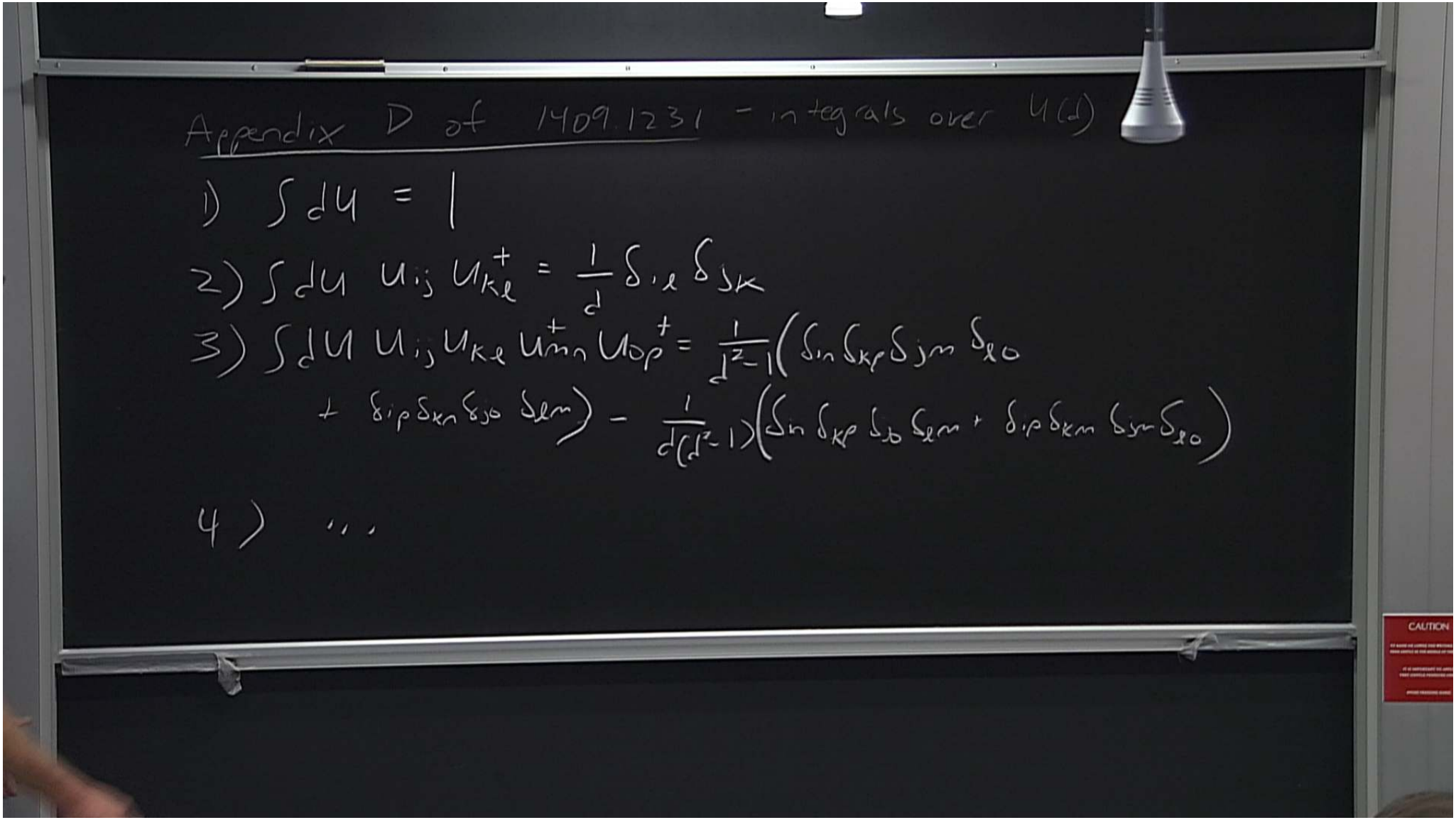
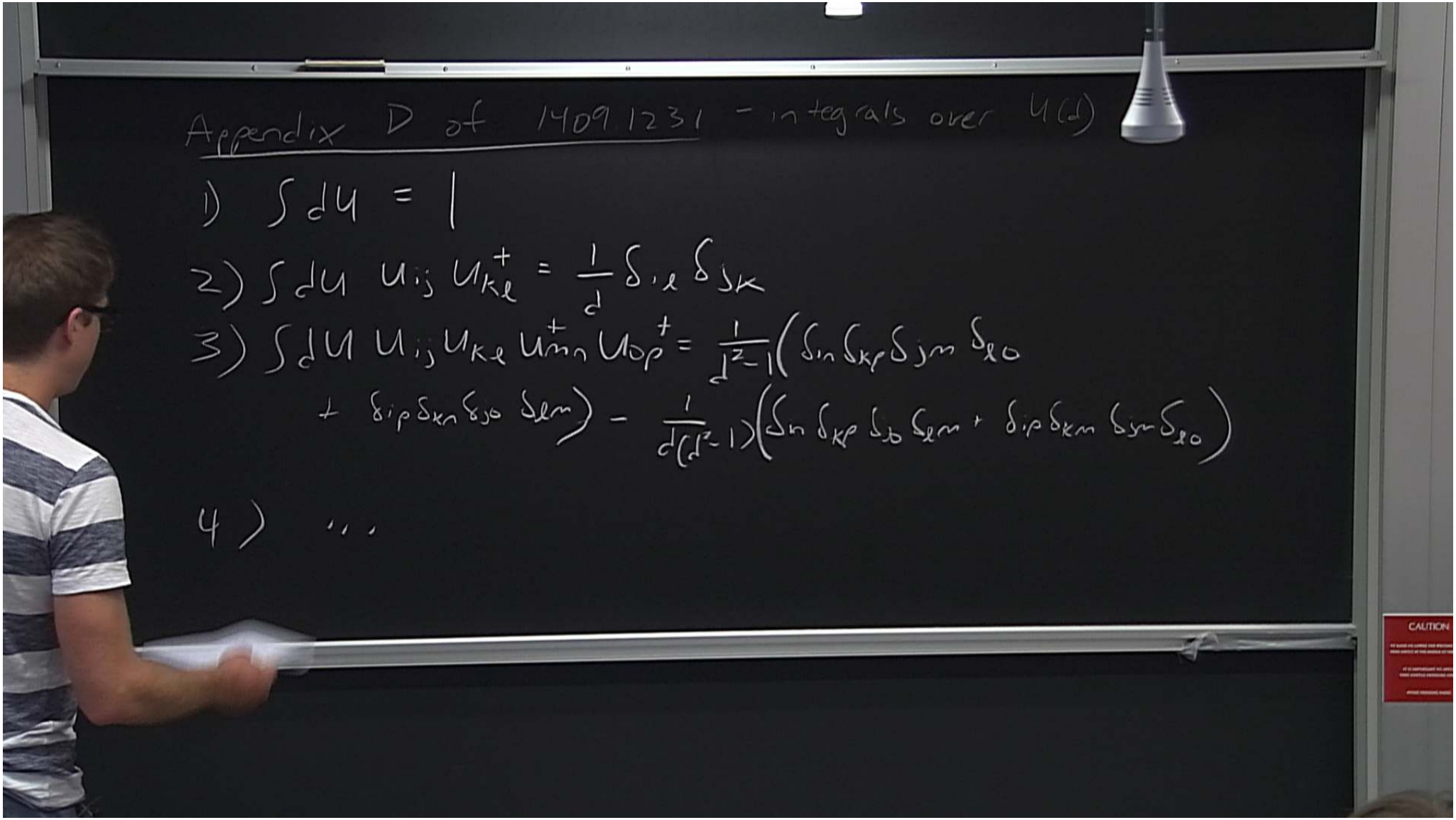
Title: Focus Lecture

Date: Jul 27, 2016 05:00 PM

URL: http://pirsa.org/16070048

Abstract:


Appendix D of 1409.1231 - integrals over U(d)
1) $\int dU = 1$
2) $\int dU\, U_{ij} U^{\dagger}_{kl} = \frac{1}{d}\delta_{il}\delta_{jk}$
3) $\int dU\, U_{ij} U_{kl} U^{\dagger}_{mn} U^{\dagger}_{op} = \frac{1}{d^2-1}\left(\delta_{in}\delta_{kp}\delta_{jm}\delta_{lo} + \delta_{ip}\delta_{kn}\delta_{jo}\delta_{lm}\right) - \frac{1}{d(d^2-1)}\left(\delta_{in}\delta_{kp}\delta_{jo}\delta_{lm} + \delta_{ip}\delta_{kn}\delta_{jm}\delta_{lo}\right)$
4) ...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CAUTION

Appendix D of 1409.1231 - integrals over $U(d)$

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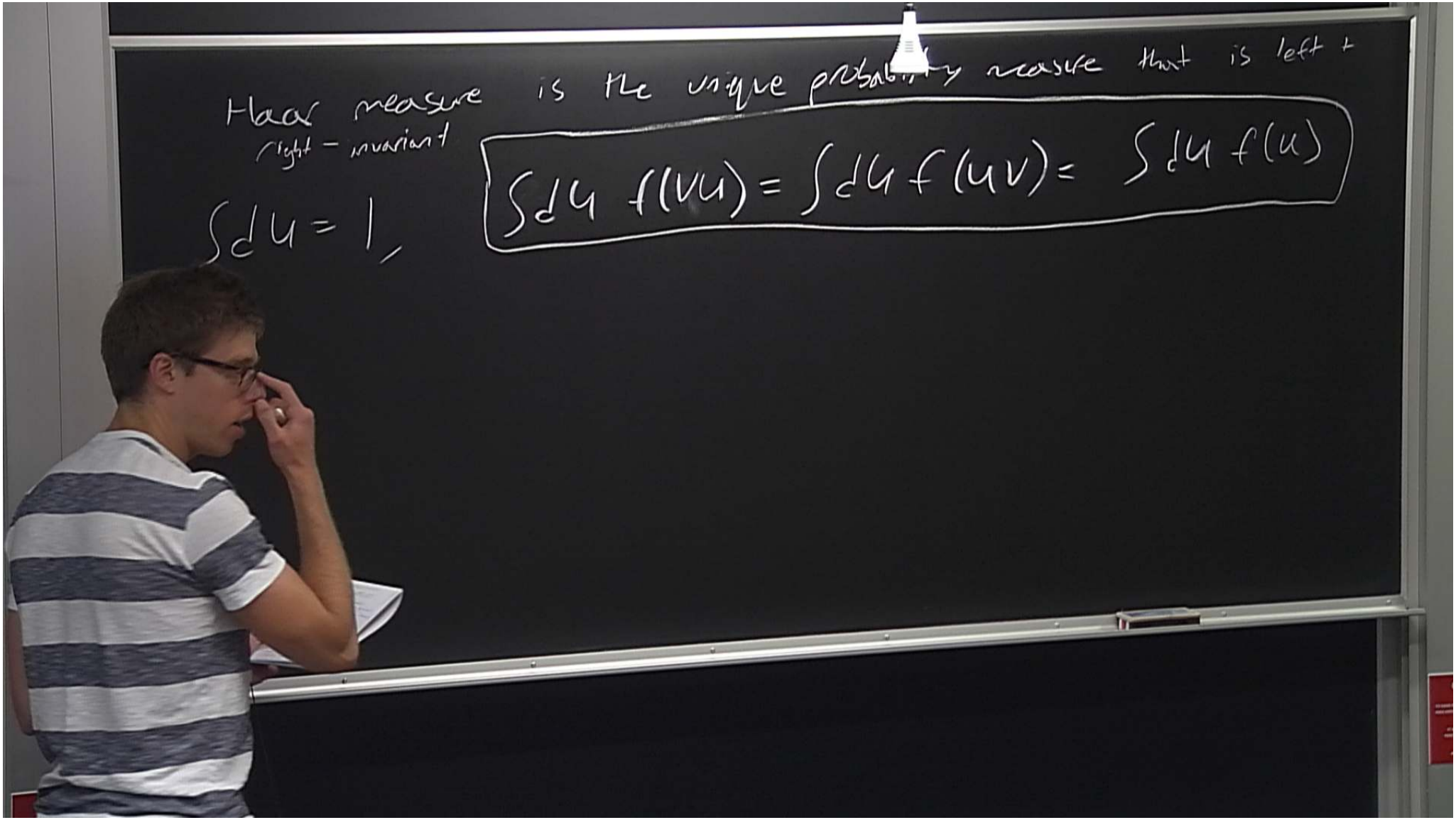
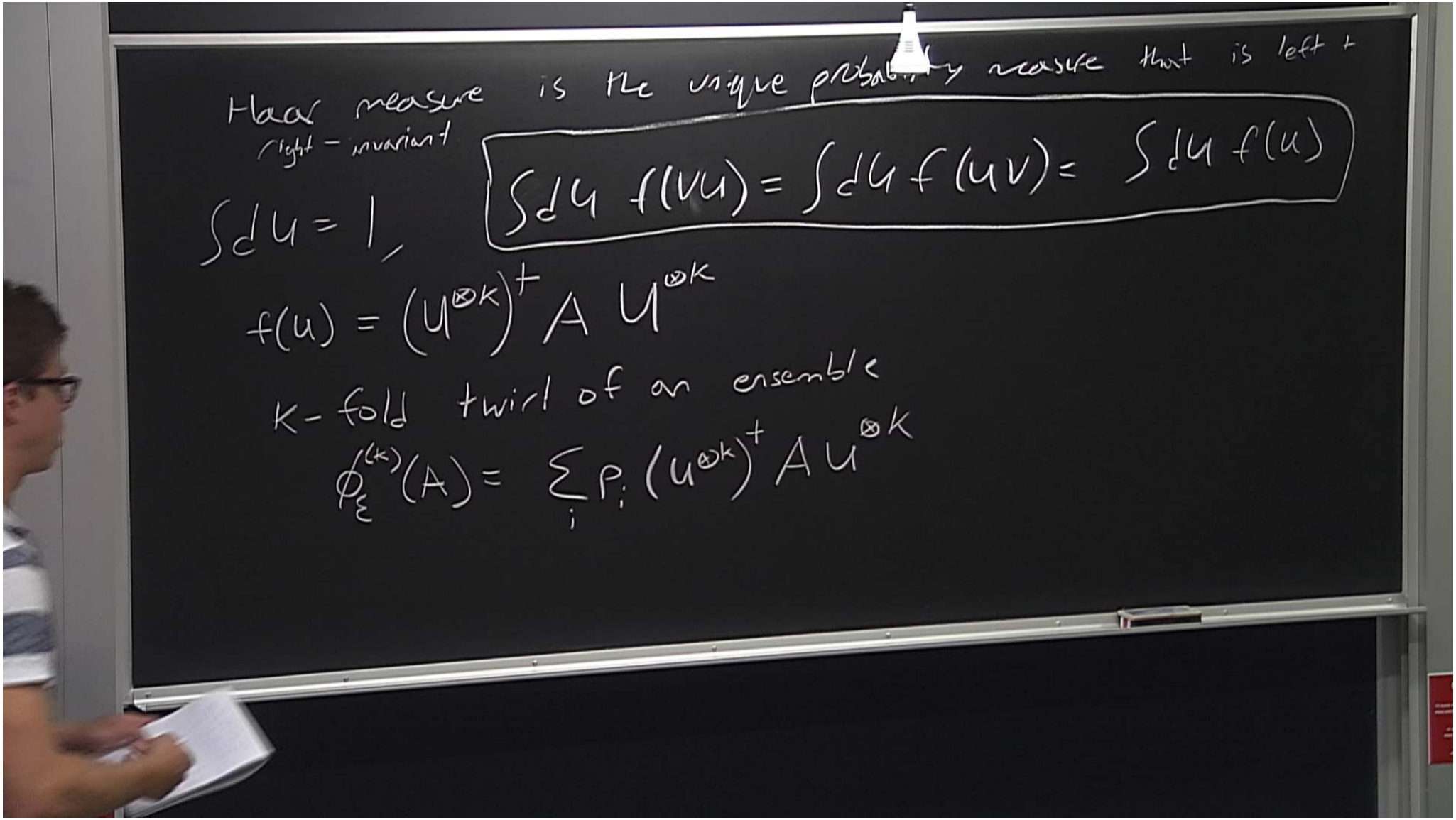
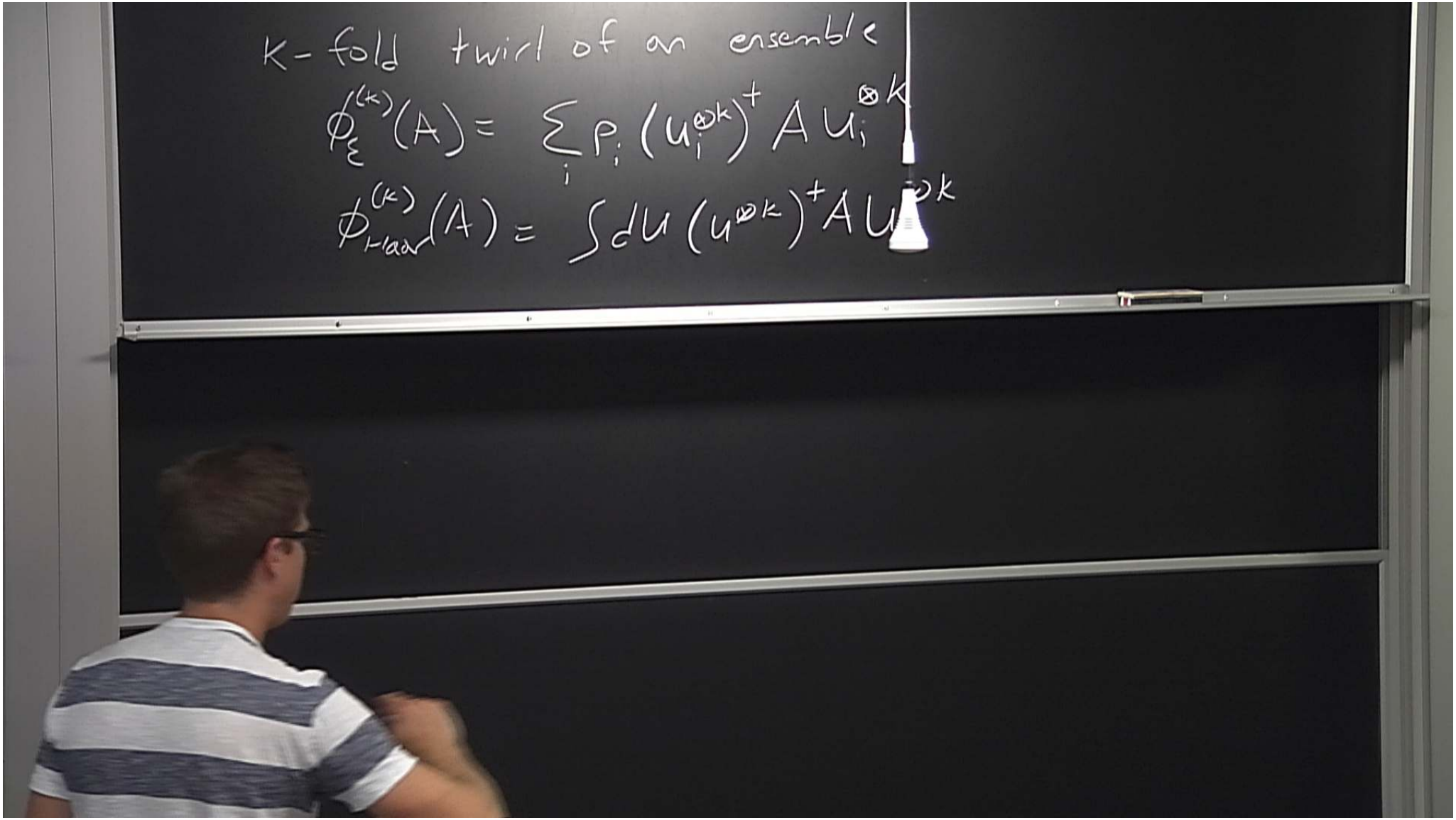

Haar measure is the unique probability measure that is left +
right - invariant
$\int dU = 1$
$\int dU\, f(VU) = \int dU\, f(UV) = \int dU\, f(U)$


Haar measure is the unique probability measure that is left +
right - invariant
$\int dU = 1$
$\int dU\, f(VU) = \int dU\, f(UV) = \int dU\, f(U)$
$f(U) = (U^{\otimes k})^{\dagger} A\, U^{\otimes k}$
k-fold twirl of an ensemble
$\phi_{\mathcal{E}}^{(k)}(A) = \sum_i p_i (U^{\otimes k})^{\dagger} A\, U^{\otimes k}$


k-fold twirl of an ensemble
$\Phi_{\mathcal{E}}^{(k)}(A) = \sum_i p_i (U_i^{\otimes k})^\dagger A U_i^{\otimes k}$
$\Phi_{\text{Haar}}^{(k)}(A) = \int dU (U^{\otimes k})^\dagger A U^{\otimes k}$

$\Phi_\mathcal{E}(A) = \sum_i p_i (U_i^{\otimes k}) A U_i$

$$\boxed{\Phi^{(k)}_{Haar}(A) = \int dU \, (U^{\otimes k})^\dagger A U^{\otimes k}}$$

Properties of $\Phi^{(k)}_{Haar}(A)$:

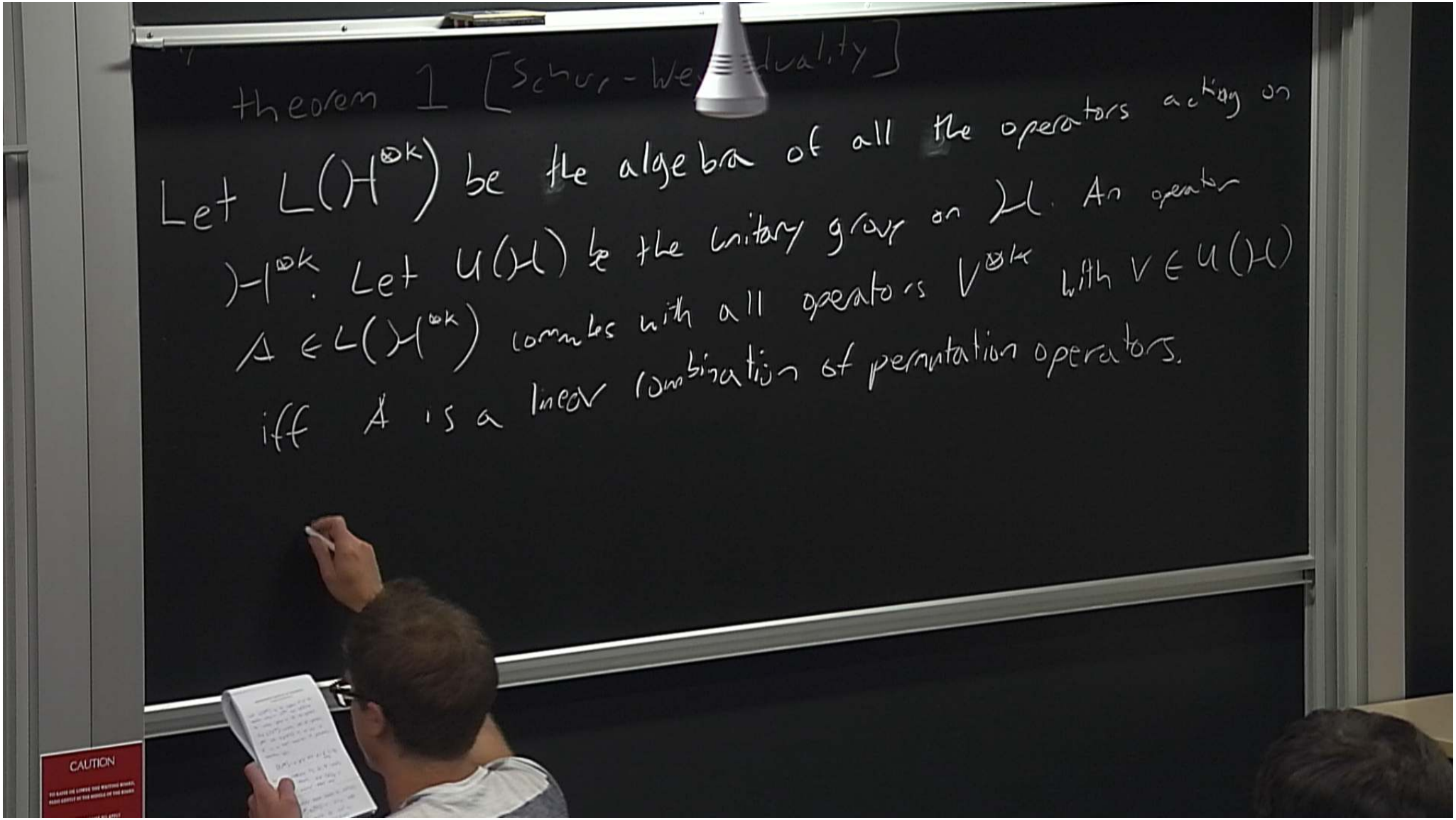
1) Invariant under unitary conjugation

$$(V^{\otimes k})^\dagger \left(\Phi^{(k)}_{Haar}(A)\right) V^{\otimes k} = \Phi^{(k)}_{Haar}(A)$$

$$= \int f(VU)\, dU = \Phi^{(k)}_{Haar}(A)$$

$$\left[V^{\otimes k}, \Phi^{(k)}_{Haar}(X)\right] = 0$$

CAUTION


theorem 1 [Schur-Weyl duality]
Let $L(\mathcal{H}^{\otimes k})$ be the algebra of all the operators acting on $\mathcal{H}^{\otimes k}$. Let $U(\mathcal{H})$ be the unitary group on $\mathcal{H}$. An operator $A \in L(\mathcal{H}^{\otimes k})$ commutes with all operators $V^{\otimes k}$ with $V \in U(\mathcal{H})$ iff A is a linear combination of permutation operators.
CAUTION

$$\Phi^{(k)}_{Haar}(A) = \int dU \, (U^{\otimes k})^\dagger A U^{\otimes k}$$

Properties of $\Phi^{(k)}_{Haar}(A)$:

1) Invariant under unitary conjugation

$$(V^{\otimes k})^\dagger \left(\Phi^{(k)}_{Haar}(A)\right) V^{\otimes k} = \Phi^{(k)}_{Haar}(A)$$

$$= \int f(VU)\, dU = \Phi^{(k)}_{Haar}(A)$$

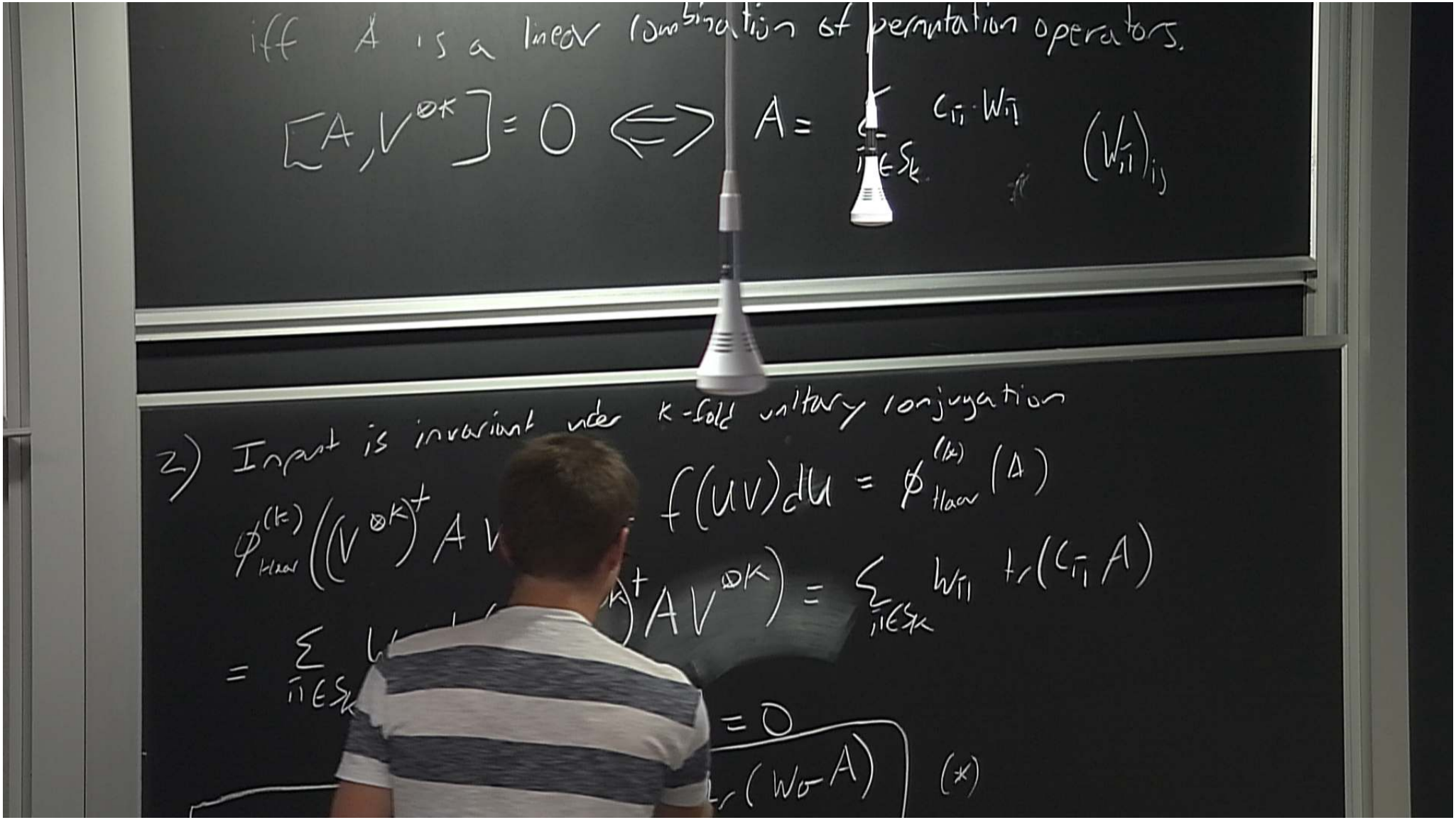
$$\left[V^{\otimes k}, \Phi^{(k)}_{Haar}(X)\right] = 0$$

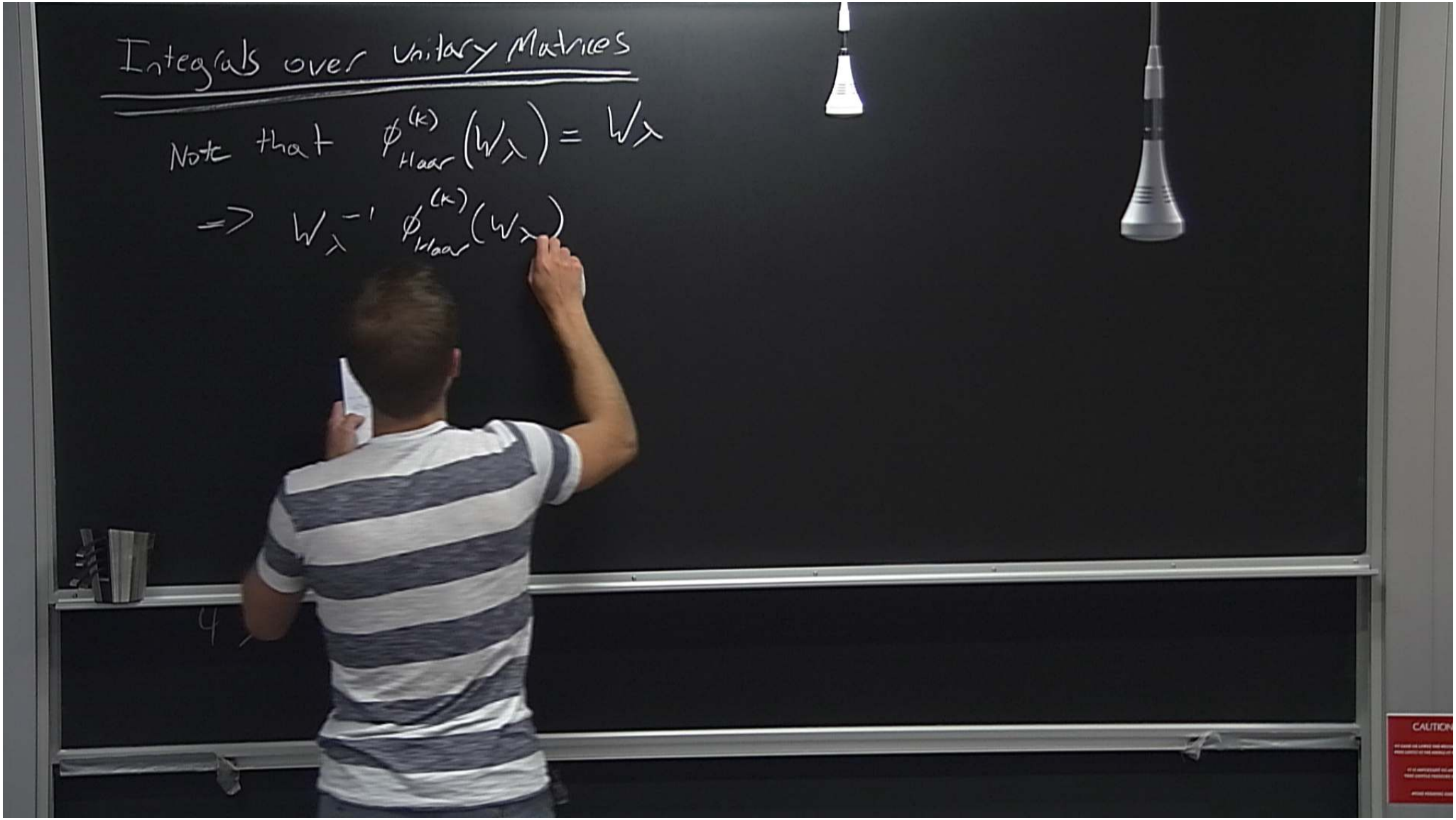
using math we can rewrite the third as

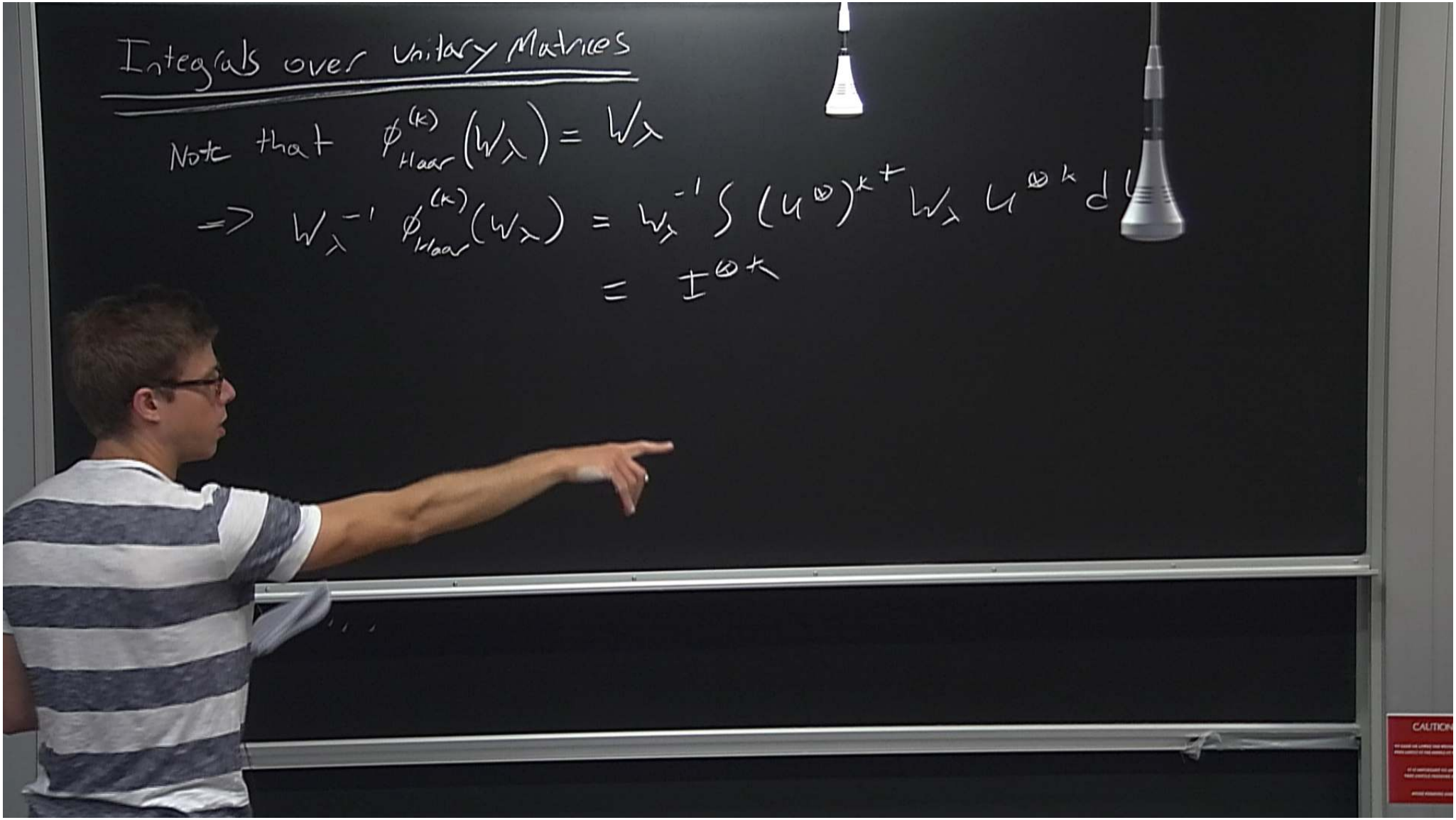
$$\Phi^{(k)}_{Haar}(A) = \sum_{\pi \in S_k} W_{ij} \cdot u_{ij}(A)$$

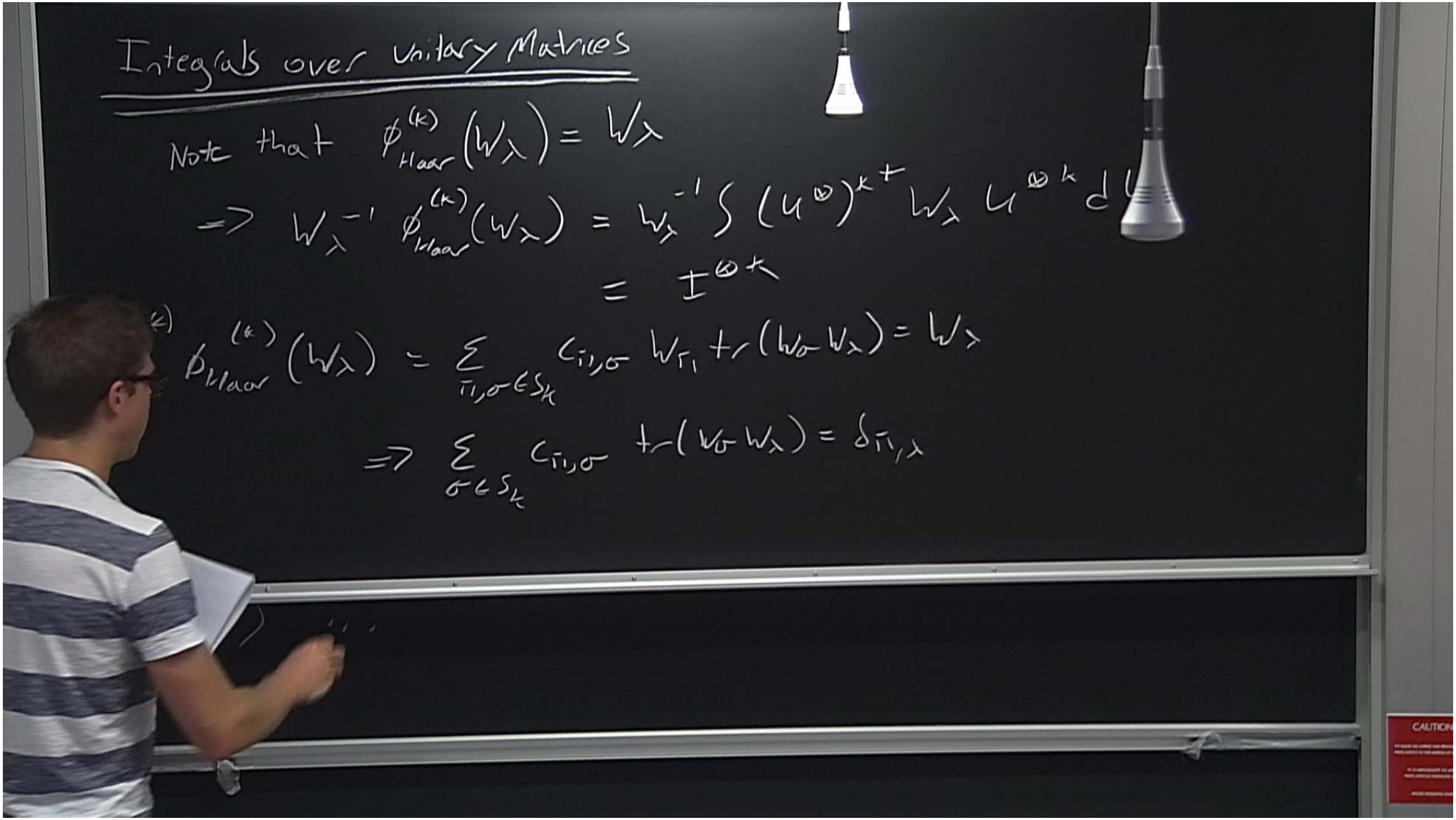
$$u_{ij}(A) = tr(C_\pi A)$$

CAUTION


iff A is a linear combination of permutation operators.
$[A, V^{\otimes k}] = 0 \iff A = \sum_{i \in S_k} c_i \cdot W_i$
$(W_i)_{ij}$
2) Input is invariant under k-fold unitary conjugation
$\phi^{(k)}_{Haar}((V^{\otimes k})^\dagger A V$
$\int f(UV)dU = \phi^{(k)}_{Haar}(A)$
$= \sum_{i \in S_k}$
$A V^{\otimes k}) = \sum_{i \in S_k} W_i \, tr(C_i A)$
$= 0$
$tr(W_\sigma A)$
(*)


Integrals over Unitary Matrices
Note that $\phi^{(k)}_{Haar}(W_\lambda) = W_\lambda$
$\Rightarrow W_\lambda^{-1}\, \phi^{(k)}_{Haar}(W_\lambda)$


Integrals over Unitary Matrices
Note that $\phi^{(k)}_{Haar}(W_\lambda) = W_\lambda$
$\Rightarrow W_\lambda^{-1}\,\phi^{(k)}_{Haar}(W_\lambda) = W_\lambda^{-1}\int (U^{\otimes})^{k\,t} W_\lambda\, U^{\otimes k}\, dU$
$= I^{\otimes k}$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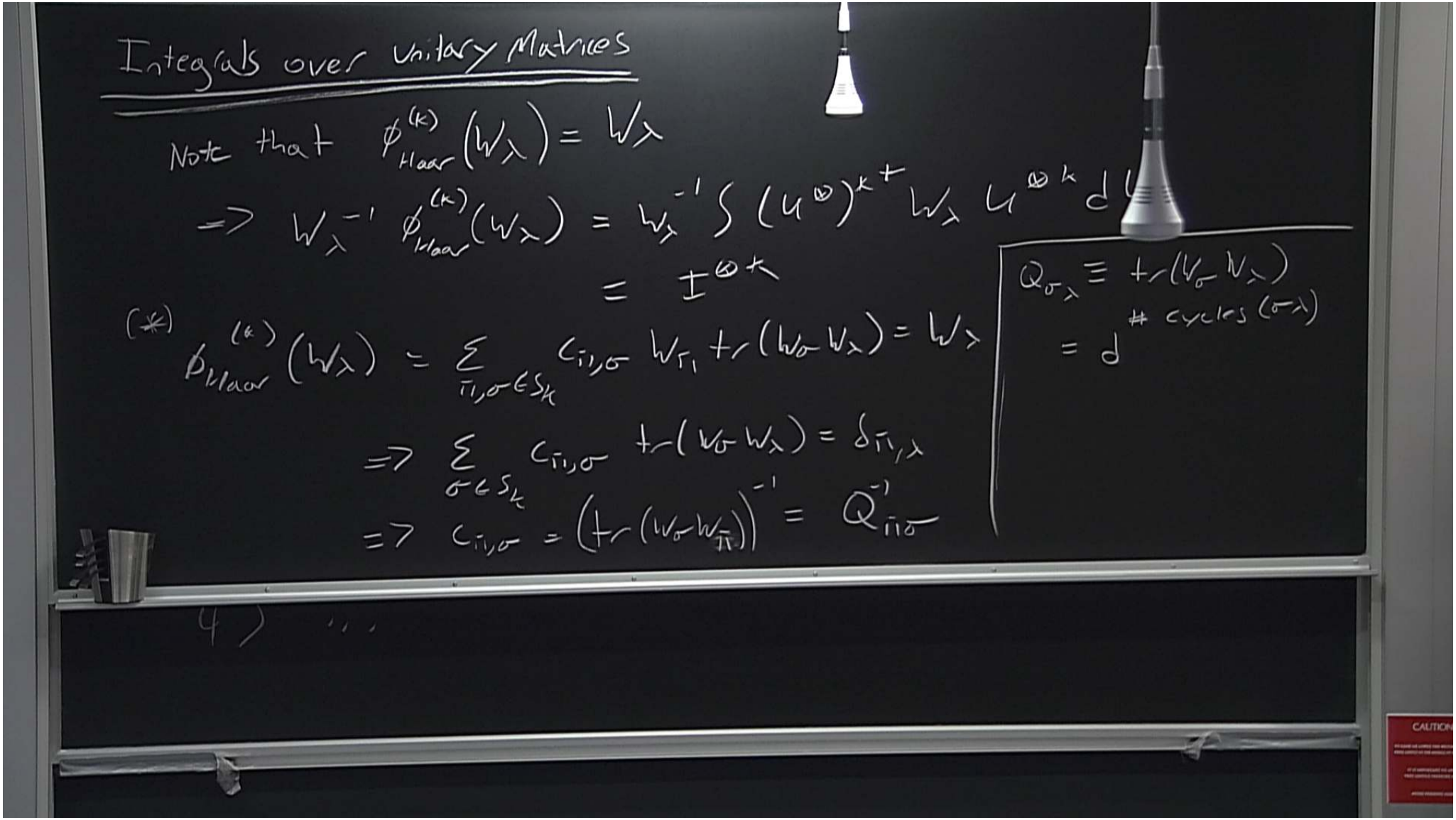
$= I^{\otimes k}$
$\Phi^{(k)}_{Haar}(W_\lambda) = \sum_{\tau,\sigma\in S_k} c_{\tau,\sigma}\, W_\tau$
$\mathrm{tr}(W_\sigma W_\lambda) = W_\lambda$
$\Rightarrow \sum_{\sigma\in S_k} c_{\tau,\sigma}\, \mathrm{tr}(W_\sigma W_\lambda) = \delta_{\tau,\lambda}$
CAUTION


Integrals over Unitary Matrices
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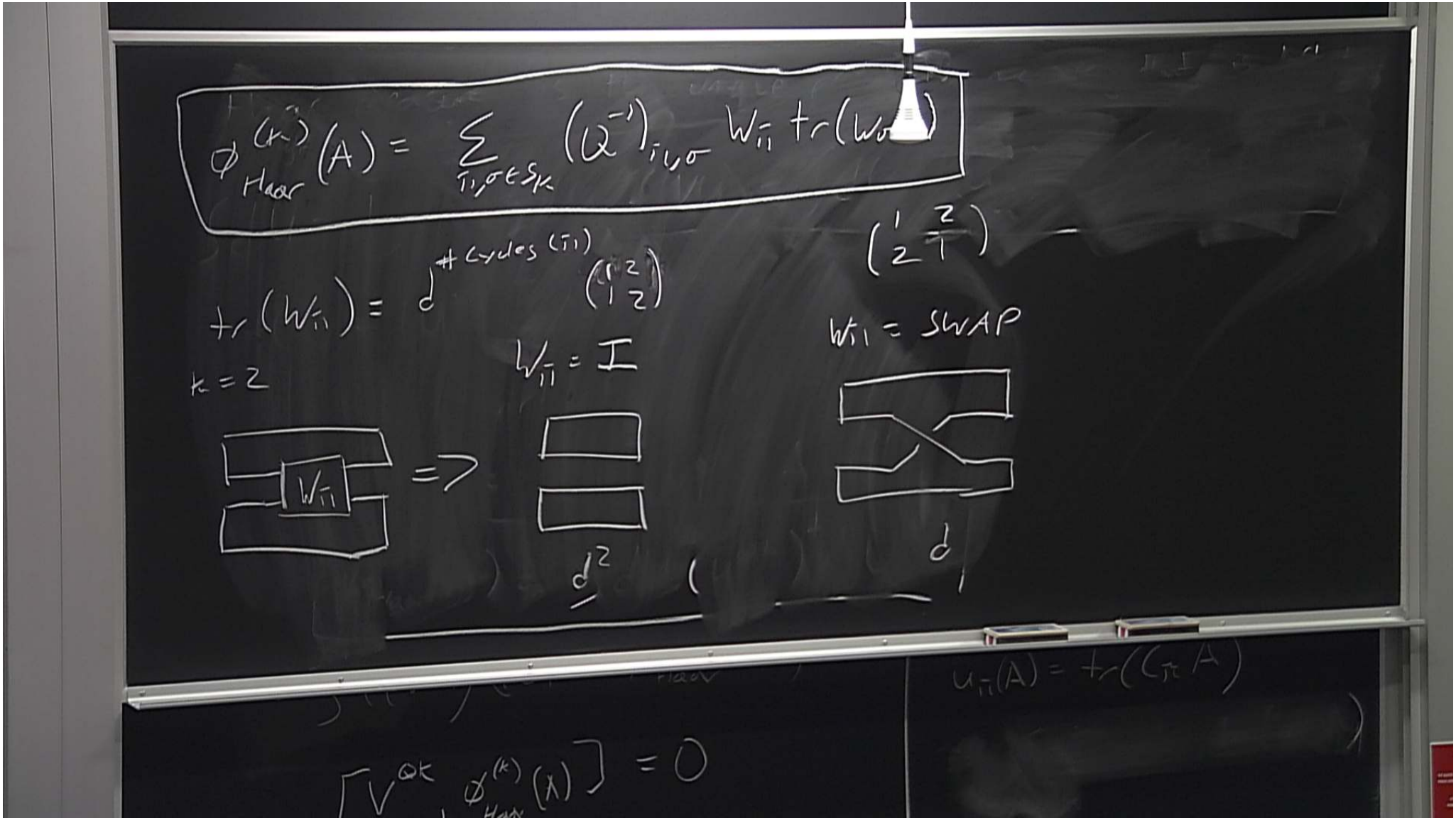
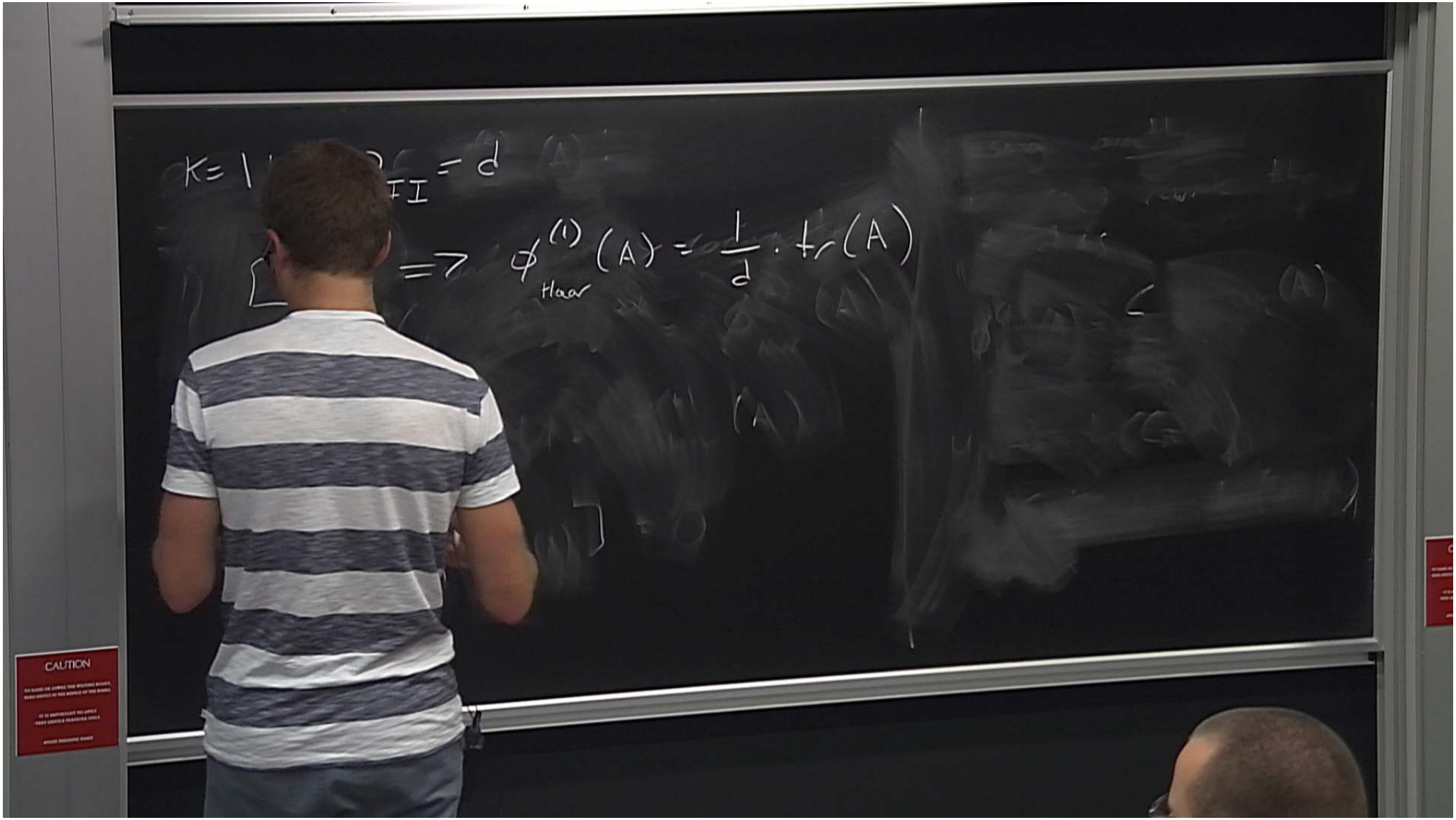
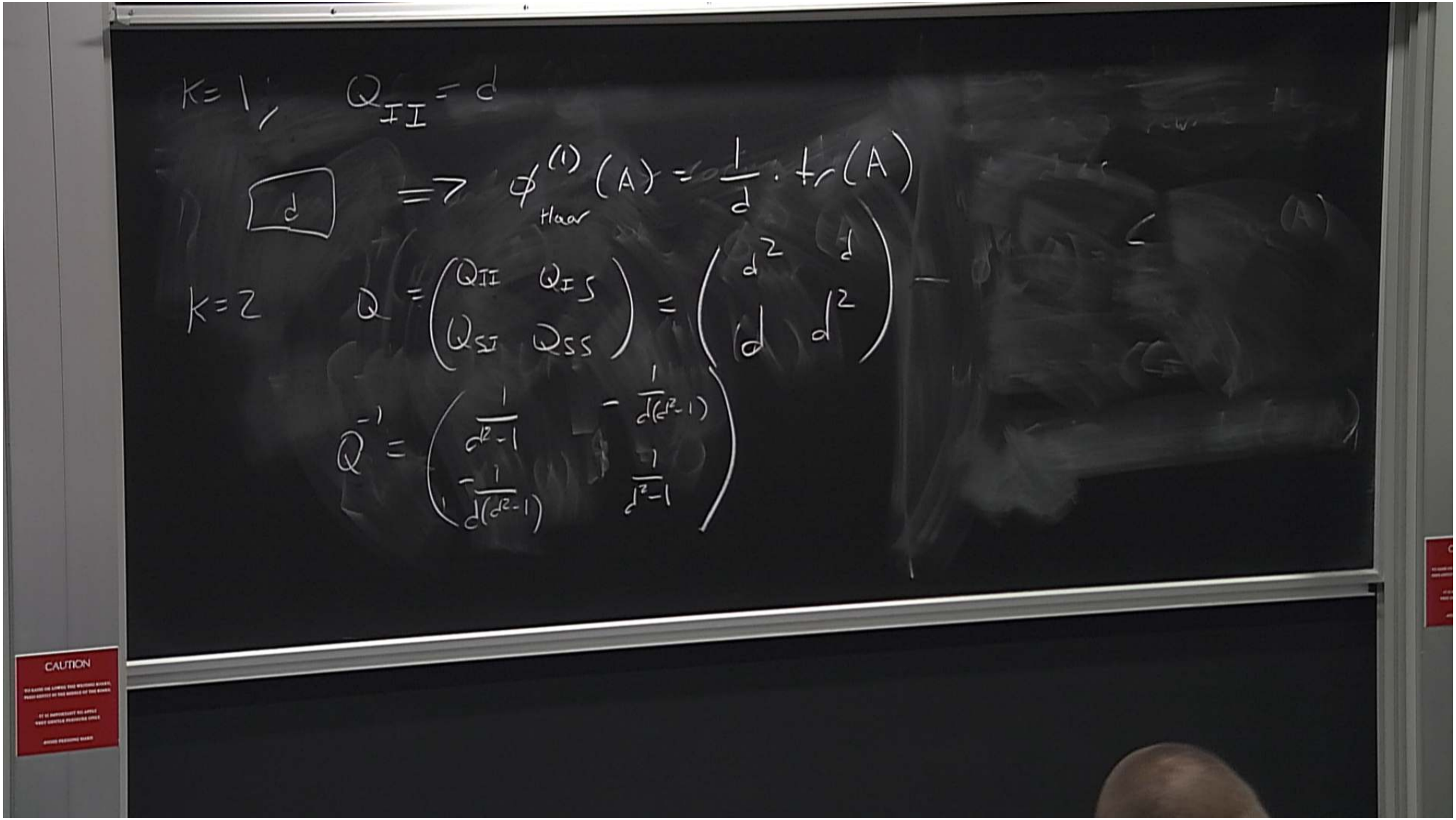
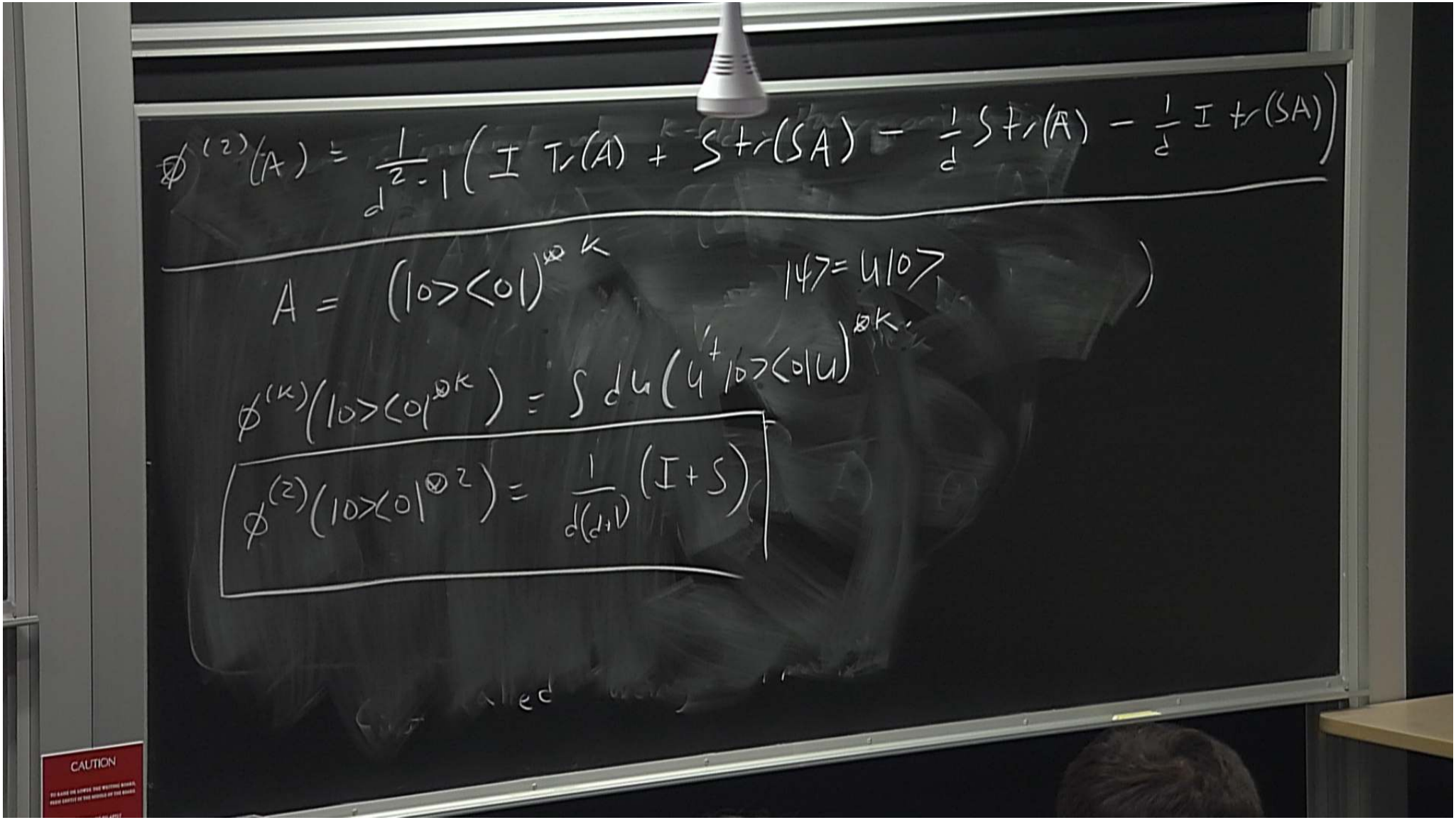
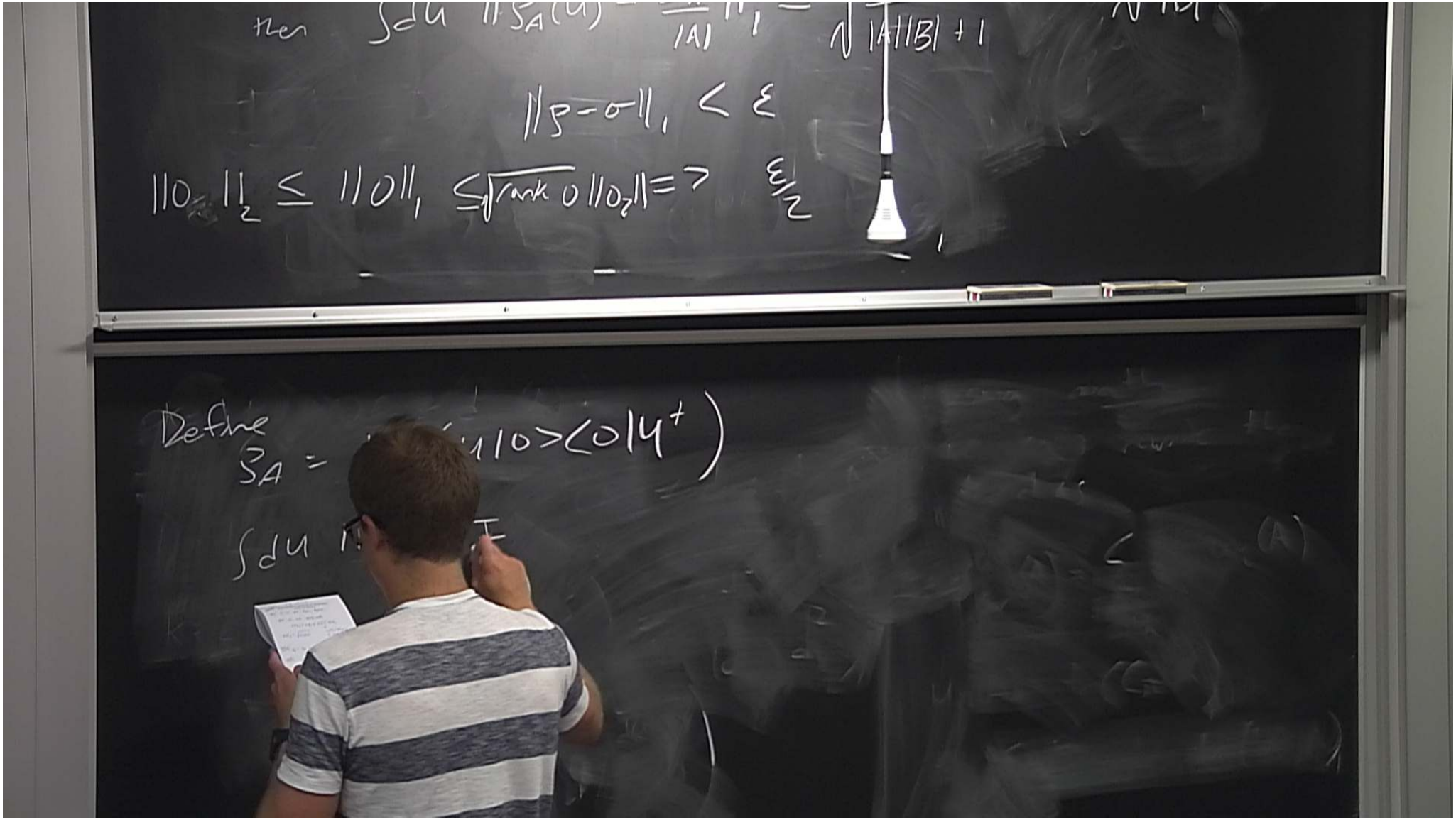
$\Rightarrow W_\lambda^{-1}\,\phi^{(k)}_{Haar}(W_\lambda) = W_\lambda^{-1}\int (U^{\otimes})^{k\,\dagger} W_\lambda\, U^{\otimes k}\, dU$
$= I^{\otimes k}$
(*) $\phi^{(k)}_{Haar}(W_\lambda) = \sum_{\tau,\sigma\in S_k} C_{\tau,\sigma}\, W_\tau\, \mathrm{tr}(W_\sigma W_\lambda) = W_\lambda$
$\Rightarrow \sum_{\sigma\in S_k} C_{\tau,\sigma}\, \mathrm{tr}(W_\sigma W_\lambda) = \delta_{\tau,\lambda}$
$\Rightarrow C_{\tau,\sigma} = \left(\mathrm{tr}(W_\sigma W_\tau)\right)^{-1} = Q^{-1}_{\tau\sigma}$
$Q_{\sigma\lambda} \equiv \mathrm{tr}(W_\sigma W_\lambda)$
$= d^{\#\text{ cycles}(\sigma\lambda)}$
4) ...

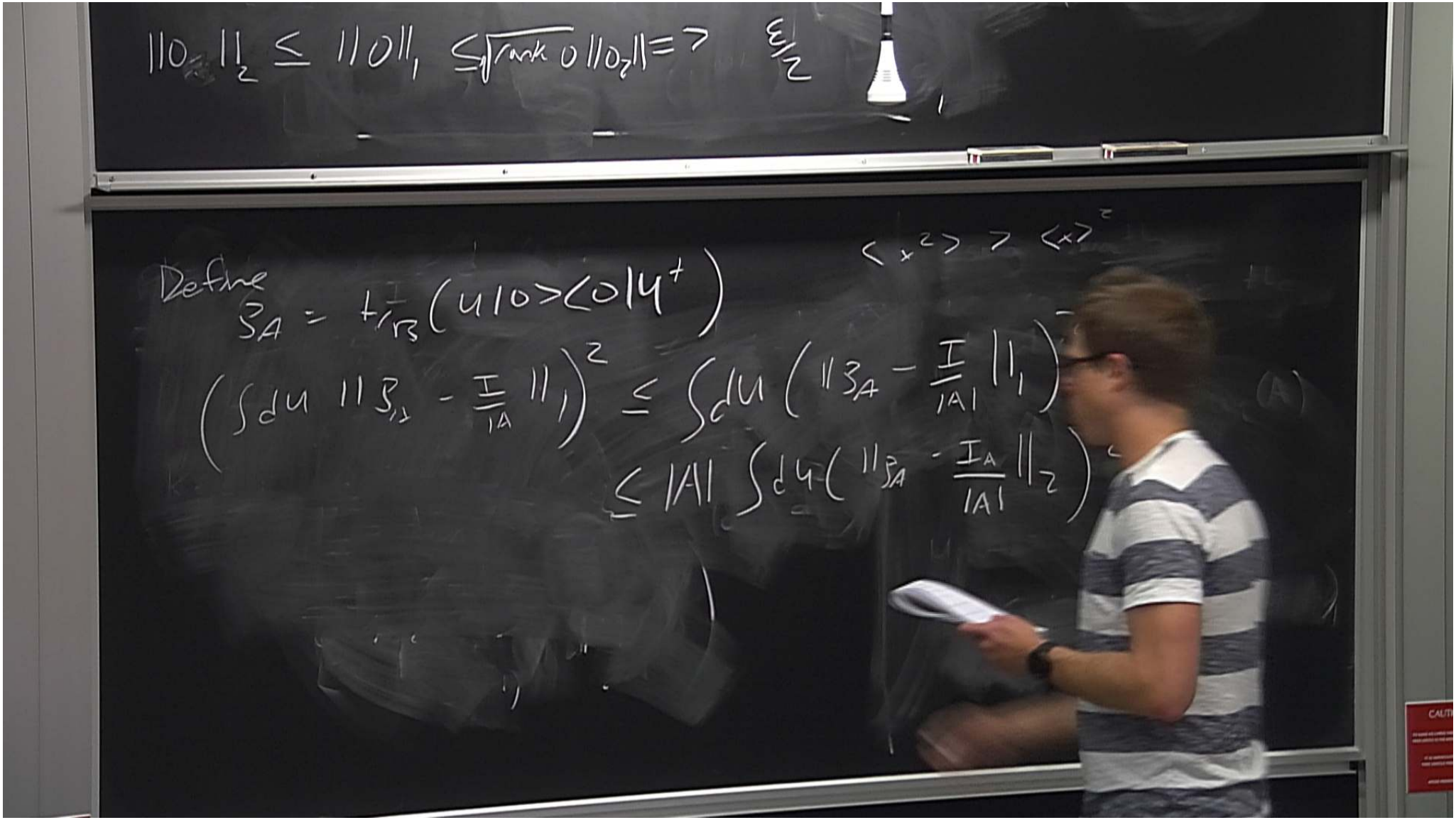

$\Phi^{(k)}_{Haar}(A) = \sum_{\tau,\sigma \in S_k} (Q^{-1})_{\tau,\sigma} W_\tau \, tr(W_\sigma A)$
$tr(W_\tau) = d^{\#\,cycles(\tau)}$
$k = 2$
$W_{(1)(2)} = I$
$W_{(12)} = SWAP$
d^2
d
$u_i(A) = tr(C_i A)$
$[V^{\otimes k}, \Phi^{(k)}_{Haar}(X)] = 0$

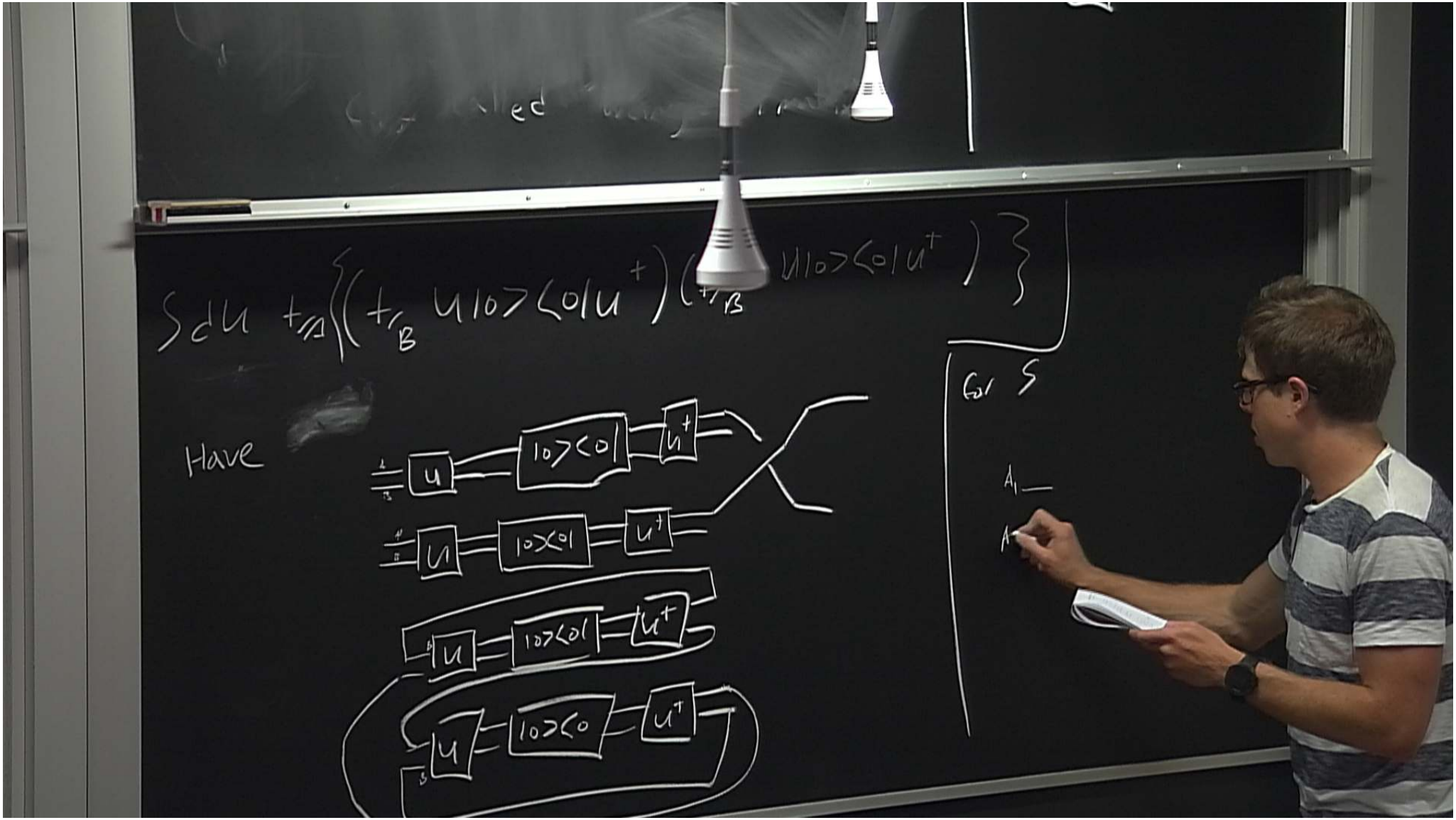

$\phi^{(1)}_{\text{Haar}}(A) = \frac{1}{d} \cdot tr(A)$
CAUTION


$k=1$; $Q_{II} = d$
d $\Rightarrow$ $\phi^{(1)}_{Haar}(A) = \frac{1}{d} \cdot tr(A)$
$k=2$ $Q = \begin{pmatrix} Q_{II} & Q_{IS} \\ Q_{SI} & Q_{SS} \end{pmatrix} = \begin{pmatrix} d^2 & d \\ d & d^2 \end{pmatrix}$
$Q^{-1} = \begin{pmatrix} \frac{1}{d^2-1} & -\frac{1}{d(d^2-1)} \\ -\frac{1}{d(d^2-1)} & \frac{1}{d^2-1} \end{pmatrix}$
CAUTION


$\Phi^{(2)}(A) = \frac{1}{d^2-1}\left(I\,\mathrm{Tr}(A) + S\,\mathrm{Tr}(SA) - \frac{1}{d}S\,\mathrm{Tr}(A) - \frac{1}{d}I\,\mathrm{Tr}(SA)\right)$
$A = (|0\rangle\langle 0|)^{\otimes k}$
$|\psi\rangle = U|0\rangle$
$\Phi^{(k)}(|0\rangle\langle 0|^{\otimes k}) = \int dU\,(U^\dagger|0\rangle\langle 0|U)^{\otimes k}$
$\Phi^{(2)}(|0\rangle\langle 0|^{\otimes 2}) = \frac{1}{d(d+1)}(I+S)$
CAUTION


then
$\|\rho - \sigma\|_1 < \varepsilon$
$\|\sigma_2\|_2 \leq \|\sigma\|_1 \leq \sqrt{\text{rank}\,\sigma}\,\|\sigma_2\|$
Define
S_A


$\|O\|_2 \le \|O\|_1 \le \sqrt{\text{rank } O}\,\|O_2\| \Rightarrow \varepsilon/2$
$\langle x^2 \rangle > \langle x \rangle^2$
Define
$\rho_A = \text{tr}_B(U|0\rangle\langle 0|U^\dagger)$
$\left(\int dU \, \|\rho_A - \frac{I}{|A|}\|_1\right)^2 \le \int dU \left(\|\rho_A - \frac{I}{|A|}\|_1\right)^2$
$\le |A| \int dU \left(\|\rho_A - \frac{I_A}{|A|}\|_2\right)^2$


Have
for S