Title: TBA

Date: Aug 03, 2016 02:00 PM

URL: http://pirsa.org/16080024

Abstract:

Higher order interference and coherence in probabilistic systems

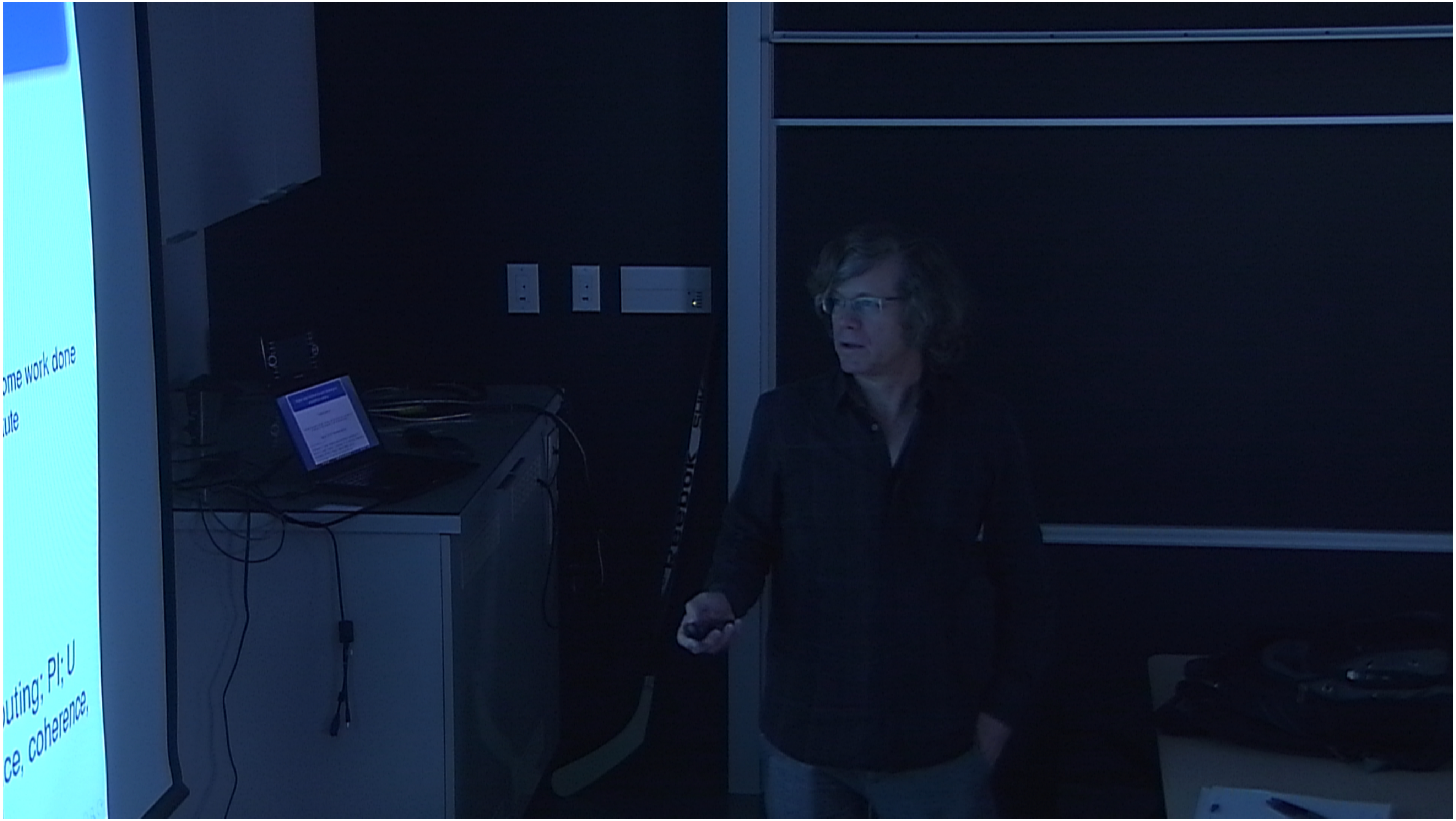
Howard Barnum

University of New Mexico and Leibniz University Hannover (current/recent), some work done at Stellenbosch Insitute of Advanced Studies and Perimeter Institute

August 3, 2016 / Perimeter Institute

hnbarnum@aol.com

Collaborators: Cozmin Ududec (Invenia Technical Computing; PI; U Waterloo), Joseph Emerson (U Waterloo, PI) (Interference, coherence, tomography, No HOI in Jordan). Markus Müller, and CU (Characterization of quantum and Jordan theories)

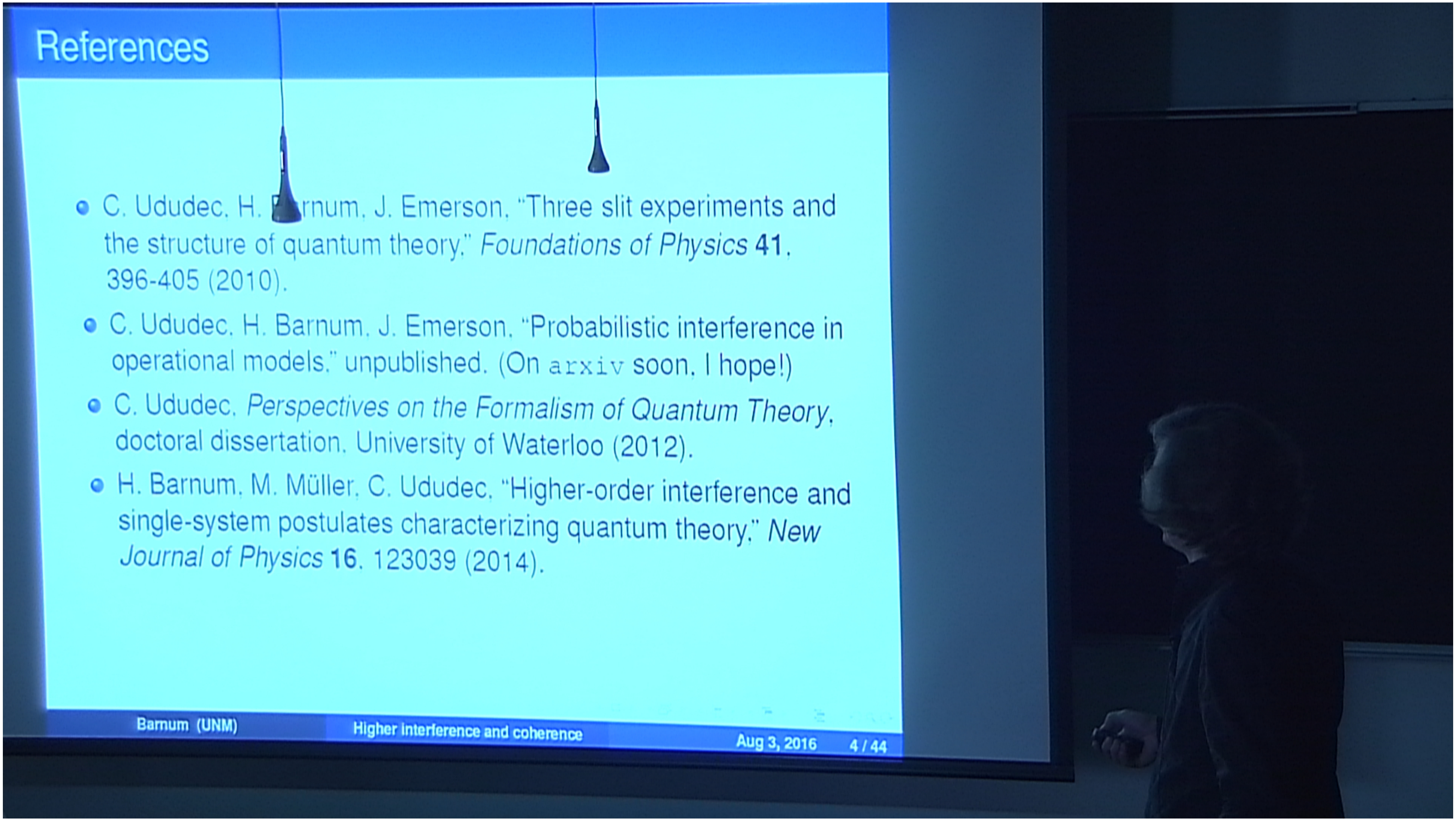

ome work done
tute
uting; PI; U
ce, coherence,

Interference: quantum and higher-order

- 2-slit experiment (and other interferometry) often thought (e.g. Feynman Lectures, atom interferometry example) essence of quantum weirdness
- As Feynman stressed, *quantum* weirdness is interference exhibited by *probabilities* rather than by definite physical quantities as in classical interference
- As Feynman also stressed, quantum interference relates to disturbance by information-gathering: which-way detectors wash it out
- Sorkin: hierarchy of interference measures for "k-slit" experiments in a "histories" framework for quantum theory.
- $I_k = 0 \implies I_{k+1} = 0$.
- Quantum theory has $I_3 = 0$ but can have $I_2 \neq 0$; classical has $I_2 = 0$.

This talk (most results w/Ududec, Emerson):

- We formulate a notion of k-slit experiment in the general probabilistic theories (GPT) framework
- We adapt Sorkin's hierarchy of interference measures to such experiments
- We formulate a notion of irreducible coherence between a set of k faces.
- Main result: k-th order interference arises from irreducible k-face coherence. Implications for state tomography.
- Also: Jordan algebraic theories have $I_3 = 0$ (for the class of *neutral* interference experiments).
- Derivation (HB, Müller, Ududec) of Jordan algebraic and quantum theory from postulates including no higher order interference.


References
C. Ududec, H. rnum, J. Emerson, "Three slit experiments and the structure of quantum theory," Foundations of Physics 41, 396-405 (2010).
C. Ududec, H. Barnum, J. Emerson, "Probabilistic interference in operational models," unpublished. (On arxiv soon, I hope!)
C. Ududec, Perspectives on the Formalism of Quantum Theory, doctoral dissertation, University of Waterloo (2012).
H. Barnum, M. Müller, C. Ududec, "Higher-order interference and single-system postulates characterizing quantum theory," New Journal of Physics 16, 123039 (2014).
Barnum (UNM)
Higher interference and coherence
Aug 3, 2016
4 / 44

References

- C. Ududec, H. Barnum, J. Emerson, “Three slit experiments and the structure of quantum theory,” *Foundations of Physics* **41**, 396-405 (2010).
- C. Ududec, H. Barnum, J. Emerson, “Probabilistic interference in operational models,” unpublished. (On `arxiv` soon, I hope!)
- C. Ududec, *Perspectives on the Formalism of Quantum Theory*, doctoral dissertation, University of Waterloo (2012).
- H. Barnum, M. Müller, C. Ududec, “Higher-order interference and single-system postulates characterizing quantum theory,” *New Journal of Physics* **16**, 123039 (2014).

Probabilistic Theories

Quantum theory doesn't have higher-order interference so we need a more general framework.

General Probabilistic Theories (GPT) framework commonly used to study informational and physical properties of quantum theory and information processing, and to formulate possible new physics, in a much broader, but flexible and probabilistically consistent framework.

Theory: Set of systems

System: Specified by bounded convex sets of allowed states, allowed measurements, allowed dynamics compatible with each measurement outcome. (Could view as a category.)

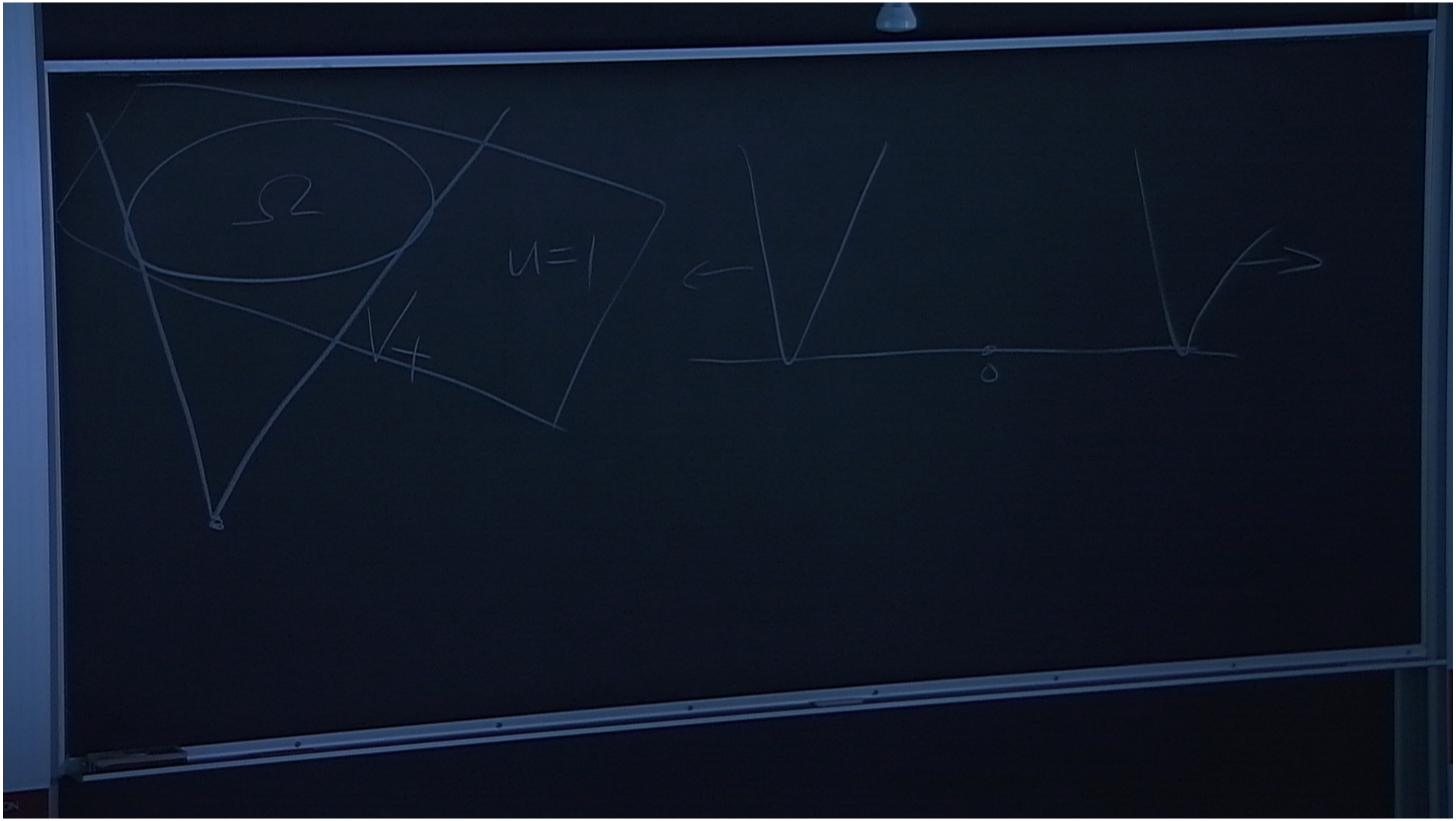
Composite systems: Rules for combining systems to get a composite system, e.g. tensor product in QM. (Could view as making it a symmetric monoidal category). Composites won't appear much in this talk.

State spaces and measurements

Normalized states of system A: Convex compact set Ω_A of dimension $d-1$, embedded in $A \simeq \mathbb{R}^d$ as the base of a regular **cone** A_+ of unnormalized states (nonnegative multiples of Ω_A). [Define cone.]
Measurement outcomes: linear functionals $A \to \mathbb{R}$ called **effects** whose values on states in Ω_A are in $[0,1]$.
Unit effect u_A has $u_A(\Omega_A) = 1$.
Measurements: Indexed sets of effects e_i with $\sum_i e_i = u_A$ (or continuous analogues).
The effects generate the **dual cone** A_+^*, of functionals nonnegative on A_+.
A_+ is **regular**: closed, generating, convex, pointed.
Define $a \geq b := a - b \in A_+$. This makes A an **ordered linear space** (inequalities can be added and multiplied by positive scalars).

Convex structure from operational considerations

- From a "phenomenological description": preparations $\times$ measurement outcomes $\rightarrow$ probabilities, get convex, bounded sets of states Ω, effects $L \subseteq [0, u]$ by "dividing out probabilistic equivalence": identify preparations with same prob for all outcomes; identify outcomes that have same prob in all states.
- Both empirics and theory involved: actual and idealized/expected phenomenology. ("operational... or aspirational?")
- For operational reasons, L is convex, full-dimensional and closed under $e \mapsto u_A - e$. For convenience, topologically closed (hence compact).
- We'll **assume** $L = [0, u_A]$. ("all mathematically consistent effects are allowed.") Desirable to relax this.
- **Important caveat:** In looking for need to extend existing theories, we may look for probabilistic distinctions between measurement/preparation procedures that existing theory might not make room for & existing experiments not have detected. Or we might need new procedures.



Ω
u=1
V+
0

Transformations, dynamics

- **Transformations** from system A to system B: linear maps $T : A \to B$ such that $T(A_+) \subseteq A_+$ and $u(Tx) \leq u(x)$ for all $x \in A_+$. (Generalizes trace non-increasing positive maps.)
- **Normalization-preserving** if $T(\Omega_A) \subseteq \Omega_B$. (Generalizes trace-preserving.)
- **Reversible** transformations $A \to A$ have $T(\Omega_A) = \Omega_A$. (Generalizes $\rho \mapsto U\rho U^\dagger$ for U unitary).
- Could specify for each A, B, a convex semigroup of “allowed" transformations (these could be the morphisms (or induced by morphisms) in a category-theoretic formulation).

Examples

Classical: A is the space of n-tuples of real numbers; $u(x) = \sum_{i=1}^{n} x_i$. So Ω_A is the probability simplex, A_+ the positive (i.e.nonnegative) orthant $x : x_i \geq 0, i \in 1, ..., n$

Quantum: $A = \mathscr{B}_h(\mathbf{H})$ = self-adjoint operators on complex (f.d.) Hilbert space $\mathbf{H}$; $u_A(X) = \mathrm{Tr}(X)$. Then Ω_A = density operators. A_+ = positive semidefinite operators.

Squit (or P/Rbit): Ω_A a square, A_+ a four-faced polyhedral cone in $\mathbb{R}^3$.

Inner-product representations: $\langle X, Y\rangle = \mathrm{tr}\, XY$ (Quantum)
$\langle x, y\rangle = \sum_i x_i y_i$ (Classical)

Quantum and classical cones are self-dual! Squit cone is not, but is isomorphic to dual.

Faces of convex sets

- **Face** of convex C: subset S such that if $x \in S$ & $x = \sum_i \lambda_i y_i$, where $y_i \in C$, $\lambda_i > 0$, $\sum_i \lambda_i = 1$, then $y_i \in S$.
- **Exposed face**: intersection of C with a supporting hyperplane.
- Faces are convex. Exposed faces are faces.
- For any subset S of a convex set C, the **exposed face generated by** S, written $\mathrm{ExpFace} S$, is the smallest exposed face containing S.
- For effects e, $e^0 := \{x \in \Omega) : e(x) = 0\}$ and $e^1 := \{x \in \Omega : e(x) = 1\}$ are exposed faces of Ω.
  I.e., exposed faces are conclusively excludable/confirmable subsets.
- *Every* exposed face is an e^0 and f^1, for some effects $e, f \in [0, u]$.

Extremal points and rays

- **Extremal points** of a convex set: ones that can't be written as a nontrivial convex combination of things in the set.
- In *any* convex set, all extremal points are faces, though not all are exposed.
- They are **atoms** of the face lattice (and if exposed, of the exposed face lattice): they are above the lattice 0, with nothing between them and 0.
- **Extremal rays** of V_+ are subsets $\mathbb{R}_+ x$, where x is extremal in a base Ω for V_+. (Only 0 is an extremal point of a pointed cone.)

The lattice of faces

- **Lattice**: partially ordered set such that every pair of elements has a least upper bound ("join") $x \vee y$ and a greatest lower bound ("meet") $x \wedge y$.
- The exposed faces of any convex set, ordered by set inclusion, form a lattice. Meet is intersection, join is $F \vee G = \mathrm{ExpFace}(F \cup G)$. (Draw some examples)
- $\emptyset$ is a face according to the above definition; if C is V_+, we will exclude it from the face lattice by convention.
- The face lattices of Ω and of V_+ are isomorphic under $F \leftrightarrow \mathbb{R}_+ F, \emptyset \leftrightarrow \{0\}$.

Finite Boolean lattices

- The set $\mathscr{P}(X)$ of subsets of a finite set X, ordered by inclusion, is a lattice. Join is union, meet is intersection.
- It has a unique complementation: $S' = X \setminus S := \{x \in X : x \notin S\}$. (It's an orthocomplementation.)
- Join distributes over meet and vice versa, so it is a *Boolean lattice* (aka Boolean algebra): a distributive, complemented lattice. (Automatically orthocomplemented.)
- These are the face lattices of finite-dimensional classical systems.

Complemented and orthocomplemented lattices

- **Bounded lattice**: Has an upper bound, called 1, and lower bound, 0.
- **Complemented lattice**: bounded lattice in which every element x has a **complement**: x' such that $x \vee x' = 1$, $x \wedge x' = 0$. (Remark: x' not necessarily unique.)
- **complementation**: choice of distinguished complement x', for every x.
- **orthocomplementation**: complementation that is involutive ($x'' = x$) and order-reversing, ($x \leq y \implies x' \geq y'$). (Remark: orthocomplementation still not necessarily unique.)
- Orthocomplemented lattices satisfy DeMorgan's laws:
  $x' \vee y' = (x \wedge y)'$
  $x' \wedge y' = (x \vee y)'$.

Finite Boolean lattices

- The set $\mathscr{P}(X)$ of subsets of a finite set X, ordered by inclusion, is a lattice. Join is union, meet is intersection.
- It has a unique complementation: $S' = X \setminus S := \{x \in X : x \notin S\}$. (It's an orthocomplementation.)
- Join distributes over meet and vice versa, so it is a *Boolean lattice* (aka Boolean algebra): a distributive, complemented lattice. (Automatically orthocomplemented.)
- These are the face lattices of finite-dimensional classical systems.

Orthomodularity

Definition

An orthocomplemented lattice is **orthomodular** if
$F \leq G \implies G = F \vee (G \wedge F')$.

Think of it as "when $F \leq G$, F has a "complement relative to G", namely $G \wedge F'$.
(Draw Boolean case.)

- The closed subspaces of a real or complex Hilbert space, or of a "quaternionic Hilbert space" are orthomodular lattices, so:
- Orthomodular lattices (OMLs) are proposed in "quantum logic" as an appropriate abstraction of this "logic" of quantum theory.
- OML's are precisely those orthocomplemented lattices that are determined by their Boolean subalgebras.

Perfectly distinguishable states

States $\omega_1, ..., \omega_n \in \Omega$ called **perfectly distinguishable** if there exist allowed effects $e_1, ..., e_n$, with $\sum_i e_i \leq u$, such that $e_i(\omega_j) = \delta_{ij}$.
Sets of states $S_1, ..., S_n$ are perfectly distinguishable if every selection $\omega_1 \in S_1, ..., \omega_n \in S_n$ is perfectly distinguishable.

Proposition (Probably folkloric)

If $S_1, ..., S_n \subseteq \Omega$ are perfectly distinguishable, then $\mathrm{ExpFace}(S_1), ..., \mathrm{ExpFace}(S_n)$ are perfectly distinguishable.

Generalized coherence

Definition

Let S be a set of not necessarily disjoint, but linearly independent faces of the positive cone V_+. There is *irreducible coherence* between the faces $F \in S$, if

$$\bigvee_{F \in S} F \not\subseteq \mathrm{lin}[\bigcup_{F \in S} F]. \quad (1)$$

Linearly independent means that no $F \in S$ lies entirely in the linear span of the others.

A state in $\bigvee_{F \in S} F$ but not in $\mathrm{lin}[\bigcup_{F \in S} F]$ "exhibits irreducible coherence" between the $F \in S$. (Extension to states not in $\bigvee_{F \in S}$ is obvious if it has a canonical component in $\mathrm{lin} \bigvee_{F \in S}$, e.g. if there is a canonical positive projection onto this.)

Interference experiments

We must associate slits/paths with *transformations*.

A *projection* P is an idempotent (i.e., $PP = P$) transformation.

Definition

A k-slit interference experiment is an indexed set $\{P_J\}_{J \subseteq I}$ of projections, where I is a finite set and $k = |I|$, satisfing the following conditions:

1. $\mathrm{im}_+ P_J$ is a face, which we'll call F_J, of V_+.
2. $P_J P_K = P_K P_J = P_{J \cap K}$
3. $F_J = \bigvee_{i \in J} F_i$
4. $J \neq \emptyset \implies P_J \neq 0$.

where the join in the third item is in the face lattice of V_+.

Do a quantum example: use mutually orthogonal set of Hilbert subspaces.

Interference experiments

We must associate slits/paths with *transformations*.

A *projection* P is an idempotent (i.e., $PP = P$) transformation.

Definition

A k-slit interference experiment is an indexed set $\{P_J\}_{J \subseteq I}$ of projections, where I is a finite set and $k = |I|$, satisfing the following conditions:

1. $\mathrm{im}_+ P_J$ is a face, which we'll call F_J, of V_+.
2. $P_J P_K = P_K P_J = P_{J \cap K}$
3. $F_J = \bigvee_{i \in J} F_i$
4. $J \neq \emptyset \implies P_J \neq 0$.

where the join in the third item is in the face lattice of V_+.

Do a quantum example: use mutually orthogonal set of Hilbert subspaces.

Interference experiments II

Proposition

The map $J \mapsto F_J$ is an injective homomorphism from the Boolean lattice of subsets of J to the face lattice of Ω.

Idempotence is a kind of weak nondisturbance; that the images are faces and we have a lattice homomorphism implies, for example that $\mathrm{im}_+ P_{J \cup K} = \mathrm{im}_+ P_J \bigvee \mathrm{im}_+ P_K$, making $P_{J \cup K}$ a "coherent coarsegraining" of the transformations P_J and P_K.

Interference expressions and patterns

Define $P^{(k)} := \sum_{l=1}^{k-1} (-1)^{l-1} \sum_{|J|=k-l} P_J$.
$P^{(k)}$ is idempotent; it is open whether it is necessarily positive.
If $I := \{1, 2\}$, $P^{(2)} = P_1 + P_2$, and if $I := \{1, 2, 3\}$,
$P^{(3)} = P_{12} + P_{13} + P_{23} - P_1 - P_2 - P_3$. (Notation: $P_{ijk} := P_{\{i,j,k\}}$.)

Definition

The k***-th order interference expression*** (for measurement outcome e with respect to the experiment $\{P_J\}_{J \subseteq I}$ on input state ω) is:

$$I_k[e, \{P_J\}_{J \subseteq I}, \omega] := e([P_I - P^{(k)}](\omega)) \equiv \langle e, [P_I - P^{(k)}]\omega \rangle. \qquad (2)$$

The k-th order ***interference pattern*** produced by state ω and final observable $\{e_j\}$ is the function $j \mapsto \langle e_k, P_I \omega \rangle - \langle e_k, P^{(k)} \omega \rangle$.

"Final observable" index j is like "position on screen".
Obviously $I_k[e, \{P_J\}_{J \subseteq I}, \omega] = 0$ is equivalent to $P^{(k)} = P_I$.

A simple example with 3rd-order interference

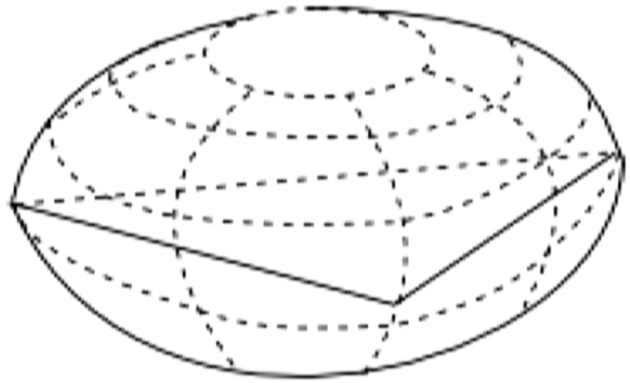


Figure: The 'Triangular Pillow' state space of Alfsen and Shultz (*Geometry of Operator Algebras*, Birkhäuser, Fig. 8.1). Every face is part of an $N = 2$ or $N = 3$ interference experiment. The $N = 3$ experiment whose faces are generated by subsets of the three vertices of the equatorial triangle (in the "xy" plane) has third-order interference, for any inital state not in the equatorial plane and any final two-outcome measurement for which the tangencies of the zero-sets of the effects are not in the equatorial plane (i.e., that is sensitive to the "z" dimension).

More simple examples with 3rd order interference

- Convex hexahedral state space (draw).
- Convex hull of the 8-affine-dimensional state space of a qutrit, and two points on either side of the 8d affine plane, lying on a line through the maximally mixed state. With slits

In all these examples, when the "slits" in the experiment correspond to three distinguishable extremal points in the "equatorial" hyperplane, $P^{(3)}$ projects onto this hyperplane *and is positive*, so represents a potential physical **"higher-order decoherence"** process.
Open question: Is $P^{(k)}$ *always* positive, when $k \geq 3$? ($P^{(2)}$ is.) If not, can we find interesting necessary and/or sufficient conditions?
This is important, especially for $P^{(3)}$ in cases when its image is standard quantum theory. Higher-order decoherence could help explain why higher-order interference is hard to observe.

More general transformations?

Can we get a reasonable notion of interference experiment involving more general transformations? Some observations:

- General substochastic maps M, with the rule $M_J = \sum_{i \in J} M_i$, can give $I_2 \neq 0$ if we naively extend the I_k definitions. *Diagonal* substochastic maps give $I_2 = 0$, though.
- Quantum possibility: experiment $\{T_J\}_{J \subseteq I}$, with $T_J : X \mapsto F_J X F_J$, where E_i for $i \in \{I\}$ are positive semidefinite, and $E_J = \sqrt{\sum_i E_i^2}$. $I_3 = 0$ if the E_i commute. (Otherwise? My guess is not.)
- Conditions like $P_J P_K = P_{J \cap K}$ and the other in the definition of interference experiment can be checked tomographically. But real-world transformations will only *approximately* satisfy them. We need to be able to recognize interference in such cases too!

More general transformations, cont'd

A relatively minor tweak to the definition would allow situations where P_I is a projection, $\mathrm{im}_+ P_I$ is not a face of V_+, but $\mathrm{im}_+ P_J$ for $J \subseteq I$ are faces of $\mathrm{im}_+ P_J$, though also not faces of V_+. Then "coherent coarsegraining" can be defined in terms of the face lattice of $\mathrm{im}_+ P_I$.

Neutral projections

Some early versions of this paper, and Cozmin's thesis, made stronger assumptions on the projections in interference experiments: that they are "filters" (see below), i.e. the duals of what Alfsen and Shultz called *compressions*. But we also observed that what we used in proving "interference arises from coherence" were only the conditions in the above definition of "interference experiment". An additional property (possessed by filters) lets us say a bit more.

A projection is **neutral** if "whatever passes it with probability one, passes it unchanged", i.e. $u(P\omega) = u(\omega) \implies P\omega = \omega$.
There exist non-neutral projections whose positive image is a face. (Examples: decohere into isomorphic subcones and then pile them on top of each other. E.g. all states to a fixed state.)

Neutral interference experiments and tomography

Proposition

If $\{P_J\}_{J \subseteq I}$ is an interference experiment with $|I| = N$, and all the P_J are neutral, then we can do complete state tomography on any state $\omega \in F_I$ by doing it on all of the $N-1$-slit "filtered" states $P_K\omega$, for all K $|K| = N-1$.

Neutrality assures that information needed for tomography is not destroyed by being disturbed by the slits, even if it doesn't interfere in the experiment.

Jordan Algebraic Systems

- Pascual Jordan, (Nachr. Akad. Göttingen, 1933):
  - **Jordan algebra:** abstracts properties of Hermitian operators.
  - Symmetric product $\bullet$ abstracts $A \bullet B = \frac{1}{2}(AB + BA)$.
  - Jordan identity: $a \bullet (b \bullet a^2) = (a \bullet b) \bullet a^2$.
  - **Formally real JA:** $a^2 + b^2 = 0 \implies a = b = 0$. Makes the cone of squares a candidate for unnormalized state space.
- Jordan, von Neumann, Wigner (Ann. Math., **35**, 29-34 (1934)): the simple f.d. formally real Jordan algebras are:
  - quantum systems (self-adjoint matrices) over $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$;
  - systems whose state space is a ball (aka "spin factors");
  - 3×3 Hermitian octonionic matrices ("exceptional" JA).
- f.d. homogeneous self-dual cones are precisely the cones of squares in f.d. formally real Jordan algebras. (Koecher 1958, Vinberg 1960)
- Simple algebras correspond to irreducible cones.

Characterization of Quantum Theory (HB, Markus Müller, Cozmin Ududec)

1. **Generalized "spectrality"**: every state is in convex hull of a set of perfectly distinguishable pure (i.e. extremal) states
2. **Symmetry**: Every set of perfectly distinguishable pure states transforms to any other such set of the same size **reversibly**.
3. **No irreducibly three-slit** (or more) **interference.**
4. **Energy observability**: Systems have nontrivial continuously parametrized reversible dynamics. Generators of one-parameter continuous subgroups ("Hamiltonians") are associated with nontrivial conserved observables.

•$1-4 \implies$ standard quantum system (over $\mathbb{C}$)

•$1-3 \implies$ irreducible Jordan algebraic systems, and classical.

•$1-2 \implies$ "projective" (every face the positive image of special kind of projection we call a **filter**), self-dual systems

Reference

New J. Phys **16** 123029 (2014). Open access. Also `arxiv:1403.4147`.

Filters

Filter (abstracts/formalizes "slit"):
normalized positive linear map $P : A \to A$: $P^2 = P$, with P and P^* both complemented.
Complemented means $\exists$ filter P' such that $\mathrm{im}_+ P = \ker_+ P'$.
Normalized means $\forall \omega \in \Omega \;\; u(P\omega) \leq 1$.

- Dual of Alfsen and Shultz' (*Geometry of State Spaces of Operator Algebras*, Birkhauser 2003) notion of **compression**.
- Filters are **neutral**: $u(P\omega) = u(\omega) \implies P\omega = \omega$. So $\mathrm{im}_+ P$ is a face.
- P' is the *unique* complementary filter.
- Ω called **projective** if every face is the positive part of the image of a filter.

Quantum example: $\rho \mapsto Q\rho Q$ where Q is the orthogonal projector onto a subspace of Hilbert space $\mathscr{H}$.

34. Characterizing Jordan algebraic systems

Theorem (Adaptation of Alfsen & Shultz, Thm 9.3.3)

Let a finite-dimensional system satisfy

(a) **Projectivity**: *there is a filter onto each face*

(b) **Symmetry of Transition Probabilities**, *and*

(c) **Filters Preserve Purity**: *if ω is a pure state, then $P\omega$ is a nonnegative multiple of a pure state.*

Then Ω is the state space of a formally real Jordan algebra.

Theorem (Barnum, Müller, Ududec)

*(***Weak Spectrality** *&* **Strong Symmetry***)* $\Longrightarrow$ **Projectivity** *&* **STP***;*
WS *&* **SS** *&* **No Higher Interference** $\Longrightarrow$ **Filters Preserve Purity***.*
Jordan algebraic system thus obtained must be either irreducible or classical. (All such satisfy **WS**, **SS**, **No HOI***.)*

34. Characterizing Jordan algebraic systems

Theorem (Adaptation of Alfsen & Shultz, Thm 9.3.3)

Let a finite-dimensional system satisfy

(a) **Projectivity**: *there is a filter onto each face*

(b) **Symmetry of Transition Probabilities**, *and*

(c) **Filters Preserve Purity**: *if ω is a pure state, then $P\omega$ is a nonnegative multiple of a pure state.*

Then Ω is the state space of a formally real Jordan algebra.

Theorem (Barnum, Müller, Ududec)

*(***Weak Spectrality** *&* **Strong Symmetry***)* $\implies$ **Projectivity** *&* **STP***;*
WS *&* **SS** *&* **No Higher Interference** $\implies$ **Filters Preserve Purity**.
Jordan algebraic system thus obtained must be either irreducible or classical. (All such satisfy **WS**, **SS**, **No HOI**.*)*