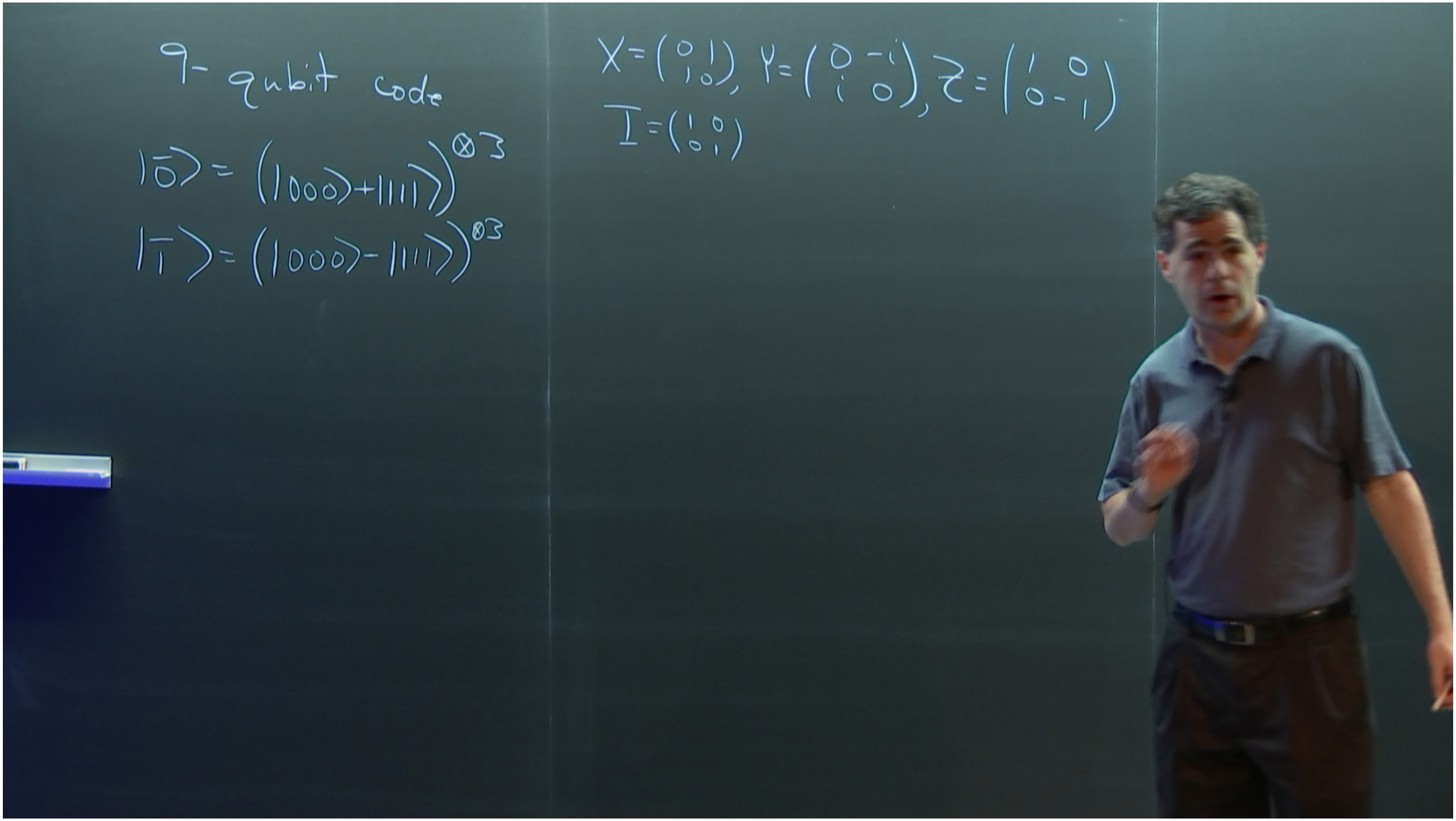
Title: Quantum Error Correction

Date: Jul 22, 2016 09:00 AM

URL: http://pirsa.org/16070029

Abstract:


9- qubit code
$|\bar{0}\rangle = (|000\rangle + |111\rangle)^{\otimes 3}$
$|\bar{1}\rangle = (|000\rangle - |111\rangle)^{\otimes 3}$
$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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Pauli group $\mathcal{P}_n$: Tensor products of $I, X, Y, \& Z$ on n qubits, global phase $\pm 1, \pm i$.

$P, Q \in \mathcal{P}_n \Rightarrow$ either $[P, Q] = 0$ or $\{P, Q\} = 0$

$X=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $Y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

Pauli group $\mathcal{P}_n$: Tensor products of $I, X, Y, \& Z$ on n qubits, global phase $\pm 1, \pm i$.

$P, Q \in \mathcal{P}_n \Rightarrow$ either $[P,Q]=0$ or $\{P,Q\}=0$

$P \in \mathcal{P}_n \Rightarrow P^2 = \pm I$

$Z\ Z$

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Z Z
Z Z
Z Z
Z Z
Z Z
Z Z

X X X X X X
X X X X X X

$$
\begin{array}{cccccccccccc}
Z & Z & & & & & & & & & & \\
 & Z & Z & & & & & & & & & \\
 & & & Z & Z & & & & & & & \\
 & & & & Z & Z & & & & & & \\
 & & & & & & Z & Z & & & & \\
 & & & & & & & Z & Z & & & \\
X & X & X & X & X & X & & & & & & \\
 & & & X & X & X & X & X & X & & &
\end{array}
$$

$$S = \{ M \in \mathcal{P}_n \mid M|\psi\rangle = |\psi\rangle \ \forall \text{codewords } |\psi\rangle \}$$

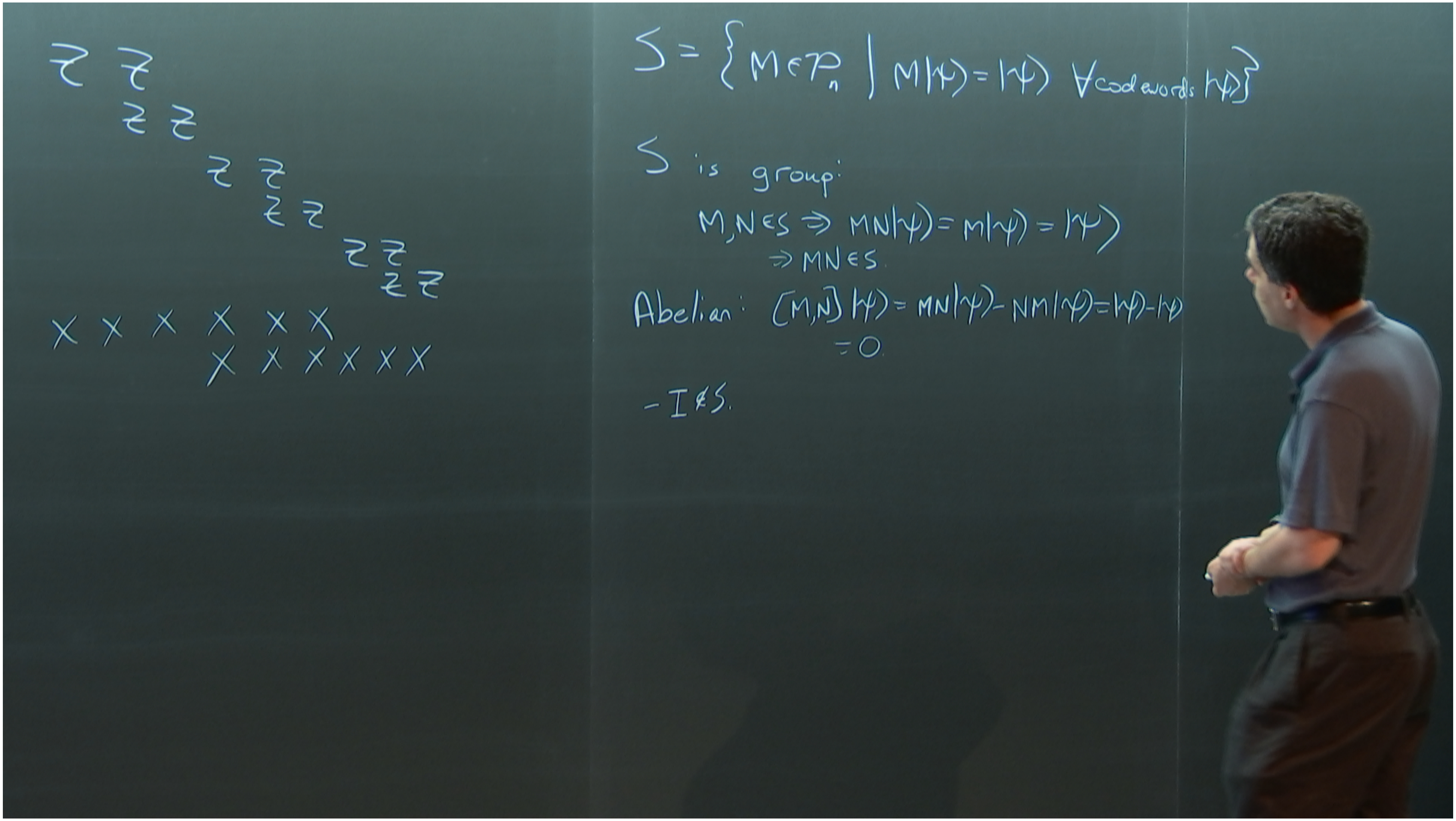
S is group:

$M, N \in S \Rightarrow MN|\psi\rangle = M|\psi\rangle = |\psi\rangle$

$\Rightarrow MN \in S$.

Abelian: $[M,N]|\psi\rangle = MN|\psi\rangle - NM|\psi\rangle = |\psi\rangle - |\psi\rangle$

$= 0.$


Z Z
Z Z
Z Z
Z Z
Z Z
Z Z
X X X X X X
X X X X X X
$S = \{ M \in \mathcal{P}_n \mid M|\psi\rangle = |\psi\rangle \ \forall \text{codewords } |\psi\rangle \}$
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$\Rightarrow MN \in S.$
Abelian: $[M,N]|\psi\rangle = MN|\psi\rangle - NM|\psi\rangle = |\psi\rangle - |\psi\rangle = 0.$
$-I \notin S.$

If stabilizer S has r generators, then $T(S)$ encodes $k = n - r$ logical qubits (n physical qubits)

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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Pauli group $\mathcal{P}_n$: Tensor products of $I, X, Y, \& Z$ on n qubits, global phase $\pm 1, \pm i$.

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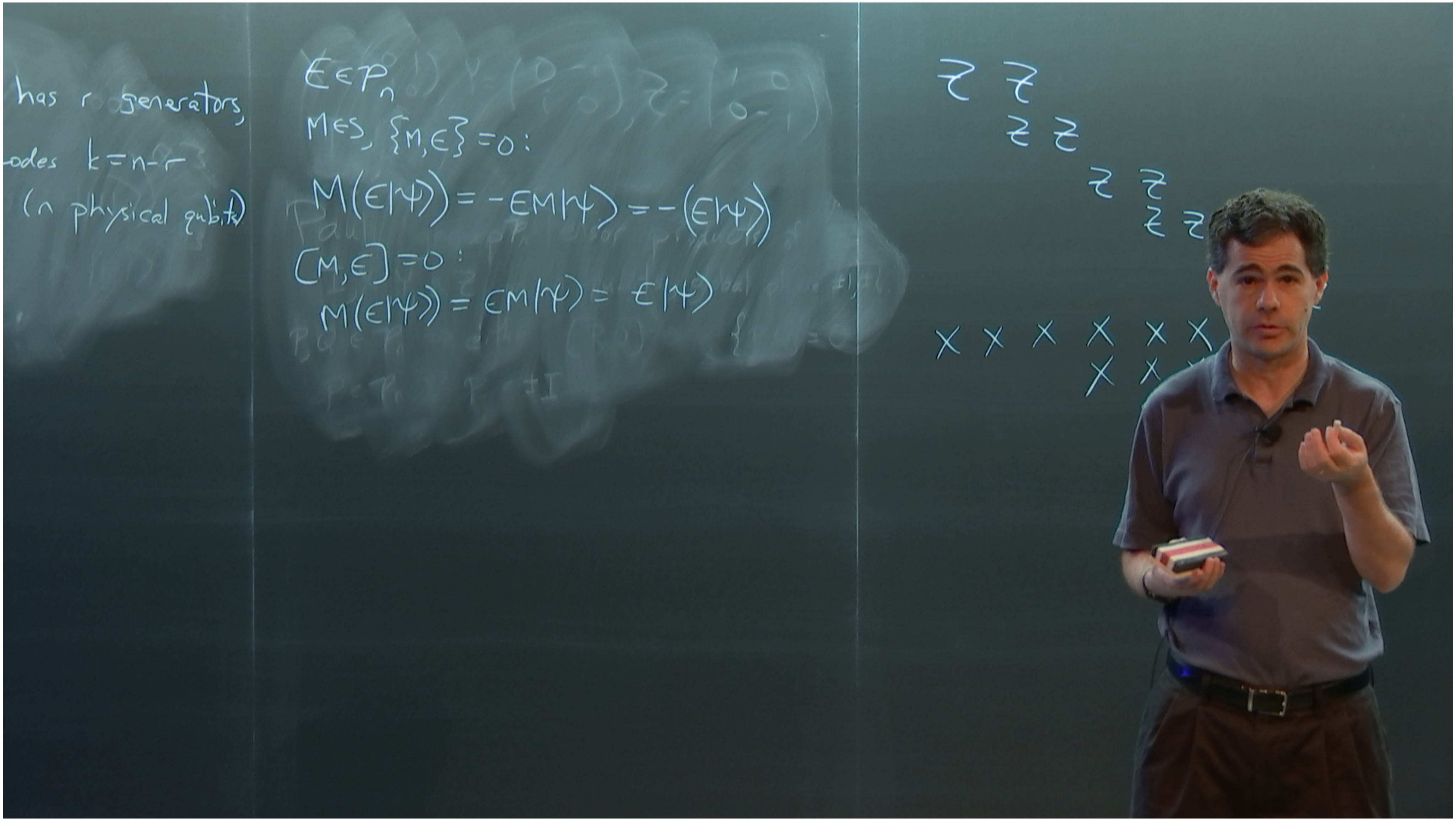
has r generators,
odes k=n-r
(n physical qubits)
E∈P_n
M∈S, {M,E}=0:
M(E|ψ⟩)=-EM|ψ⟩=-(E|ψ⟩)
[M,E]=0:
M(E|ψ⟩)=EM|ψ⟩= E|ψ⟩
Z Z
Z Z
Z Z
Z Z
X X X X X X
X X

rators,

qubits)

$E \in \mathcal{P}_n$

$M \in S$, $\{M, E\} = 0$:

$M(E|\psi\rangle) = -EM|\psi\rangle = -(E|\psi\rangle)$

$[M, E] = 0$:

$M(E|\psi\rangle) = EM|\psi\rangle = E|\psi\rangle$

$N(S) = \{N \in \mathcal{P}_n \mid [M, N] = 0 \ \forall M \in S\}$

$S =$

1) S is

M

2) Abelian

3) $-I \notin$

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Thm: QECC with stabilizer S detects errors outside $N(S) \setminus S$.

$S = \{M$

1) S is g

$M, N \in$

2) Abelian:

3)

$|S| = 2^r$

$N(S) = \{ N \in \mathcal{P}_n \mid [M,N] = 0 \ \forall M \in S \}$

<u>Thm:</u> QECC with stabilizer S detects errors outside $N(S) \setminus S$.

$S \leq N(S)$

$N(S)/S \cong \mathcal{P}_k$

Set $\mathcal{E}$ of errors.

1) E has unique error syndrome iff commutes w/ unique set of generators.

E & F have same error syndrome if they commute w/ same generators.

Pick S & f

Let $T(S) =$

For any S

(i.e. S is an A

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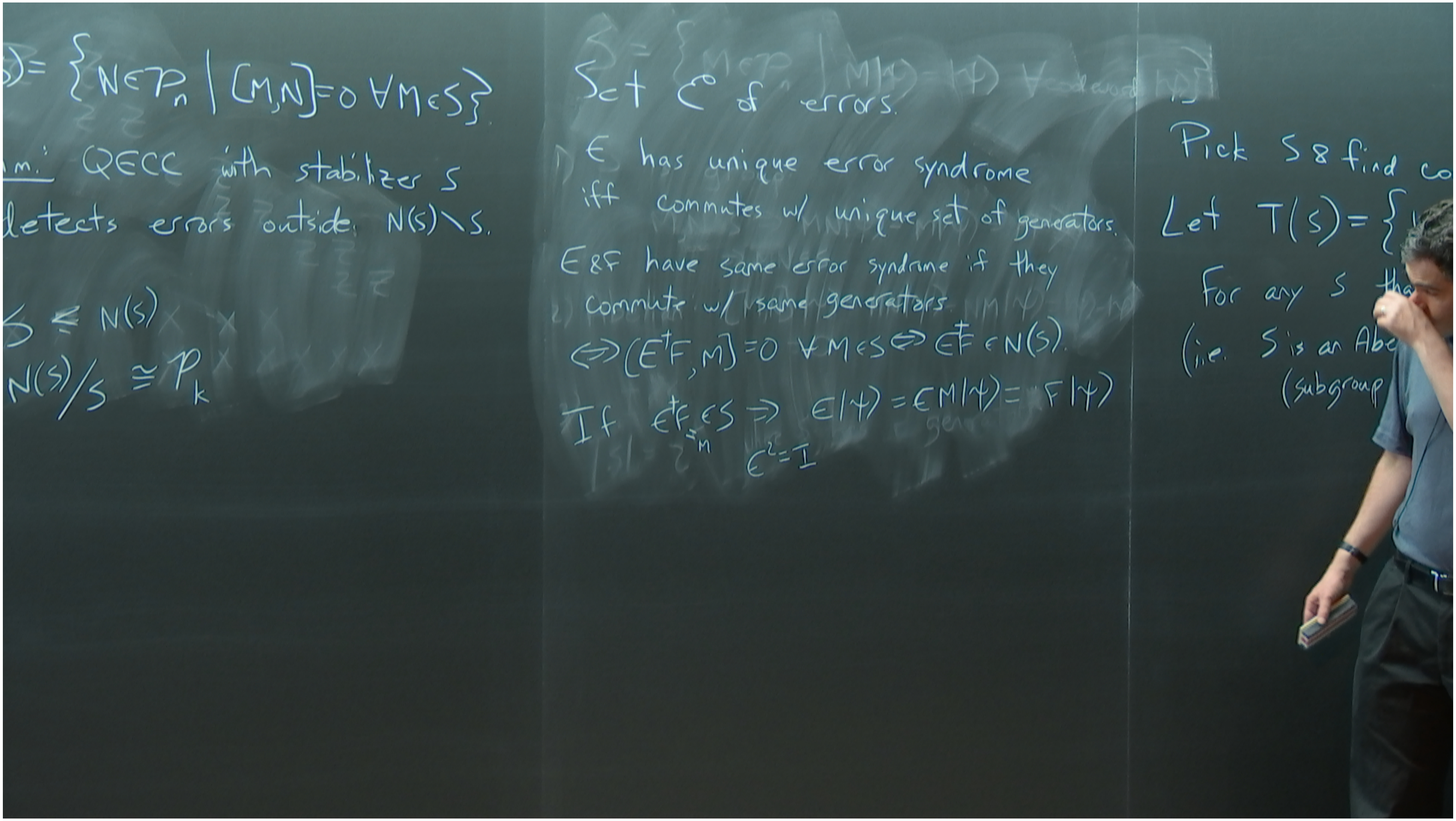
)= { N ∈ P_n | [M,N]=0 ∀ M∈S }
m.: QECC with stabilizer S
detects errors outside N(S)\S.
S ≤ N(S)
N(S)/S ≅ P_k
Set E of errors.
1) E has unique error syndrome
iff commutes w/ unique set of generators.
E & F have same error syndrome if they
2) commute w/ same generators
⇔ [E†F, M]=0 ∀ M∈S ⇔ E†F ∈ N(S).
If E†F ∈ S ⇒ E|ψ⟩ = EM|ψ⟩ = F|ψ⟩
E² = I
Pick S & find co
Let T(S) = {
For any S tha
(i.e. S is an Abe
(subgroup

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Thm: QECC with stabilizer S detects errors outside $N(S) \setminus S$.

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E has unique error syndrome iff commutes w/ unique set of generators.

E & F have same error syndrome if they commute w/ same generators.

$\Leftrightarrow [E^\dagger F, M] = 0 \ \forall M \in S \Leftrightarrow E^\dagger F \in N(S)$.

If $\underbrace{E^\dagger F}_{=M} \in S \Rightarrow E|\psi\rangle = EM|\psi\rangle = F|\psi\rangle$

$E^2 = I$

Pick S & fi

$T(S) =$

For any

S is an Ab

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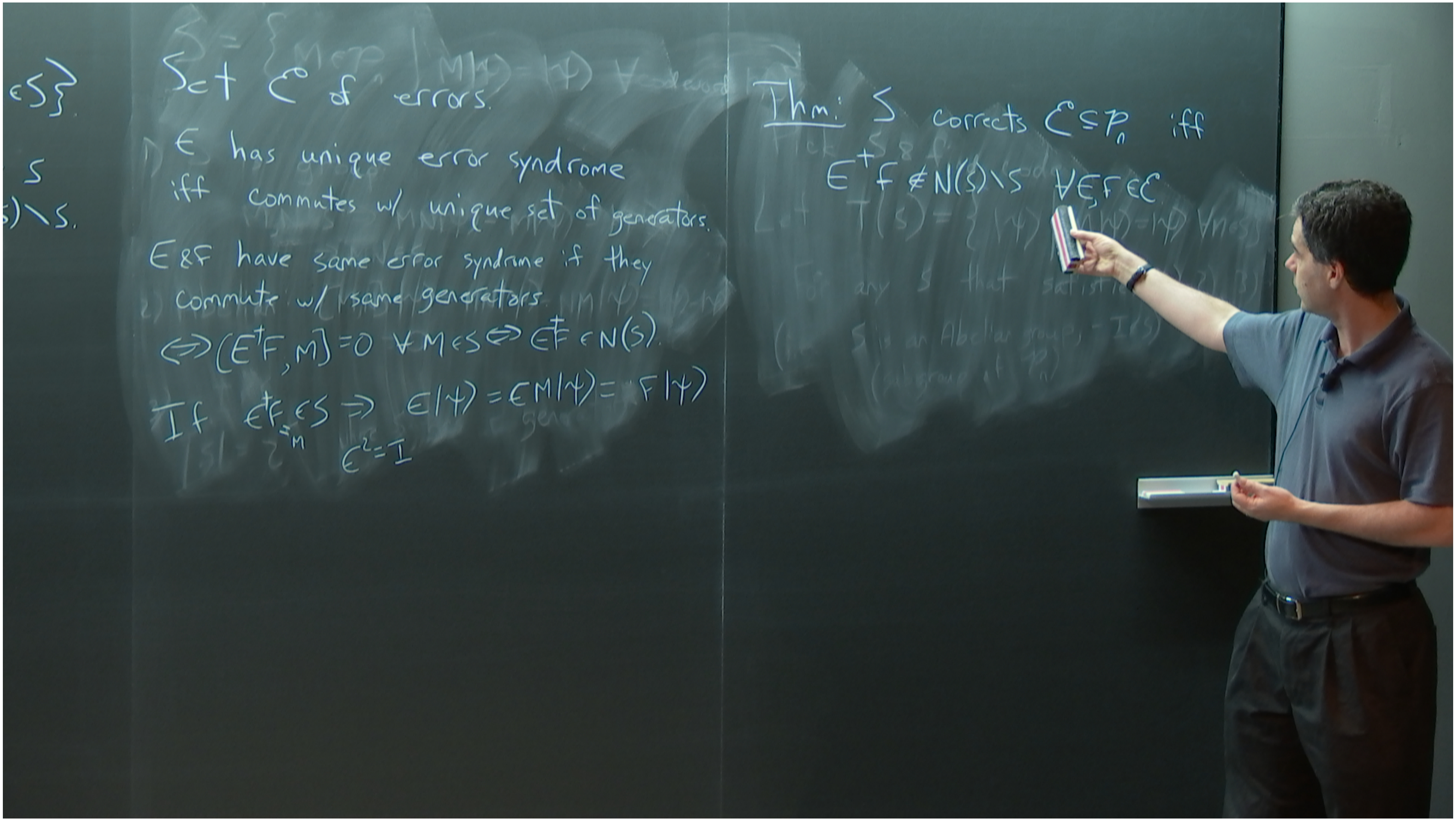
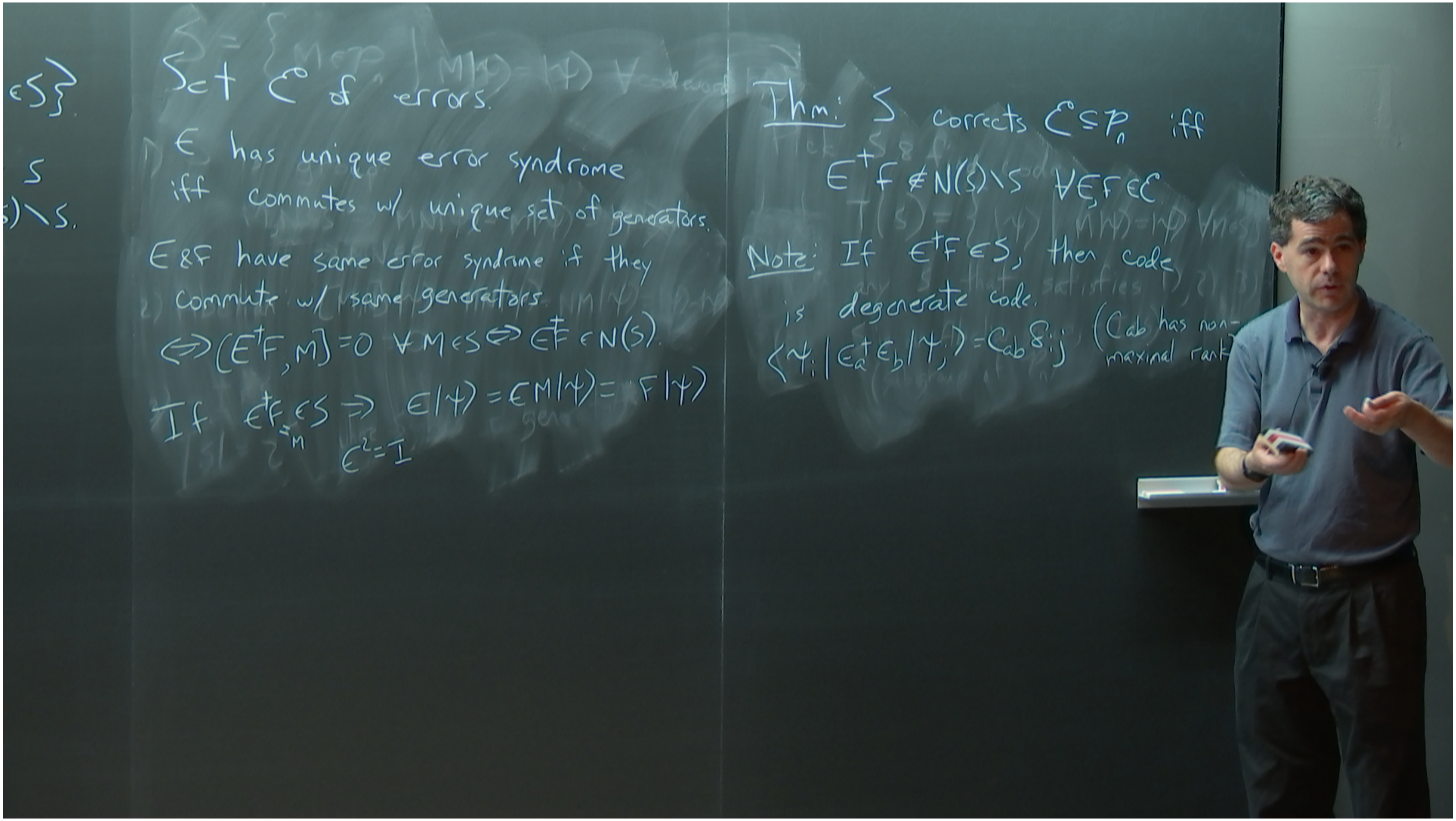

Set ℰ of errors.
E has unique error syndrome
iff commutes w/ unique set of generators.
E & F have same error syndrome if they
commute w/ same generators.
⇔ [E†F, M] = 0 ∀ M ∈ S ⇔ E†F ∈ N(S).
If E†F ∈ S ⇒ E|ψ⟩ = EM|ψ⟩ = F|ψ⟩
E² = I
Thm: S corrects ℰ ⊆ Pn iff
E†F ∉ N(S)\S ∀ E,F ∈ ℰ


Set ℰ of errors.
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E & F have same error syndrome if they
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⟺ [E†F, M] = 0 ∀ M ∈ S ⟺ E†F ∈ N(S).
If E†F ∈ S ⇒ E|ψ⟩ = EM|ψ⟩ = F|ψ⟩
E² = I
Thm: S corrects ℰ ⊆ Pₙ iff
E†F ∉ N(S)\S ∀ E,F ∈ ℰ
Note: If E†F ∈ S, then code
is degenerate code.
⟨ψᵢ|Eₐ†E_b|ψⱼ⟩ = C_ab δᵢⱼ
(C_ab has non-maximal rank)

Set $\mathcal{E}$ of errors.

1) E has unique error syndrome iff commutes w/ unique set of generators.

E & F have same error syndrome if they commute w/ same generators.

$$\Leftrightarrow [E^\dagger F, M] = 0 \ \forall M \in S \Leftrightarrow E^\dagger F \in N(S).$$

$$\text{If } \underbrace{E^\dagger F}_{=M} \in S \Rightarrow E|\psi\rangle = EM|\psi\rangle = F|\psi\rangle$$

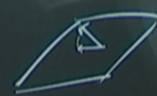
$$E^2 = I$$

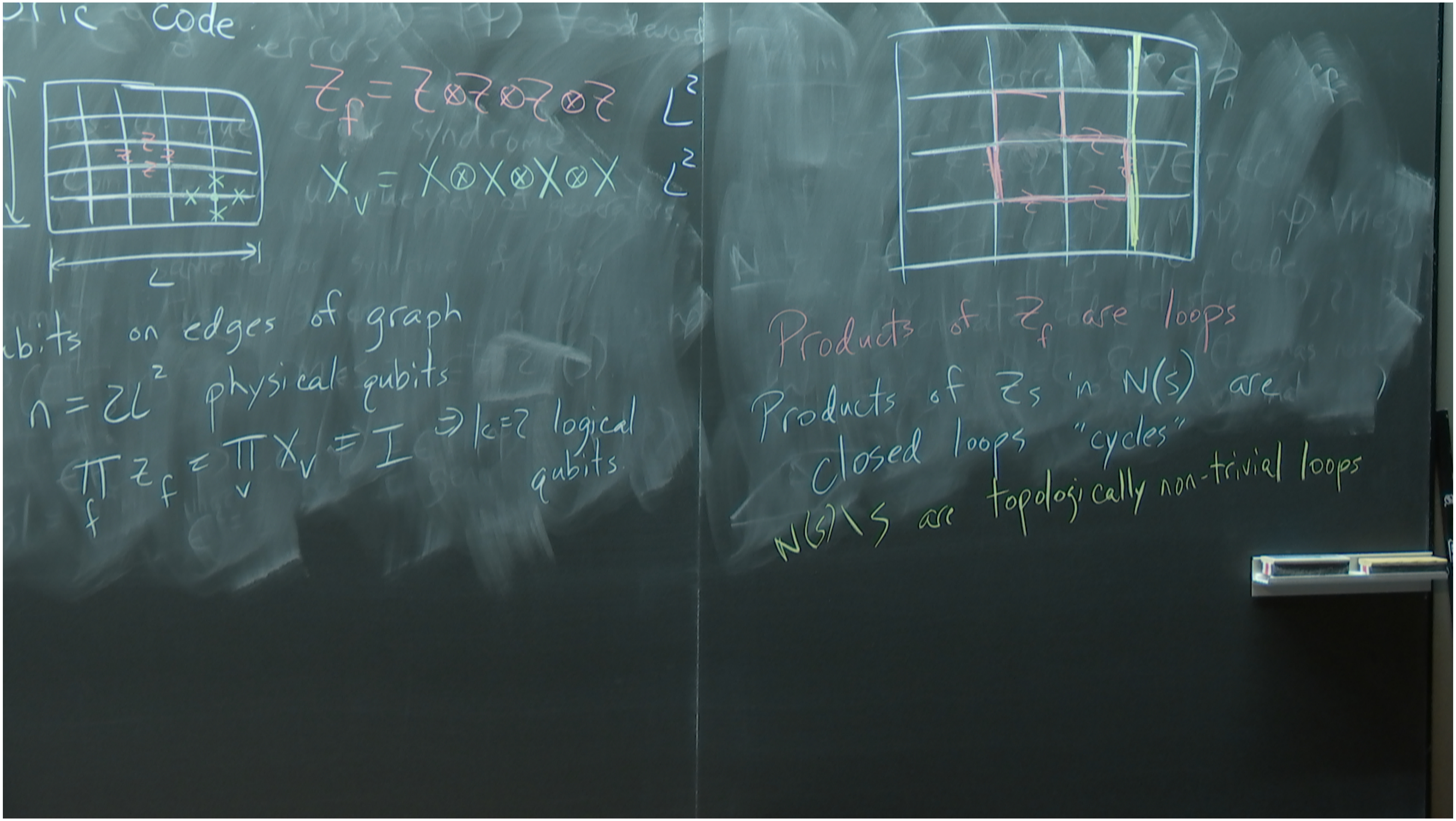
Thm: S corrects $\mathcal{E} \subseteq \mathcal{P}_n$ iff

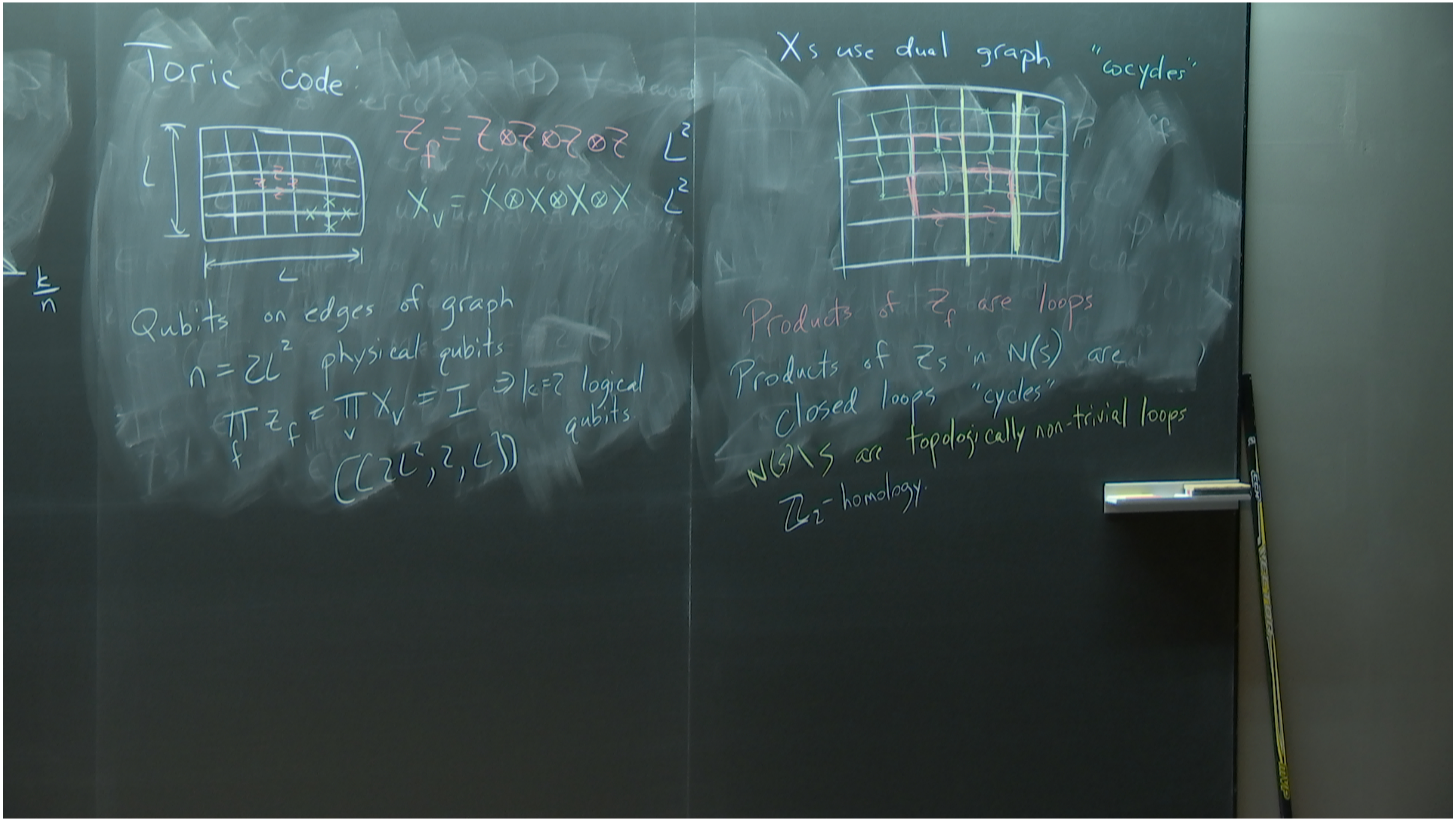
$$E^\dagger F \notin N(S) \setminus S \quad \forall E, F \in \mathcal{E}$$

Note: If $E^\dagger F \in S$, then code is degenerate code.

$$\langle \psi_i | E_a^\dagger E_b | \psi_j \rangle = C_{ab} \delta_{ij} \quad (C_{ab} \text{ has non-maximal rank})$$




code
Z_f = Z⊗Z⊗Z⊗Z L^2
X_v = X⊗X⊗X⊗X L^2
L
bits on edges of graph
n = 2L^2 physical qubits
Π_f Z_f = Π_v X_v = I ⇒ k=2 logical qubits
Products of Z_f are loops
Products of Z_s in N(S) are closed loops "cycles"
N(S)\S are topologically non-trivial loops


Toric code:
L
L
$Z_f = Z\otimes Z\otimes Z\otimes Z$ L^2
$X_v = X\otimes X\otimes X\otimes X$ L^2
$\frac{k}{n}$
Qubits on edges of graph
$n = 2L^2$ physical qubits
$\prod_f Z_f = \prod_v X_v = I \Rightarrow k=2$ logical qubits
$[[2L^2, 2, L]]$
Xs use dual graph "cocycles"
Products of Z_f are loops
Products of Zs in $N(S)$ are closed loops "cycles"
$N(S)\setminus S$ are topologically non-trivial loops
$\mathbb{Z}_2$-homology.