Title: Universal features of Lifshitz Greenâ€™s functions--- from holography and field theory

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Abstract: In this talk, we examine the behavior of the retarded Greenâ€™s function in theories with Lifshitz scaling symmetry, both through dual gravitational models and a direct field theory approach. In contrast with the case of a relativistic CFT, where the Greenâ€™s function is fixed (up to normalization) by symmetry, the generic Lifshitz Greenâ€™s function can a priori depend on an arbitrary function Nevertheless, we demonstrate that the imaginary part of the retarded Greenâ€™s function (i.e. the spectral function) of scalar operators is exponentially suppressed in a window of frequencies near zero. This behavior is universal in all Lifshitz theories without additional constraining symmetries. On the gravity side, this result is robust against higher derivative corrections, while on the field theory side we present two z>1 examples where the exponential suppression arises from summing the perturbative expansion to infinite order, as a consequence of the energy-momentum conservation.

Critical scaling and dynamic critical exponent

- For quantum critical systems with (emergent) space-rotational symmetries, the quantum dynamics is described by the dynamic critical exponent z

$$\vec{x} \to \Lambda \vec{x}, \qquad t \to \Lambda^z t,$$

- For $z = 1$, the critical system shows Lorentz and conformal symmetry
  - Symmetries enforce strong constrains, which lead to universal scaling behavior
  - For example: Green's function for some real scalar field (bosons)
    $$G(k,\omega) =< \varphi(k,\omega)\varphi(-k,-\omega) >= (k^2-\omega^2)^\nu$$
  - Universally determined by scaling dimensions of the quantum operator
- For $z > 1$, the critical system has lower symmetry
  - Scaling behaviors are more complicated and less universal
  - If $\varphi(k,\omega)$ is a scalar, $\tilde{\varphi}(k,\omega) = f(\omega^2/k^{2z})\varphi(k,\omega)$ is also a scalar with the same scaling dimensions.
  - But $\varphi(k,\omega)$ and $\tilde{\varphi}(k,\omega)$ obviously have different Green's functions
  - $G_{\tilde{\varphi}}(k,\omega) = f^2(\omega^2/k^{2z})G_\varphi(k,\omega)$
  - (Exceptions, e.g. certain systems with z=2 has Galilean invariance)
- Strictly speaking, Lifshitz means z=2. But here we consider general z>1.

Sachdev, Quantum Phase Transitions

Green's function for $z > 1$

- Scaling symmetry: $\omega \to \lambda^z \omega$ and $\vec{k} \to \lambda \vec{k}$
- Retarded Green's function: $G_R(\omega, \vec{k}\,) = |\vec{k}\,|^{2\nu z} \mathcal{G}(\hat{\omega})$
  - ν is the scaling dimension of $\varphi(k, \omega)$
  - $\widehat{\omega} = \omega / k^z$
  - $\mathcal{G}(\widehat{\omega})$ is an arbitrary function
- How to determine $\mathcal{G}(\widehat{\omega})$?

Green's function: holographic approach

- Using the magical dictionary for gravity-Field theory correspondence,
- Gravity background: $$ds^2_{d+2} = \frac{-dt^2 + d\rho^2}{\rho^2} + \frac{d\vec{x}^2}{\rho^{2/z}}$$
- With a probe scalar: $$\phi(t, \vec{x}, \rho)$$
- Equation of motion: $$(\Box - m^2)\phi = 0$$
  - Taking: $$\phi(t, \vec{x}, \rho) = e^{i(\vec{k}\cdot\vec{x} - \omega t)}\rho^{d/2z}\psi(\rho)$$
  - Define dimension less $$\hat{\rho} = \rho|\vec{k}|^z$$
- The EOM becomes: $$-\psi''(\hat{\rho}) + \hat{U}_0(\hat{\rho})\psi(\hat{\rho}) = 0$$
  - where $$\hat{U}_0(\hat{\rho}) = \frac{\nu^2 - 1/4}{\hat{\rho}^2} + \frac{1}{\hat{\rho}^{2-2/z}} - \hat{\omega}^2$$

Son and Starinets, 2002
Kachru, Liu, Mulligan, 2008

Green's function: holographic approach

- At the boundary: $\psi(\hat\rho \to 0) \sim A\hat\rho^{\frac{1}{2}-\nu} + B\hat\rho^{\frac{1}{2}+\nu}$
- At the horizon: $\psi(\hat\rho \to \infty) \sim a e^{i\hat\omega\hat\rho} + b e^{-i\hat\omega\hat\rho}$
- Infalling boundary conditions: $b = 0$ at the horizon
- The Green's function: $\mathcal{G}(\hat\omega) = \left.\frac{B}{A}\right|_{b=0}$

Compare with field theory:

- Field Theory: Green's function for $z > 1$ is NOT expected to be universal
  - If $\varphi(k,\omega)$ is a scalar, $\tilde\varphi(k,\omega) = f(\omega^2/k^{2z})\varphi(k,\omega)$ is also a scalar with the same scaling dimensions.
  - $G_{\tilde\varphi}(k,\omega) = f^2\left(\frac{\omega^2}{k^{2z}}\right) G_\varphi(k,\omega)$

Green's function: holographic approach

- Using the magical dictionary for gravity-CFT correspondence,
- Gravity background: $ds^2_{d+2} = \frac{-dt^2 + d\rho^2}{\rho^2} + \frac{d\vec{x}^2}{\rho^{2/z}}$
- With a probe scalar: $\phi(t, \vec{x}, \rho)$
- Equation of motion: $(\Box - m^2)\phi = 0$

No Lorentz symmetry. There is no reason for the EOM to take this form.

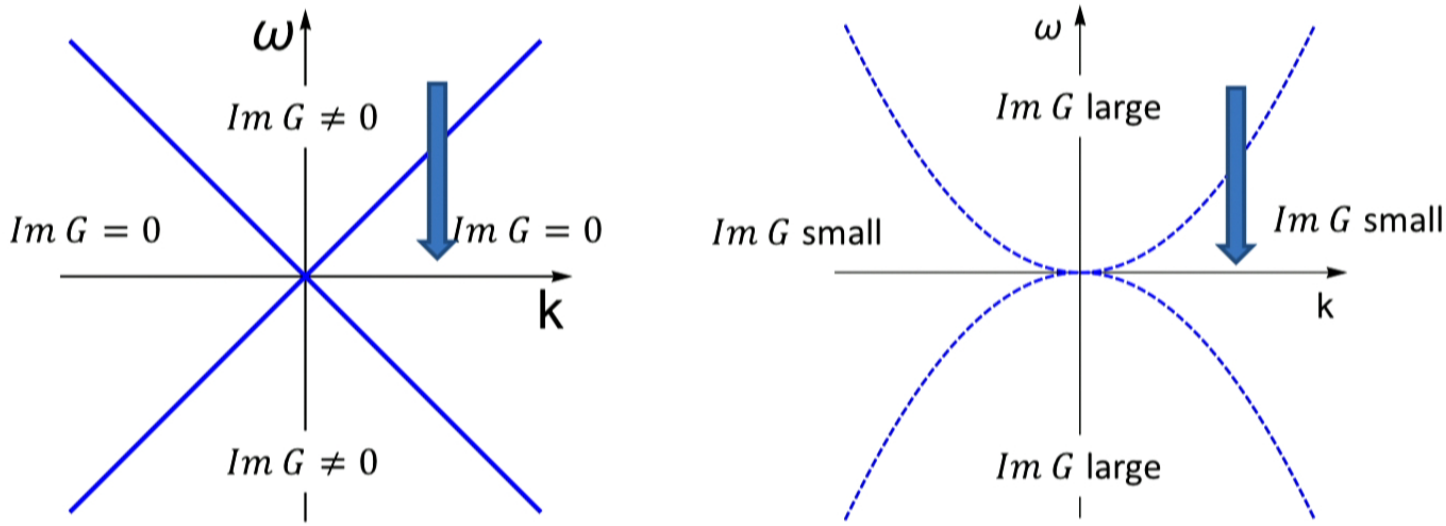
- New (derivative) terms can be introduced to the EOM
- EOM can be write: $-\psi''(\hat{\rho}) + \hat{U}_0(\hat{\rho})\psi(\hat{\rho}) = 0$

Effective potential: $\hat{U} = \frac{\nu^2 - 1/4}{\hat{\rho}^2} + \frac{1}{\hat{\rho}^{2-2/z}} - \hat{\omega}^2 + \sum_{i+j>2} \lambda_{i,j}\hat{\omega}^i \hat{\rho}^{i+j/z-2}$

With higher order terms: different Green's function solutions (consistent with QFT)

Universal properties

- The Green's function is NOT universal.
- But $Im\ G$ shows universal asymptotic form (gravity approach)



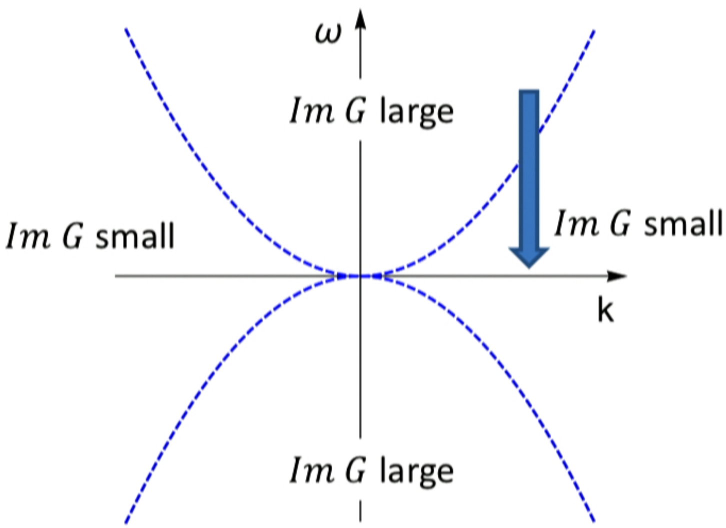
$z = 1$: a hard boundary

$Im\ G$=0 for $\omega \ll k$

$z = 2$: a soft boundary

$Im\ G \sim \exp(-const.\frac{k^2}{\omega})$ for $\omega \ll k$

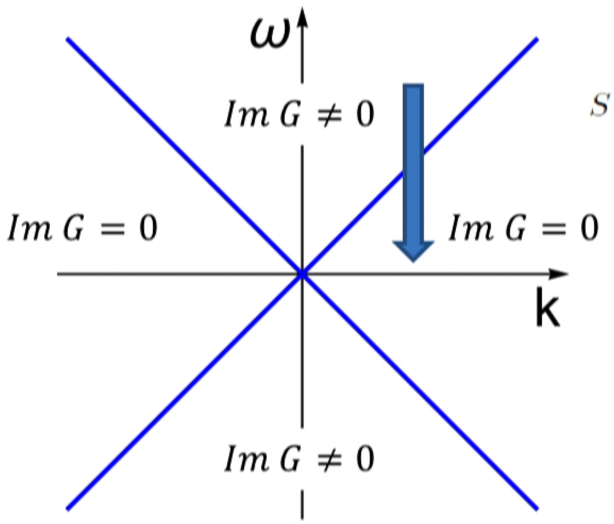
z>1



$z = 2$: a soft boundary

For general $z > 1$, $Im\ G \approx \exp[-const.\,(\frac{k^z}{\omega})^{\frac{1}{z-1}}]$ for $\omega \ll k$

Field theory: energy-momentum conservation



$z = 1$: a hard light cone

$Im\ G$=0 for $\omega < k$

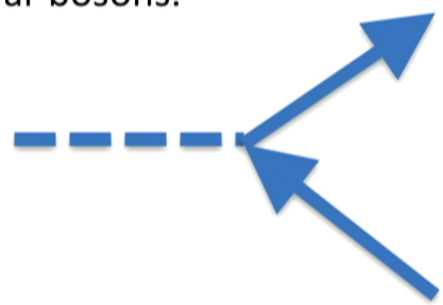
Why?

Consider a Dirac system ($z = 1$)

$$S = \int dt d^d\mathbf{r}\ \bar{\Psi} i \left(\gamma_0 \partial_0 - \gamma_1 \partial_x - \gamma_2 \partial_y\right) \Psi - g \int dt d^d\mathbf{r} \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1$$

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \gamma_0 = \sigma_y \quad \gamma_1 = i\sigma_x \quad \gamma_2 = -i\sigma_z$$

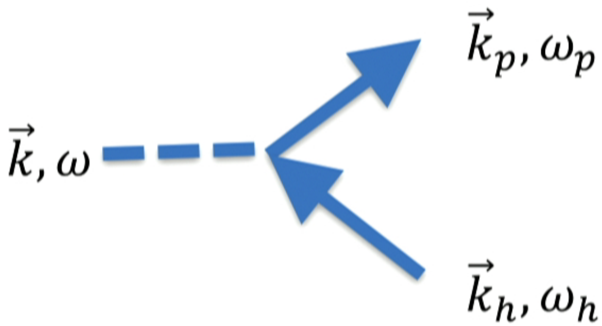
Scalar bosons:



Study the Green's function of this boson

Field theory: energy-momentum conservation

- Imaginary part of Green's function measures the decay rate
  - $Im\ G = 0$: boson cannot decay
  - $Im\ G \neq 0$: boson can decay
- How to determine whether a decay is possible or not?
  - Look at the energy-momentum conservation law



$$\vec{k} = \vec{k}_p \text{-} \vec{k}_h$$

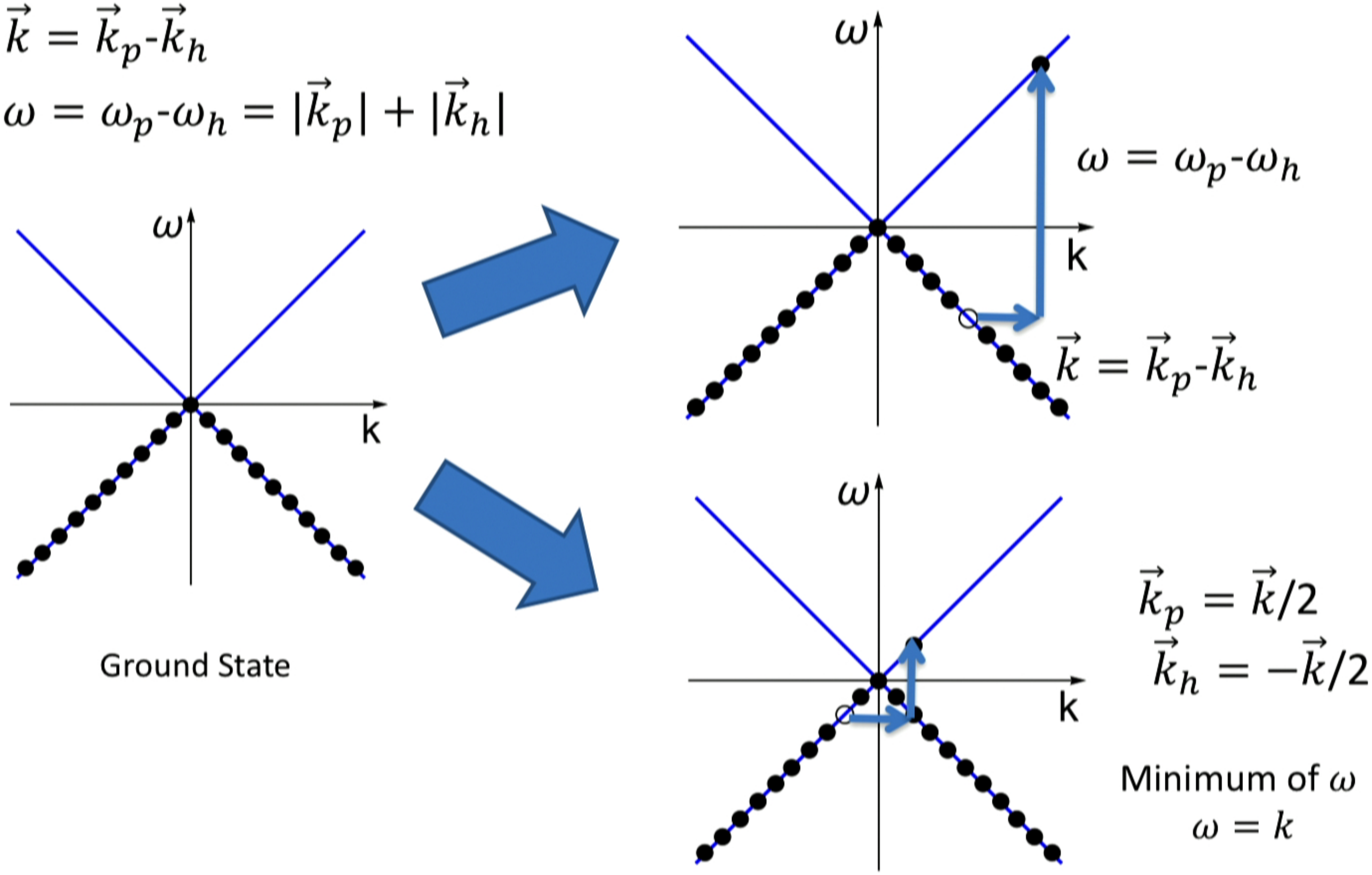
$$\omega = \omega_p \text{-} \omega_h = |\vec{k}_p| + |\vec{k}_h|$$

- If we fix $\vec{k}$, to satisfy the conservation law, $\omega > |\vec{k}|$.

Energy-momentum conservation

$\vec{k} = \vec{k}_p\text{-}\vec{k}_h$

$\omega = \omega_p\text{-}\omega_h = |\vec{k}_p| + |\vec{k}_h|$



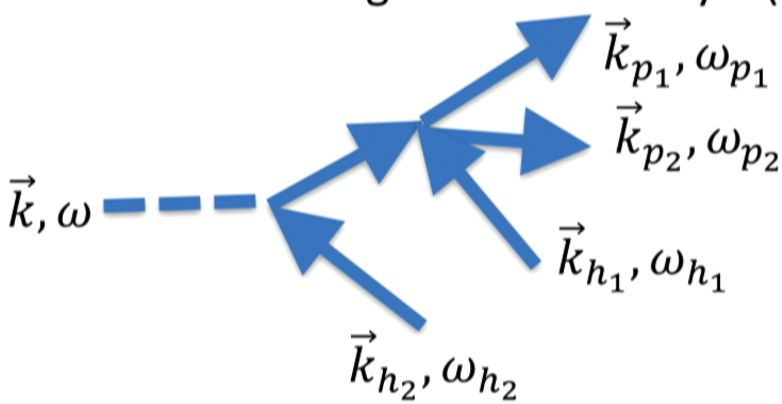
Higher order decay

- Conservation law: decay only for $\omega > |\vec{k}|$: **particle-hole continuum**

$$\vec{k} = \vec{k}_p - \vec{k}_h$$

$$\omega = \omega_p - \omega_h = |\vec{k}_p| + |\vec{k}_h|$$

- How about higher order decay? (n particles + n holes)



$$\vec{k} = \vec{k}_{p_1} + \vec{k}_{p_2} - \vec{k}_{h_1} - \vec{k}_{h_2}$$

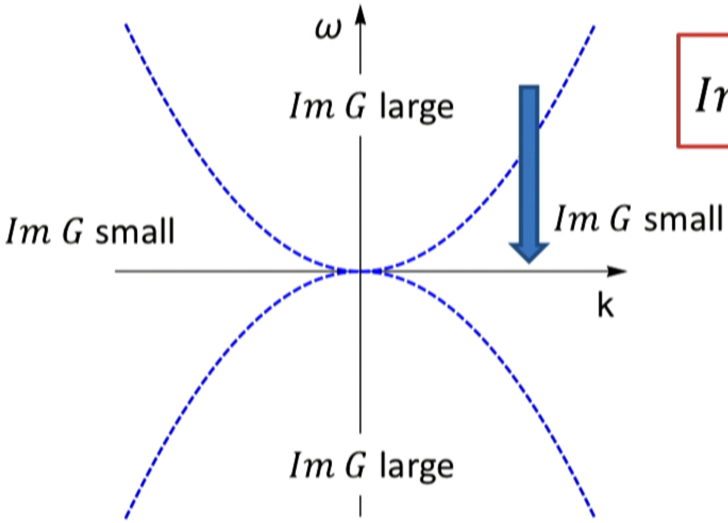
$$\omega = \omega_{p_1} + \omega_{p_2} - \omega_{h_1} - \omega_{h_2}$$

Same bound for decay:
$\omega > |\vec{k}|$:

$\vec{k}_p = \vec{k}/2\text{n}$ and $\vec{k}_h = -\vec{k}/2\text{n}$

Im G=0 for $\omega < |\vec{k}|$, even to infinite order in diagrammatic expansions

z= 2



$z = 2$: a soft boundary

$$Im\ G \approx \exp(-const.\frac{k^2}{\omega}) \text{ for } \omega \ll k$$

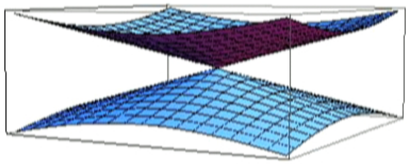
A model for $z = 2$

Quadratic band crossing point (QBCP)

- Start from Dirac theory:

$$S = \int dt d^d\mathbf{r}\ \bar{\Psi} i\left(\gamma_0\partial_0 - \gamma_1\partial_x - \gamma_2\partial_y\right)\Psi - g\int dt d^d\mathbf{r}\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1$$

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad \begin{aligned}\gamma_0 &= \sigma_y\\ \gamma_1 &= i\sigma_x\\ \gamma_2 &= -i\sigma_z\end{aligned}$$

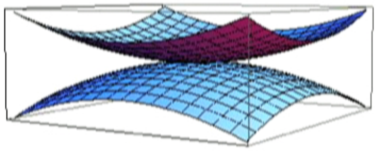


- Modify the spatial derivatives (2nd order and d-wave symmetry)

$$S = \int dt d^d\mathbf{r}\ \bar{\Psi}\left[\gamma_0\left(i\partial_0 + t_0\nabla^2\right) + \gamma_1 t_1\left(\partial_x^2 - \partial_y^2\right) + \gamma_2\left(2\partial_x\partial_y\right)\right]\Psi - g\int dt d^d\mathbf{r}\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1$$

$$\omega = k^2\left(t_0 \pm \sqrt{\frac{t_1^2 + 1 + (t_1^2 - 1)\cos(4\theta)}{2}}\right)$$

- d-wave symmetry: $\partial_x^2 - \partial_y^2$ and $2\partial_x\partial_y$
- $t_0 \neq 0$: breaks particle-hole
- $t_1 \neq 1$: breaks SO(2) to C4



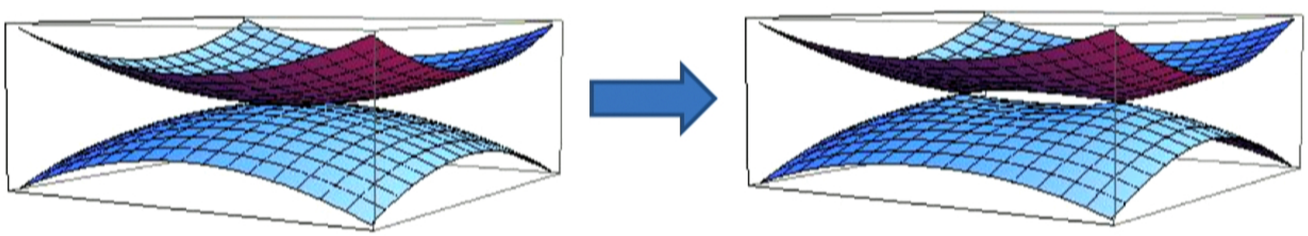
KS, Hong Yao, Eduardo Fradkin and Steven A. Kivelson, ,PRL 103, 046811 (2009)

A model for $z = 2$

$$S = \int dt d^d\mathbf{r}\ \bar{\Psi}\left[\gamma_0\left(i\partial_0 + t_0\nabla^2\right) + \gamma_1 t_1\left(\partial_x^2 - \partial_y^2\right) + \gamma_2\left(2\partial_x\partial_y\right)\right]\Psi - g\int dt d^d\mathbf{r}\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1$$

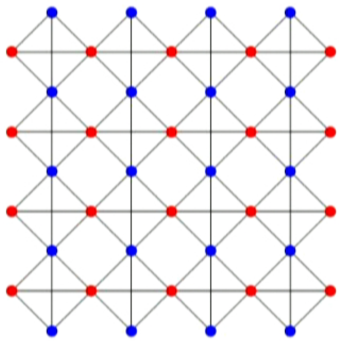
$$\omega = k^2\left(t_0 \pm \sqrt{\frac{t_1^2 + 1 + (t_1^2 - 1)\cos(4\theta)}{2}}\right)$$

- No fermion doubling (possible to have a single QBCP)
- mass term is prohibited by time-reversal/chiral symmetry
  - Critical, as long as these symmetries are preserved
- Rotational symmetry needed: C4, C6 or SO(2)
  - Otherwise, a QBCP will split into two Dirac points

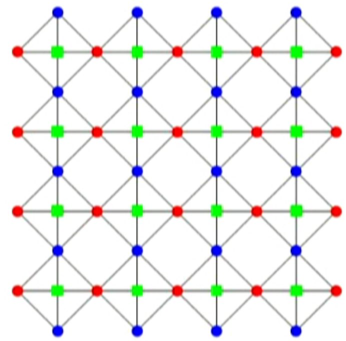


KS, Hong Yao, Eduardo Fradkin and Steven A. Kivelson, ,PRL 103, 046811 (2009)

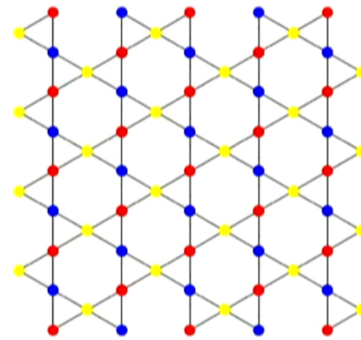
Lattice models



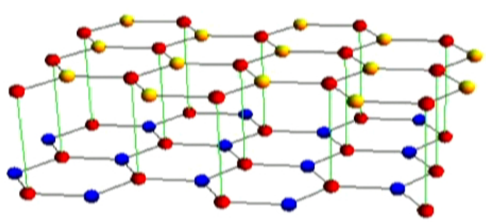
Checkerboard Lattice
1QBCP



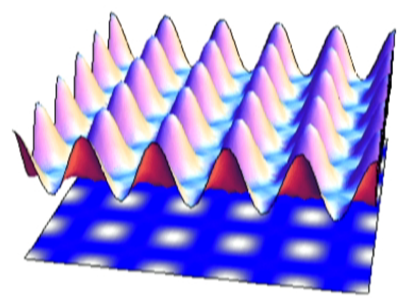
Emery/ Lieb Lattice
1QBCP



Kagome Lattice
1QBCP or 2 Dirac



Bilayer Graphene
2QBCP



Square lattice (with multi-orbitals)
1 or 2 QBCP(s)

RG analysis

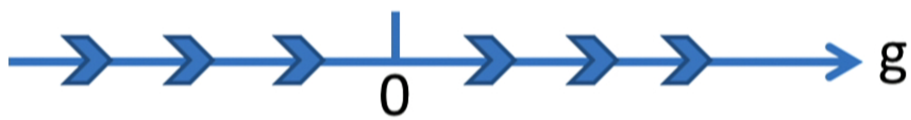
$$S=\int dtd^d\mathbf{r}\;\bar{\Psi}\left[\gamma_0\left(i\partial_0+t_0\nabla^2\right)+\gamma_1 t_1\left(\partial_x^2-\partial_y^2\right)+\gamma_2\left(2\partial_x\partial_y\right)\right]\Psi-g\int dtd^d\mathbf{r}\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1$$

- Dimension counting: for $d+1$ space time

  $$[\omega]=2,\ [k]=1,\ [\Psi]=d/2,\ \text{and}\ [g]=2-d$$

- Critical dimension: $d=2$, where $[g]=0$
  - Dirac: $[g]=1-d$, so critical dimension is $d=1$ *(Wilson, **PRD**, 1973)*
  - At tree-level, a $d=2$ QBCP is similar to a $d=1$ Dirac system
- Beyond tree-level, $d=2$ QBCPs very different from $d=1$ Dirac
  - 1+1D Dirac: exact marginal (a Luttinger liquid)
  - 2+1D QBCP g does flow

marginally irrelevant (IR) marginally relevant (IR)



KS, Hong Yao, Eduardo Fradkin and Steven A. Kivelson, ,PRL 103, 046811 (2009)

Repulsive g in 2+1D

$$S = \int dt d^d \mathbf{r}\, \bar{\Psi} \left[\gamma_0 \left(i\partial_0 + t_0 \nabla^2\right) + \gamma_1 t_1 \left(\partial_x^2 - \partial_y^2\right) + \gamma_2 \left(2\partial_x \partial_y\right)\right] \Psi - g \int dt d^d \mathbf{r} \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1$$

marginally irrelevant (IR) marginally relevant (IR)

QBPC becomes unstable at low T (IR)

❑ Spontaneously break T-reversal: topological insulator

- Ising universality class
- Massive (introduce a mass)
- Typically favored at weak coupling (small g>0)

❑ Spontaneously break rotational symmetry: a nemtic phase

- Ising/U(1) universality class (depending on details of the model)
- Massive or massless (Dirac) (depending on details of the model)
- Typically favored at strong coupling (large g>0)
- For the case of U(1) and massless, (gravity and field theory)

$$q_x \to \lambda^p q_x\,, \qquad q_y \to \lambda^q q_y \qquad \text{and} \quad \omega \to \lambda^q \omega$$

KS, Hong Yao, Eduardo Fradkin and Steven A. Kivelson, ,PRL 103, 046811 (2009)
Sera Cremonini, Xi Dong, Junchen Rong, **KS**, JHEP (2015) 2015: 82.

Attractive g in 2+1D

$$S = \int dt d^d \mathbf{r}\ \bar{\Psi} \left[\gamma_0 \left(i\partial_0 + t_0 \nabla^2\right) + \gamma_1 t_1 \left(\partial_x^2 - \partial_y^2\right) + \gamma_2 \left(2\partial_x \partial_y\right)\right] \Psi - g \int dt d^d \mathbf{r} \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1$$

marginally irrelevant (IR) marginally relevant (IR)

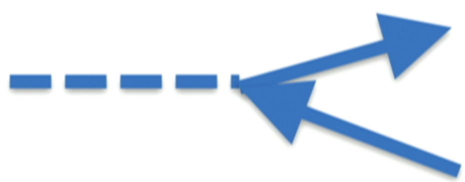
- Focus on g<0 or d>2, where a QBCP is stable
  - Introduce a scalar boson and study its Green's function
  - e.g. some fermion bilinears

$n = \langle \Psi^\dagger \Psi \rangle = \langle \bar{\Psi} \gamma_0 \Psi \rangle = \langle \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 \rangle$ Density

$Q_1 = \langle \Psi^\dagger \sigma_z \Psi \rangle = \langle \bar{\Psi} \gamma_1 \Psi \rangle = \langle \psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2 \rangle$ Nematic main-axis

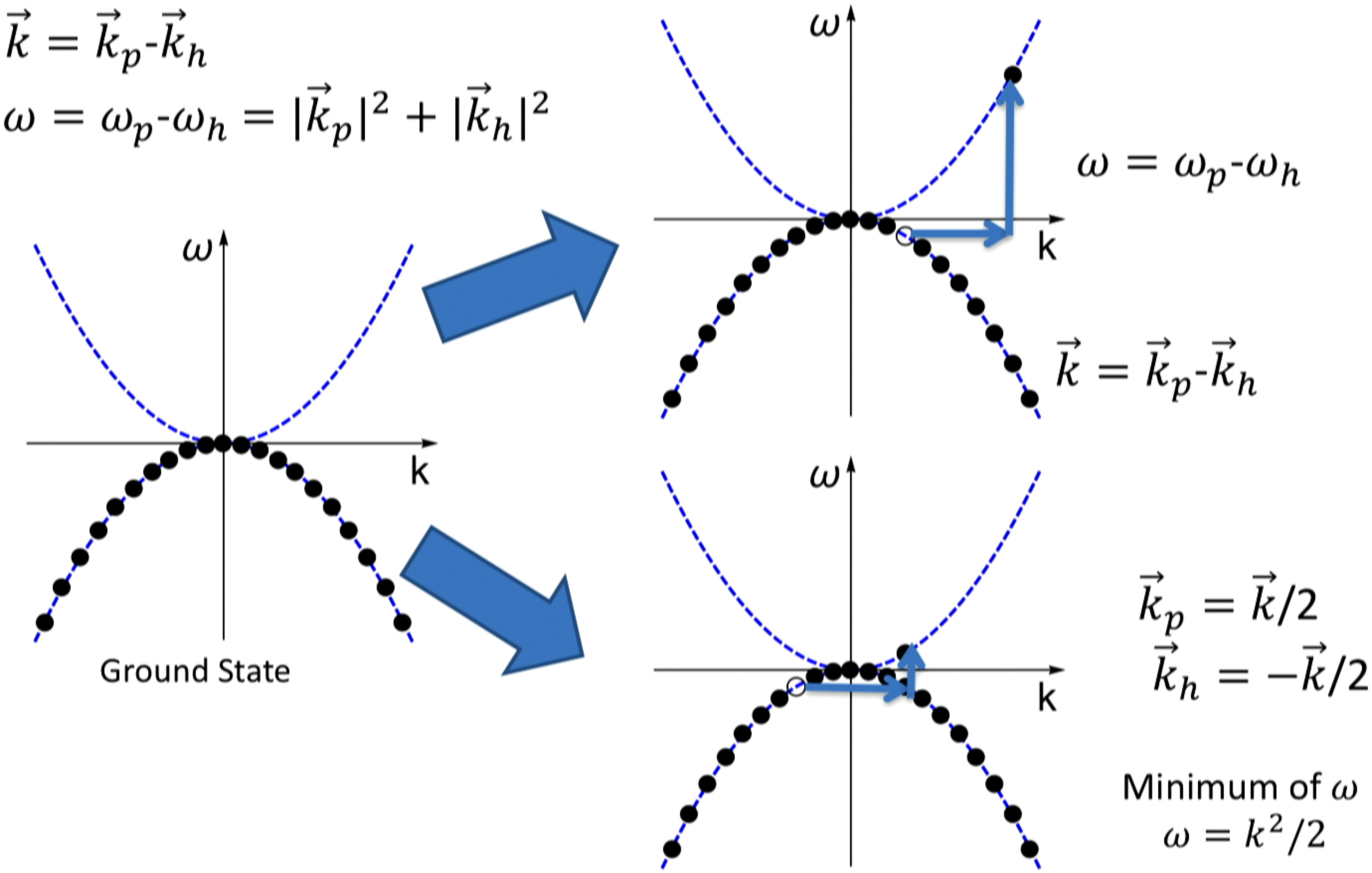
$Q_2 = \langle \Psi^\dagger \sigma_x \Psi \rangle = \langle \bar{\Psi} \gamma_2 \Psi \rangle = \langle \psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1 \rangle$ Nematic diagonal

$\Phi = \langle \Psi^\dagger \sigma_y \Psi \rangle = \langle \bar{\Psi} \Psi \rangle = i \langle \psi_1^\dagger \psi_2 - \psi_2^\dagger \psi_1 \rangle$ Quantum anomalous Hall



Cynthia Keeler, Gino Knodel, James T. Liu, **KS**, JHEP (2015) 2015: 57.

Energy-momentum conservation

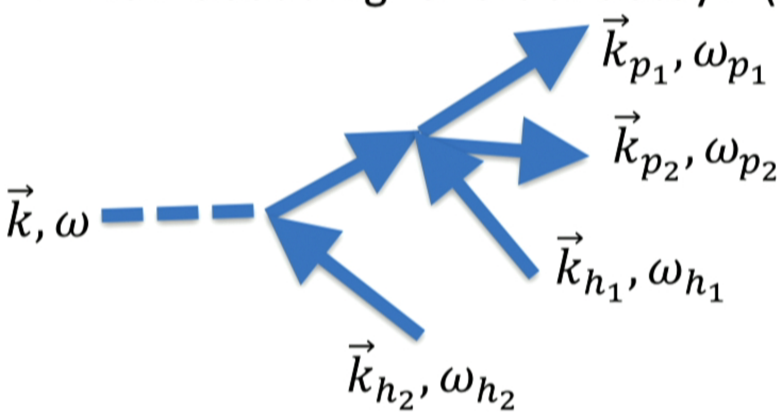


Higher order decay

- Conservation law: decay only for $\omega > k^2/2$: **particle-hole continuum**

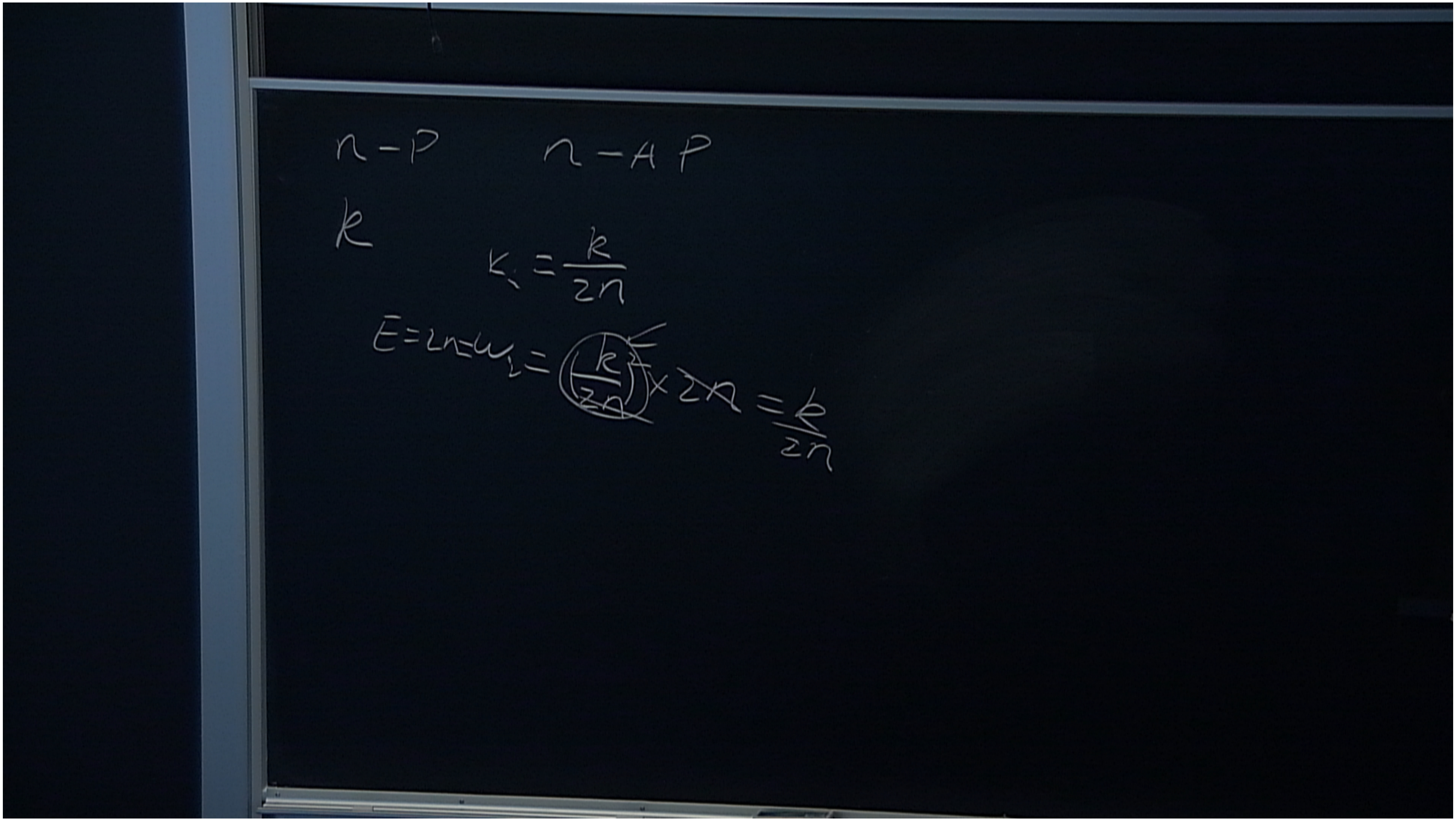
$$\vec{k} = \vec{k}_p \text{-} \vec{k}_h$$
$$\omega = \omega_p \text{-} \omega_h$$

- How about higher order decay? (n particles + n holes)

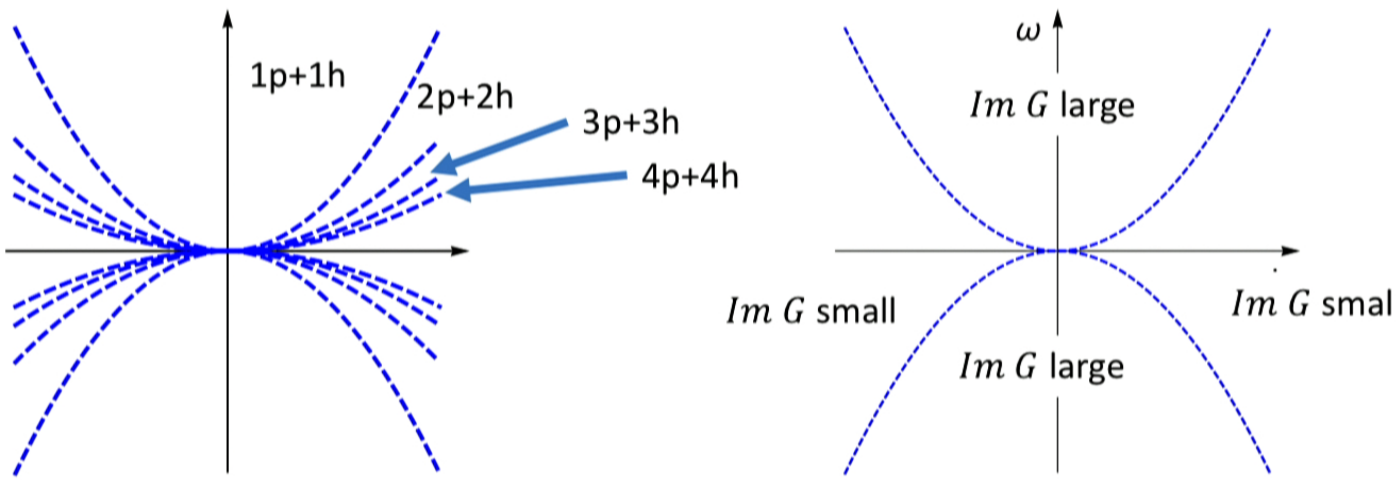


$$\vec{k} = \vec{k}_{p_1} + \vec{k}_{p_2} \text{-} \vec{k}_{h_1} \text{-} \vec{k}_{h_2}$$
$$\omega = \omega_{p_1} + \omega_{p_2} \text{-} \omega_{h_1} \text{-} \omega_{h_2}$$
$$\vec{k}_p = \vec{k}/2\text{n} \text{ and } \vec{k}_h = -\vec{k}/2\text{n}$$

The bound for decay changes:
$$\boldsymbol{\omega > k^2/2\mathbf{n}}$$


n - P
n - A P
k
k_i = k/2n
E = 2n ω_i = (k/2n)^2 × 2n = k/2n

Quantitative comparison



- boson can decay at any $\omega \neq 0$
- So, $Im\ G \sim$decay probablity, is never zero for $\omega \neq 0$

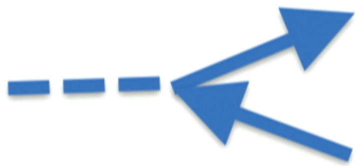
Consistent with gravity approach (no hard cone for z>1)

- At lower ω, decay can only happen via higher order
- Higher order decay has lower probability
- $Im\ G$ becomes smaller at low ω

Consistent with gravity approach

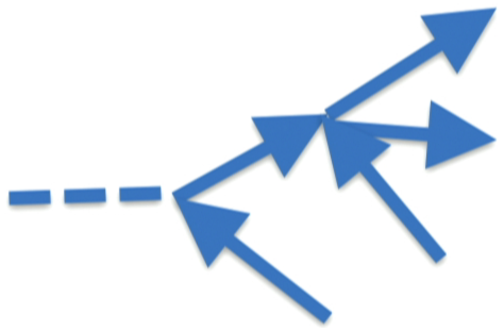
Higher order decay

- 1 particle +1 hole: no vertex



Decay rate $\sim O(g^0)$

- 2 particles +2 holes: one four-Fermi vertex

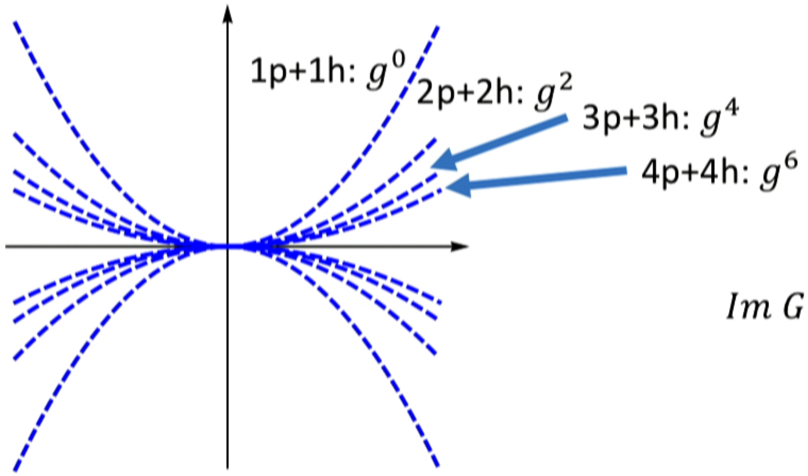


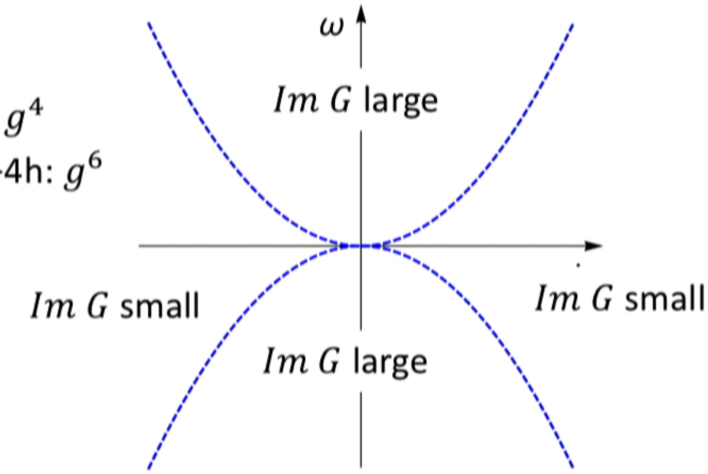
Decay rate $\sim O(g^2)$

- n particles + n holes

Decay rate $\sim O(g^{2(n-1)})$

Quantitative comparison





- Decays to n particle-hole pairs can happen only for $\omega > k^2/2\mathrm{n}$
- At low ω, possible decay channel must have $\mathrm{n} > k^2/2\omega$

  Dominate decay channel: $\mathrm{n} \sim k^2/2\omega$
- For decay channel n, the probability is

$$Im\ G \sim g^{2n-2} \sim g^{k^2/\omega\ -2} \sim \exp(-const. \frac{k^2}{\omega})$$

- Gravity approach: $Im\ G \approx \exp(-const. \frac{k^2}{\omega})$ for $\omega \ll k$

Higher z

- Conversation law (decay to n particles + n holes)

$$\vec{k} = \sum_i^n \vec{k}_{p_i} - \sum_i^n \vec{k}_{h_i}$$

$$\omega = \sum_i^n \omega_{p_i} - \sum_i^n \omega_{h_i} = \sum_i^n (k_{p_i})^z + \sum_i^n (k_{h_i})^z$$

- For fixed $\vec{k}$, ω has a lower bound:
  - $\vec{k}_p = \vec{k}/2\mathrm{n}$ and $\vec{k}_h = -\vec{k}/2\mathrm{n}$
  - $\omega_{min} = \frac{k^z}{(2n)^{z-1}}$
- Decay only happens if $\omega > \frac{k^z}{(2n)^{z-1}}$ or say if $n > \frac{1}{2}(\frac{k^z}{\omega})^{\frac{1}{z-1}}$
- Decay rate $Im\ G \sim g^{2n-2} \sim g^{(\frac{k^z}{\omega})^{\frac{1}{z-1}}-2} \sim \exp[-const.\left(\frac{k^z}{\omega}\right)^{\frac{1}{z-1}}]$
- Gravity approach: $Im\ G \approx \exp[-const.(\frac{k^z}{\omega})^{\frac{1}{z-1}}]$

Quantum Lifshitz model

$$S = \int d\vec{x}dt \left[(\partial_0 \Phi)^2 - \left(\nabla^2 \Phi\right)^2 - m\Phi^2 - g\Phi^4 \right]$$

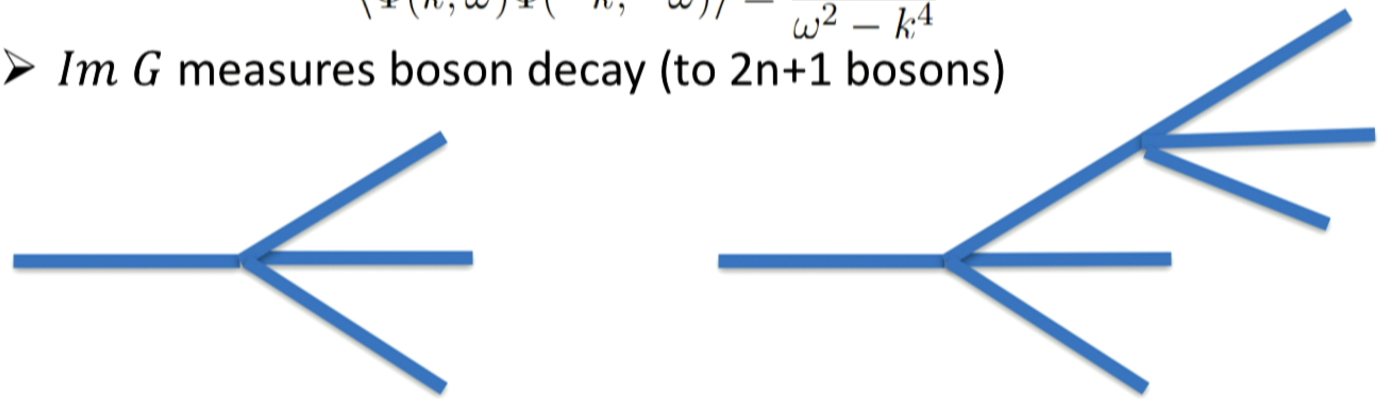
- Dimension counting

$$[\omega] = 2, \qquad [k] = 1, \qquad [\Phi] = \frac{d-2}{2}, \qquad [g] = 6 - d,$$

- (free) Green's function

$$\langle \Phi(k,\omega)\Phi(-k,-\omega) \rangle = \frac{1}{\omega^2 - k^4}$$

- *Im G* measures boson decay (to 2n+1 bosons)



Ardonne, Fendley and E. Fradkin, Annals Phys. 310, 493 (2004)

Decay rate

- Conservation law (decay to $2n + 1$ bosons)

$$\vec{k} = \sum_{i}^{2n+1} \vec{k}_i$$

$$\omega = \sum_{i}^{2n+1} \omega_i = \sum_{i}^{2n+1} (k_i)^2$$

- For fixed $\vec{k}$, ω has a lower bound:
  - $\vec{k}_i = \vec{k}/(2n+1)$
  - $\omega_{min} = \frac{k^2}{2n+1}$
- Decay only happens if $\omega > \frac{k^2}{2n+1}$ or say if $n > \frac{k^2}{2\omega} - \frac{1}{2}$
- Decay rate $Im\ G \sim g^{2n-2} \sim g^{\frac{k^2}{\omega}-3} \sim \exp(-const. \frac{k^2}{\omega})$
- Gravity approach: $Im\ G \approx \exp(-const. \frac{k^2}{\omega})$

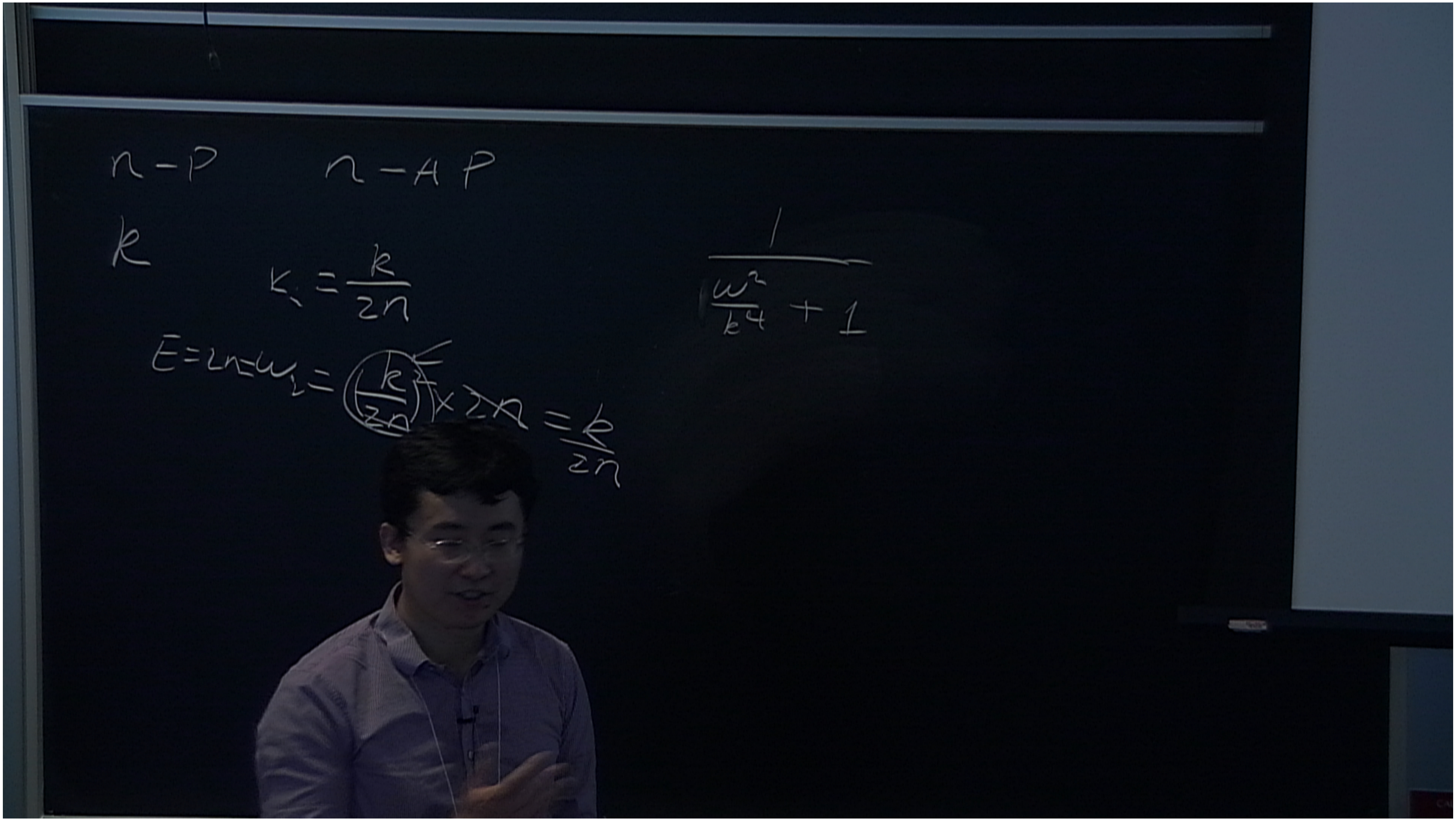
Conclusions

- z>1 critical system: more diversified.
  - Absence of Lorentz/conformal symmetry
  - Green's function cannot be uniquely determined (by dimensions)
  - In gravity theory, diversified behavior comes from the EOM

- However, for $Im\ G$, universal behaviors arises at $\omega \ll k$

$$Im\ G \approx \exp[-const.\,(\frac{k^z}{\omega})^{\frac{1}{z-1}}]$$

- Field theory: same conclusion from energy-momentum conservation


n − P
n − A P
k
k_i = k / 2n
1 / (ω^2/k^4 + 1)