Title: Bootstrapping 3D CFTs

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Abstract: I will review recent results from applying the conformal bootstrap to 3D CFTs, including precise determinations of critical exponents and in the 3D Ising and O(N) vector models, new constraints on 3D Gross-Neveu models, and general bounds on correlation function coefficients of currents and stress tensors.

Bootstrapping 3D CFTs

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August 26, 2016

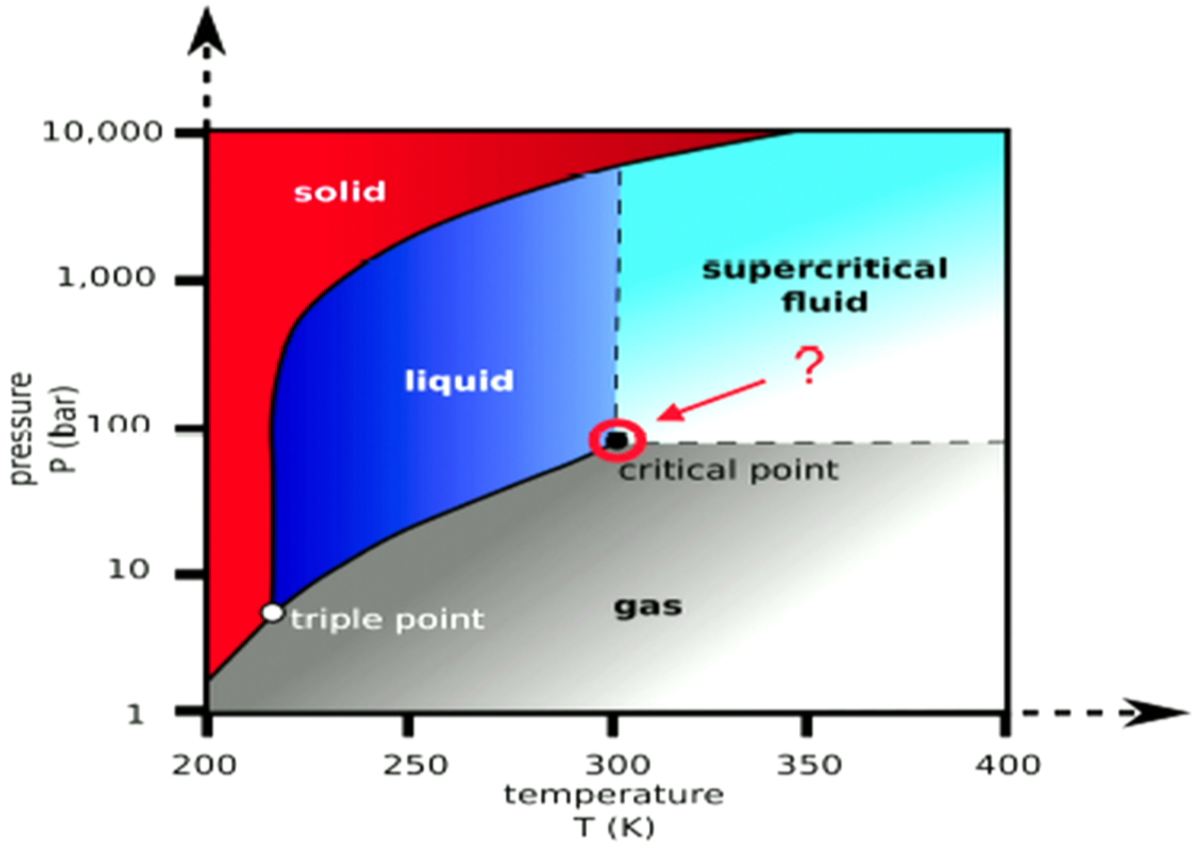
Low Energy Challenges for High Energy Physicists II

Why Study CFTs?

There are many interesting applications of conformal field theories:

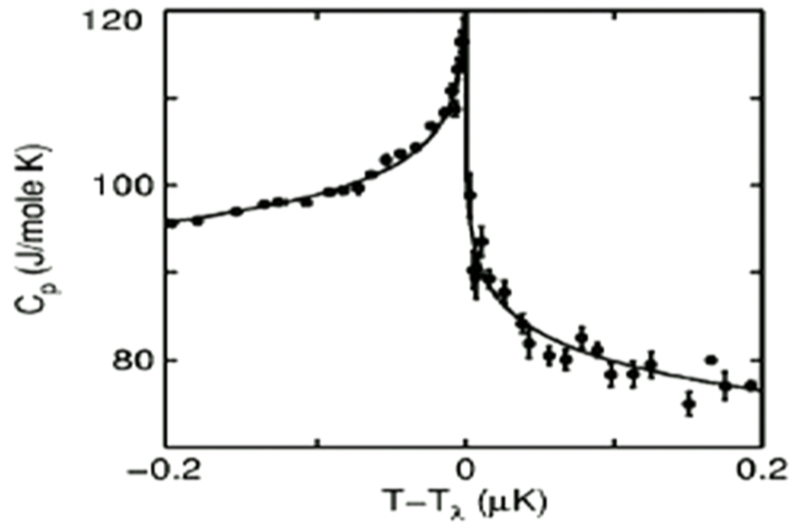
- 2D: String Theory
- 2D/3D: Statistical and Condensed Matter Systems
- 4D: Scenarios for Physics Beyond the Standard Model
- 6D: Mysterious $(2,0)$ Theory
- Holography and AdS/CFT: Study Quantum Gravity with CFTs

Phase Diagram for Fluids



3D Ising CFT $\leftrightarrow \mathcal{L}_{\text{Ising}} \sim \lambda_c \phi^4$

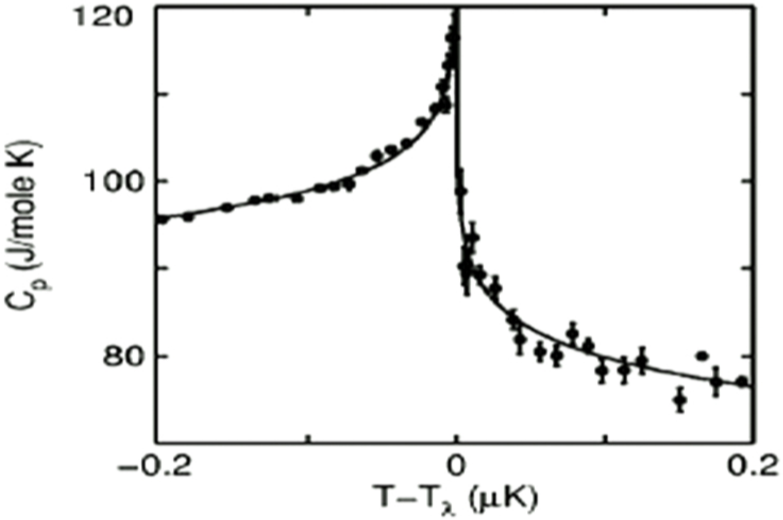
3D O(N) Models



Going to N scalars is also interesting! $\rightarrow \mathcal{L}_{O(N)} \sim \lambda_c(\phi_i\phi^i)^2$

- $N = 2$: Superfluid (λ) transition in ^{4}He [Lipa et al, '96; '03]
  Quantum critical point in (2+1)D superconductors

3D O(N) Models



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  Quantum critical point in (2+1)D superconductors
- $N=3$: Isotropic ferromagnets (Fe, Co, Ni, ...)
- Large N: Solvable in $1/N$ expansion

Main Goal

We would like to map out the space of CFTs and predict their observables

Conformal Bootstrap

- The conformal bootstrap aims to use mathematical consistency conditions to map out and solve the space of CFTs
  - Conformal Symmetry
  - Crossing Symmetry
  - Unitarity / Reflection Positivity

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- Beautiful success story in 2D
  [Ferrara, Gatto, Grillo '73; Polyakov '74; Belavin, Polyakov, Zamolodchikov '83]
- Exciting progress in $D > 2$ starting in 2008
  [Rattazzi, Rychkov, Tonni, Vichi '08; ...]

Conformal Block Expansion

Can probe spectrum by expanding 4-point functions in conformal blocks:

$$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\sigma} x_{34}^{2\Delta_\sigma}} \sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2\, g_{\Delta,\ell}(u,v)$$

- Blocks $g_{\Delta,\ell}(u,v)$ are known functions of $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$
- Similar to expansion in spherical harmonics Y_ℓ^m, but for CFTs

Scalar Conformal Blocks

Explicit formulas in even D [Dolan, Osborn '00; '03]:

$$
\begin{aligned}
g^{2D}_{\Delta,\ell}(u,v) &= k_{\Delta+\ell}(z)k_{\Delta-\ell}(\bar{z}) + z \leftrightarrow \bar{z} \\
g^{4D}_{\Delta,\ell}(u,v) &= \frac{z\bar{z}}{z-\bar{z}}[k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\bar{z}) - z \leftrightarrow \bar{z}] \\
k_\beta(x) &= x^{\beta/2}{}_2F_1(\beta/2,\beta/2,\beta;x)
\end{aligned}
$$

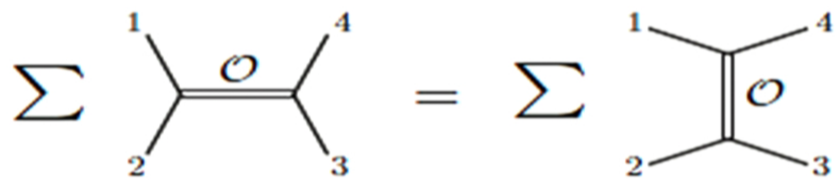
where $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$

- Conformal blocks are eigenfunctions of $SO(D+1,1)$ Casimir
- Outside of even D, can be computed recursively to arbitrary precision [El-Showk, Paulos, DP, Rychkov, Simmons-Duffin, Vichi; Kos, DP, Simmons-Duffin '13; '14]

Crossing Symmetry

$\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle$ is symmetric under permutations of x_i:

- Switching $x_1 \leftrightarrow x_3$ gives the crossing symmetry condition:



$$u^{-\Delta_\sigma} \sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(u,v) \quad = \quad v^{-\Delta_\sigma} \sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2 g_{\Delta,\ell}(v,u)$$

- Only unknowns are set of scaling dimensions and coefficents: $\{\Delta, \lambda_{\mathcal{O}}\}$

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Numerical Approach

- By applying clever linear functionals α one can prove that some assumptions on $\{\Delta, \lambda_{\mathcal{O}}\}$ are incompatible with crossing + unitarity:

$$0 = \sum_{\Delta,\ell} \lambda_{\mathcal{O}}^2 \alpha[u^{-\Delta_\sigma} g_{\Delta,\ell}(u,v) - v^{-\Delta_\sigma} g_{\Delta,\ell}(v,u)] > 0$$

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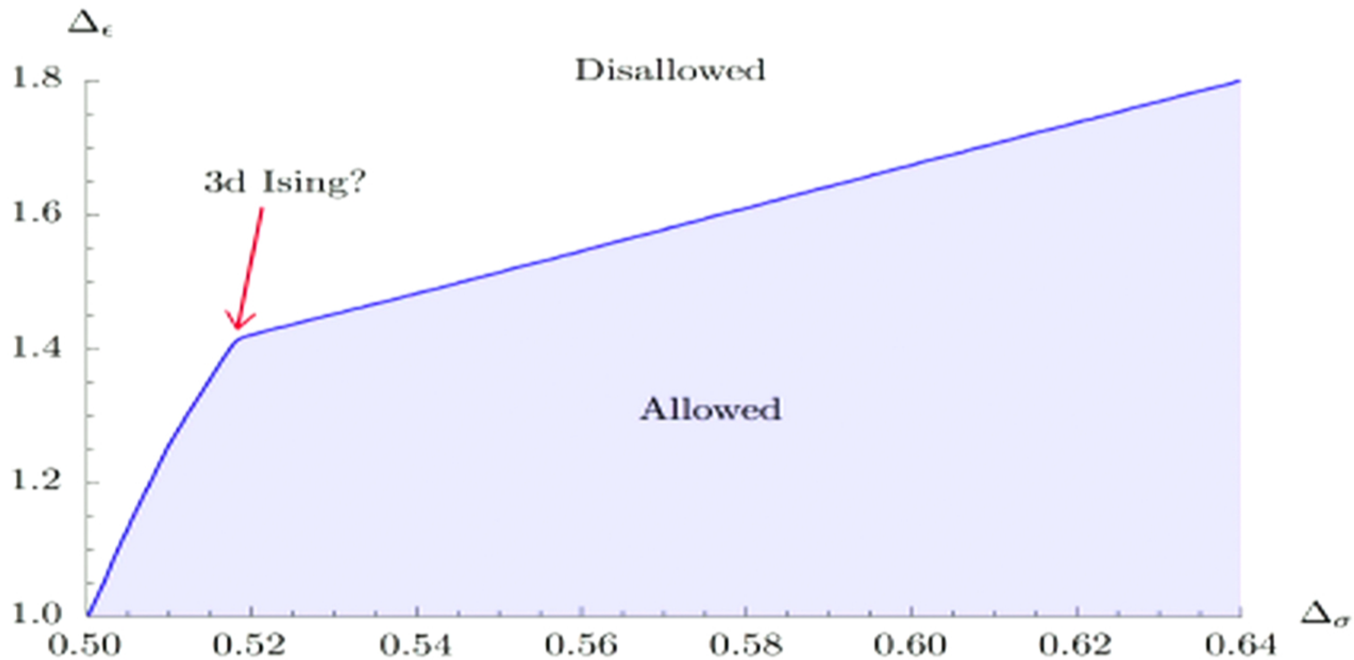
- Find $\alpha \sim \sum_{mn} a_{mn} \partial_z^m \partial_{\bar{z}}^n \big|_{1/2,1/2}$ using linear/semidefinite programming [Rattazzi, Rychkov, Tonni, Vichi '08; DP, Simmons-Duffin, Vichi '11]
  - Functional search space ranges from ~ 20 to ~ 1200 components
  - Each plot $\leftrightarrow$ Solve $\mathcal{O}(1000)$ optimization problems on HPC clusters
  - State of the art: SDPB [Simmons-Duffin '15]

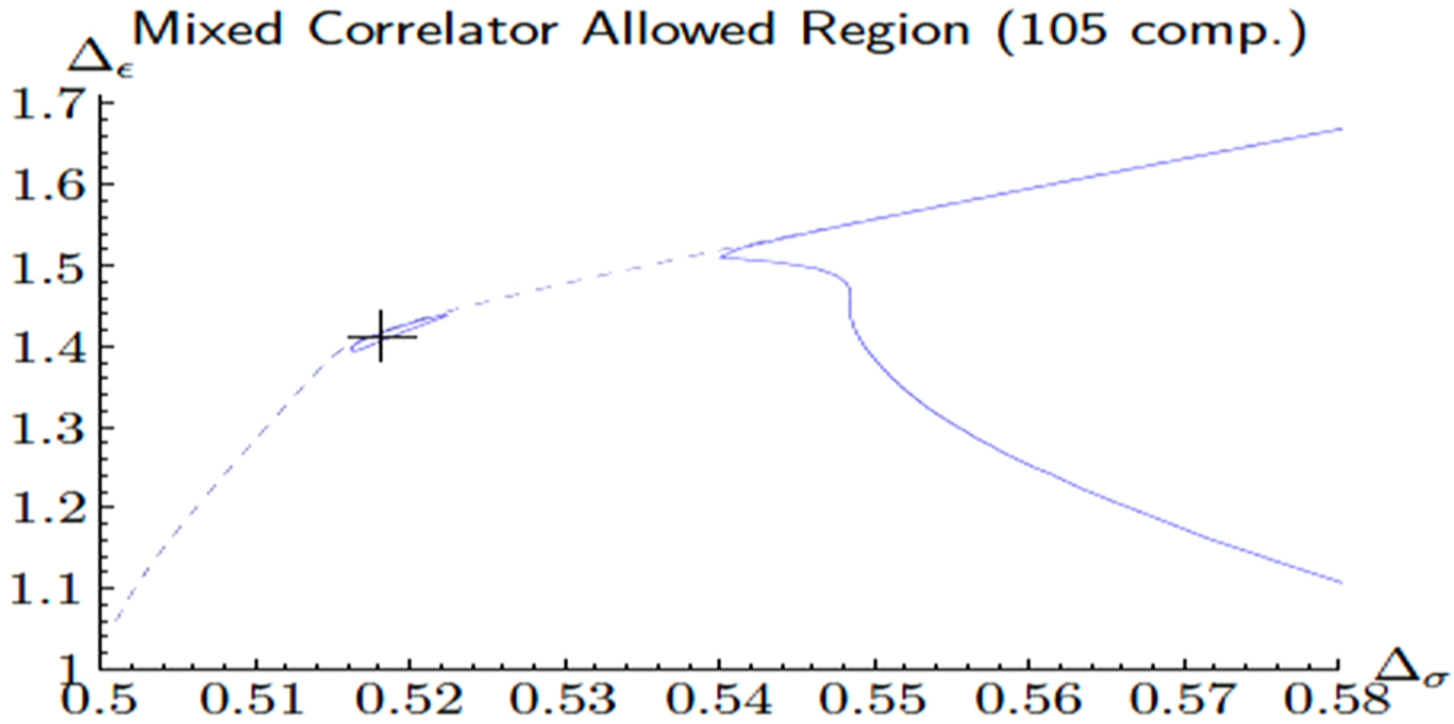
3D Dimension Bounds



[El-Showk, Paulos, DP, Rychkov, Simmons-Duffin, Vichi, '12; '14]

- Bound on leading scalar in $\sigma \times \sigma \sim \mathbb{1} + \epsilon + \ldots$
- 3D Ising (Lattice): $\Delta_\sigma \simeq 0.51813(5)$, $\Delta_\epsilon \simeq 1.41275(25)$ [Hasenbusch '10]
  (Leading critical exponents: $\eta = 2\Delta_\sigma - 1$, $\nu = \frac{1}{3-\Delta_\epsilon}$)

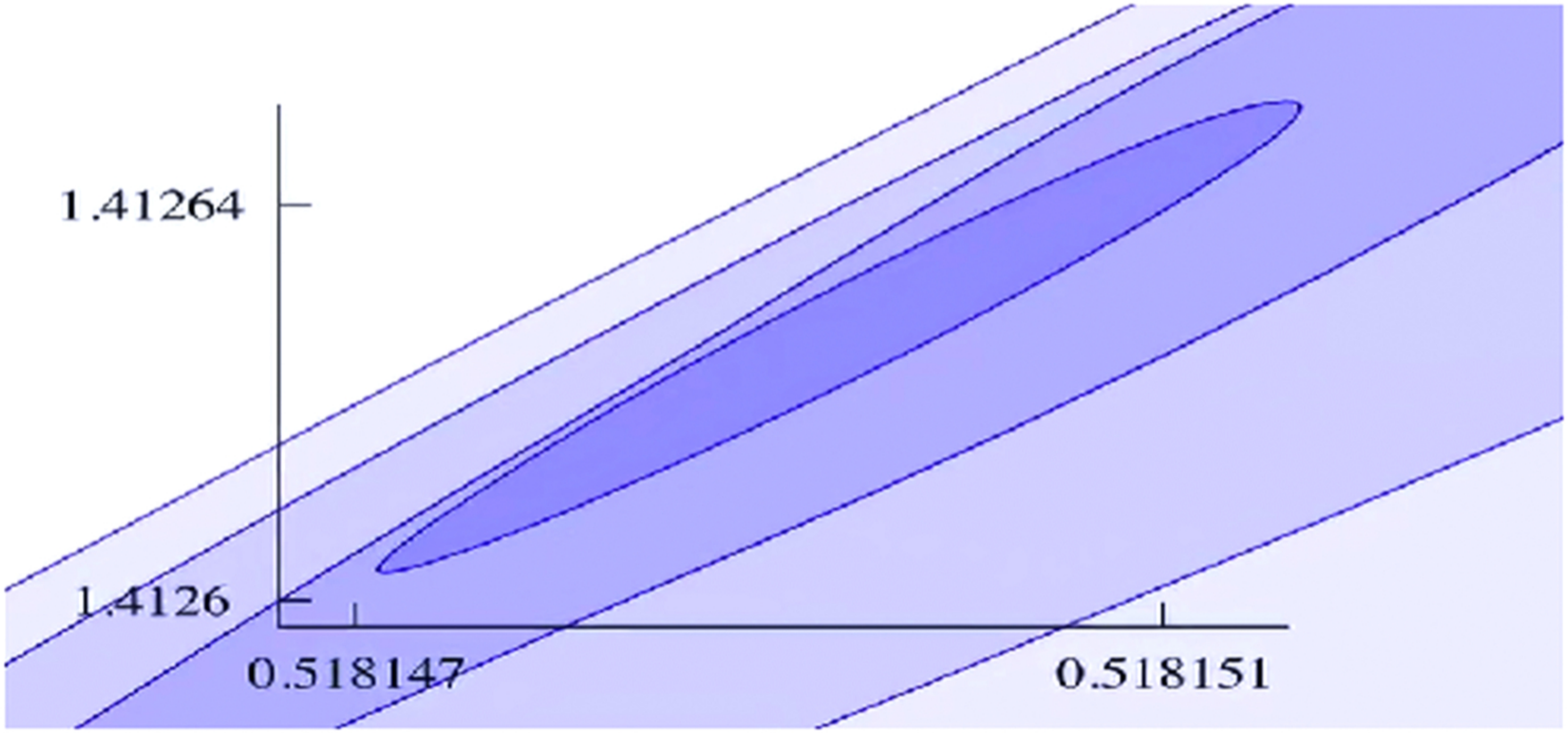
Mixed Correlator Islands



[Kos, DP, Simmons-Duffin '14]

- Combining constraints from $\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle$, can impose that σ and ϵ are only relevant ($\Delta < 3$) operators, yielding a rigorous island!

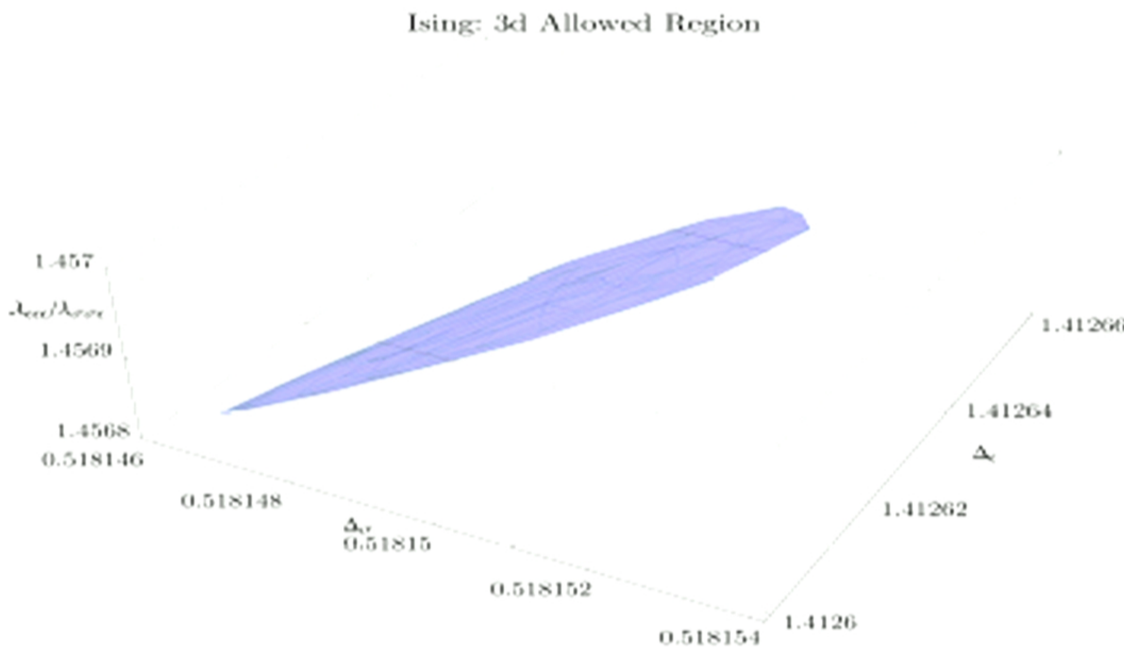
Mixed Correlator Islands



[Kos, DP, Simmons-Duffin '14; Simmons-Duffin '15; Kos, DP, Simmons-Duffin, Vichi '16]

- Increasing to 1265 components using SDPB, region keeps shrinking!

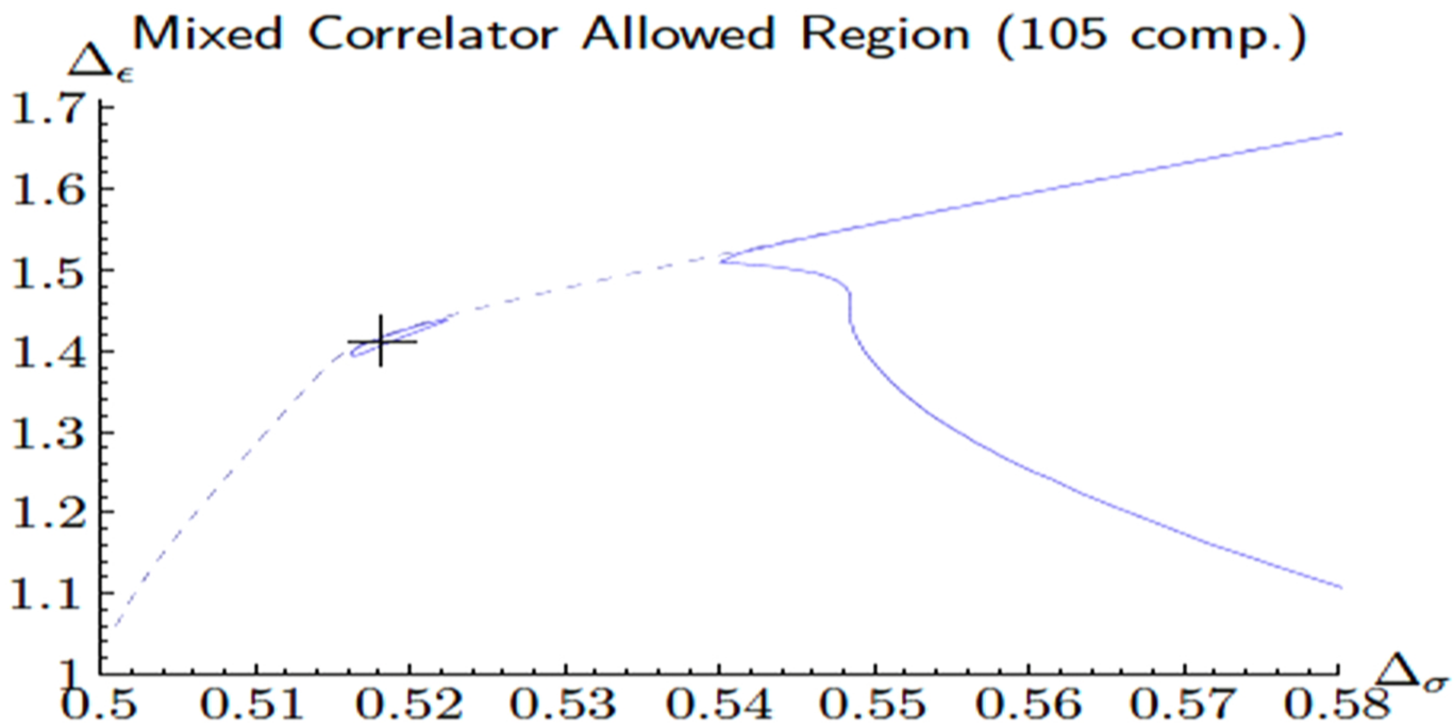
Mixed Correlator Islands



[Kos, DP, Simmons-Duffin, Vichi '16]

- Best bounds: first map out a 3d Island in $\{\Delta_\sigma, \Delta_\epsilon, \lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon}\}$
- Since the functional can be different for each choice of $\lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon}$, the $(\Delta_\sigma, \Delta_\epsilon)$ projection is smaller than having no assumption on $\lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon}$

Mixed Correlator Islands



[Kos, DP, Simmons-Duffin '14]

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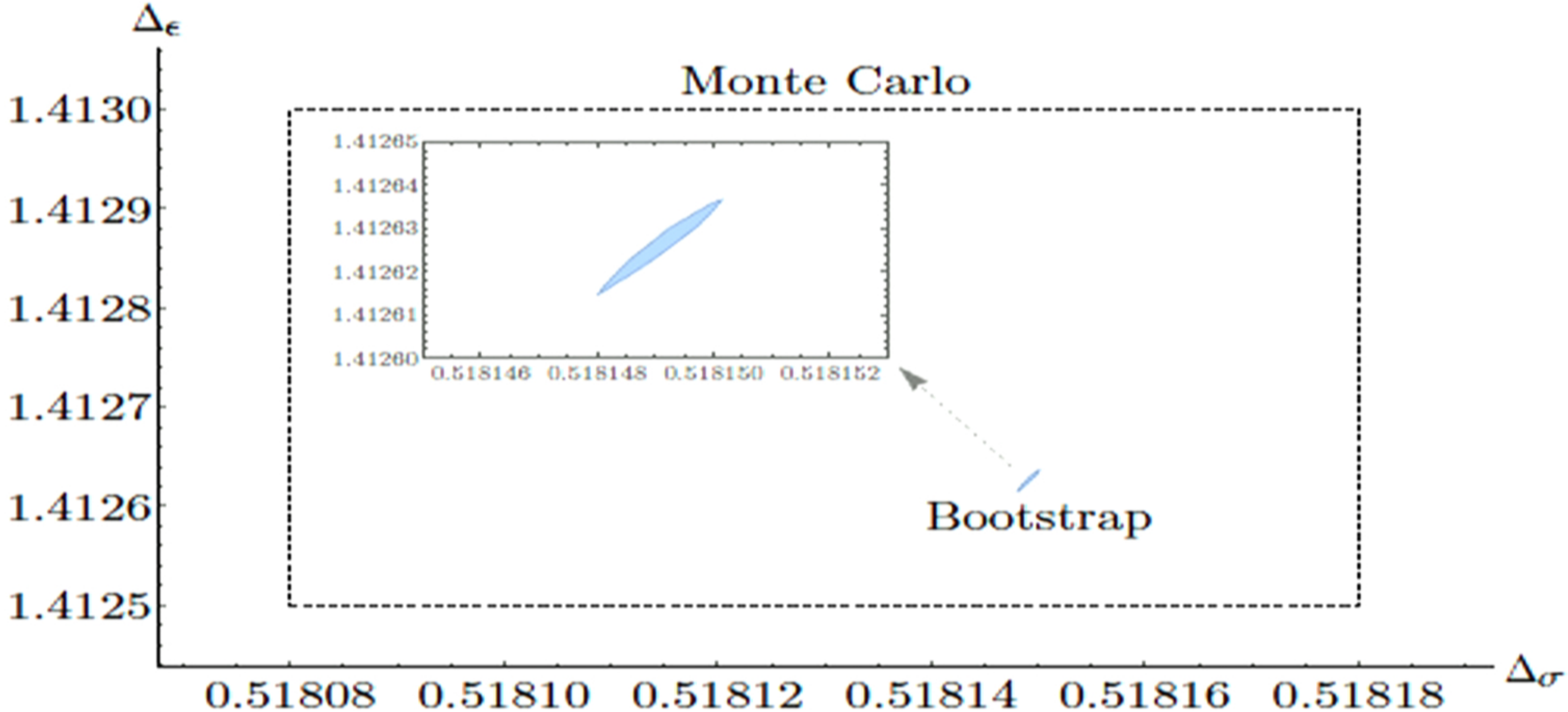
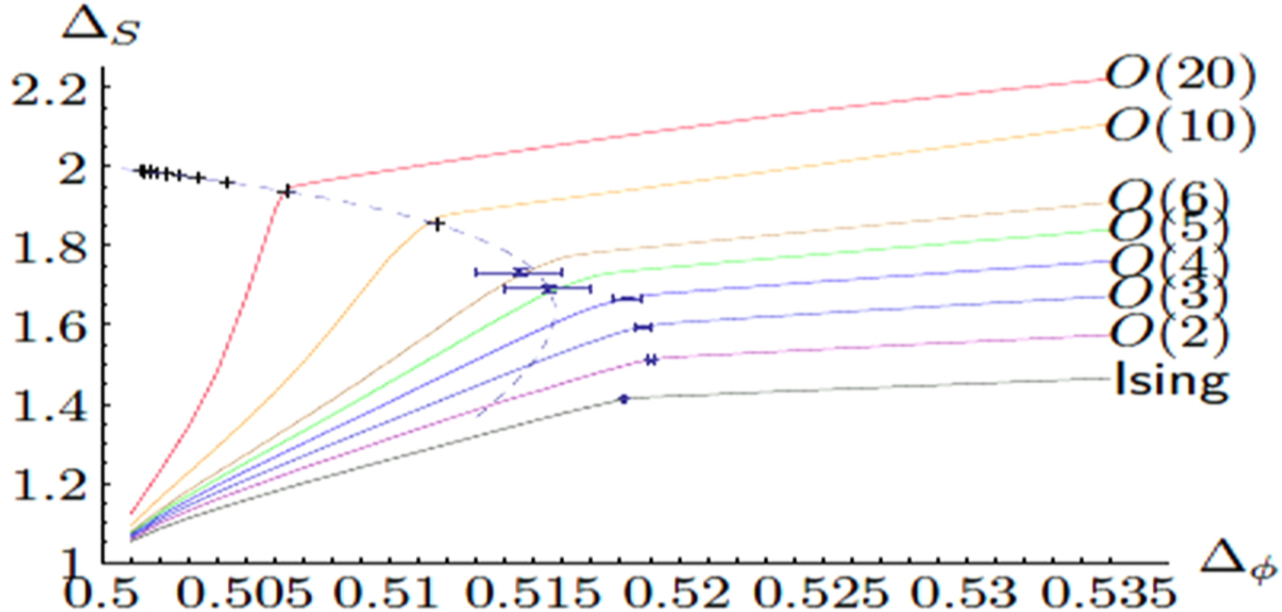


[Kos, DP, Simmons-Duffin, Vichi '16]

$$\{\Delta_\sigma, \Delta_\epsilon\} = \{0.518149(1), 1.412625(10)\}$$
$$\{\lambda_{\sigma\sigma\epsilon}, \lambda_{\epsilon\epsilon\epsilon}\} = \{1.0518537(41), 1.532435(19)\}$$

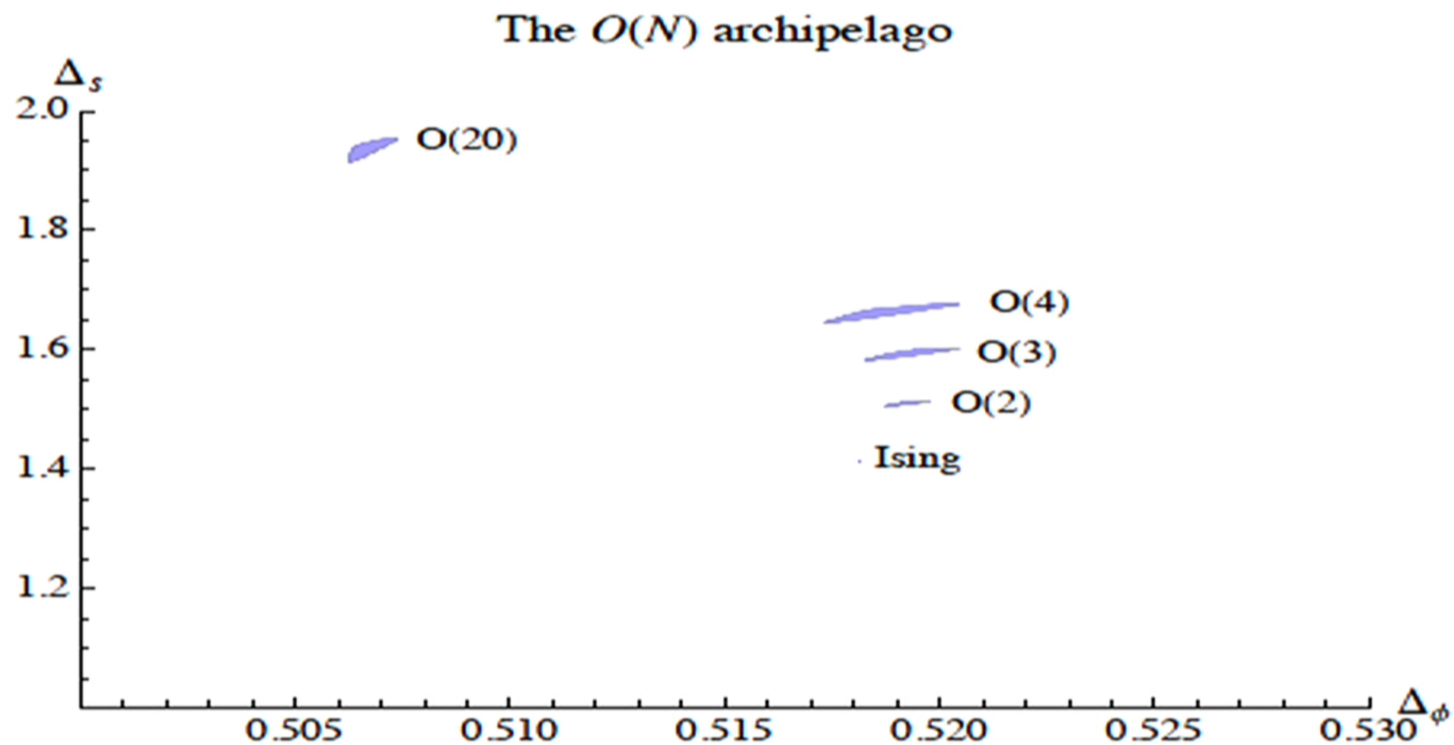
3D $O(N)$ Bounds



[Kos, DP, Simmons-Duffin '13]

- Extension to $\langle \phi_i \phi_j \phi_k \phi_l \rangle$, where ϕ_i is $O(N)$ vector
- Large N: matches $1/N$ expansion, Small N: matches experiment!

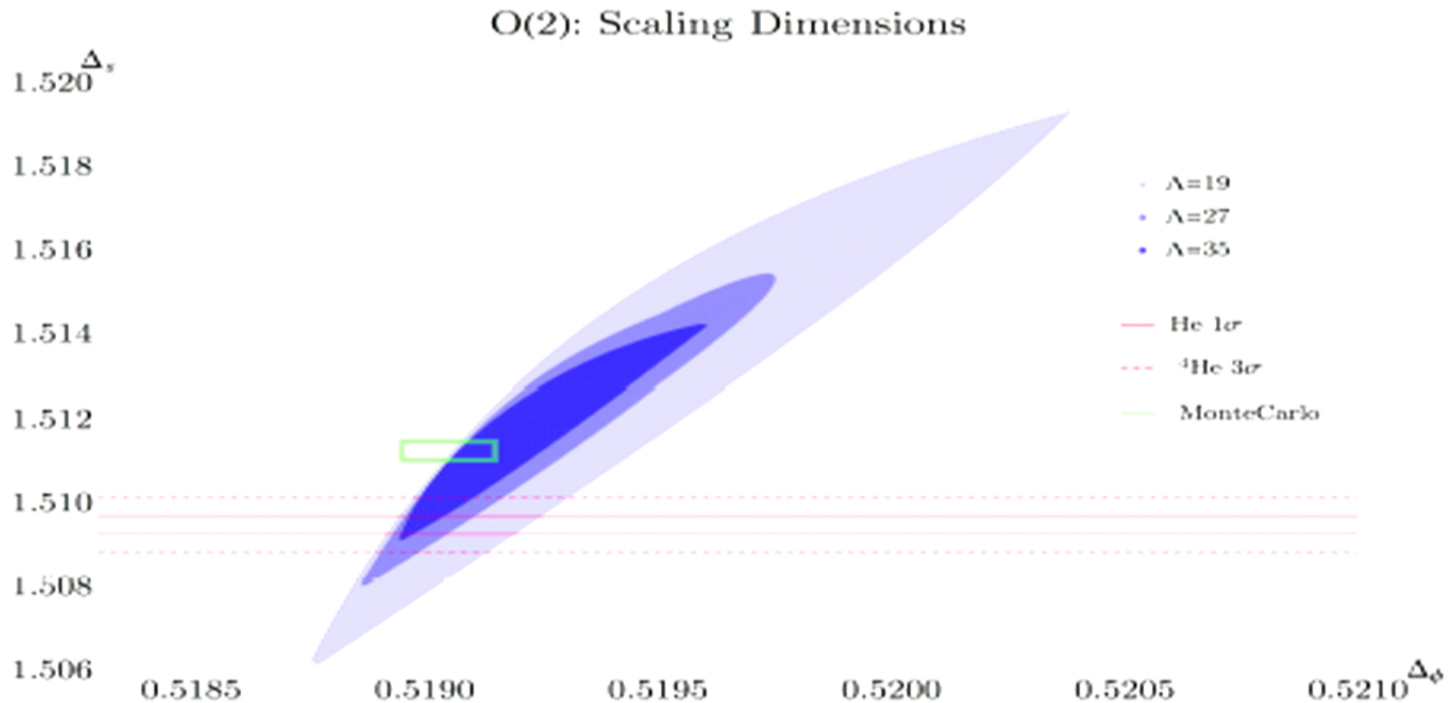
$O(N)$ Archipelago from Mixed Correlators



[Kos, DP, Simmons-Duffin, Vichi '15; '16]

- Mixed system assuming only one relevant $O(N)$ vector ϕ_i and singlet s

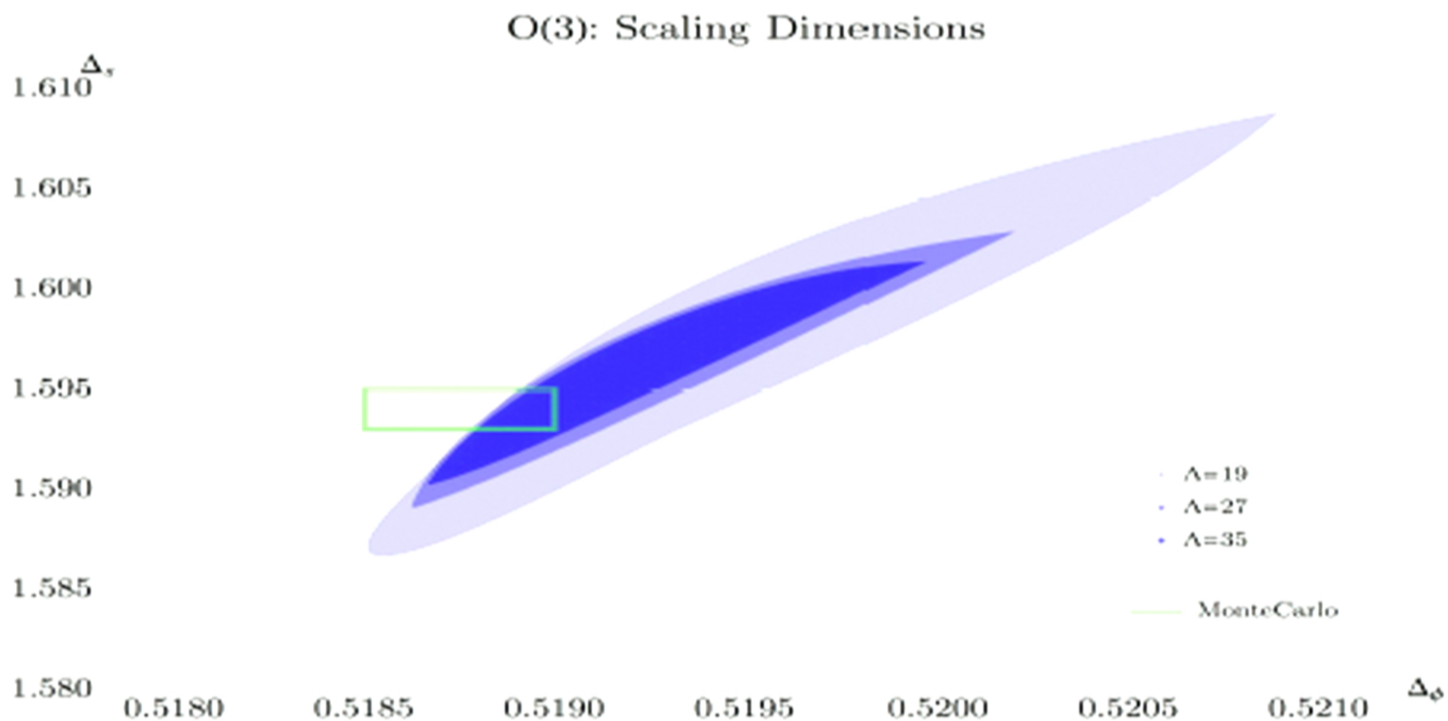
$O(2)$ Zoom



[Kos, DP, Simmons-Duffin, Vichi '15; '16]

- $\{\Delta_\phi, \Delta_s, \lambda_{\phi\phi s}, \lambda_{sss}\} = \{.51926(32), 1.5117(25), .68726(65), .8286(60)\}$
- Close to resolving 8σ discrepancy between lattice and expt

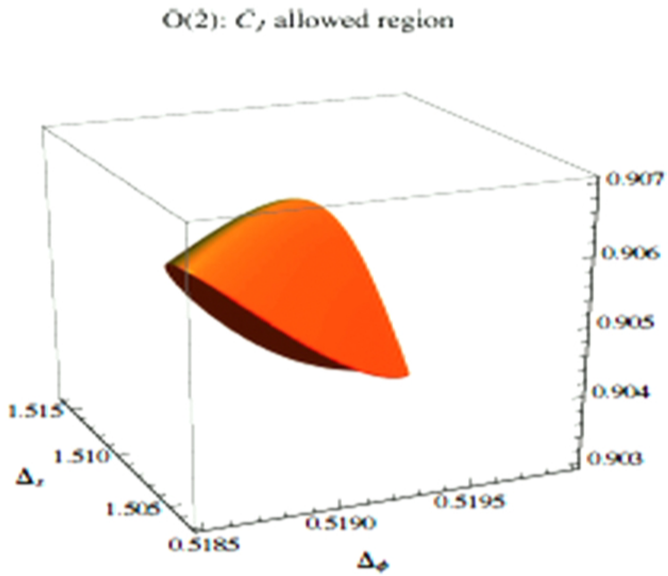
$O(3)$ Zoom

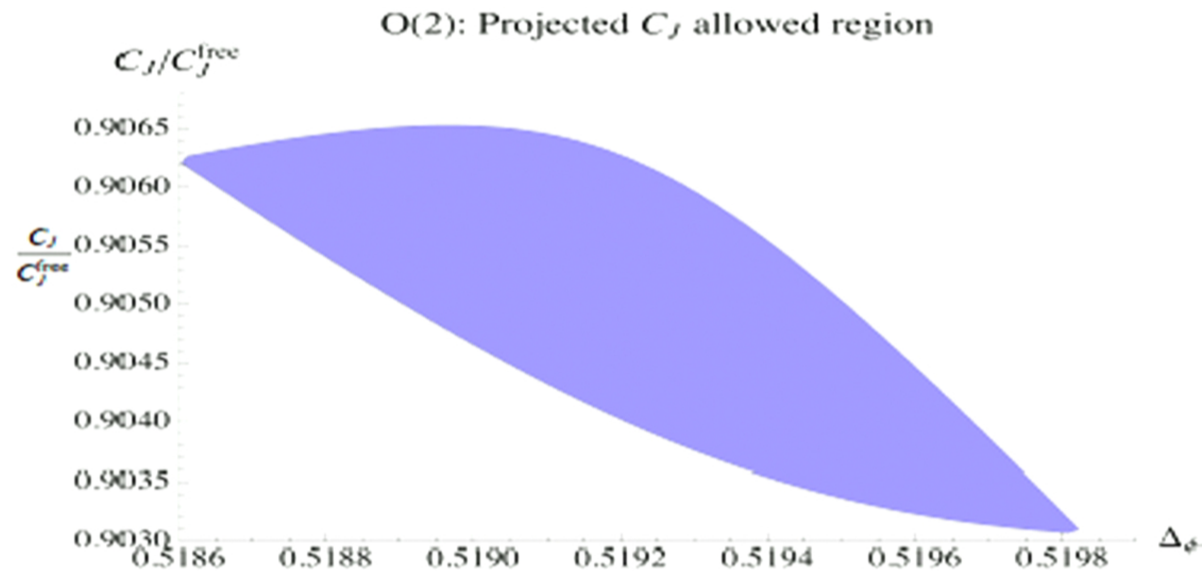


[Kos, DP, Simmons-Duffin, Vichi '15; '16]

- $\{\Delta_\phi, \Delta_s, \lambda_{\phi\phi s}, \lambda_{sss}\} = \{.51928(62), 1.5957(55), .5244(11), .499(12)\}$

$O(2)$ Conductivity

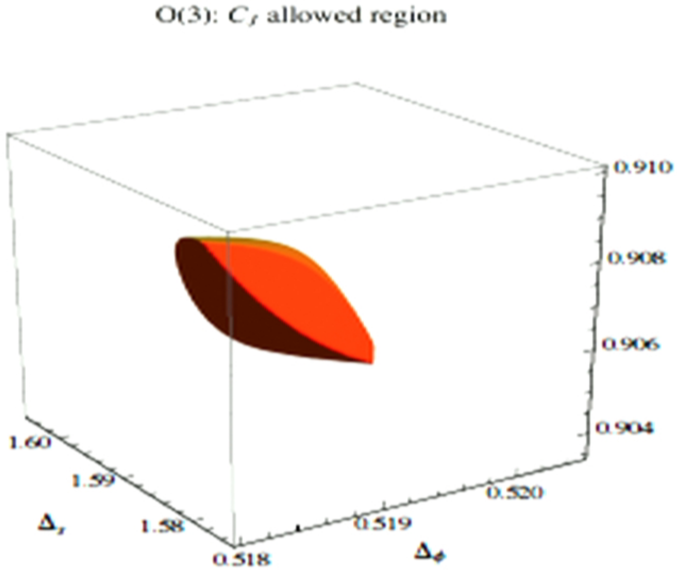


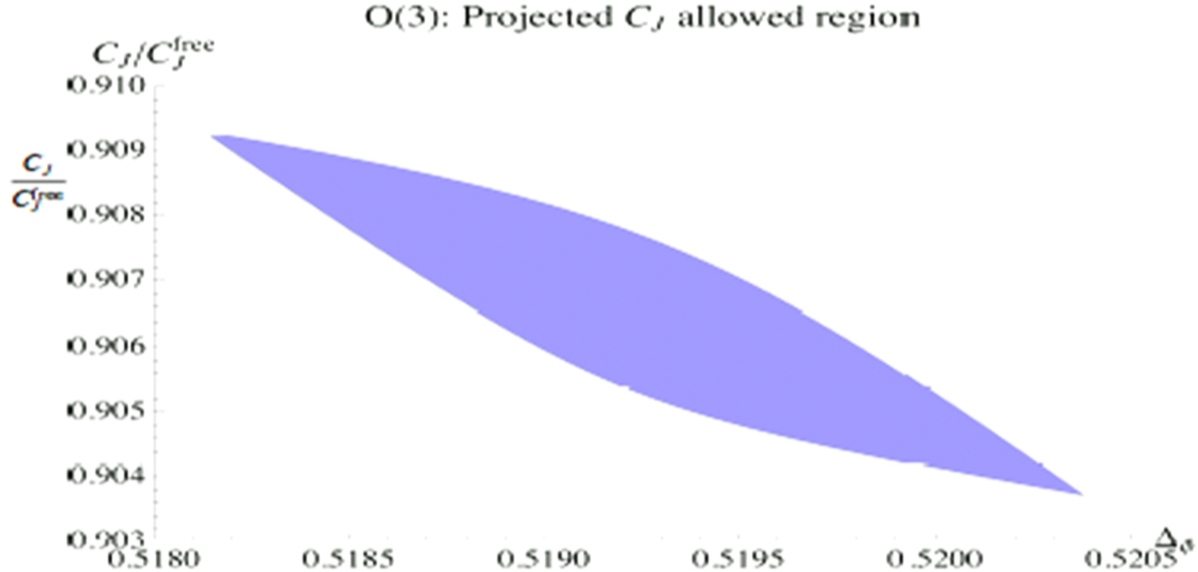


[Kos, DP, Simmons-Duffin, Vichi '15]

- Rigorous determination of $\langle JJ \rangle \propto C_J \propto \sigma_\infty$, giving high-frequency conductivity in $(2+1)$D superconductors: $2\pi\sigma_\infty = 0.3554(6)$

$O(3)$ Conductivity





[Kos, DP, Simmons-Duffin, Vichi '15]

- Similar determinations at large N, e.g. at $N = 20$:

$$C_J/C_J^{free}|_{\text{bootstrap}} = 0.9674(8),$$

$$C_J/C_J^{free}|_{\text{Large N}} \approx 1 - \frac{32}{9\pi^2}\frac{1}{N} = 0.964$$

Fermion Bootstrap

- Generalize to 4-point functions $\langle\psi\psi\psi\psi\rangle$ of a Majorana fermion in 3D ($SO(2,1) \simeq Sp(2,\mathbb{R}) \rightarrow$ real two-component spinors)
- We will also assume a parity symmetry: $(x, y) \rightarrow (-x, y)$
- To classify 3-point and 4-point structures, we can work in an embedding space, where $SO(3,2) \simeq Sp(4,\mathbb{R})$ is linearly realized

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Results:

- $\langle\psi\psi\mathcal{O}^{(\ell \text{ even})}\rangle$ has two structures of even parity and one of odd parity
- $\langle\psi\psi\mathcal{O}^{(\ell \text{ odd})}\rangle$ has one structure of odd parity
- $\langle\psi\psi\psi\psi\rangle$ has 5 independent tensor structures

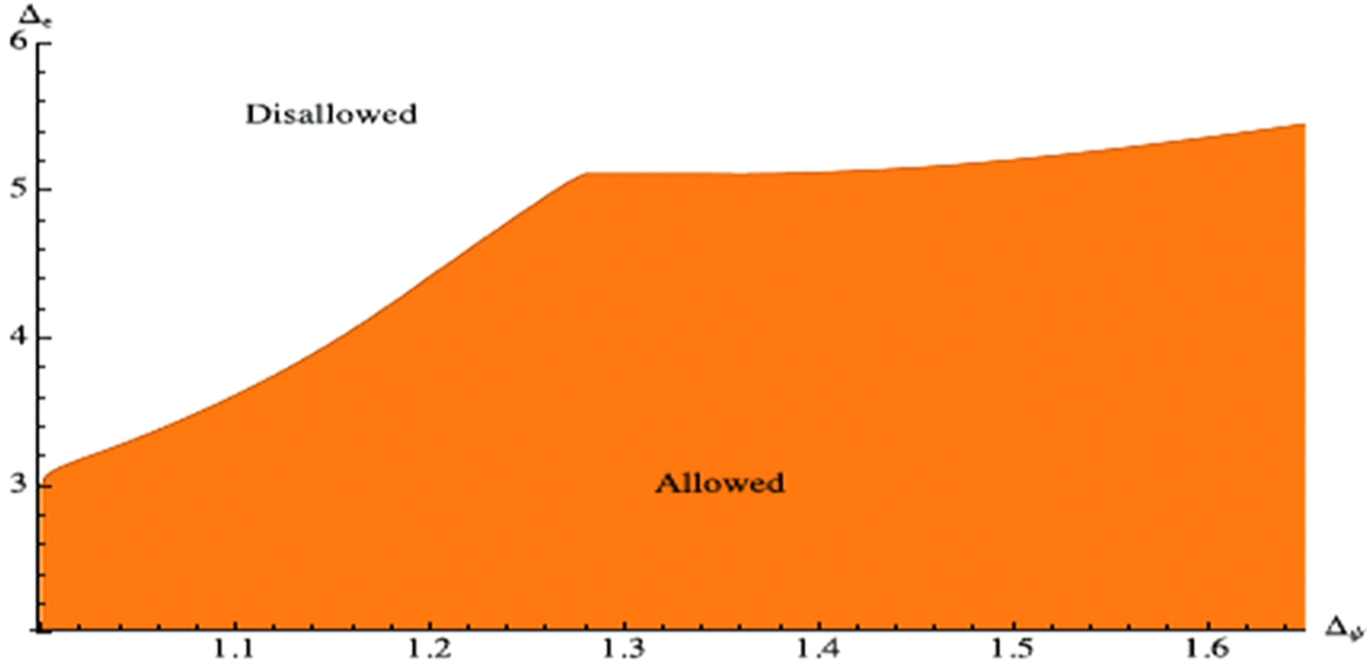
Fermion Bootstrap

- Crossing symmetry leads to a 5-vector of sum rules:

$$0 = \sum_{\mathcal{O}_+, \ell_+} \begin{pmatrix} \lambda^1_{\mathcal{O}_+} & \lambda^2_{\mathcal{O}_+} \end{pmatrix} \vec{F}^{2\times 2}_{++,\Delta,\ell}(u,v) \begin{pmatrix} \lambda^1_{\mathcal{O}_+} \\ \lambda^2_{\mathcal{O}_+} \end{pmatrix} + \sum_{\mathcal{O}_-, \ell_+} (\lambda^3_{\mathcal{O}_-})^2 \vec{F}_{-+,\Delta,\ell}(u,v) + \sum_{\mathcal{O}_-, \ell_-} (\lambda^4_{\mathcal{O}_-})^2 \vec{F}_{--,\Delta,\ell}(u,v),$$

- To calculate the conformal blocks, we express $\langle\psi\psi\mathcal{O}\rangle_a = D_a\langle\phi\phi\mathcal{O}\rangle$, which lets us relate $\int\langle\psi\psi\mathcal{O}\rangle_a\langle\tilde{\mathcal{O}}\psi\psi\rangle_b$ to $\int\langle\phi\phi\mathcal{O}\rangle\langle\tilde{\mathcal{O}}\phi\phi\rangle$
- Bounds follow from applying functionals $\vec{\alpha}$ (SDP is manditory)

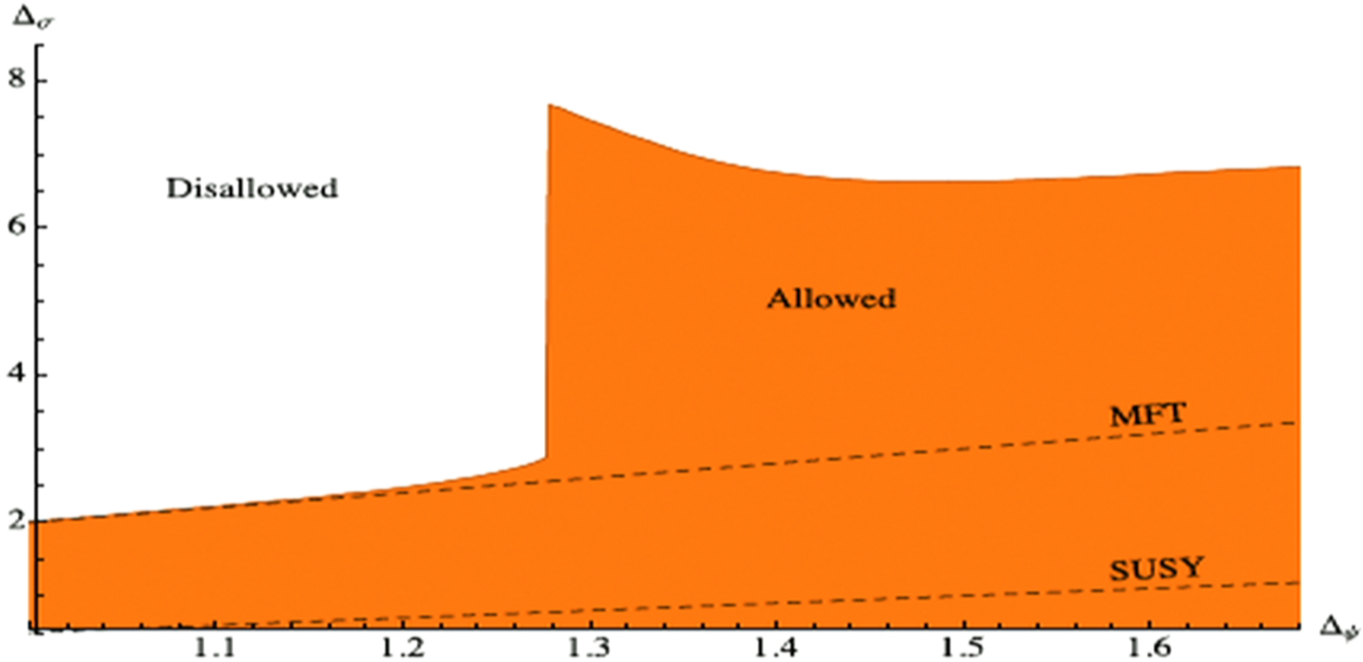
3D Fermion Bootstrap



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby '15]

- Bound on leading parity-even scalar in $\psi \times \psi \sim \epsilon + \ldots$

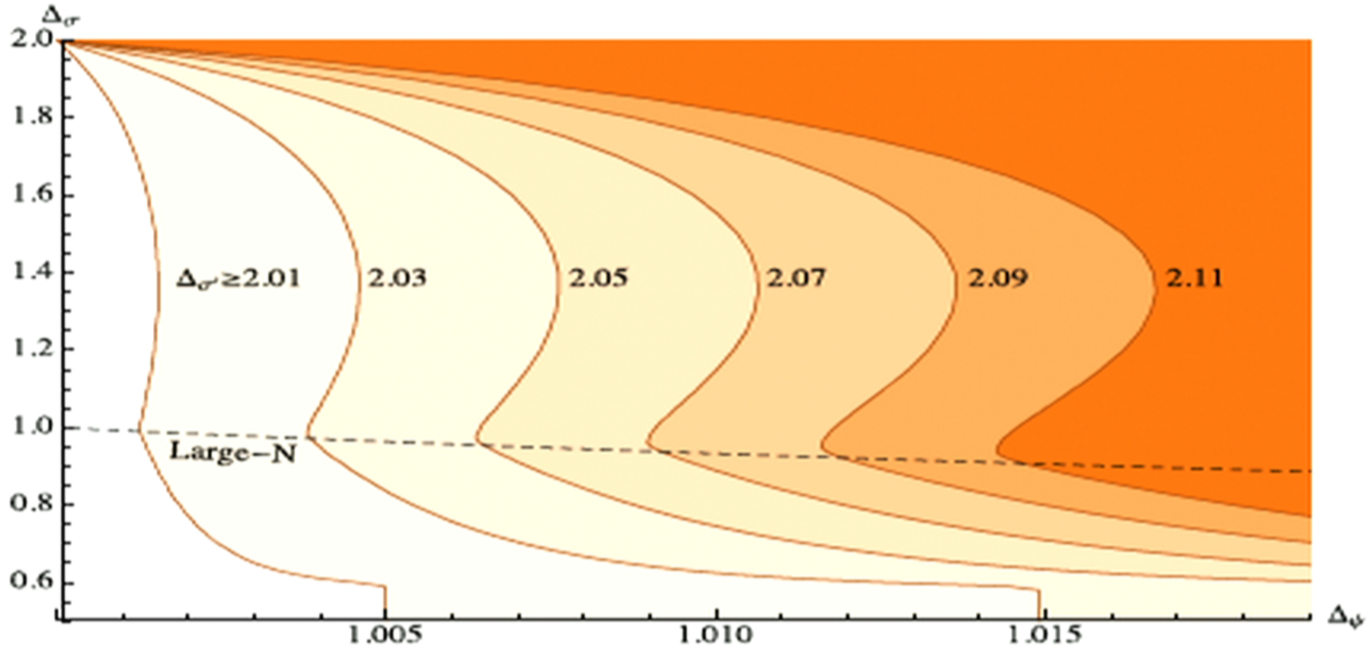
3D Fermion Bootstrap



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby '15]

- Bound on leading parity-odd scalar in $\psi \times \psi \sim \sigma + \ldots$
- Haven't yet found known CFT that coincides with jump

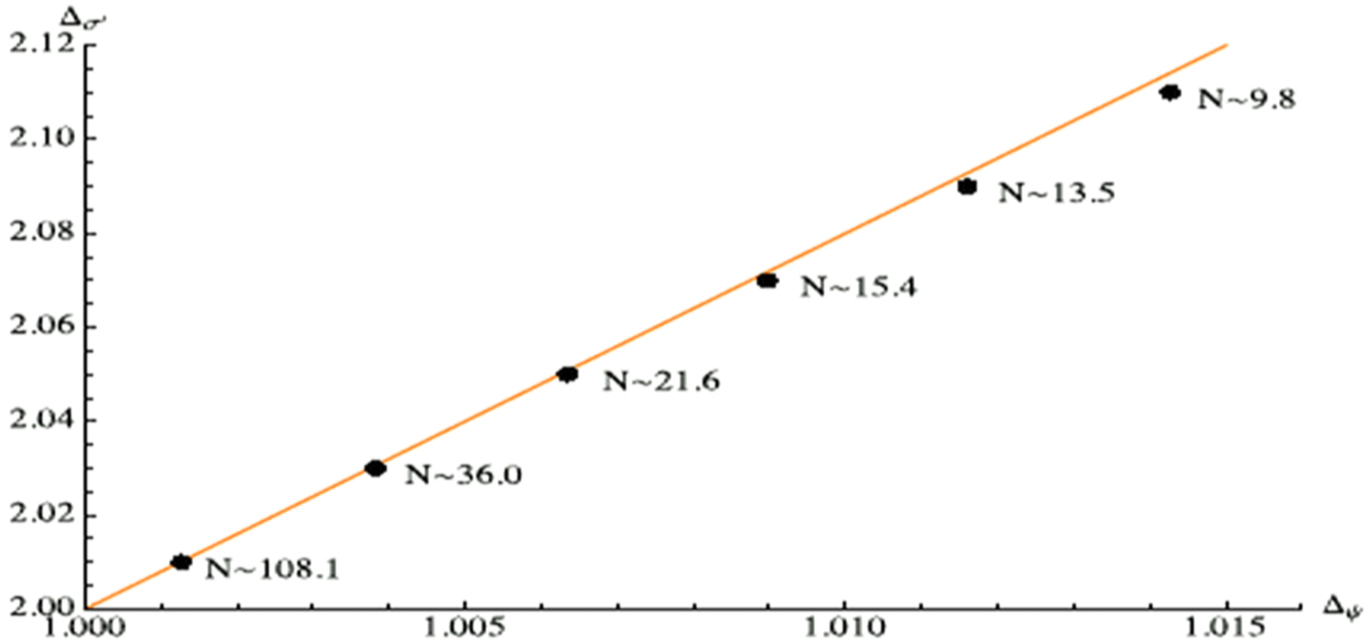
3D Fermion Bootstrap



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby '15]

- However, adding a gap until 2nd parity-odd scalar σ' reveals features that coincide with Large N Gross-Neveu-Yukawa models ($\mathcal{L} \sim \sigma \overline{\psi}_i \psi^i$):

3D Fermion Bootstrap

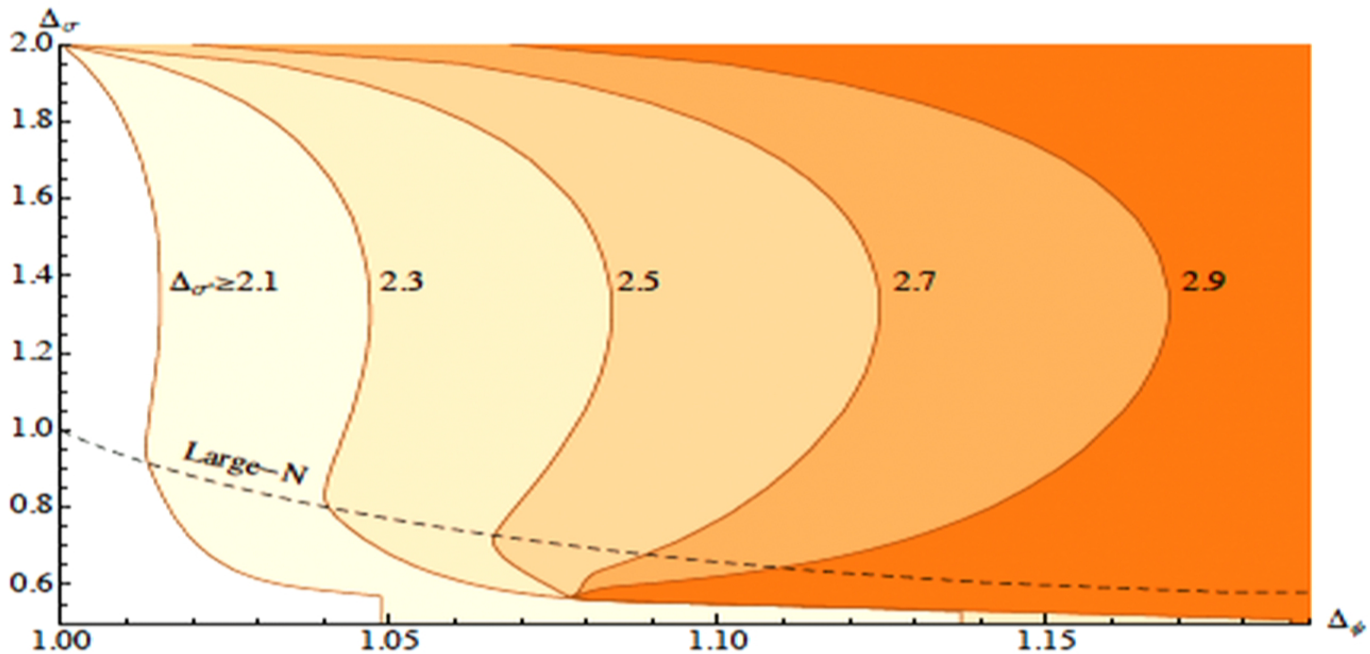


[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby '15]

Large N Gross-Neveu-Yukawa:

- $\Delta_\psi = 1 + \frac{4}{3\pi^2 N}, \quad \Delta_\sigma = 1 - \frac{32}{3\pi^2 N}, \quad \Delta_{\bar{\psi}\psi} = 2 + \frac{32}{3\pi^2 N}$

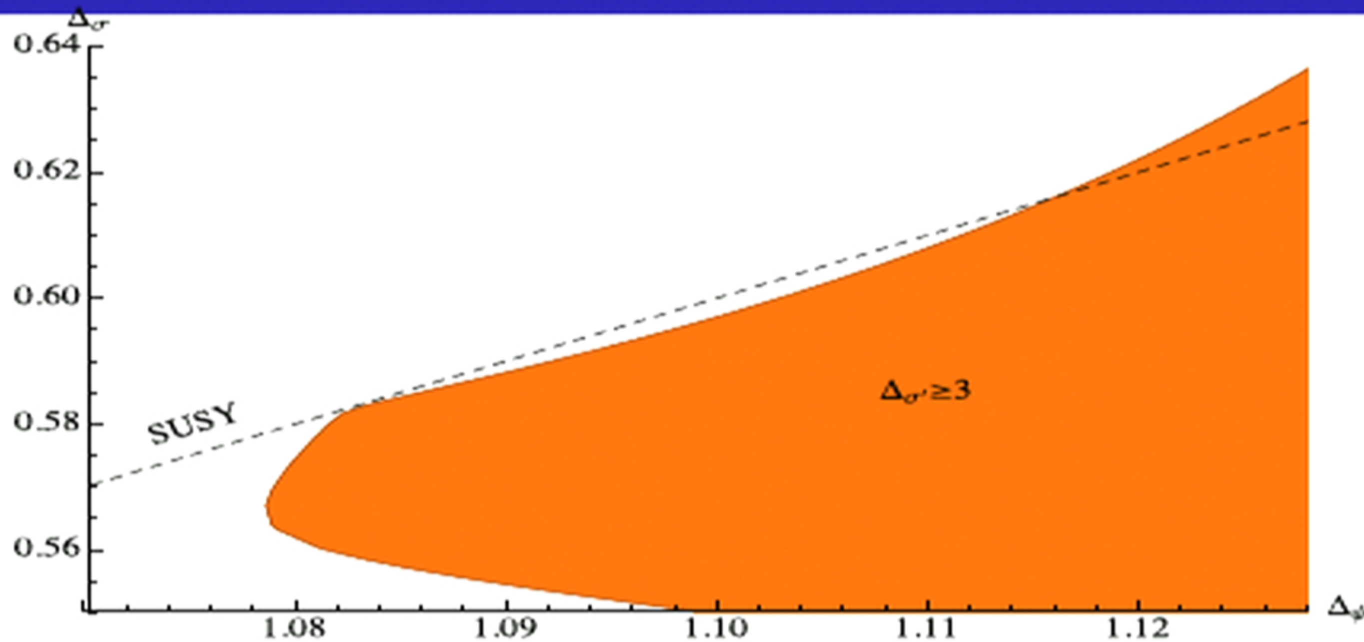
3D Fermion Bootstrap



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby '15]

- Larger gaps in $\Delta_{\sigma'}$ probe the theories with smaller values of N
- Taking $N \to 1$ should reveal 3D $\mathcal{N} = 1$ SUSY Ising model ($W = \Phi^3$)

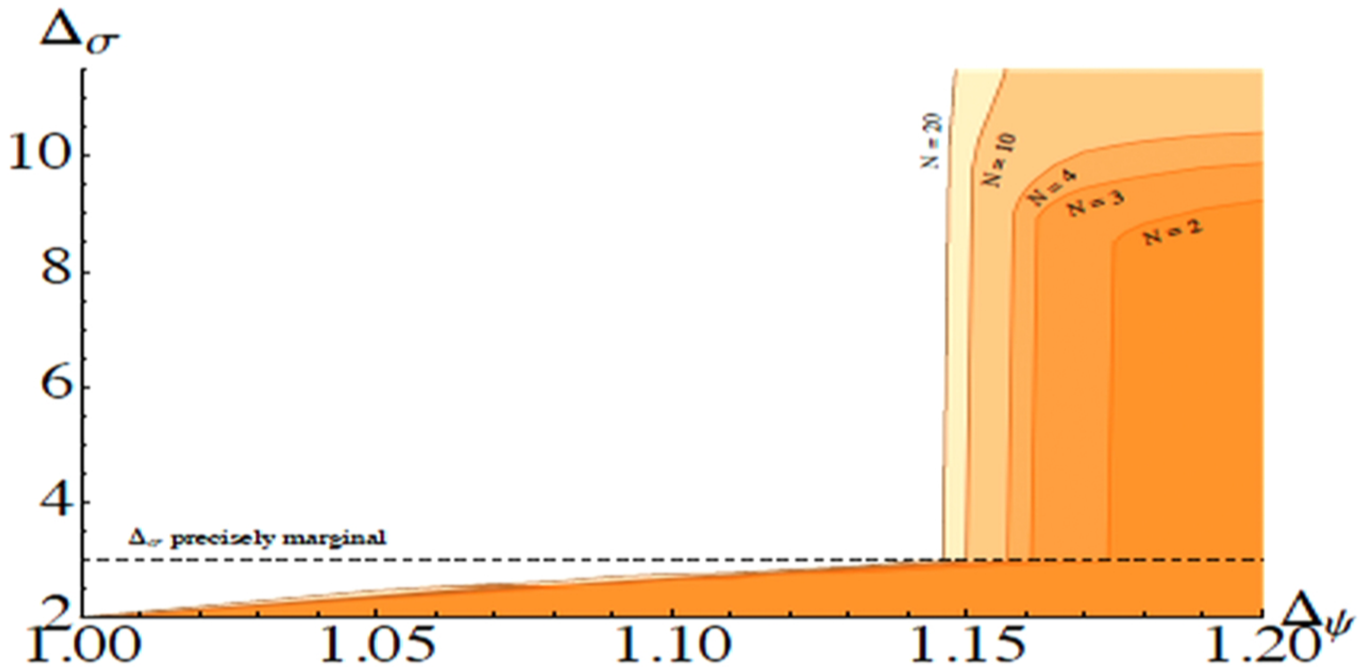
3D Fermion Bootstrap



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby '15]

- Conjecture is that this theory sits at point where kink passes through SUSY line $\Delta_\psi = \Delta_\sigma + 1/2$ (occurs around $\Delta_\psi \sim 1.082, \Delta_{\sigma'} \sim 2.95$)
- Near ϵ-expansion estimate ($\Delta_\psi \approx 1.09$) [Fei, Giombi, Klebanov, Tarnopolsky '16]
- Boundary of topological superconductors? [Grover, Sheng, Vishwanath '13]

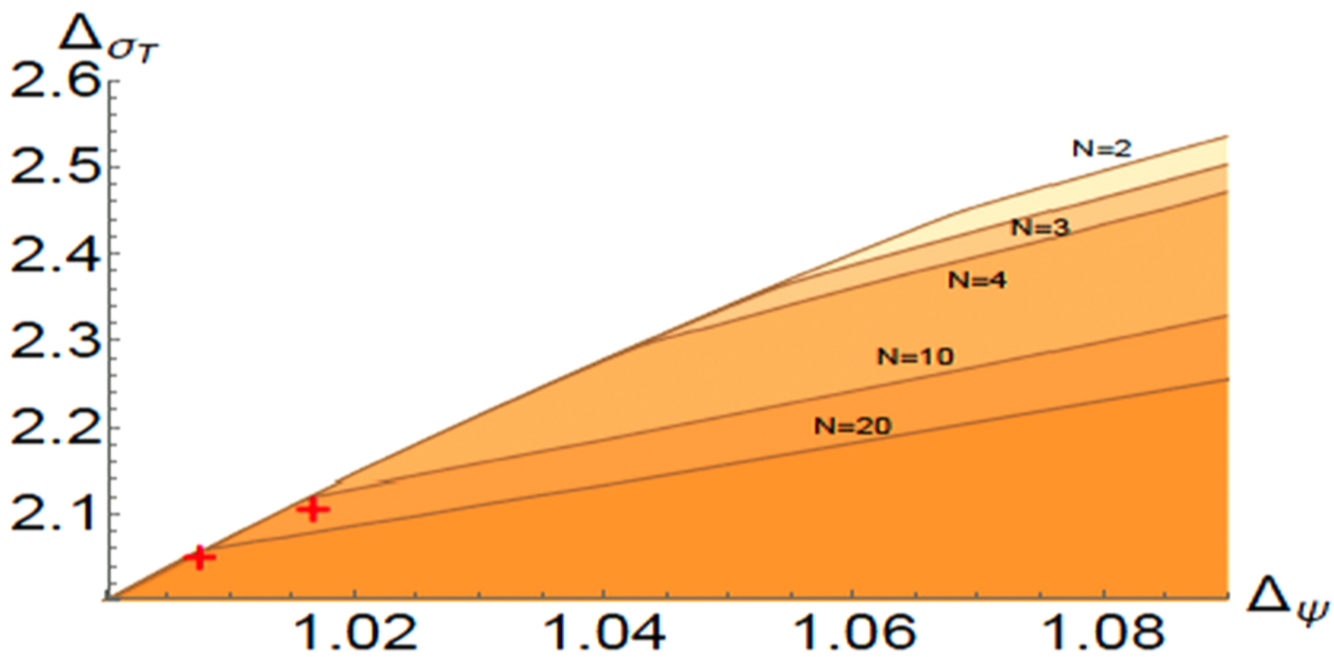
3D Fermions with O(N) Symmetry



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby, to appear]

- Generalization to $\langle\psi_i\psi_j\psi_k\psi_\ell\rangle$ where ψ_i is fundamental of $O(N)$
- Bound on leading parity-odd singlet σ still has mysterious jumps

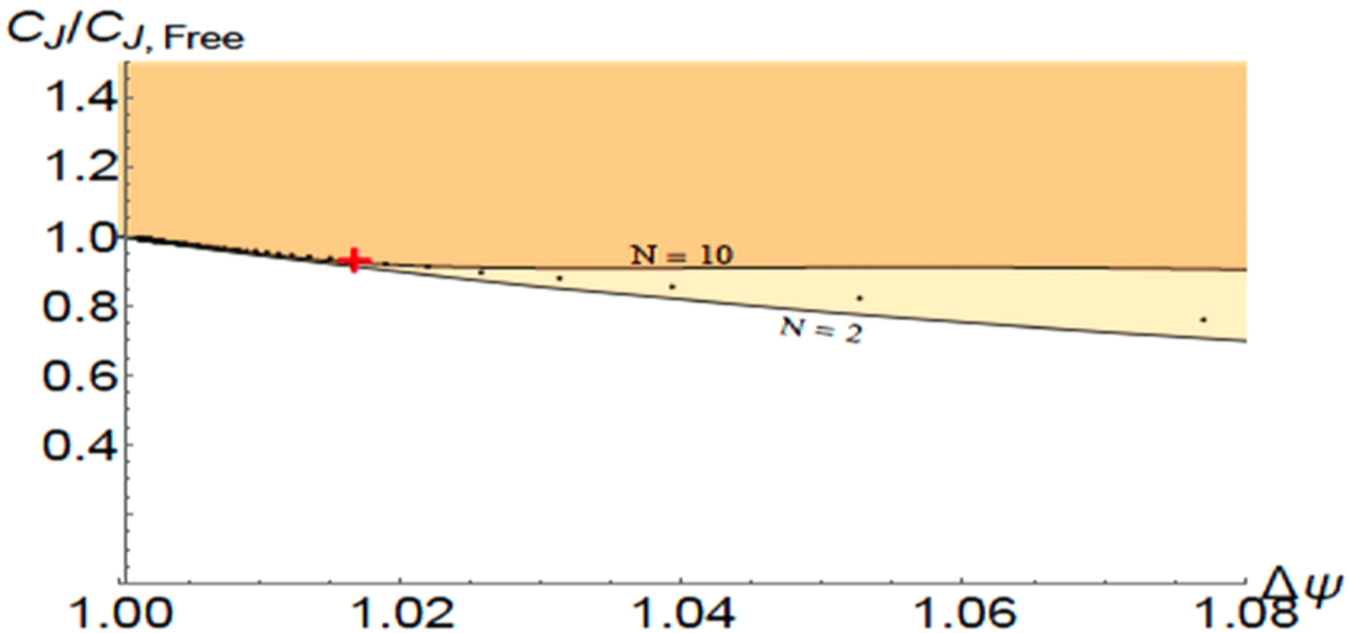
3D Fermions with O(N) Symmetry



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby, to appear]

- Bound on leading parity-odd symmetric tensor σ_T seems promising!
- Looks plausible that all models saturate $N = 2$ bound

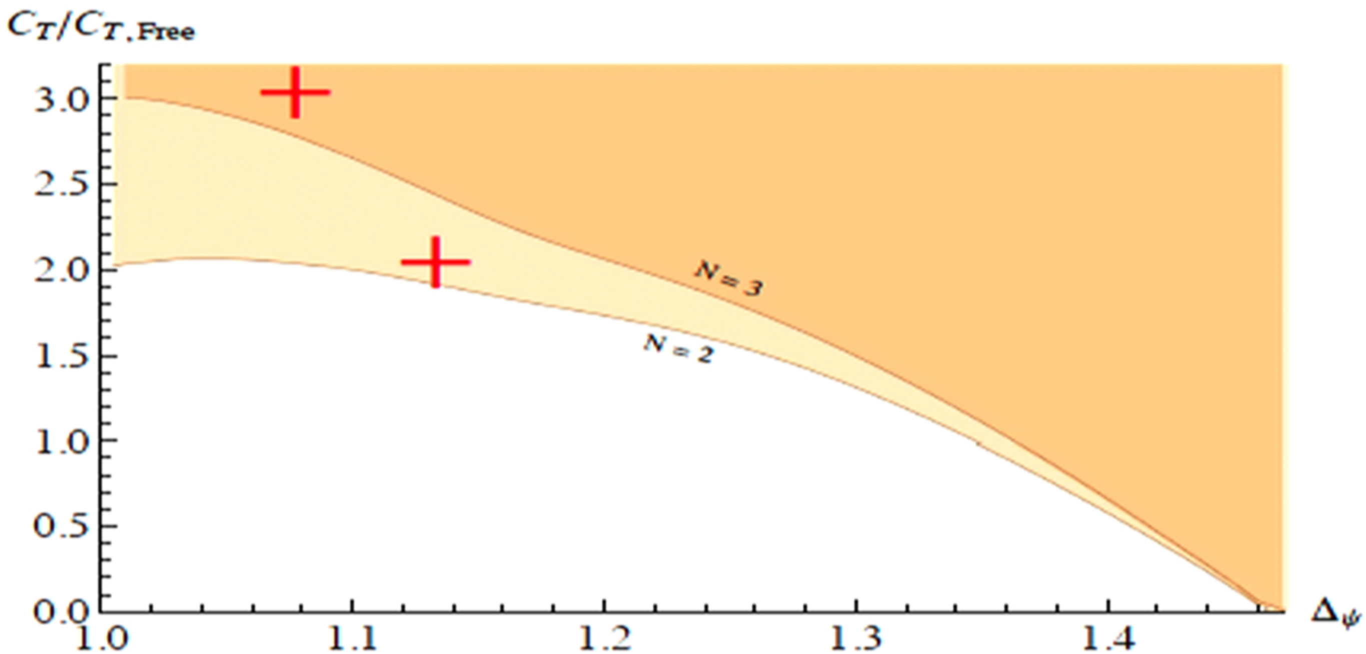
3D Fermions with O(N) Symmetry



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby, to appear]

- Preliminary bounds on $\langle JJ \rangle \propto C_J$ also follow large N curve
- Could again be possible that all models saturate $N = 2$ bound

3D Fermions with O(N) Symmetry



[Iliesiu, Kos, DP, Pufu, Simmons-Duffin, Yacoby, to appear]

- Preliminary bounds on $\langle TT \rangle \propto C_T$ grow linearly with N, as expected

3D Bootstrap Future

Where do we go from here?

- Make O(N) model predictions more precise (resolve 8σ discrepancy!)
  - Extend mixed correlator bootstrap to include external t^{ij}?
  - Higher spectrum ($\{\phi', s', t'\}$, higher $O(N)$ reps, leading twist trajectory)

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- Find rigorous islands for fermionic CFTs and gauge theories
  - 3D fermions: Study mixed $\{\psi, \sigma\}$ system
    - Gross-Neveu models, Hubbard model, $\mathcal{N} = 1, 2$ Ising model, ...
  - 3D QED, 3D QCD, Chern-Simons + matter, ...
  - Can we classify all the bootstrap solutions???

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- Bootstrap currents and stress tensor

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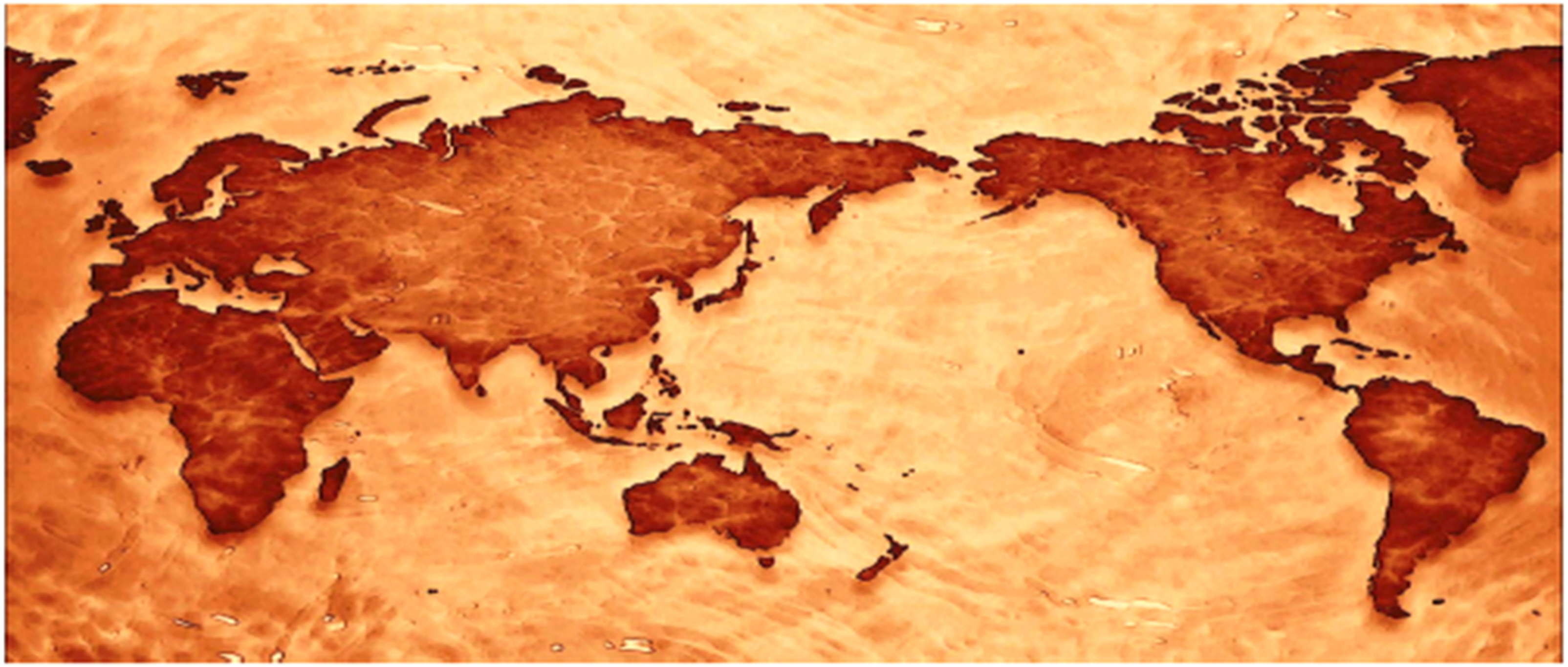
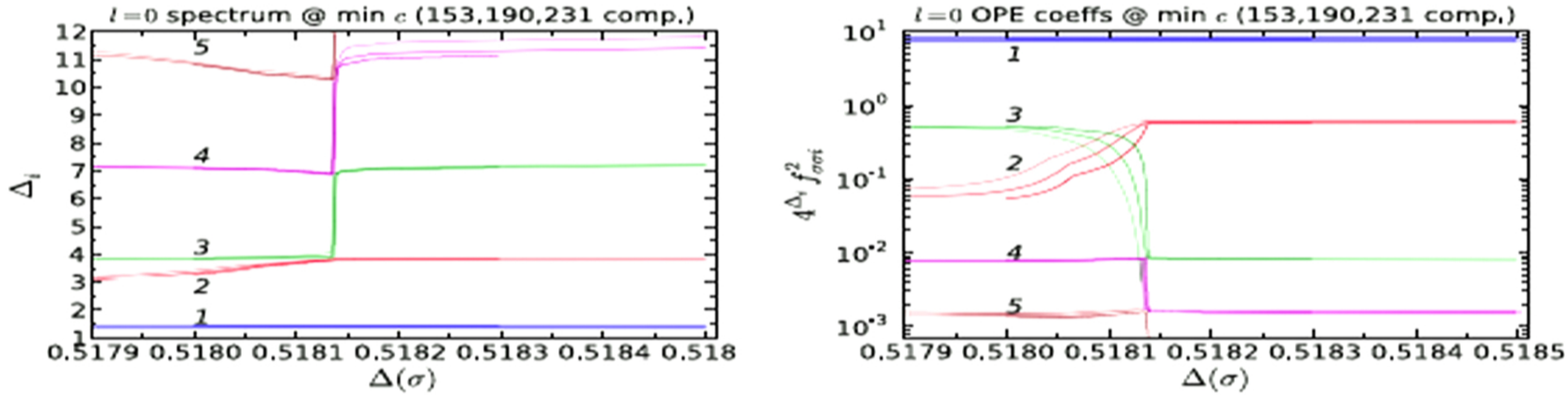
- Find rigorous islands for fermionic CFTs and gauge theories
  - 3D fermions: Study mixed $\{\psi, \sigma\}$ system
    - Gross-Neveu models, Hubbard model, $\mathcal{N} = 1, 2$ Ising model, ...
  - 3D QED, 3D QCD, Chern-Simons + matter, ...
  - Can we classify all the bootstrap solutions???
- Bootstrap currents and stress tensor
  - Analytic Bootstrap for $\langle JJ\phi\phi\rangle$ and $\langle TT\phi\phi\rangle$
    $\rightarrow$ Sum rules for coefficients in $\langle JJT\rangle \propto \hat{n}_{s,f}$ and $\langle TTT\rangle \propto n_{s,f,t}$
    $\rightarrow$ Proof of Hofman-Maldacena bounds $\hat{n}_{s,f} \geq 0$ and $n_{s,f,t} \geq 0$
    [Hartman, Jain, Kundu '15; '16; Hofman, Li, Meltzer, DP, Rejon-Barrera '16]
  - Can they be strengthened? Interplay with numerical studies?

Bootstrap Future



- With more work I believe we can create a detailed map of the space of conformal field theories...we may even discover a new world!

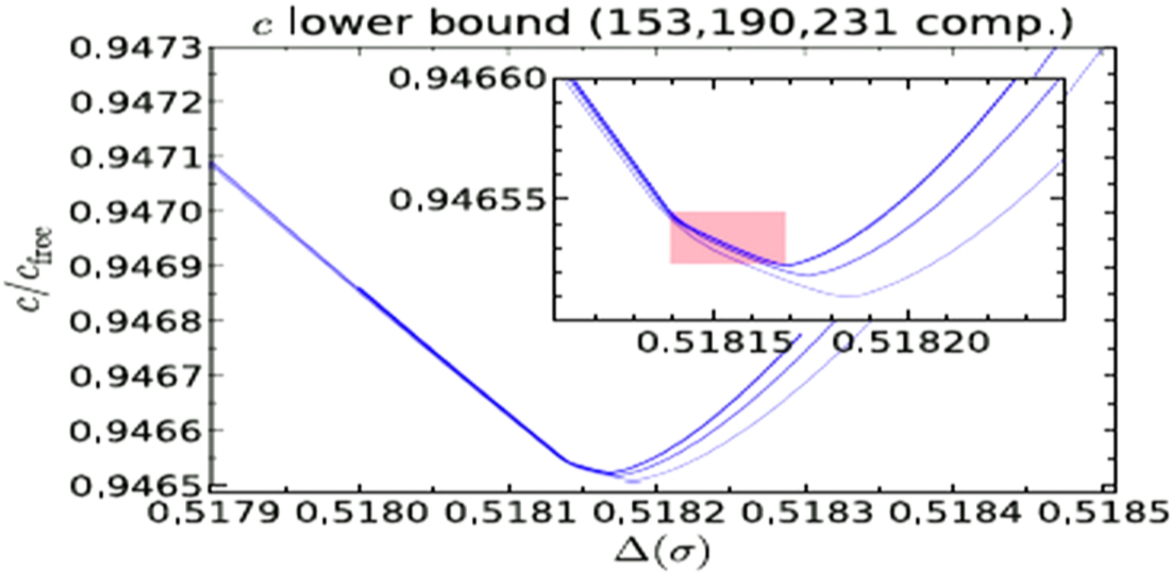
c-minimization and Non-rigorous Spectrum



[El-Showk, Paulos, DP, Rychkov, Simmons-Duffin, Vichi, '14]

- "Kink" $\leftrightarrow$ operators merge and disappear from spectrum!
- Reminiscent of null states in 2D or equations of motion in $(4-\epsilon)D$ $\rightarrow$ Non-perturbative equation of motion?
- E.g., in ϕ^4 theory, expect $\partial^2\phi \sim \phi^3 \rightarrow$ gap in $\mathbb{Z}_2$-odd spectrum...

c-minimization and Non-rigorous Spectrum



[El-Showk, Paulos, DP, Rychkov, Simmons-Duffin, Vichi, '14]

- Under the conjecture that the central charge $\langle TT \rangle \propto c$ is minimized, a precise spectrum in $\sigma \times \sigma \sim \mathbb{1} + \epsilon + \epsilon' + \ldots$ can be extracted:

$$\Delta_\sigma \simeq 0.518154(15),\ \Delta_\epsilon \simeq 1.41267(13),\ \Delta_{\epsilon'} = 3.8303(18),\ \ldots$$