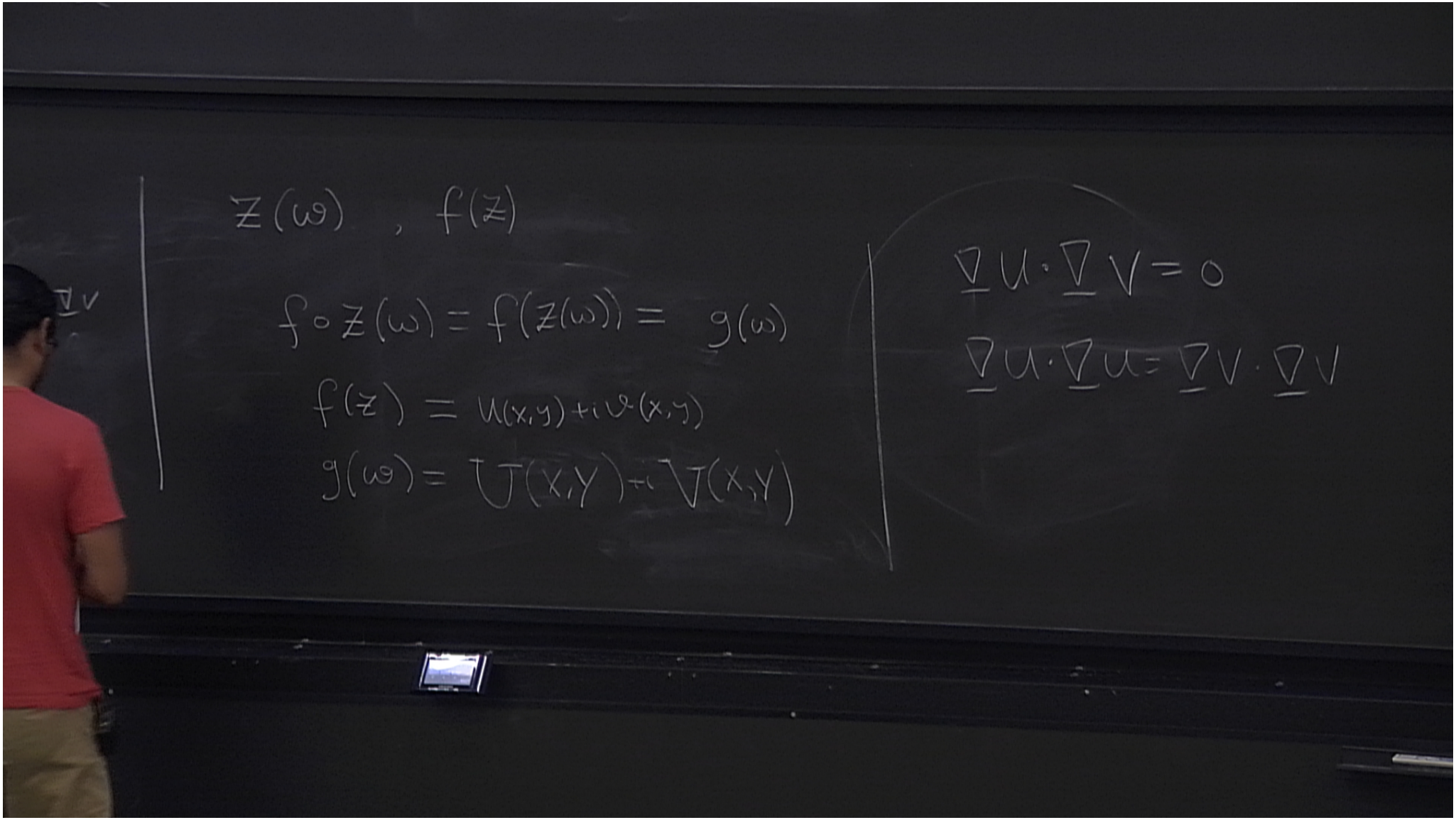


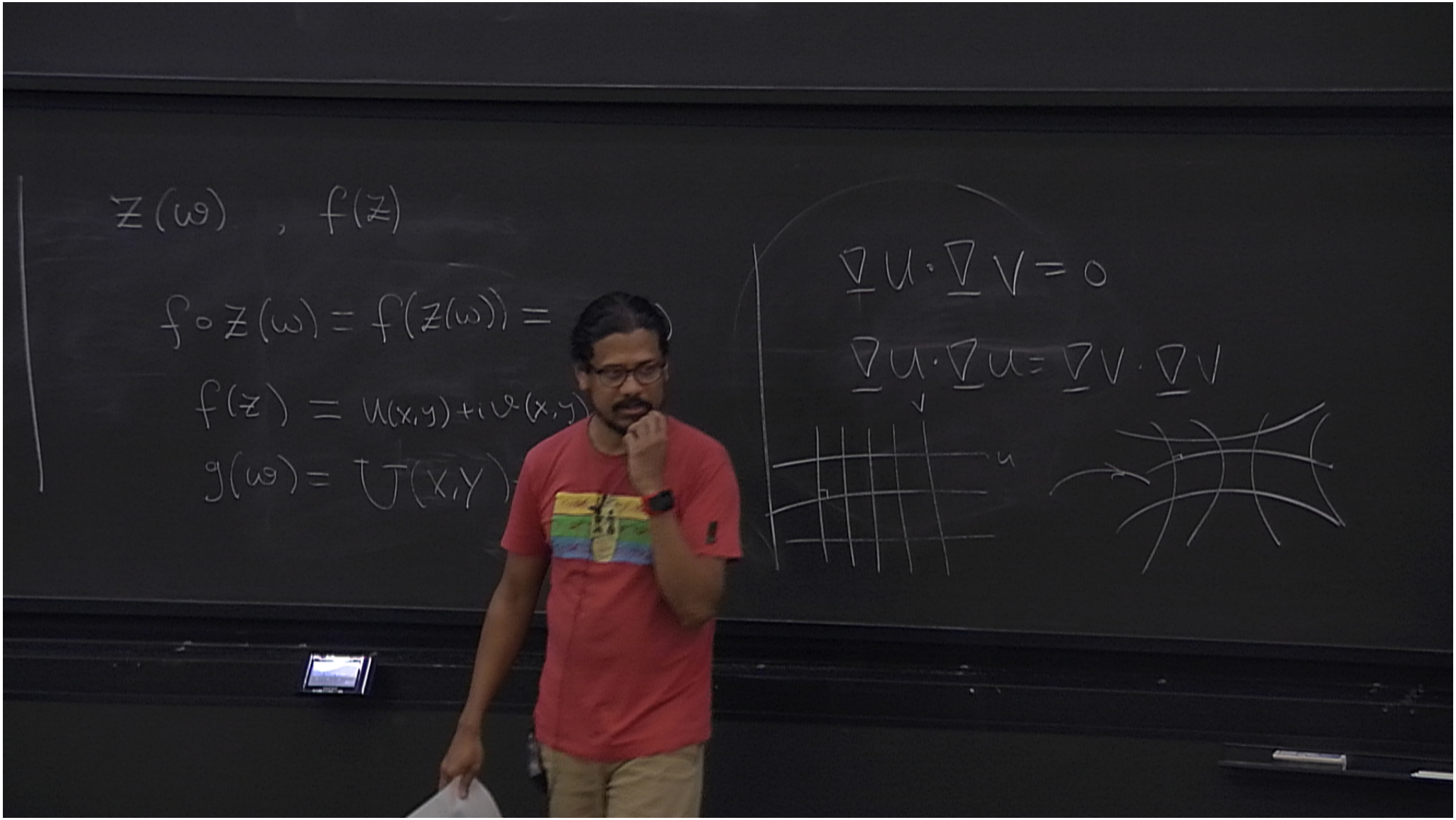

$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{xy}{x^2+y^2} \end{cases}$

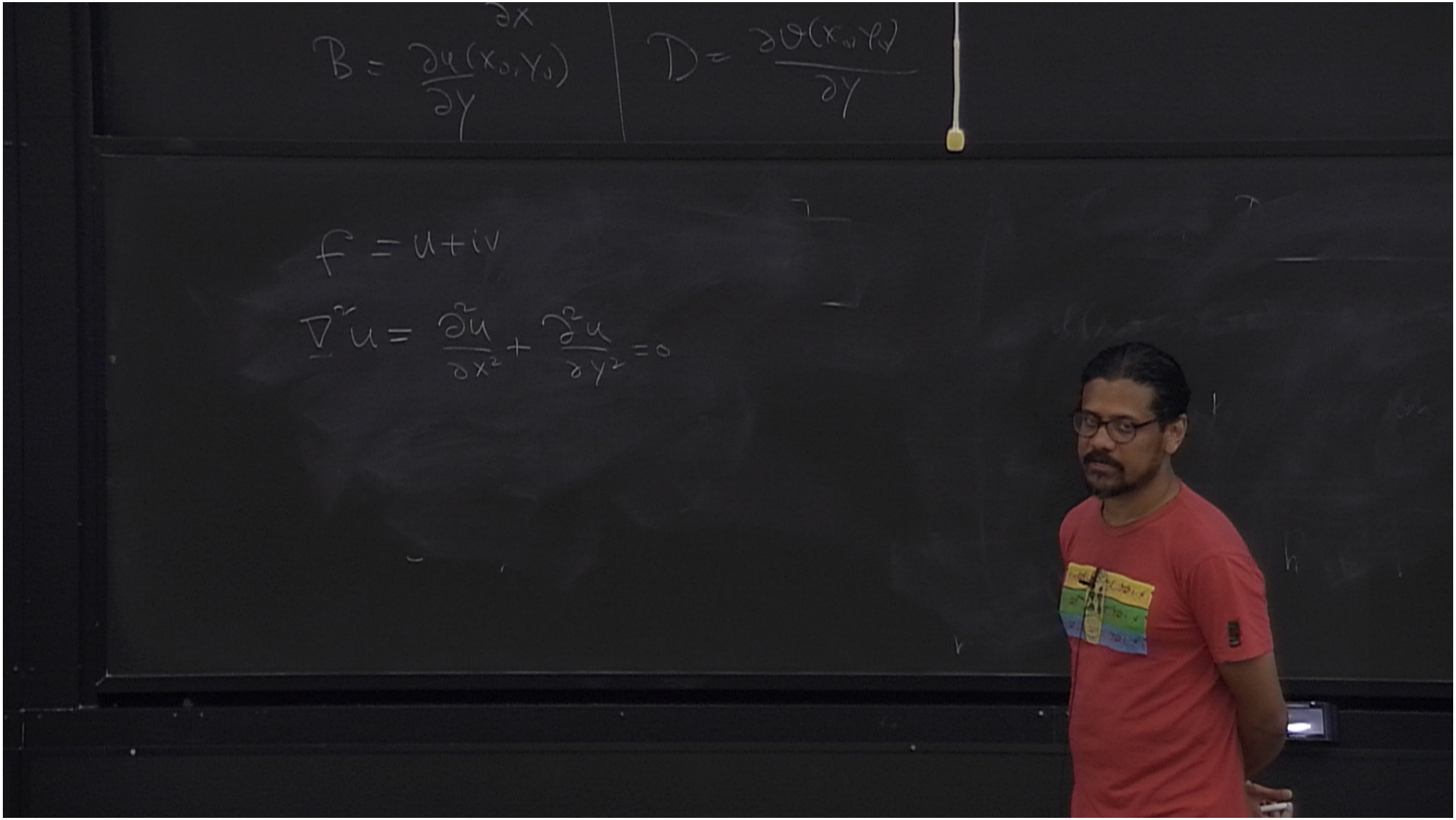

$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{xy}{x^2+y^2} \end{cases}$
$z = x+iy$
$\bar{z} = x-iy$
$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}$
$\frac{\partial f}{\partial y} = i\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right)$
$\frac{\partial f}{\partial \bar{z}} = 0$
$f = f(z)$


f = u + iv
∇u
∇v
∇u · ∇v = 0
∇u · ∇u = ∇v · ∇v


f = u + iv
∇u
∇v
∇u · ∇v = 0
∇u · ∇u = ∇v · ∇v


$z(\omega)$, $f(z)$
$f \circ z(\omega) = f(z(\omega)) = g(\omega)$
$f(z) = u(x,y) + iv(x,y)$
$g(\omega) = U(x,y) + iV(x,y)$
$\nabla u \cdot \nabla v = 0$
$\nabla u \cdot \nabla u = \nabla v \cdot \nabla v$


$z(\omega)$, $f(z)$
$f \circ z(\omega) = f(z(\omega)) =$
$f(z) = u(x,y) + iv(x,y)$
$g(\omega) = U(x,y)$
$\nabla u \cdot \nabla v = 0$
$\nabla u \cdot \nabla u = \nabla v \cdot \nabla v$


$\frac{\partial}{\partial x}$
$B = \frac{\partial u(x_0, y_0)}{\partial y}$
$D = \frac{\partial v(x_0, y_0)}{\partial y}$
$f = u + iv$
$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$B = \frac{\partial u(x_0, y_0)}{\partial y} \qquad D = \frac{\partial v(x_0, y_0)}{\partial y}$$

$$f = u + iv$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u = 4xy\,(y^2 - x^2)$$

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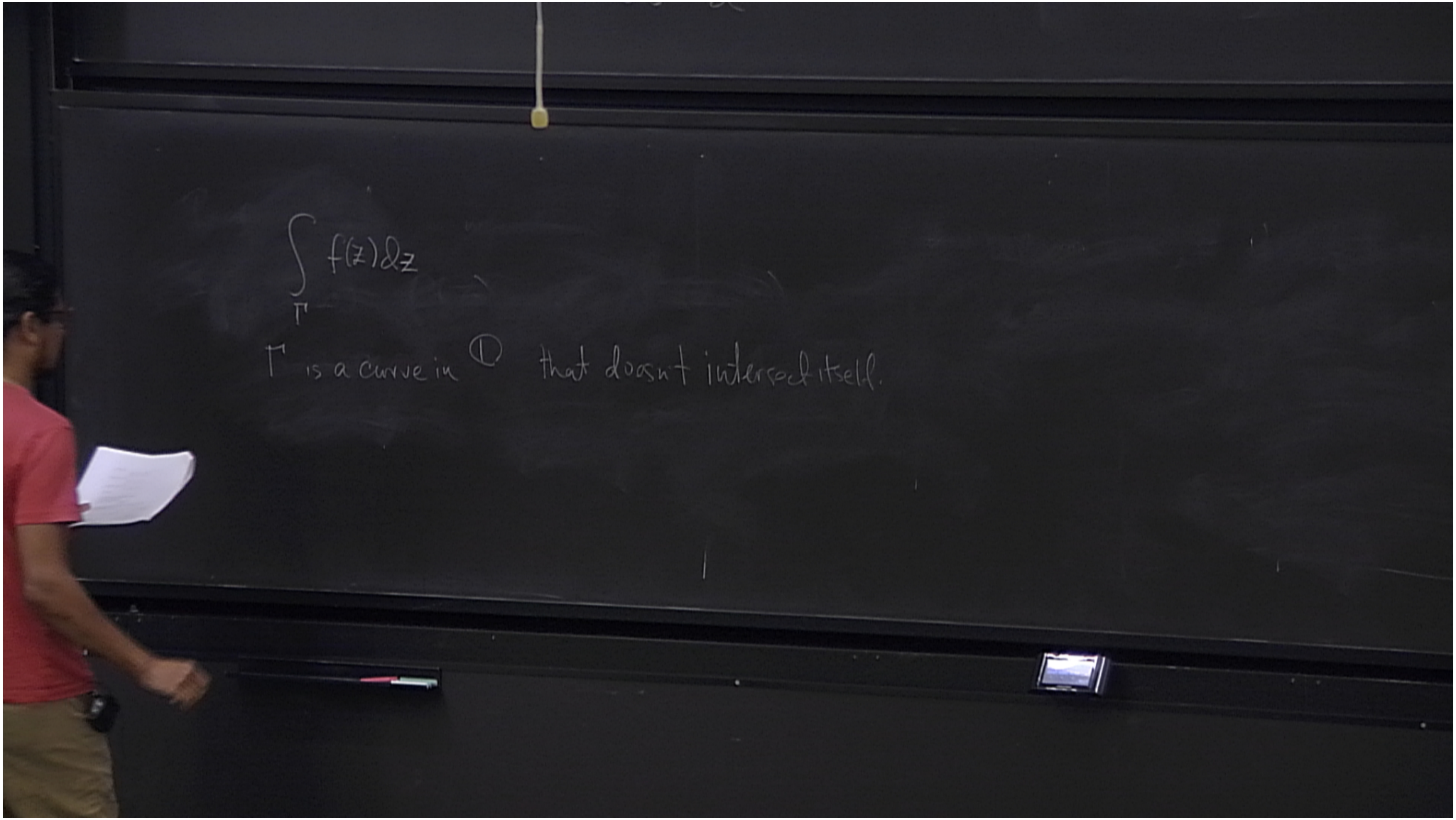
$\frac{\partial u}{\partial x} = 4y(y^2 - x^2) - 8x^2y = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = 4x(y^2 - x^2) + 8xy^2 = -\frac{\partial v}{\partial x}$

$v = y^4 - 6x^2y^2 + C_1(x)$

$v = x^4 - 6x^2y^2 + C_2(y)$

$v = x^4 + y^4 - 6x^2y^2$

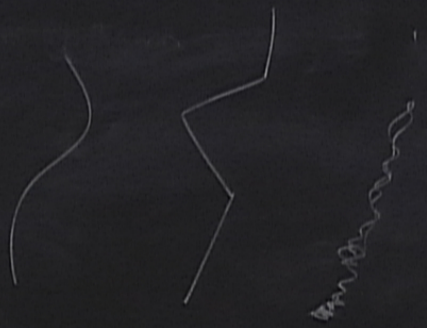

$\int_\Gamma f(z)\,dz$
Γ is a curve in $\mathbb{C}$ that doesn't intersect itself.

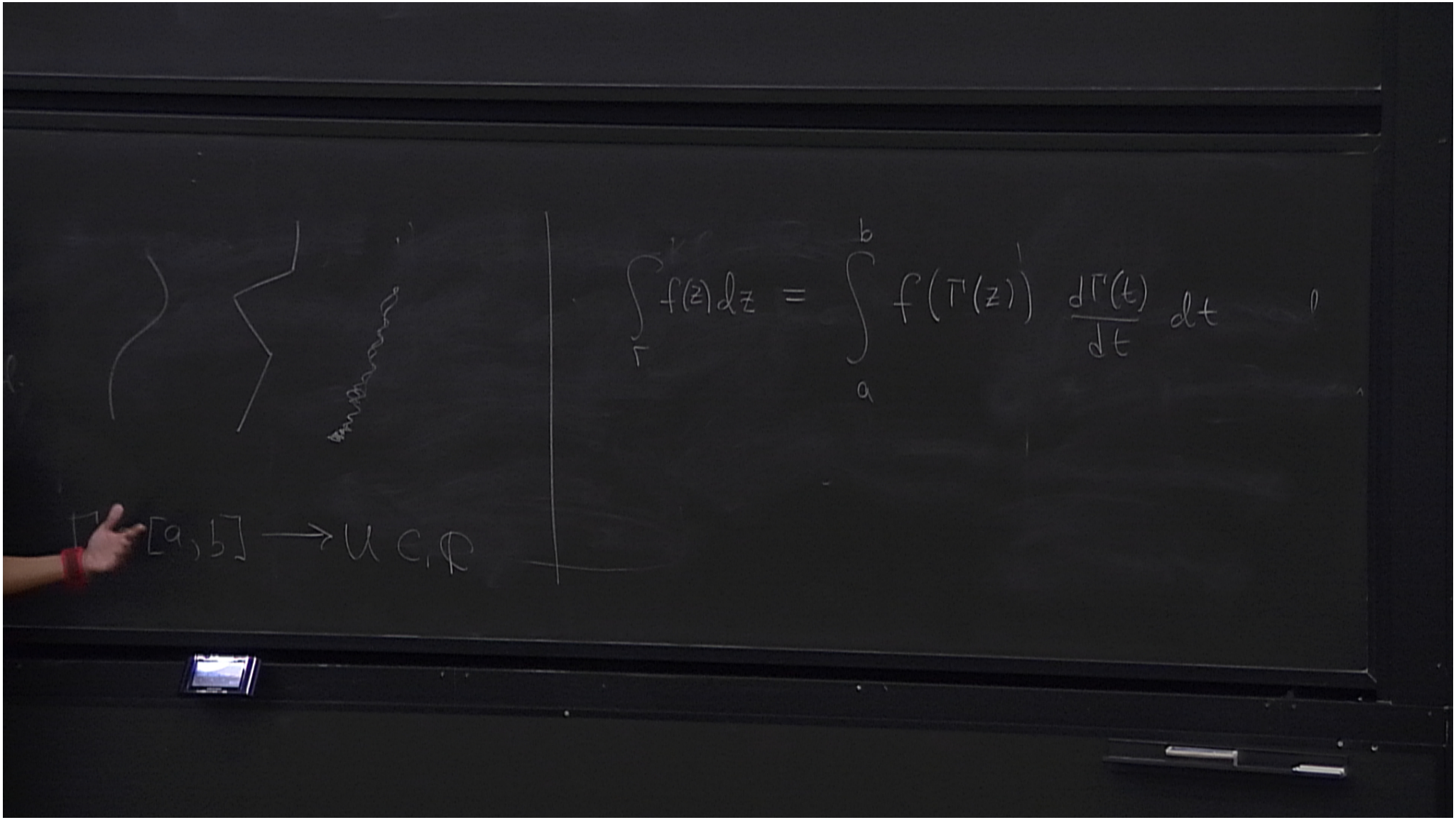
$\int_\Gamma f(z)\,dz$

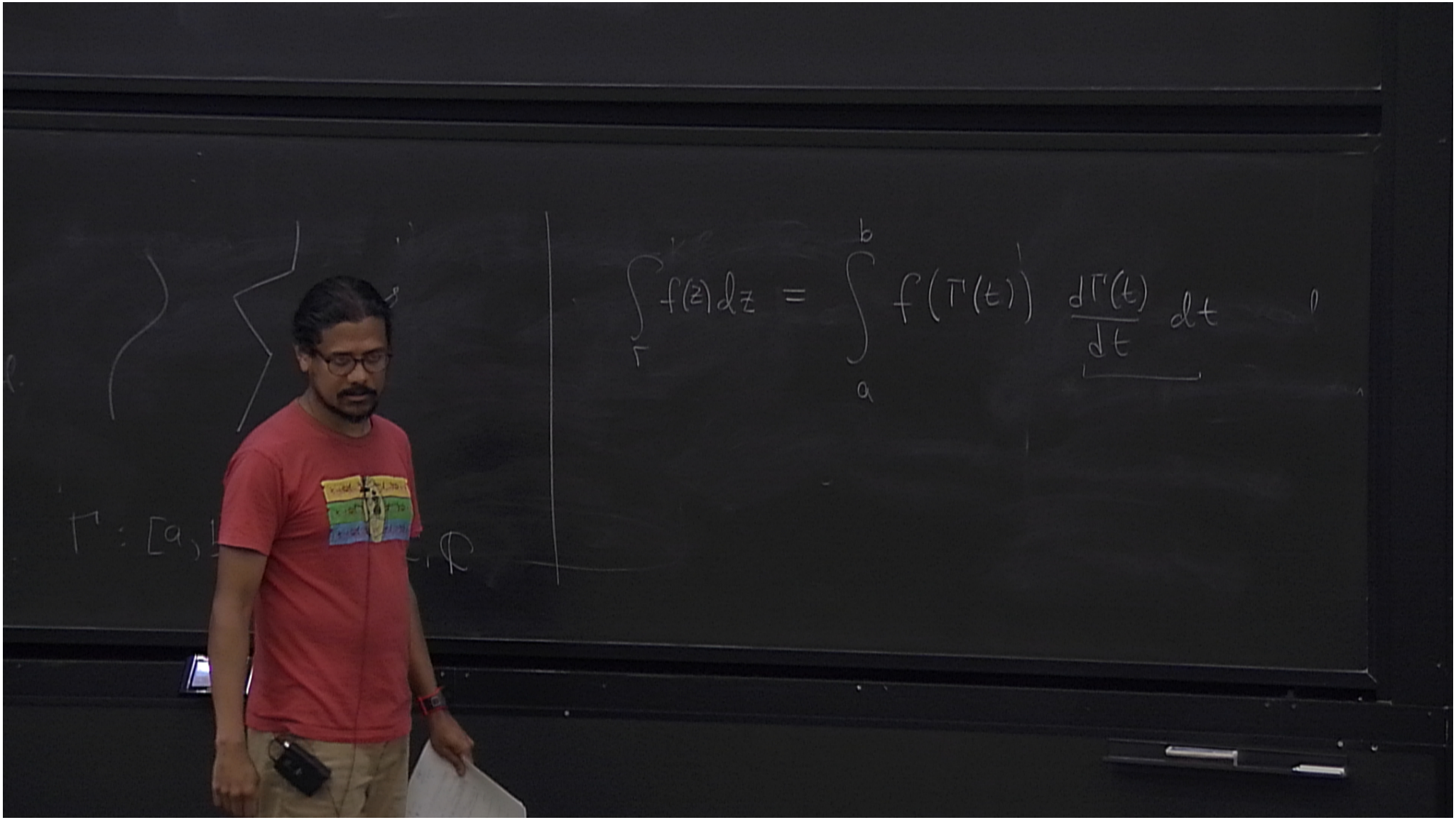
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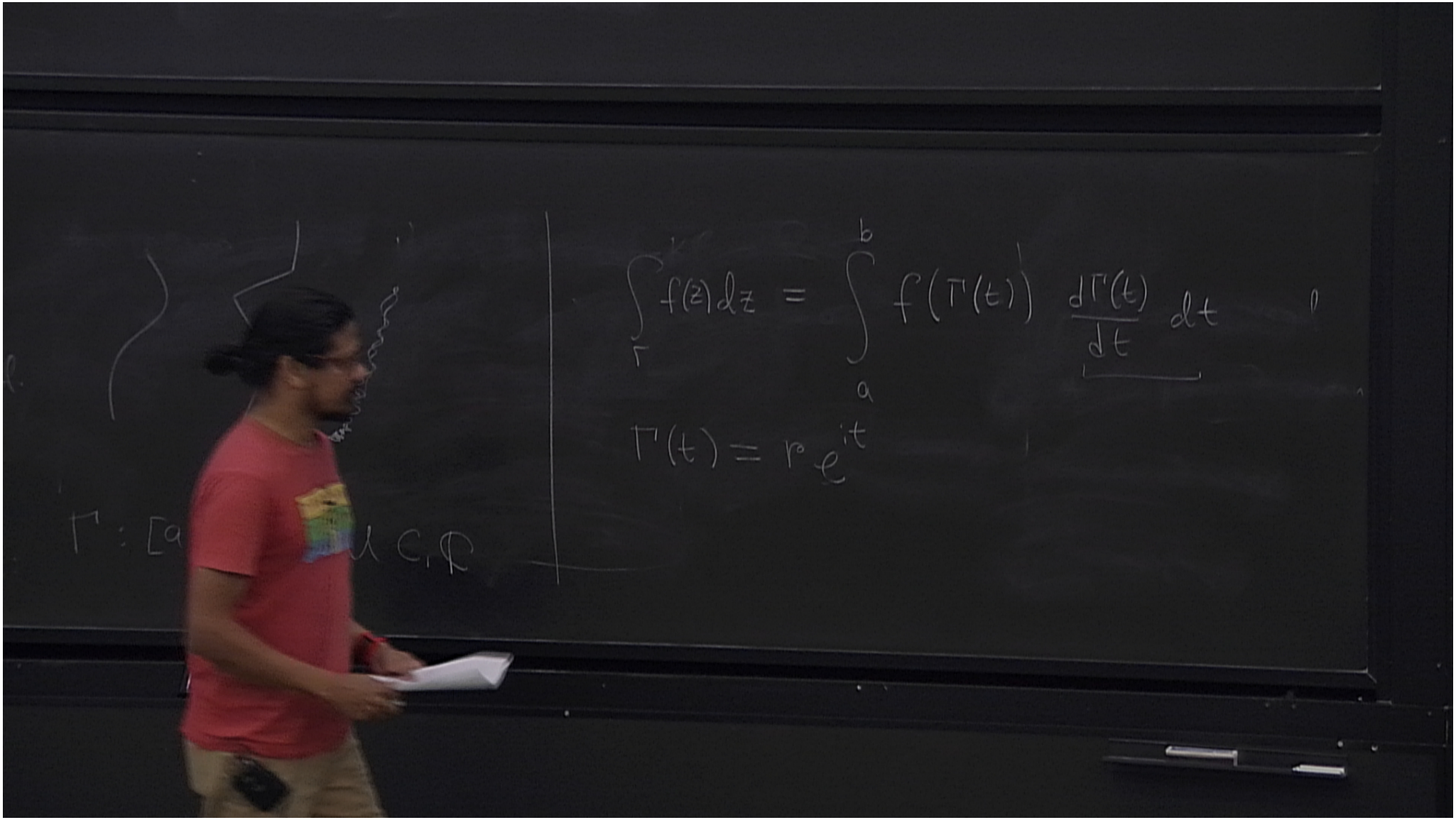
Simple or Jordan curves

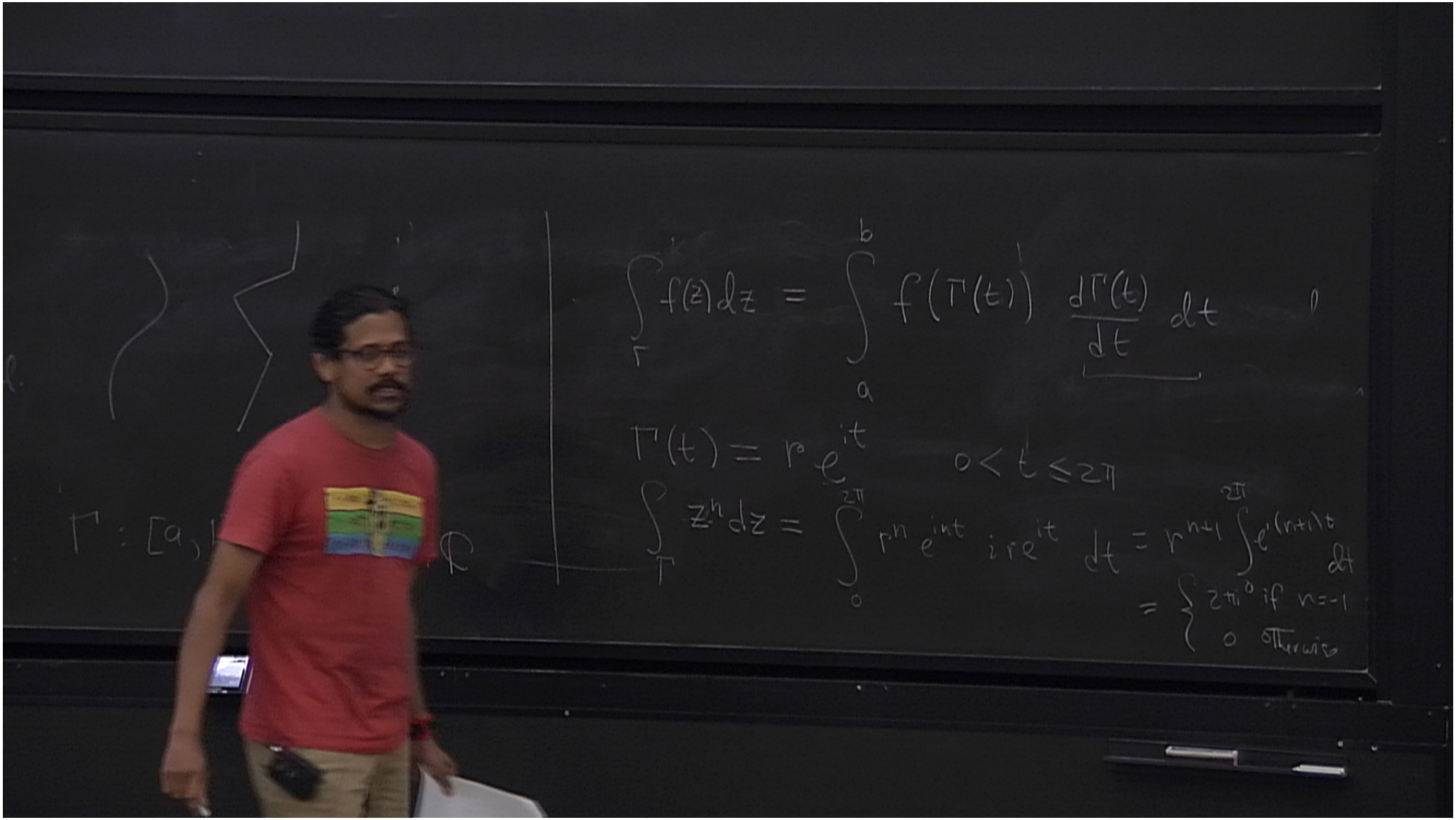
Contour is a piecewise smooth, simple curve $\Gamma : [a,b] \to U \subset \mathbb{C}$




$\int_\Gamma f(z)\,dz = \int_a^b f(\Gamma(z))\,\frac{d\Gamma(t)}{dt}\,dt$
$\Gamma : [a,b] \to U \subset \mathbb{R}$


$\int_\Gamma f(z)\,dz = \int_a^b f(\Gamma(t))\,\frac{d\Gamma(t)}{dt}\,dt$


$\int_\Gamma f(z)\,dz = \int_a^b f(\Gamma(t))\,\frac{d\Gamma(t)}{dt}\,dt$
$\Gamma(t) = r e^{it}$


$\int_\Gamma f(z)\,dz = \int_a^b f(\Gamma(t))\,\frac{d\Gamma(t)}{dt}\,dt$
$\Gamma(t) = re^{it} \quad 0 < t \le 2\pi$
$\int_\Gamma z^n dz = \int_0^{2\pi} r^n e^{int}\, ire^{it}\, dt = r^{n+1} \int_0^{2\pi} e^{i(n+1)t}\, dt$
$= \begin{cases} 2\pi i & \text{if } n=-1 \\ 0 & \text{otherwise} \end{cases}$