Title: Theories of non-Fermi liquids

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URL: http://pirsa.org/16080047

Abstract: I will review and compare numerous models for metallic states without quasiparticle excitations. The solvable SYK model provides a useful starting point, and also has remarkable holographic connections to the quantum gravity of black holes in AdS2. Quantum critical states of two-dimensional metals are obtained by coupling the fermions to fluctuating bosonic order parameters or gauge fields: I will discuss their physical properties and possible connections to experiments.

What is a non-Fermi liquid ?

- A compressible phase at $T = 0$: the density $\mathcal{Q}$ varies smoothly as a function of μ. Global U(1) symmetry is unbroken.

- No quasiparticle excitations

- Shortest possible local-equilibration/de-phasing/ transition-to-quantum-chaos with

$$\tau_\varphi \geq C\frac{\hbar}{k_B T}$$

S. Sachdev, *Quantum Phase Transitions* (1999)

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$$

P. Kovtun, D.T. Son, and A.O. Starinets, PRL **94**, 111601 (2005)

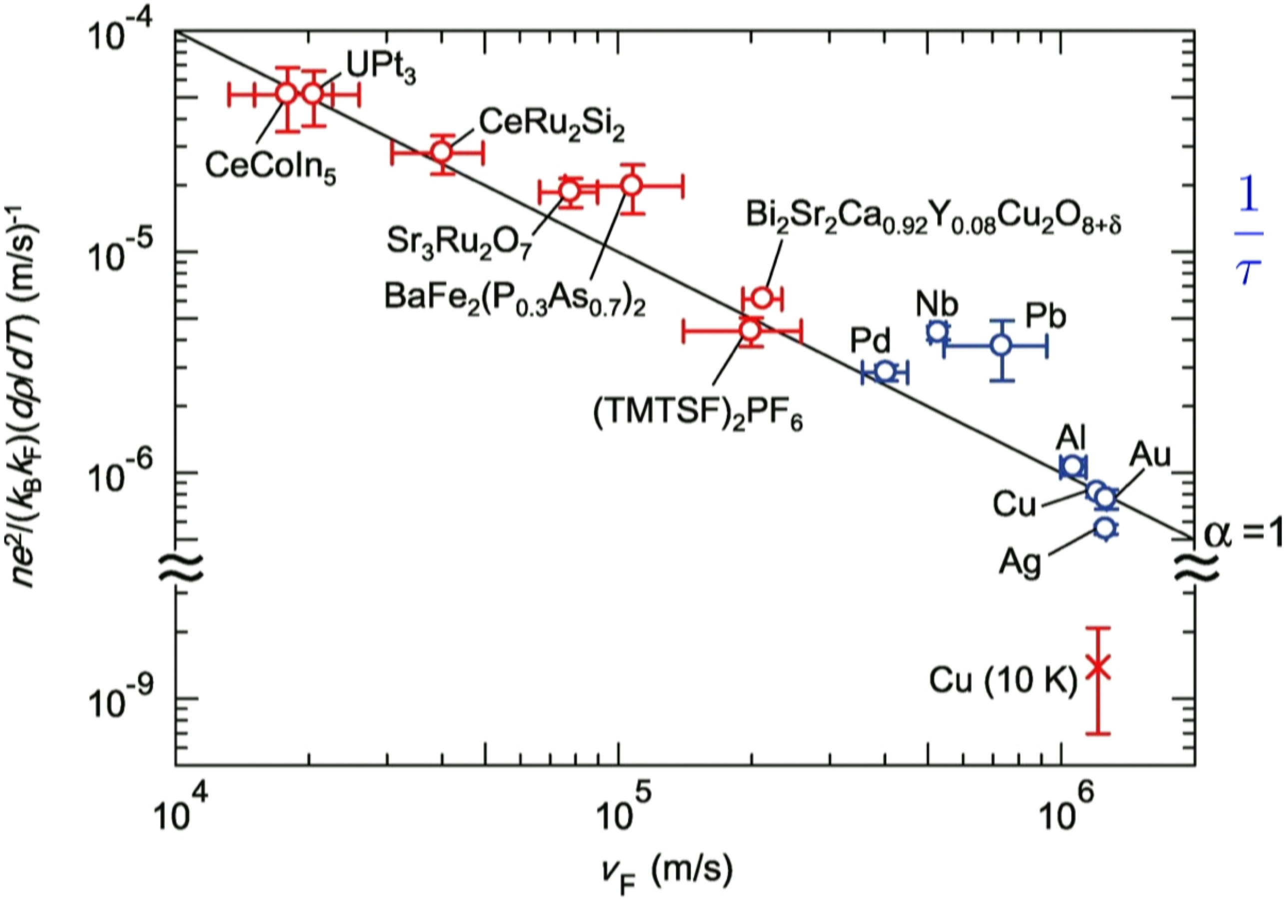
$$\frac{D}{v_b^2} \geq \tilde{C}\frac{\hbar}{k_B T}$$

Saturation requires fixed point with disorder *and* interactions
S.A. Hartnoll, Nature Physics **11**, 54 (2015)
M. Blake, PRL **117**, 091601 (2016)

$$\tau_L \geq \frac{1}{2\pi}\frac{\hbar}{k_B T}$$

J. Maldacena, S. H. Shenker and D. Stanford, JHEP 08 (2016)106

In Fermi liquids, $\tau \sim 1/T^2$; in gapped systems, $\tau \sim e^{\Delta/T}$.



$$\frac{1}{\tau} = \alpha \frac{k_B T}{\hbar}$$

J. A. N. Bruin, H. Sakai, R. S. Perry, A. P. Mackenzie, *Science*. **339,** 804 (2013)

Theories of non-Fermi liquids

- Sachdev-Ye-Kitaev (SYK) model
- Ising-nematic criticality in $d=2$
- Higgs criticality in the cuprates

Infinite-range model with quasiparticles

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j + \ldots$$

$$c_i c_j + c_j c_i = 0 \quad , \quad c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$$

$$\frac{1}{N} \sum_i c_i^\dagger c_i = \mathcal{Q}$$

t_{ij} are independent random variables with $\overline{t_{ij}} = 0$ and $\overline{|t_{ij}|^2} = t^2$

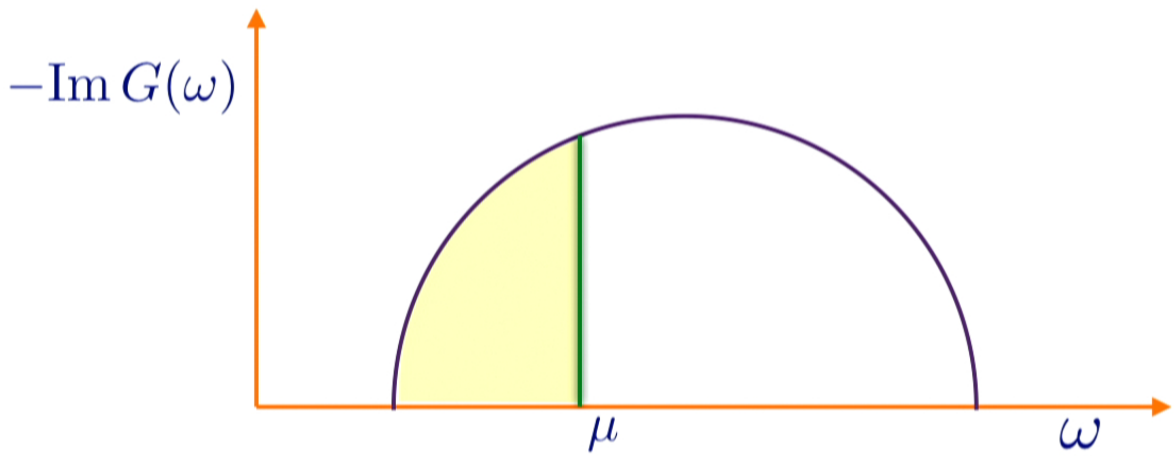
Fermions occupying the eigenstates of a
N x *N* random matrix

Infinite-range model with quasiparticles

Feynman graph expansion in $t_{ij..}$, and graph-by-graph average, yields exact equations in the large N limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = t^2 G(\tau)$$
$$G(\tau = 0^-) = \mathcal{Q}.$$

$G(\omega)$ can be determined by solving a quadratic equation.



Infinite-range model with quasiparticles

Now add weak interactions

$$H = \frac{1}{(N)^{1/2}} \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j + \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$

$J_{ij;k\ell}$ are independent random variables with $\overline{J_{ij;k\ell}} = 0$ and $\overline{|J_{ij;k\ell}|^2} = J^2$. We compute the lifetime of a quasiparticle, τ_α, in an exact eigenstate $\psi_\alpha(i)$ of the free particle Hamitonian with energy E_α. By Fermi's Golden rule, for E_α at the Fermi energy

$$\begin{aligned}\frac{1}{\tau_\alpha} &= \pi J^2 \rho_0^3 \int dE_\beta dE_\gamma dE_\delta f(E_\beta)(1 - f(E_\gamma))(1 - f(E_\delta))\delta(E_\alpha + E_\beta - E_\gamma - E_\delta) \\ &= \frac{\pi^3 J^2 \rho_0^3}{4} T^2\end{aligned}$$

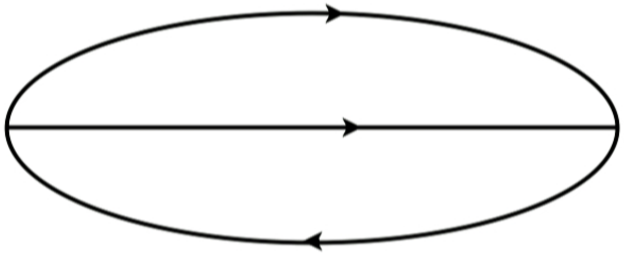
where ρ_0 is the density of states at the Fermi energy.

Fermi liquid state: Two-body interactions lead to a scattering time of quasiparticle excitations from in (random) single-particle eigenstates which diverges as $\sim T^{-2}$ at the Fermi level.

SYK model

Feynman graph expansion in $J_{ij..}$, and graph-by-graph average, yields exact equations in the large N limit:

$$G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)} \quad , \quad \Sigma(\tau) = -J^2 G^2(\tau) G(-\tau)$$
$$G(\tau = 0^-) = \mathcal{Q}.$$

$\Sigma =$ 

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

SYK model

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$$G(\tau = 0^-) = \mathcal{Q}.$$

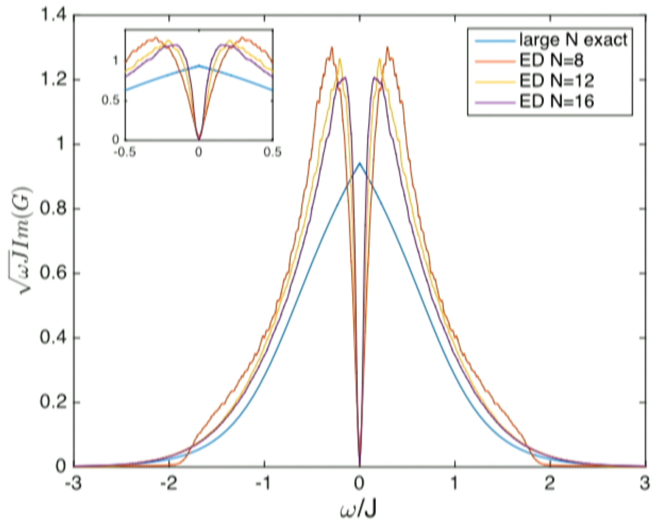
Low frequency analysis shows that the solutions must be gapless and obey

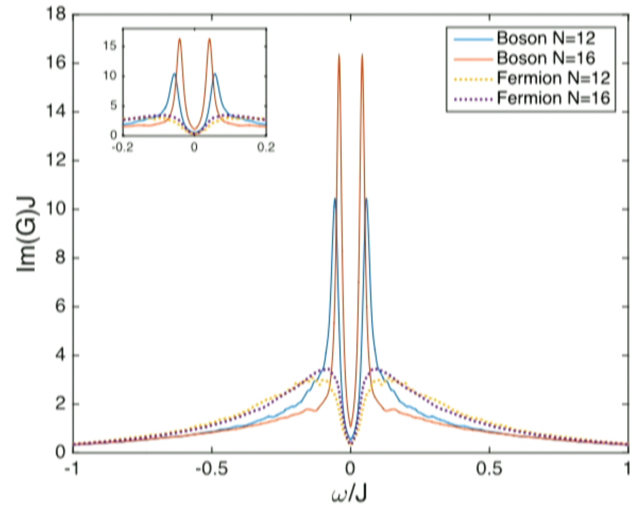
$$\Sigma(z) = \mu - \frac{1}{A}\sqrt{z} + \dots \quad , \quad G(z) = \frac{A}{\sqrt{z}}$$

for some complex A. The ground state is a non-Fermi liquid, with a continuously variable density $\mathcal{Q}$.

S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993)

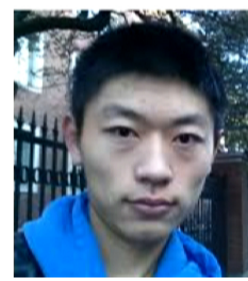
SYK model





Large N solution of equations for G and Σ agree well with exact diagonalization of the finite N Hamiltonian $\Rightarrow$ no spin-glass order

However, exact diagonalization of the same model with hard-core bosons indicates the presence of spin-glass order in the ground state.



W. Fu and S. Sachdev, arXiv: 1603.05246

It would be nice to have a solvable model of holography.

theory	bulk dual	anom. dim.	chaos	solvable in $1/N$	black hole
SYM	Einstein grav.	large	maximal	no	yes
$O(N)$	Vasiliev	$1/N$	$1/N$	yes	no
SYK	"$\ell_s \sim \ell_{AdS}$"	$O(1)$	maximal	yes	yes

column added by SS

Slide by D. Stanford at Strings 2016, Beijing

SYK model

- $T = 0$ Green's function $G \sim 1/\sqrt{\tau}$
- $T > 0$ Green's function implies conformal invariance $G \sim 1/(\sin(\pi T \tau))^{1/2}$
- Non-zero entropy as $T \to 0$, $S(T \to 0) = N S_0 + \ldots$

A. Georges, O. Parcollet, and S. Sachdev, Phys. Rev. B **63**, 134406 (2001)

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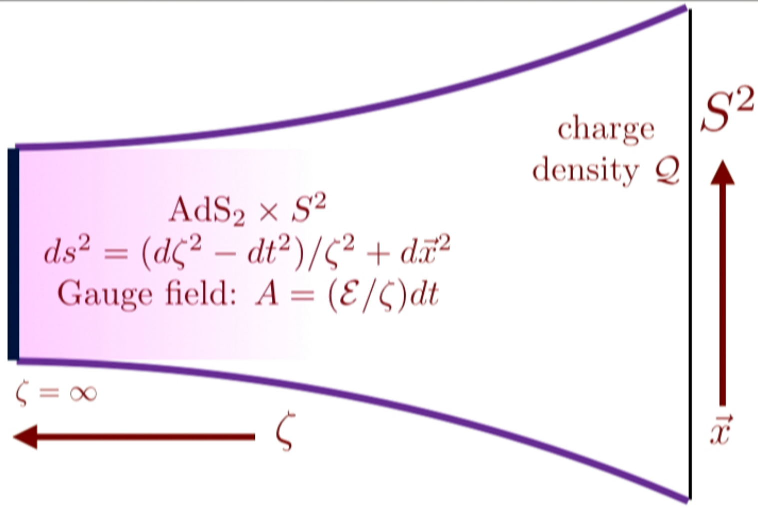
- Non-zero entropy as $T \to 0$, $S(T \to 0) = NS_0 + \ldots$
- These features indicate that the SYK model is dual to the low energy limit of a quantum gravity theory of black holes with AdS_2 near-horizon geometry. The Bekenstein-Hawking entropy is NS_0.

S. Sachdev, PRL **105**, 151602 (2010)

SYK and AdS_2



PHYSICAL REVIEW LETTERS **105, 151602 (2010)**

Holographic Metals and the Fractionalized Fermi Liquid

Subir Sachdev

Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

(Received 23 June 2010; published 4 October 2010)

We show that there is a close correspondence between the physical properties of holographic metals near charged black holes in anti–de Sitter (AdS) space, and the fractionalized Fermi liquid phase of the lattice Anderson model. The latter phase has a "small" Fermi surface of conduction electrons, along with a spin liquid of local moments. This correspondence implies that certain mean-field gapless spin liquids are states of matter at nonzero density realizing the near-horizon, $AdS_2 \times R^2$ physics of Reissner-Nordström black holes.

SYK model

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- $T > 0$ Green's function implies conformal invariance $G \sim 1/(\sin(\pi T \tau))^{1/2}$
- Non-zero entropy as $T \to 0$, $S(T \to 0) = NS_0 + \ldots$
- These features indicate that the SYK model is dual to the low energy limit of a quantum gravity theory of black holes with AdS_2 near-horizon geometry. The Bekenstein-Hawking entropy is NS_0.
- There is a scalar zero mode associated with the breaking of reparameterization invariance down to SL(2,R). The same pattern of symmetries is present in gravity theories on AdS_2.

A. Kitaev, KITP talk, 2015

SYK model

- The dependence of S_0 on the density $\mathcal{Q}$ matches the behavior of the Wald-Bekenstein-Hawking entropy of AdS_2 horizons in a large class of gravity theories.

- The scalar zero mode leads to a linear-in-T specific heat

$$S(T \to 0) = NS_0 + N\gamma T + \ldots.$$

An identical scalar zero model is also present in the low energy limit of theories of quantum gravity on AdS_2.

J. Maldacena and D. Stanford, arXiv:1604.07818

SYK model

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An identical scalar zero model is also present in the low energy limit of theories of quantum gravity on AdS_2.

- The Lyapunov time to quantum chaos saturates the lower bound both in the SYK model and in quantum gravity.

$$\tau_L = \frac{1}{2\pi} \frac{\hbar}{k_B T}$$

A. Kitaev, KITP talk, 2015

J. Maldacena and D. Stanford, arXiv:1604.07818

SYK model

After integrating the fermions, the partition function can be written as a path integral with an action S analogous to a Luttinger-Ward functional

$$Z = \int \mathcal{D}G(\tau_1, \tau_2)\mathcal{D}\Sigma(\tau_1, \tau_2) \exp(-NS)$$

A. Georges, O. Parcollet, and S. Sachdev, Phys. Rev. B **63**, 134406 (2001)

$$S = \ln\det\left[\delta(\tau_1 - \tau_2)(\partial_{\tau_1} + \mu) - \Sigma(\tau_1, \tau_2)\right] + \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2)\left[G(\tau_2, \tau_1) + (J^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right]$$

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$$+ \int d\tau_1 d\tau_2 \Sigma(\tau_1, \tau_2)\left[G(\tau_2, \tau_1) + (J^2/2)G^2(\tau_2, \tau_1)G^2(\tau_1, \tau_2)\right]$$

A. Georges, O. Parcollet, and S. Sachdev, Phys. Rev. B **63**, 134406 (2001)

At frequencies $\ll J$, the time derivative in the determinant is less important, and without it the path integral is invariant under the reparametrization and gauge transformations

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4}\frac{g(\sigma_1)}{g(\sigma_2)}G(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4}\frac{g(\sigma_1)}{g(\sigma_2)}\Sigma(\sigma_1, \sigma_2)$$

A. Georges and O. Parcollet PRB **59**, 5341 (1999)
A. Kitaev, unpublished
S. Sachdev, PRX **5**, 041025 (2015)

where $f(\sigma)$ and $g(\sigma)$ are arbitrary functions.

SYK model

Let us write the large N saddle point solutions of S as

$$G_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-1/2} \quad , \quad \Sigma_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-3/2}.$$

These are not invariant under the reparametrization symmetry but are invariant only under a SL(2,R) subgroup under which

$$f(\tau) = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1.$$

So the (approximate) reparametrization symmetry is spontaneously broken.

Reparametrization zero mode

Expand about the saddle point by writing

$$G(\tau_1, \tau_2) = [f'(\tau_1) f'(\tau_2)]^{1/4} G_s(f(\tau_1) - f(\tau_2))$$

(and similarly for Σ) and obtain an effective action for $f(\tau)$. This action does not vanish because of the time derivative in the determinant which is not reparameterization invariant.

J. Maldacena and D. Stanford, arXiv:1604.07818

See also A. Kitaev, unpublished, and J. Polchinski and V. Rosenhaus, arXiv:1601.06768

SYK model

However the effective action must vanish for SL(2,R) transformations because G_s, Σ_s are invariant under it. In this manner we obtain the effective action as a Schwarzian

$$N S_{\text{eff}} = -\frac{N\gamma}{4\pi^2} \int d\tau \, \{f, \tau\} \quad , \quad \{f, \tau\} \equiv \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 ,$$

where the specific heat, $\mathcal{C} = N\gamma T$.

The Schwarzian effective action implies that the SYK model *saturates* the lower bound on the Lyapunov time

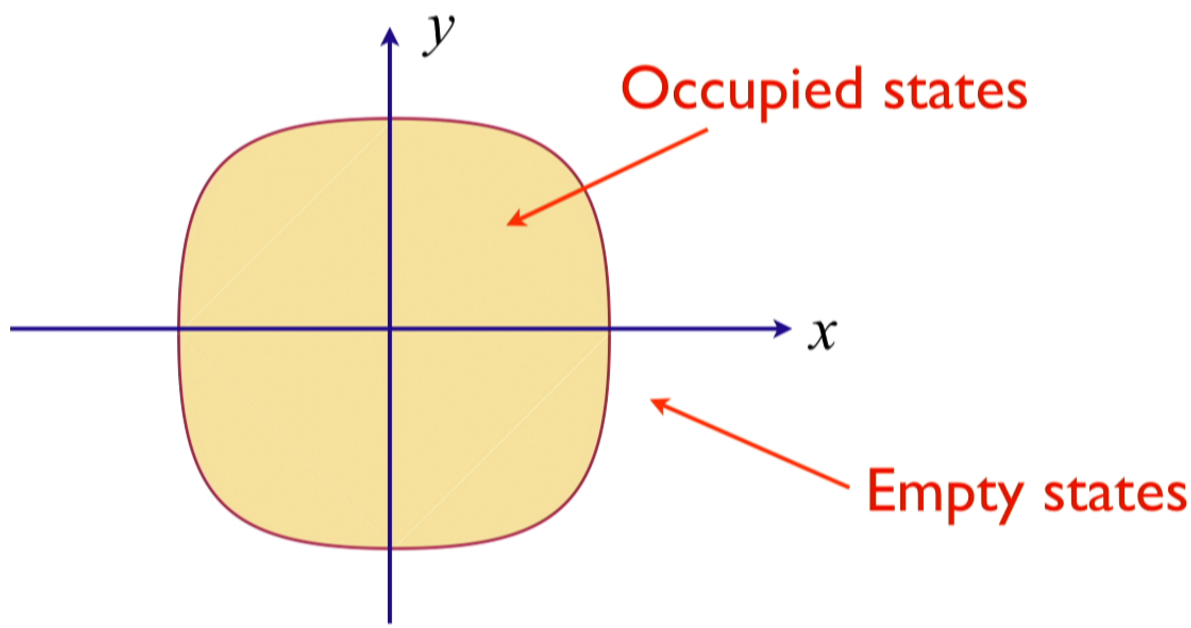
$$\tau_L = \frac{1}{2\pi} \frac{\hbar}{k_B T}$$

J. Maldacena and D. Stanford, arXiv:1604.07818
See also A. Kitaev, unpublished, and J. Polchinski and V. Rosenhaus, arXiv:1601.06768

Theories of non-Fermi liquids

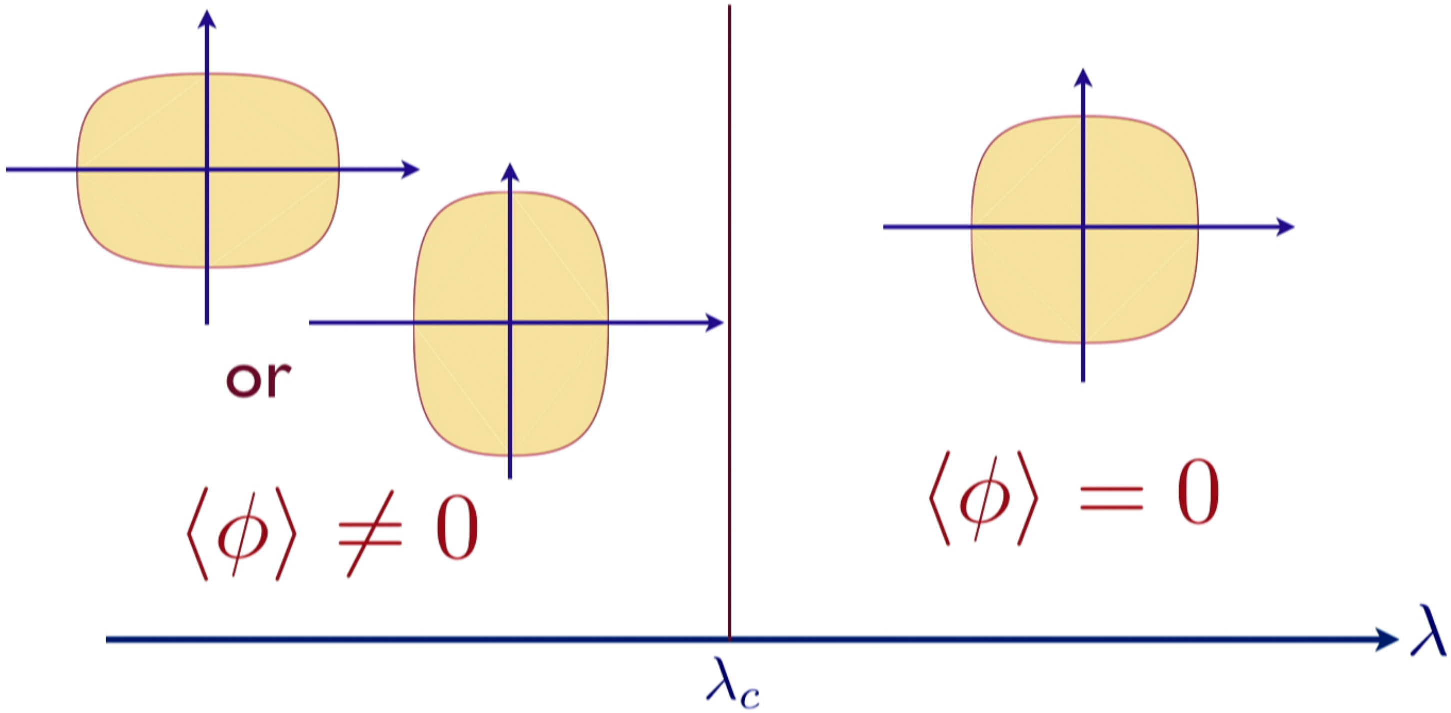
- Sachdev-Ye-Kitaev (SYK) model
- Ising-nematic criticality in d=2
- Higgs criticality in the cuprates

Quantum criticality of Ising-nematic ordering in a metal



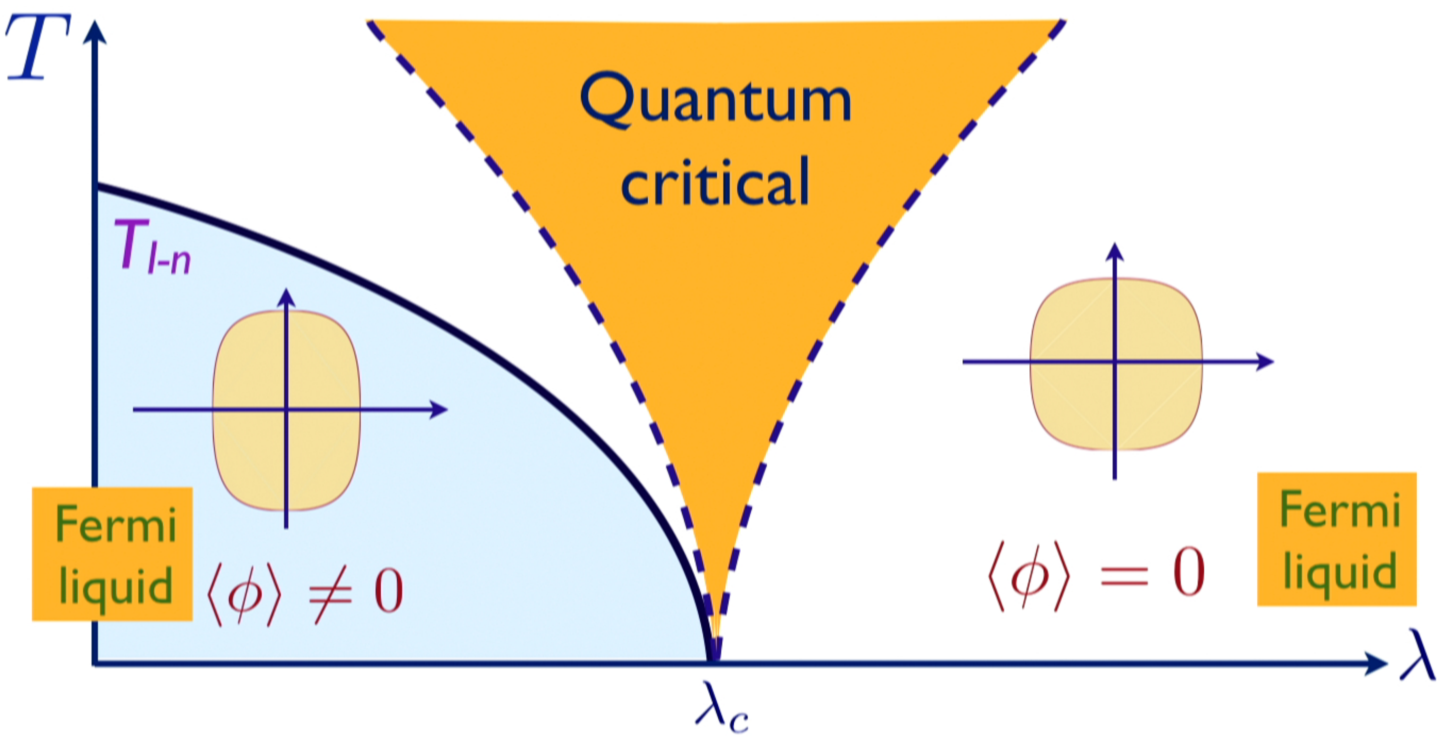
A metal with a Fermi surface
with full square lattice symmetry

Quantum criticality of Ising-nematic ordering in a metal



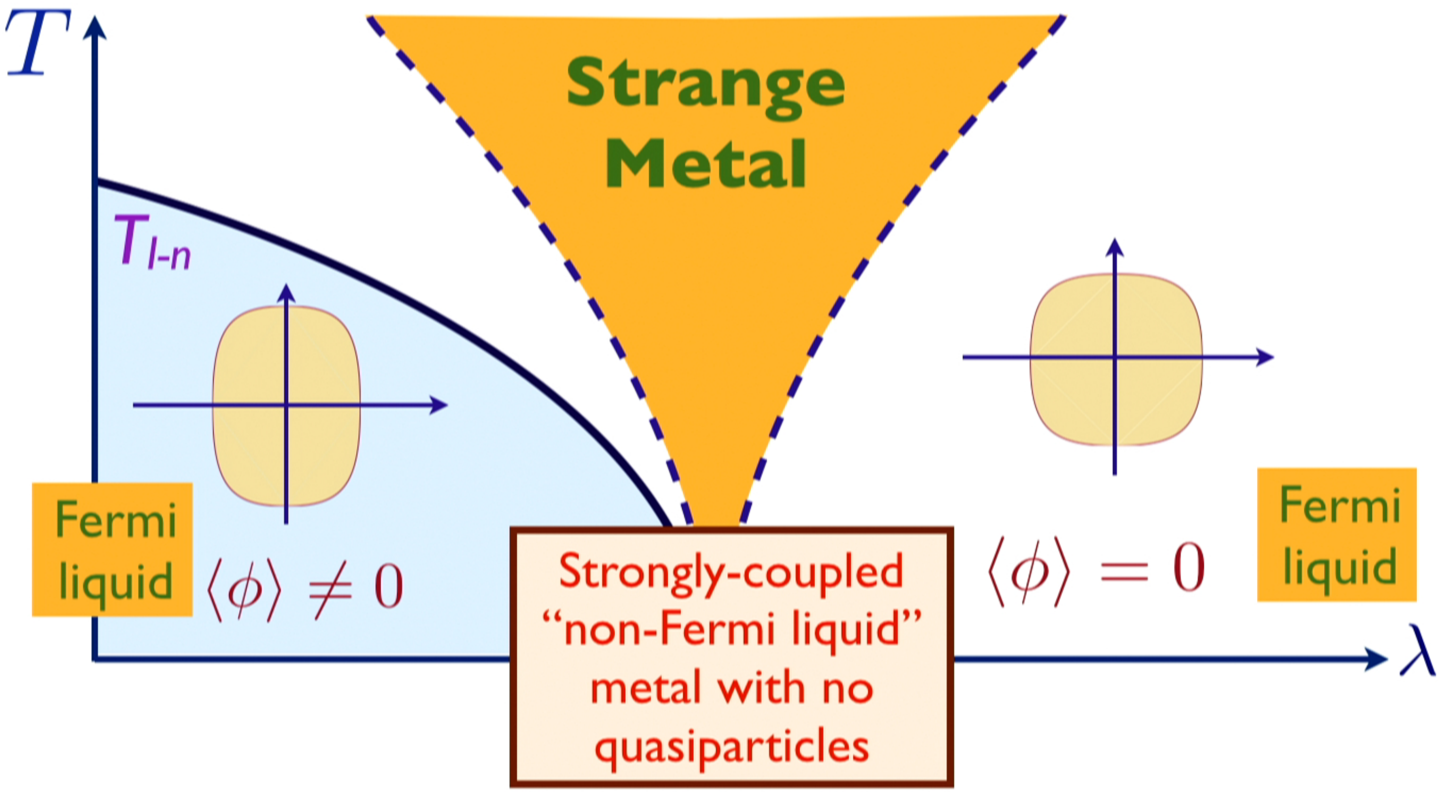
Pomeranchuk instability as a function of coupling λ

Quantum criticality of Ising-nematic ordering in a metal



Phase diagram as a function of T and λ

<u>Quantum criticality of Ising-nematic ordering in a metal</u>



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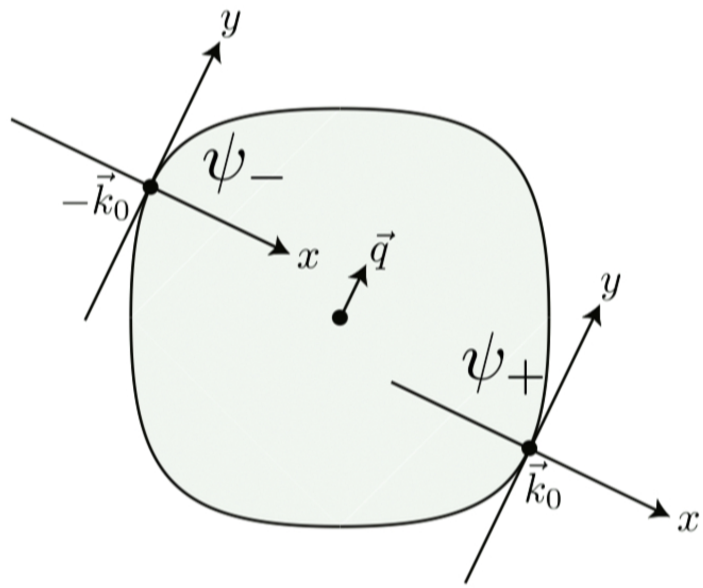


- Exact solution for some exponents for the 'one-sided' chiral model (Shouvik Sur and Sung-Sik Lee, PRB **90**, 045121 (2014)).
- Expansion in $\epsilon = 5/2 - d$ (D. Dalidovich and Sung-Sik Lee, PRB **88**, 245106 (2013)).

$$\mathcal{L}[\psi_\pm, \phi] =$$

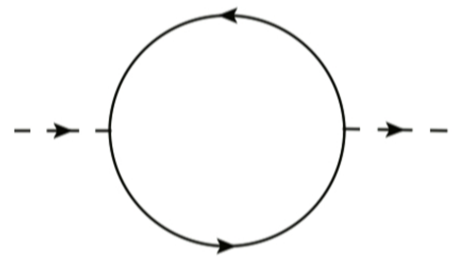
$$\psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right) \psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right) \psi_-$$

$$-\phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_-\right) + \frac{1}{2g^2} \left(\partial_y \phi\right)^2$$

M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010)

Quantum criticality of Ising-nematic ordering in a metal

$$\mathcal{L} = \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y\phi\right)^2$$



One loop ϕ self-energy with N_f fermion flavors:

$$\Sigma_\phi(\vec{q},\omega) = N_f \int \frac{d^2k}{4\pi^2}\frac{d\Omega}{2\pi}\frac{1}{\left[-i(\Omega+\omega)+k_x+q_x+(k_y+q_y)^2\right]\left[-i\Omega-k_x+k_y^2\right]} = \frac{N_f}{4\pi}\frac{|\omega|}{|q_y|}$$

Landau-damping

Quantum criticality of Ising-nematic ordering in a metal

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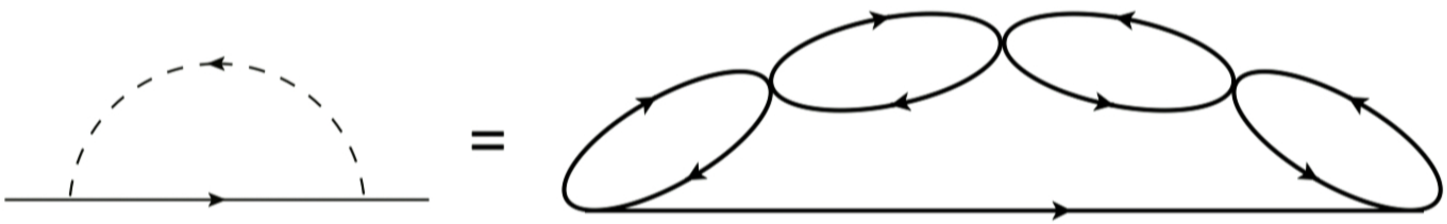


Electron self-energy at order $1/N_f$:

Diagrams similar to SYK!

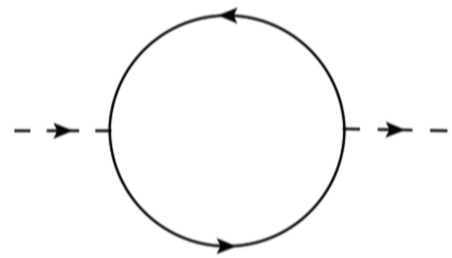
$$\Sigma(\vec{k},\Omega) = -\frac{1}{N_f}\int \frac{d^2q}{4\pi^2}\frac{d\omega}{2\pi}\frac{1}{\left[-i(\omega+\Omega)+k_x+q_x+(k_y+q_y)^2\right]\left[\frac{q_y^2}{g^2}+\frac{|\omega|}{|q_y|}\right]}$$

$$= -i\frac{2}{\sqrt{3}N_f}\left(\frac{g^2}{4\pi}\right)^{2/3}\mathrm{sgn}(\Omega)|\Omega|^{2/3}$$

$\sim |\Omega|^{d/3}$ in dimension d.

Quantum criticality of Ising-nematic ordering in a metal

$$\mathcal{L} = \psi_+^\dagger \left(\partial_\tau - i\partial_x - \partial_y^2\right)\psi_+ + \psi_-^\dagger \left(\partial_\tau + i\partial_x - \partial_y^2\right)\psi_- - \phi\left(\psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-\right) + \frac{1}{2g^2}\left(\partial_y\phi\right)^2$$



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$$= \frac{N_f}{4\pi}\frac{|\omega|}{|q_y|}$$

Landau-damping

Quantum criticality of Ising-nematic ordering in a metal

$$\mathcal{L} = \psi_+^\dagger \left(\cancel{\partial_\tau} - i\partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left(\cancel{\partial_\tau} + i\partial_x - \partial_y^2 \right) \psi_- - \phi \left(\psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} \left(\partial_y \phi \right)^2$$

Simple scaling argument for $z = 3/2$.

Under the rescaling $x \to x/s$, $y \to y/s^{1/2}$, and $\tau \to \tau/s^z$, we find invariance provided

$$\begin{aligned} \phi &\to \phi\, s \\ \psi &\to \psi\, s^{(2z+1)/4} \\ g &\to g\, s^{(3-2z)/4} \end{aligned}$$

So the action is invariant provided $z = 3/2$.

Quantum criticality of Ising-nematic ordering in a metal

The entropy density, s, obeys

$$s \sim T^{(d-\theta)/z} \sim T^{2/3}$$

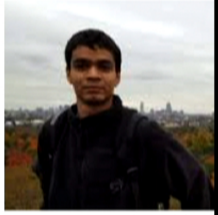
Y. B. Kim, A. Furusaki, X.-G. Wen, and P.A. Lee, PRB **50**, 17917 (1994)

where $z = 3/2$ is the dynamic critical exponent for fermionic excitations dispersing normal to the Fermi surface, and $d-\theta = 1$ is the number of dimensions normal to the Fermi surface.

A RG analysis using a dimensionality expansion below $d = 5/2$ shows that the opitcal conductivity obeys

$$\sigma \sim \omega^{(d-\theta-2)/z} \sim \omega^{-2/3}$$

A. Eberlein, I. Mandal, and S. Sachdev, PRB **94**, 045133 (2016)



We also computed the shear viscosity and found

$$\eta \sim T^{(d-\theta-2)/z} \sim T^{-2/3}$$

Note that $\eta/s \sim T^{-2/z}$ does not scale to a constant: so for the viscosity we do not have a system of reduced dimensionality, a consequence of the anisotropic scaling between the directions normal and parallel to the Fermi surface.

A. A. Patel, A. Eberlein, and S. Sachdev, arXiv:1607.03894

SU(2) gauge theory by transformation to a rotating reference frame

Field	Symbol	Statistics	$SU(2)_{\text{gauge}}$	$SU(2)_{\text{spin}}$	$U(1)_{\text{e.m.charge}}$
Electron	c	fermion	$\mathbf{1}$	$\mathbf{2}$	-1
AF order	Φ	boson	$\mathbf{1}$	$\mathbf{3}$	0
Chargon	ψ	fermion	$\mathbf{2}$	$\mathbf{1}$	-1
Spinon	R or z	boson	$\bar{\mathbf{2}}$	$\mathbf{2}$	0
Higgs	H	boson	$\mathbf{3}$	$\mathbf{1}$	0

S. Sachdev, M. A. Metlitski, Y. Qi, and C. Xu, PRB **80**, 155129 (2009); D. Chowdhury and S. Sachdev, PRB **91**, 115123 (2015); S. Sachdev and D. Chowdhury, arXiv:1605.03579

Non-Fermi liquids

- Shortest possible "phase coherence" time, fastest possible local equilibration time, or fastest possible Lyapunov time towards quantum chaos, all of order $$\frac{\hbar}{k_B T}$$
- Realization in solvable SYK model, which saturates the lower bound on the Lyapunov time. Its properties have some similarities to non-rational, large central charge CFT2s.
- Remarkable match between SYK and quantum gravity of black holes with AdS_2 horizons, including a SL(2,R)-invariant Schwarzian effective action for thermal energy fluctuations.
- Transport of non-Fermi liquids described by collective flow of fluid around impurities. Non-Fermi liquids with a critical Fermi surface have a divergent η/s as $T \to 0$. This is a consequence of the spatial anisotropy in the vicinity of a Fermi surface point, and unlike existing holographic models.
- Higgs criticality is an attractive model for the strange metal in the cuprates, consistent with recent measurements of the Hall effect as a function of doping.