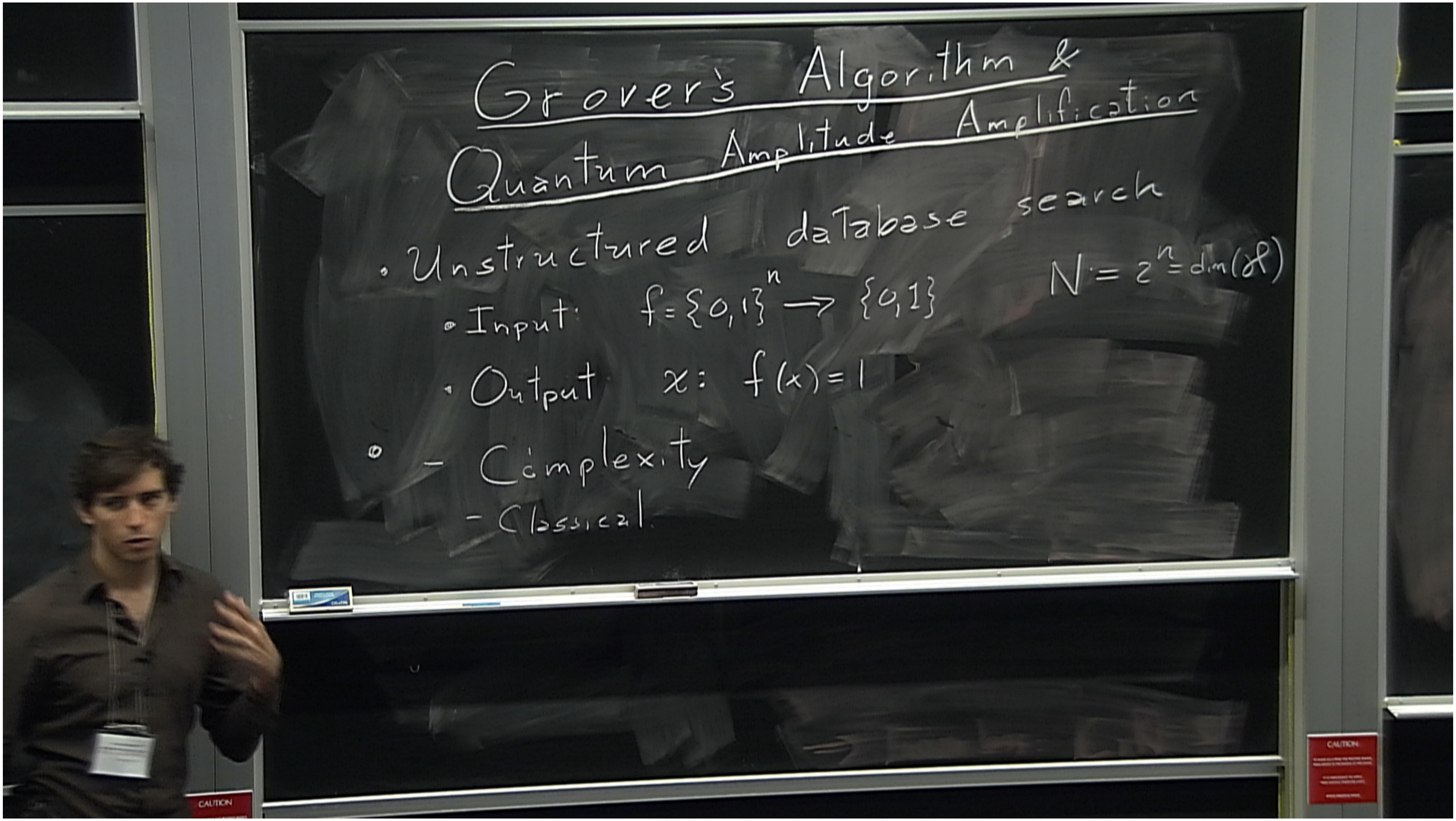
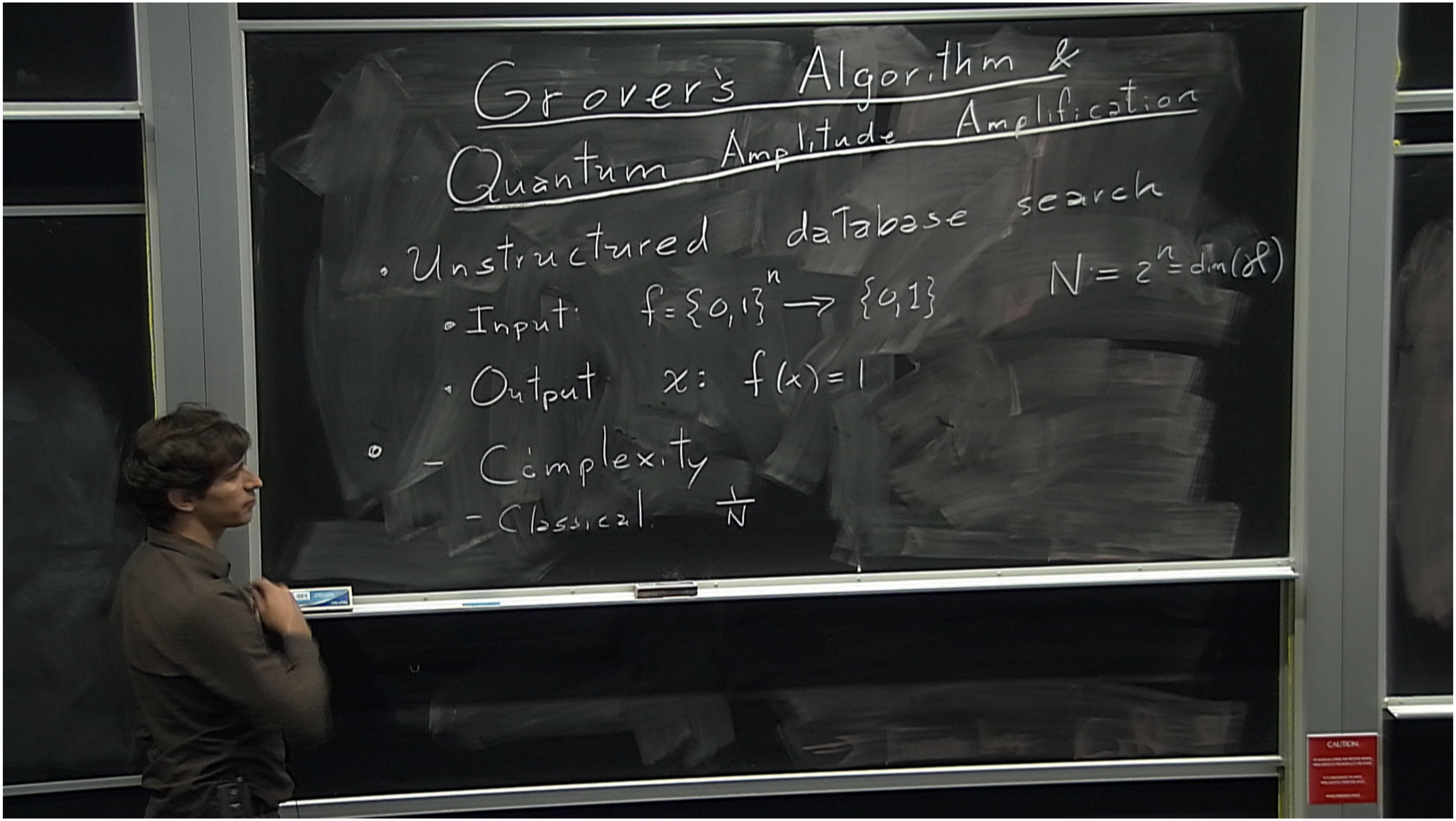
Title: Groverâ€™s Algorithm

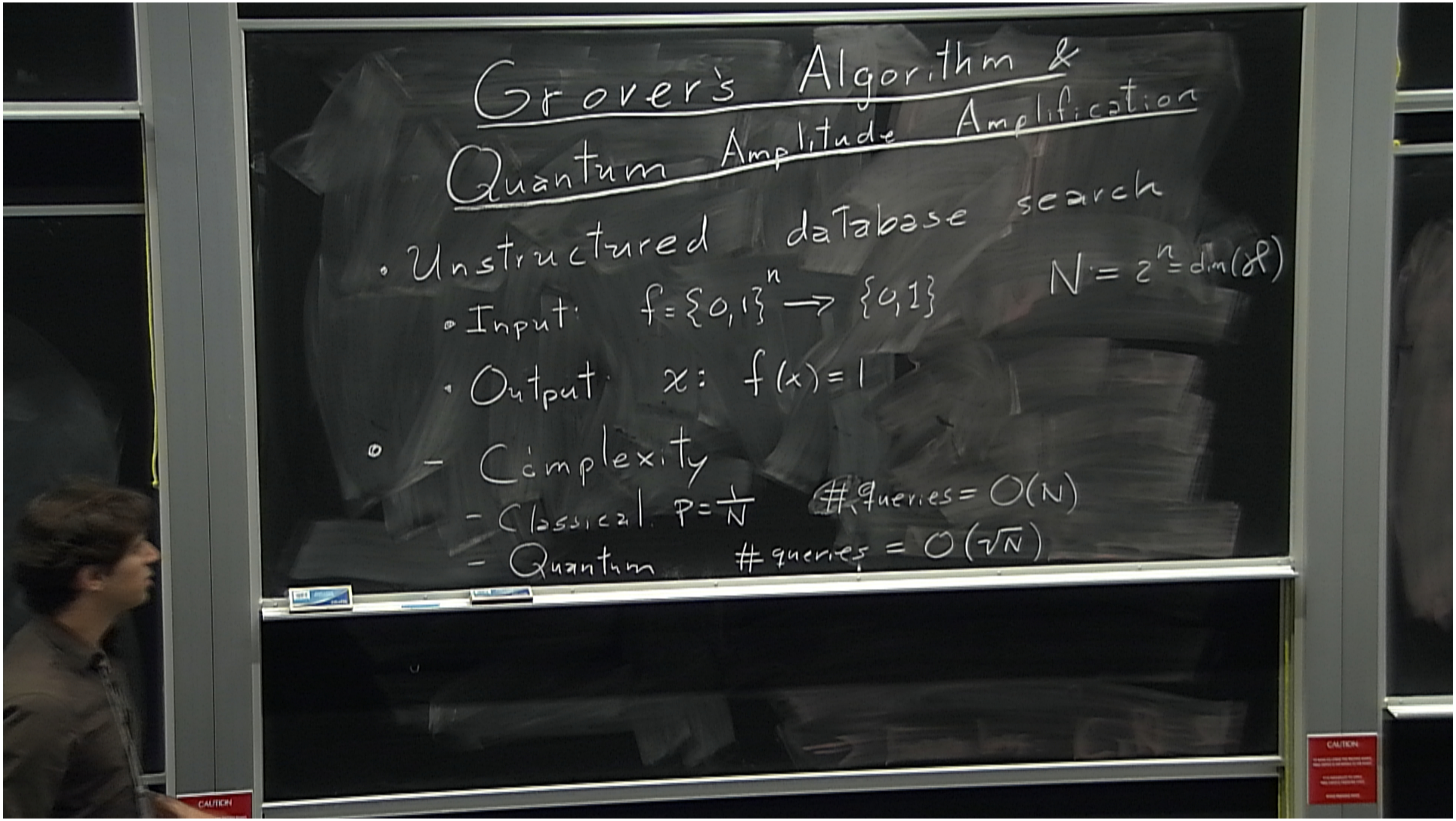
Date: Jul 19, 2016 05:00 PM

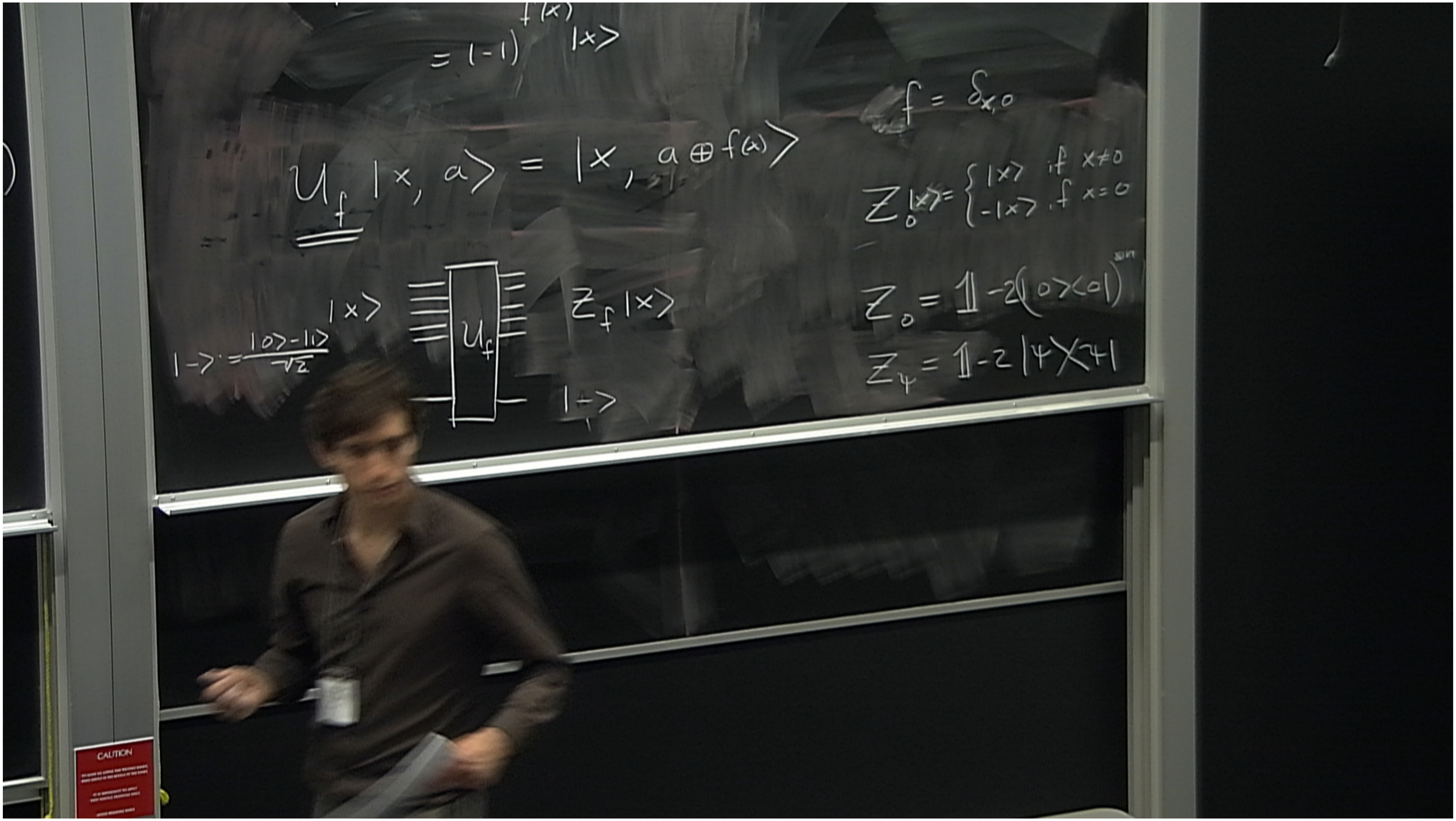
URL: http://pirsa.org/16070016

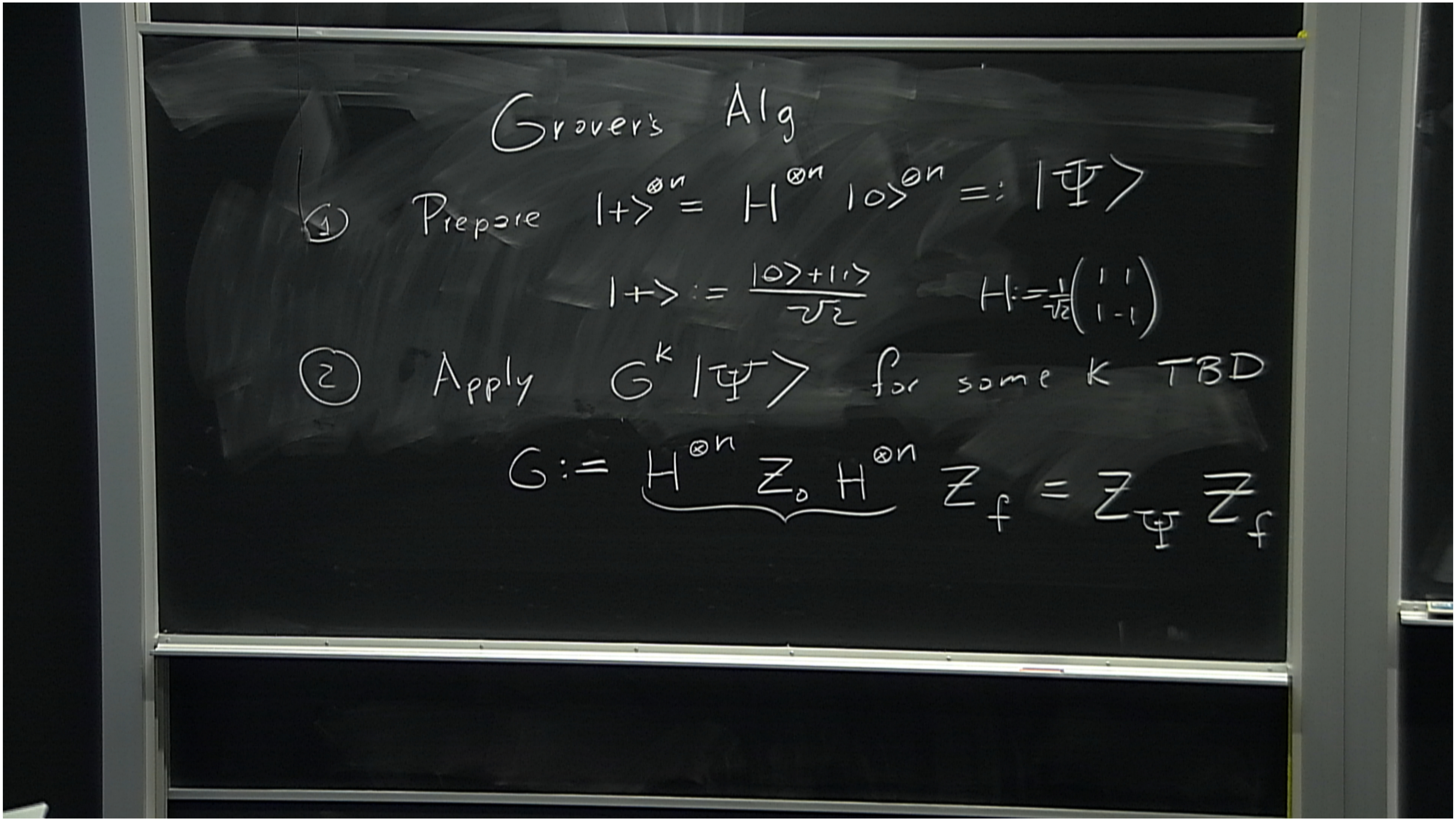
Abstract:

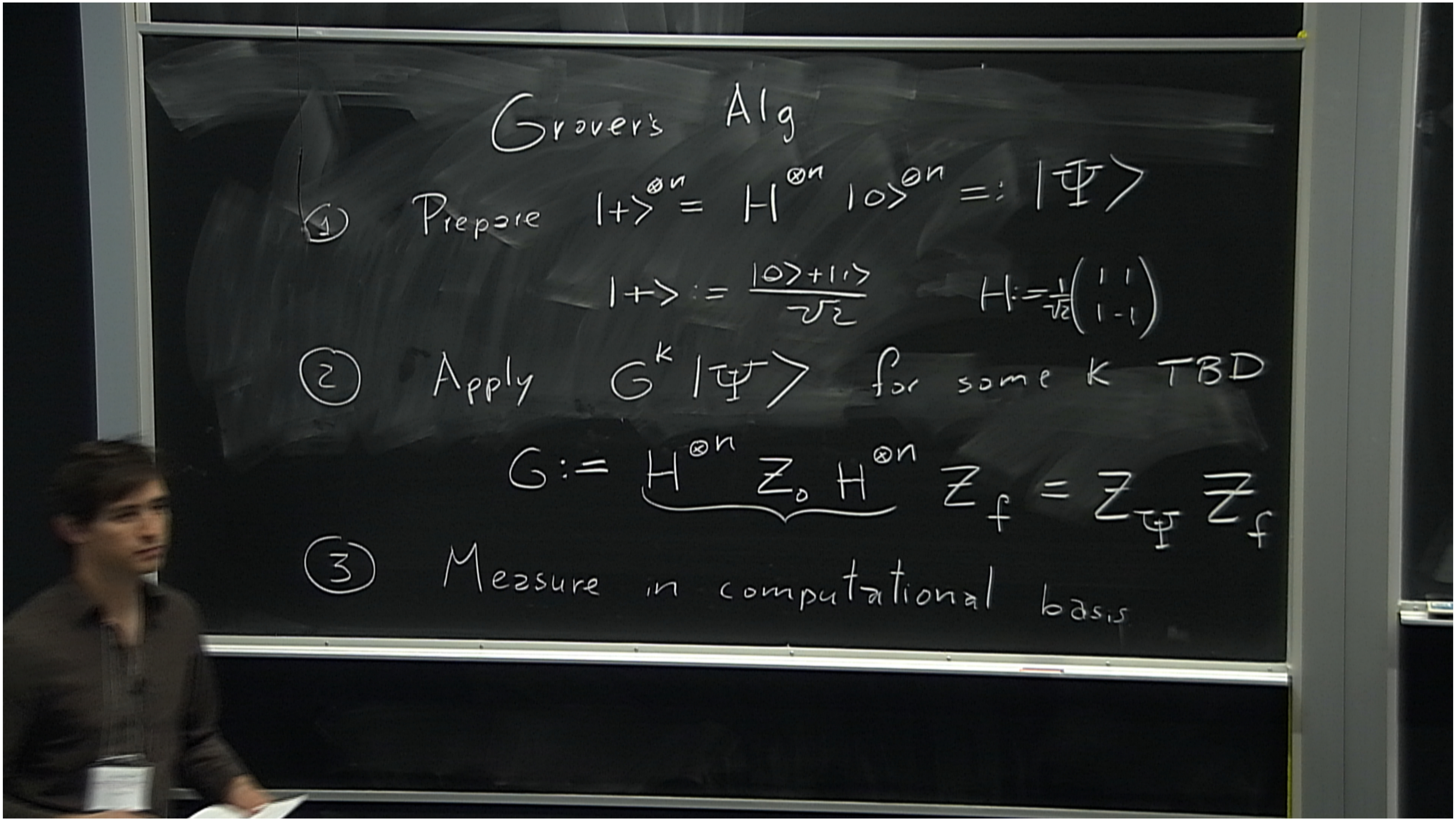

Grover's Algorithm &
Quantum Amplitude Amplification
• Unstructured database search
N = 2^n = dim(H)
• Input: f = {0,1}^n → {0,1}
• Output x: f(x) = 1
• – Complexity
– Classical

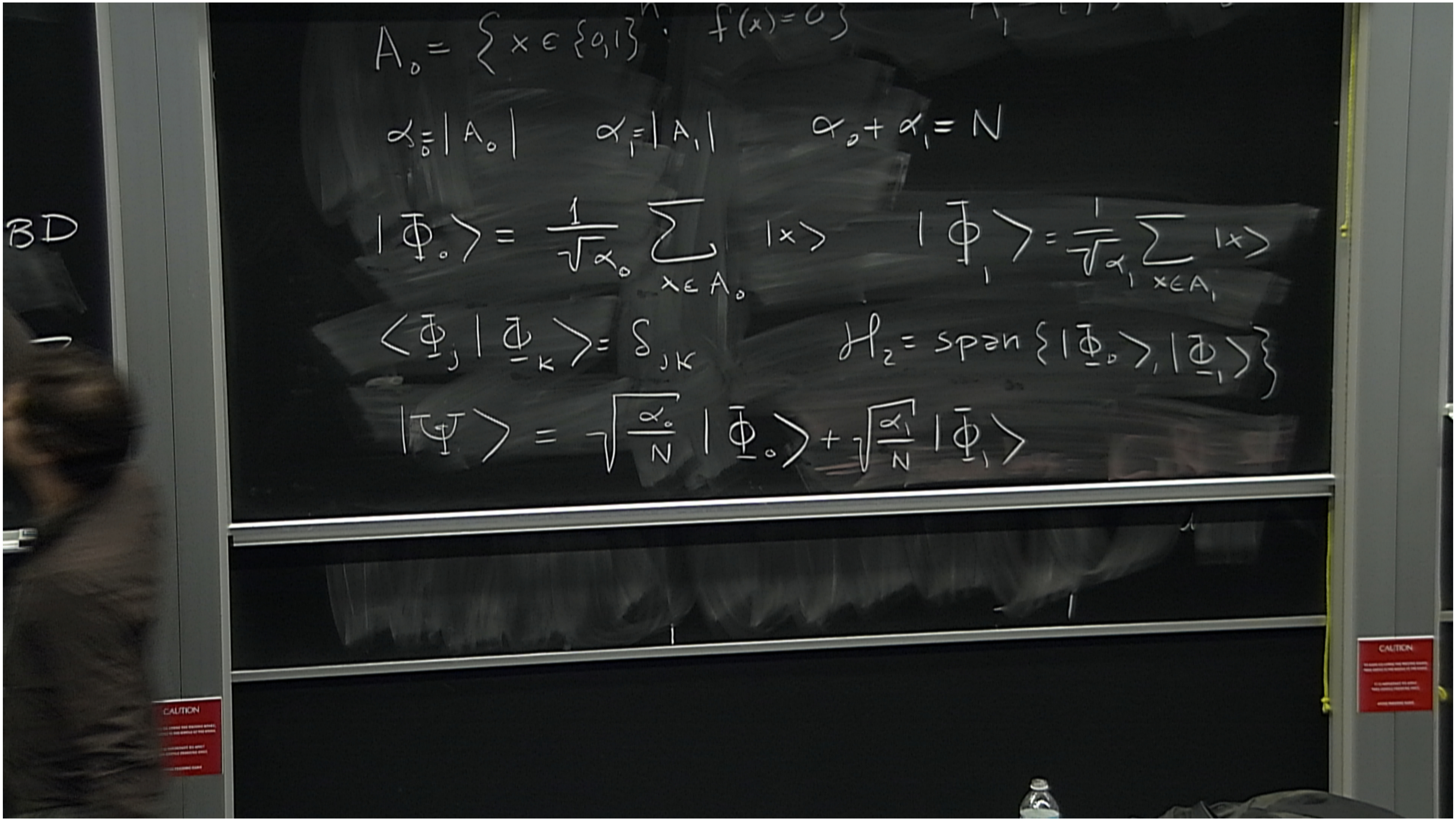

Grover's Algorithm &
Quantum Amplitude Amplification
• Unstructured database search
N = 2^n = dim(H)
• Input: f = {0,1}^n → {0,1}
• Output x: f(x) = 1
• - Complexity
- Classical 1/N

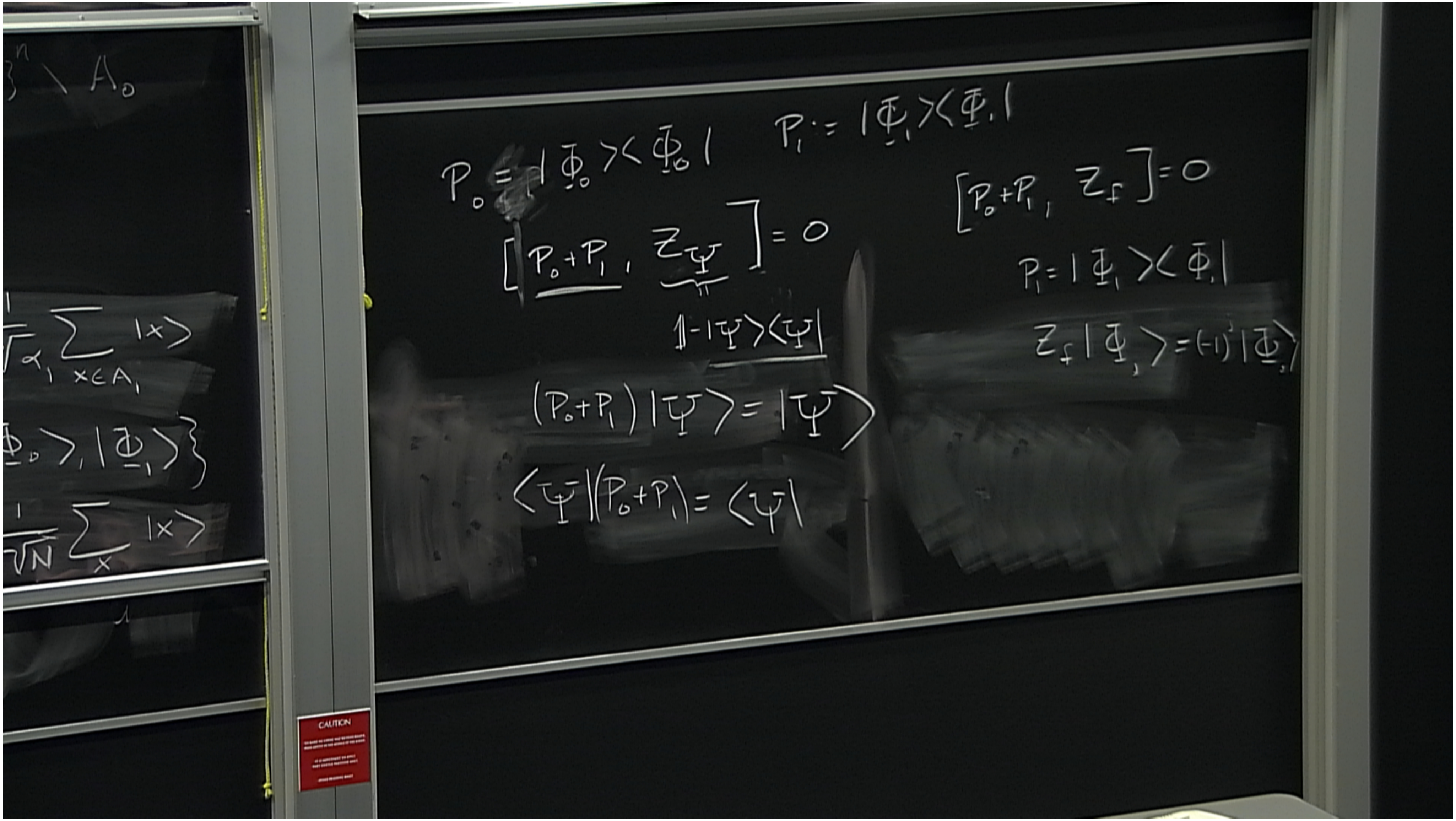

Grover's Algorithm &
Quantum Amplitude Amplification
• Unstructured database search
N = 2^n = dim(ℋ)
• Input: f = {0,1}^n → {0,1}
• Output: x: f(x) = 1
• – Complexity
– Classical p = 1/N   # queries = O(N)
– Quantum   # queries = O(√N)


$= (-1)^{f(x)} |x\rangle$
$U_f |x, a\rangle = |x, a \oplus f(x)\rangle$
$f = \delta_{x,0}$
$Z_0 |x\rangle = \begin{cases} |x\rangle & \text{if } x \neq 0 \\ -|x\rangle & \text{if } x = 0 \end{cases}$
$Z_0 = \mathbb{1} - 2(|0\rangle\langle 0|)^{\otimes n}$
$Z_\psi = \mathbb{1} - 2|\psi\rangle\langle\psi|$
$|x\rangle$
U_f
$Z_f |x\rangle$
$|-\rangle := \frac{|0\rangle - |1\rangle}{\sqrt{2}}$
$|-\rangle$
CAUTION

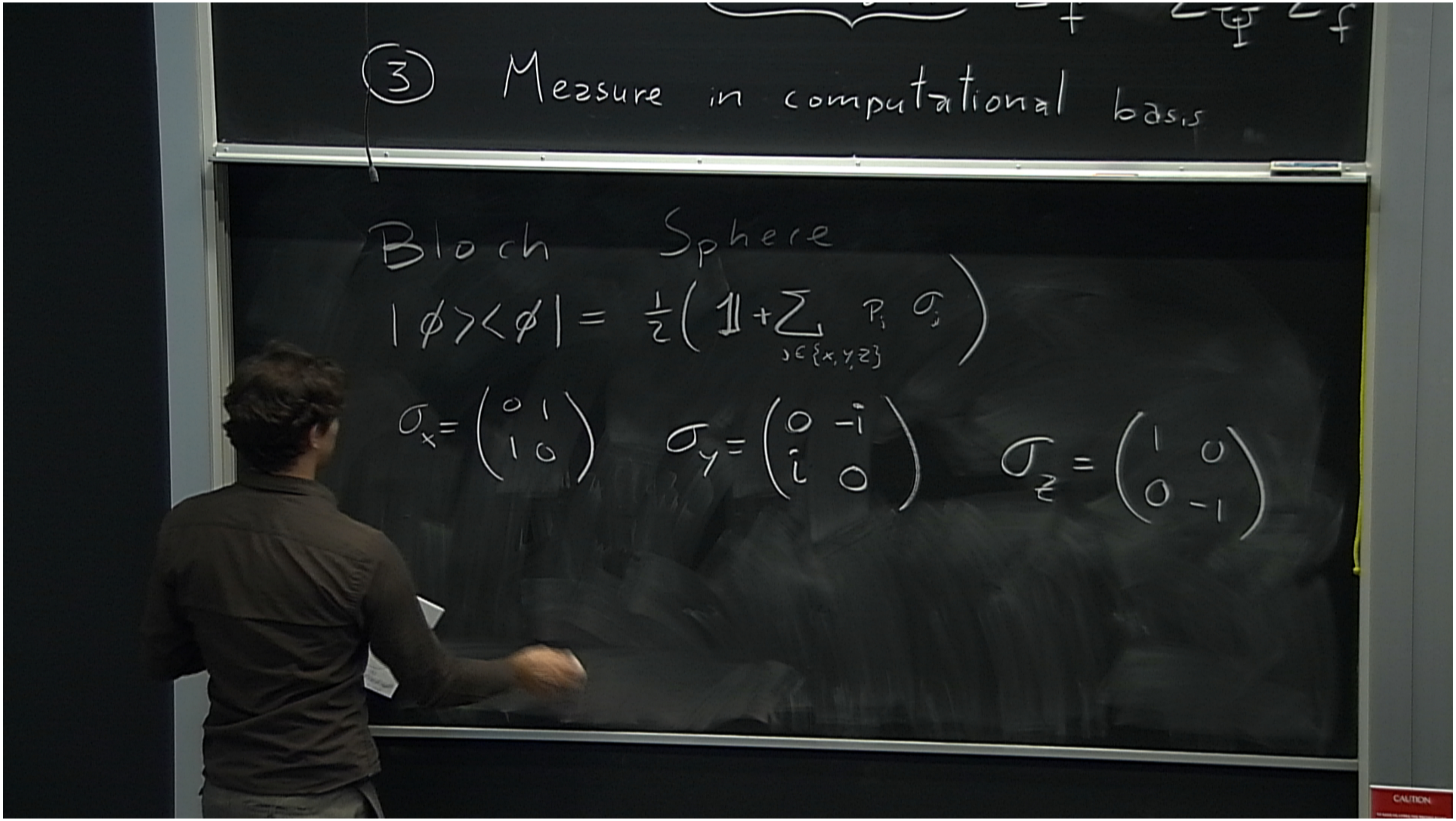

Grover's Alg
① Prepare $|+\rangle^{\otimes n} = H^{\otimes n} |0\rangle^{\otimes n} =: |\Psi\rangle$
$|+\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}}$
$H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
② Apply $G^k |\Psi\rangle$ for some k TBD
$G := \underbrace{H^{\otimes n} Z_0 H^{\otimes n}} Z_f = Z_\Psi Z_f$

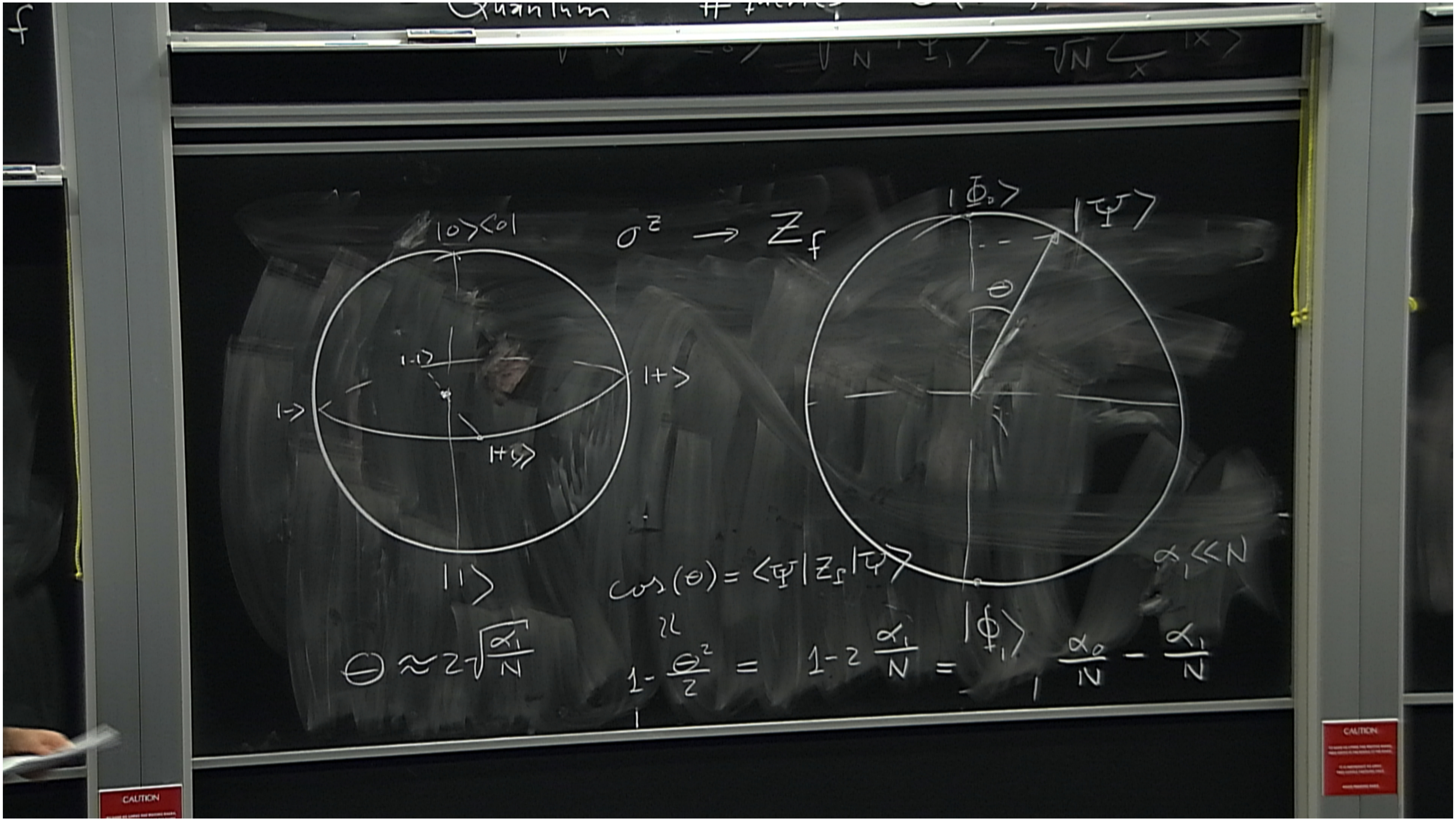

Grover's Alg
① Prepare $|+\rangle^{\otimes n} = H^{\otimes n} |0\rangle^{\otimes n} =: |\Psi\rangle$
$|+\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ $H := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
② Apply $G^k |\Psi\rangle$ for some k TBD
$G := \underbrace{H^{\otimes n} Z_0 H^{\otimes n}} Z_f = Z_\Psi Z_f$
③ Measure in computational basis

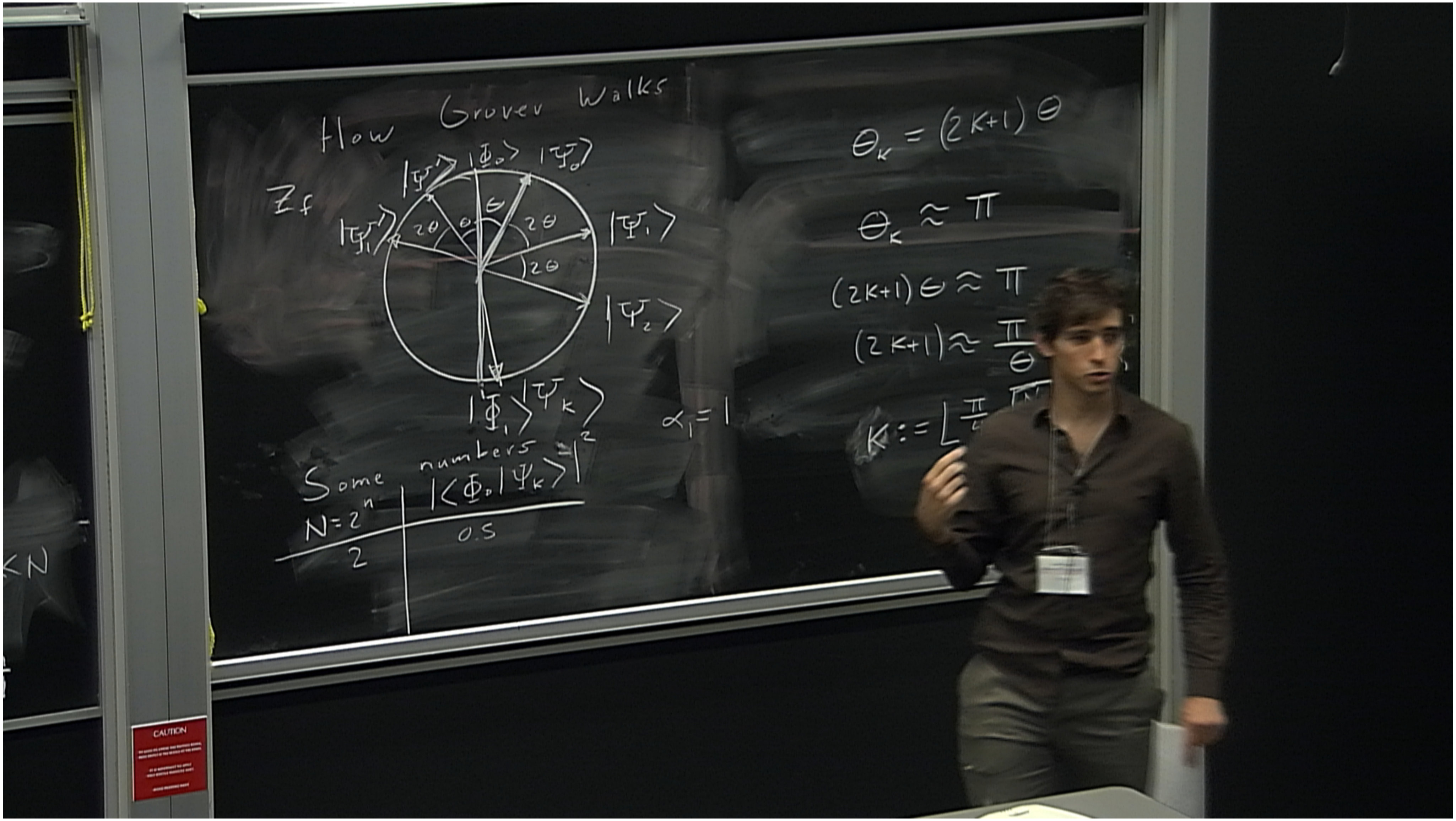

BD
$A_0 = \{ x \in \{0,1\}^n : f(x) = 0 \}$
$\alpha_0 = |A_0|$ $\alpha_1 = |A_1|$ $\alpha_0 + \alpha_1 = N$
$|\Phi_0\rangle = \frac{1}{\sqrt{\alpha_0}} \sum_{x \in A_0} |x\rangle$ $|\Phi_1\rangle = \frac{1}{\sqrt{\alpha_1}} \sum_{x \in A_1} |x\rangle$
$\langle \Phi_j | \Phi_k \rangle = \delta_{jk}$ $\mathcal{H}_2 = \mathrm{span}\{ |\Phi_0\rangle, |\Phi_1\rangle \}$
$|\Psi\rangle = \sqrt{\frac{\alpha_0}{N}} |\Phi_0\rangle + \sqrt{\frac{\alpha_1}{N}} |\Phi_1\rangle$

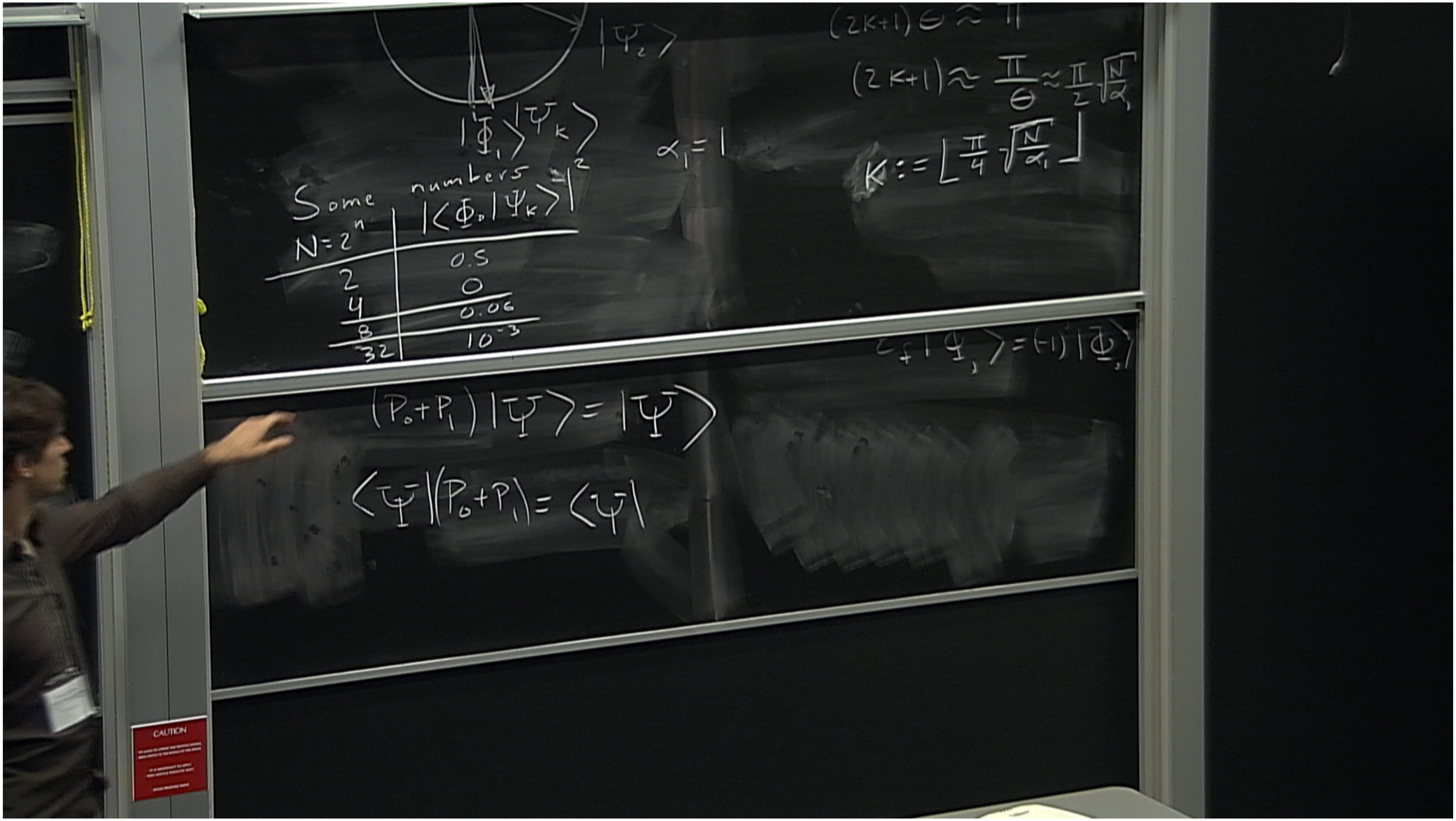


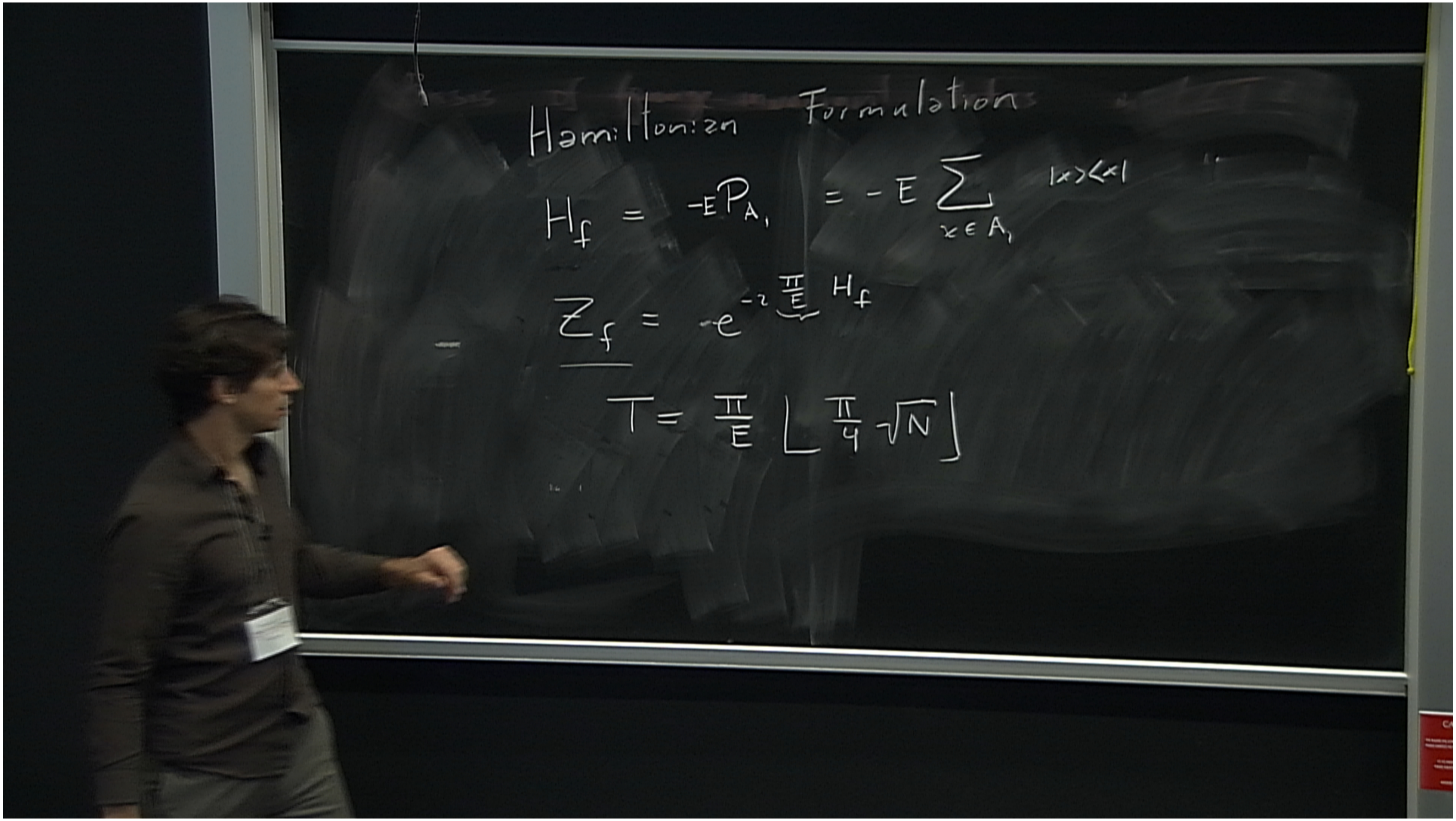
$P_0 = |\Phi_0\rangle\langle\Phi_0|$
$P_1 := |\Phi_1\rangle\langle\Phi_1|$
$[P_0+P_1, Z_f] = 0$
$[P_0+P_1, Z_\Psi] = 0$
$P_1 = |\Phi_1\rangle\langle\Phi_1|$
$\mathbb{1} - |\Psi\rangle\langle\Psi|$
$Z_f|\Phi_j\rangle = (-1)^j|\Phi_j\rangle$
$(P_0+P_1)|\Psi\rangle = |\Psi\rangle$
$\langle\Psi|(P_0+P_1) = \langle\Psi|$
CAUTION

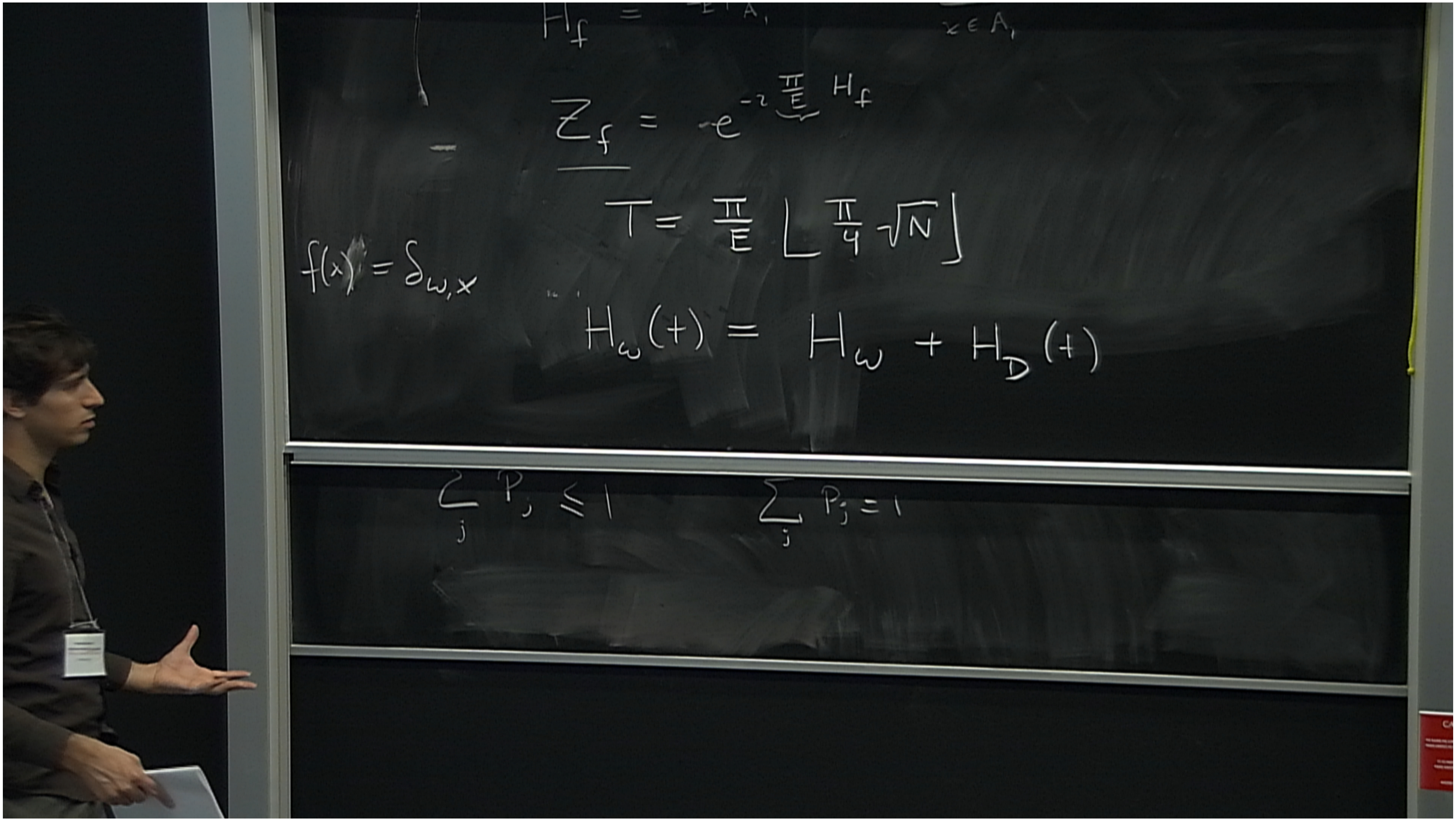

③ Measure in computational basis
Bloch Sphere
$|\phi\rangle\langle\phi| = \frac{1}{2}\left(\mathbb{1} + \sum_{i\in\{x,y,z\}} p_i \sigma_i\right)$
$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

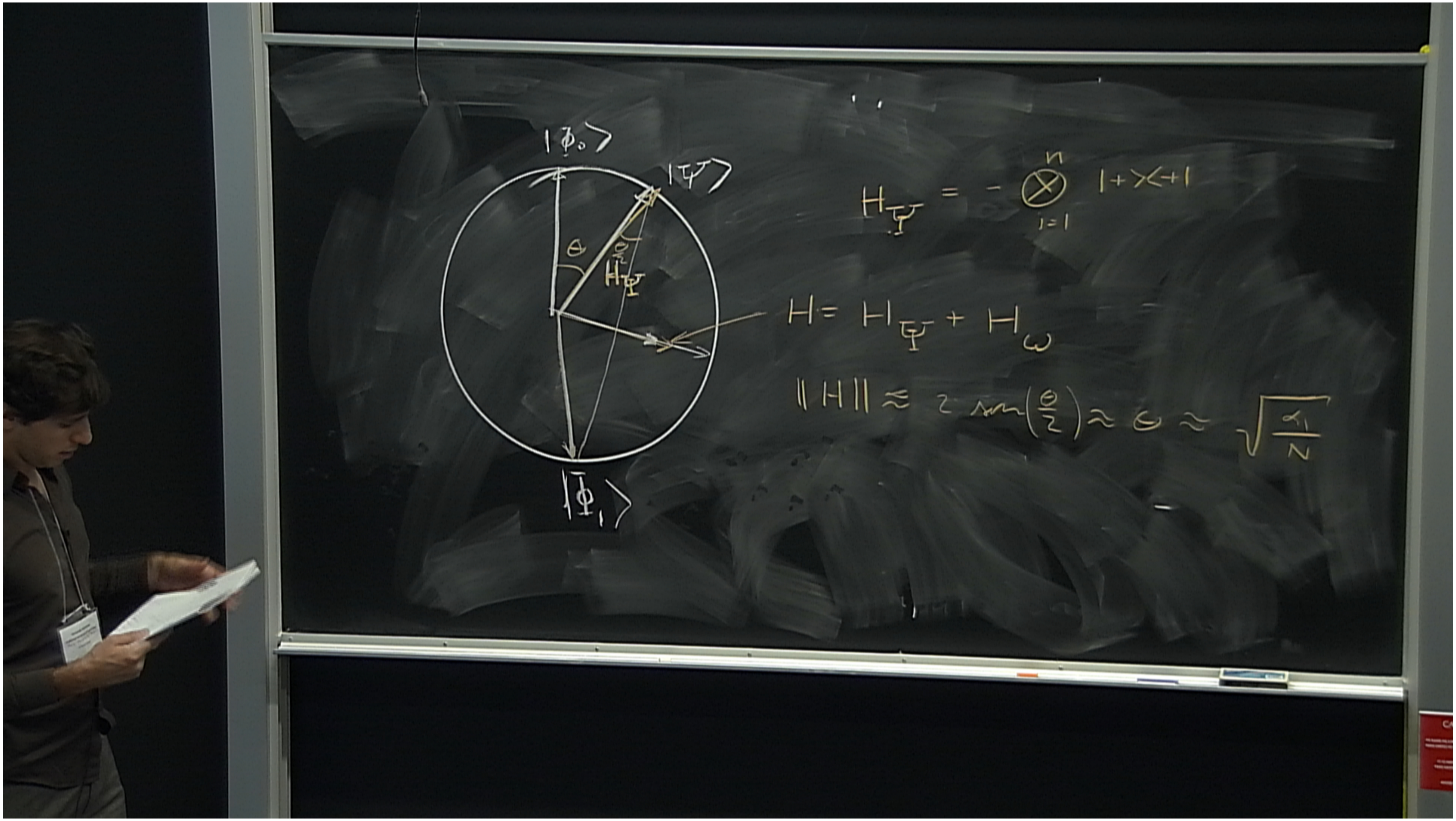

|0⟩⟨0|
σ^z → Z_f
|+⟩
|−⟩
|+y⟩
|1⟩
|Φ_0⟩
|Ψ⟩
θ
cos(θ) = ⟨Ψ|Z_f|Ψ⟩
α_1 << N
|ϕ_1⟩
θ ≈ 2√(α_1/N)
1 − θ²/2 = 1 − 2 α_1/N = α_0/N − α_1/N

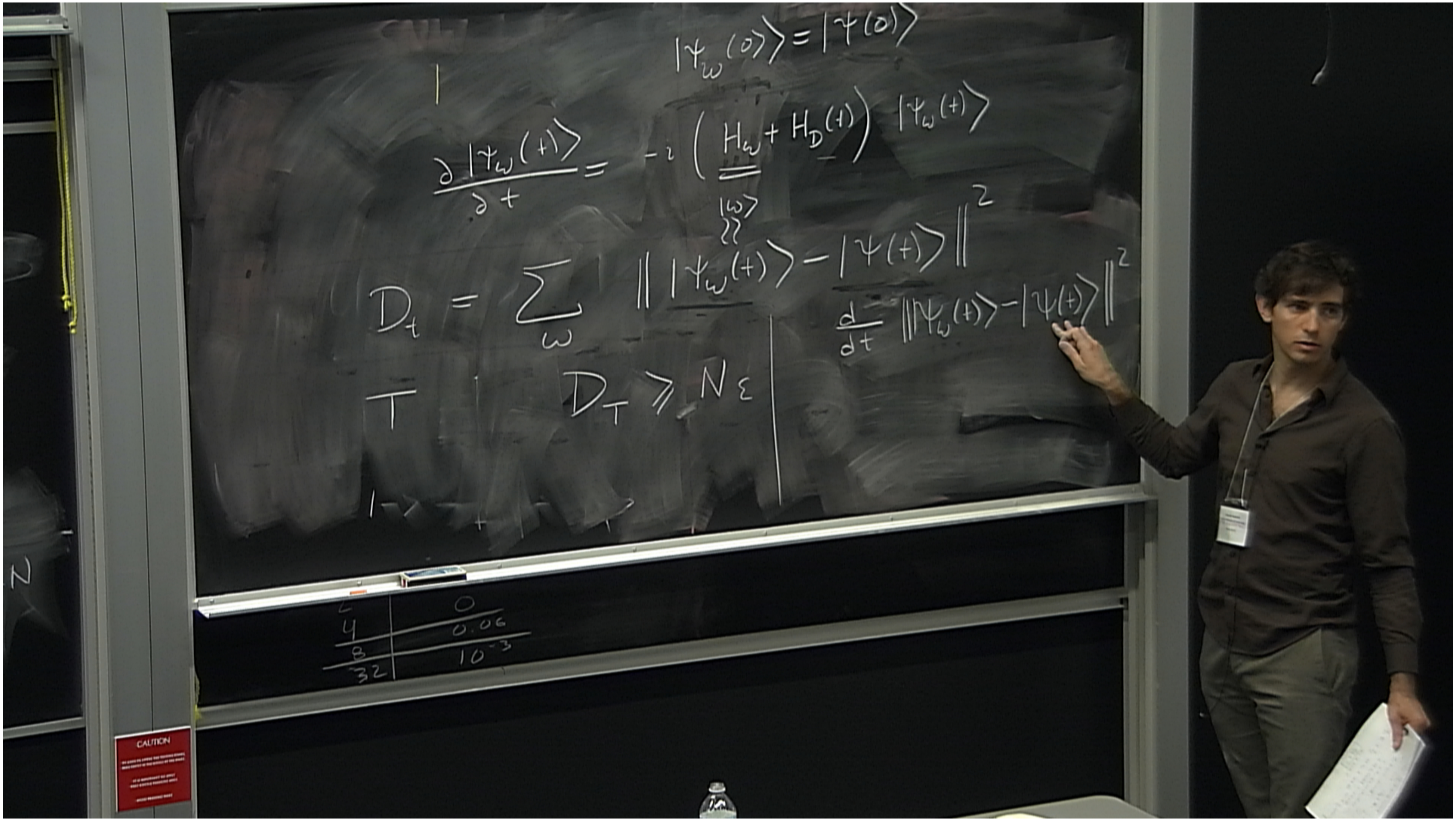

How Grover Walks
Z_f
$|\Phi_0\rangle$ $|\Psi_0\rangle$
$|\Psi_1\rangle$
$|\Psi_2\rangle$
$|\Phi_1\rangle$ $|\Psi_k\rangle$
2θ
$\alpha_1 = 1$
$\theta_k = (2k+1)\theta$
$\theta_k \approx \pi$
$(2k+1)\theta \approx \pi$
Some numbers
$N=2^n$ | $|\langle\Phi_0|\Psi_k\rangle|^2$
2 | 0.5


Some numbers
N=2^n
|⟨Φ₀|Ψ_K⟩|²
2
0.5
4
0
8
0.06
32
10^-3
α₁=1
(2K+1)≈ π/θ ≈ π/2 √(N/α₁)
K:=⌊π/4 √(N/α₁)⌋
(P₀+P₁)|Ψ⟩=|Ψ⟩
⟨Ψ|(P₀+P₁)=⟨Ψ|
CAUTION

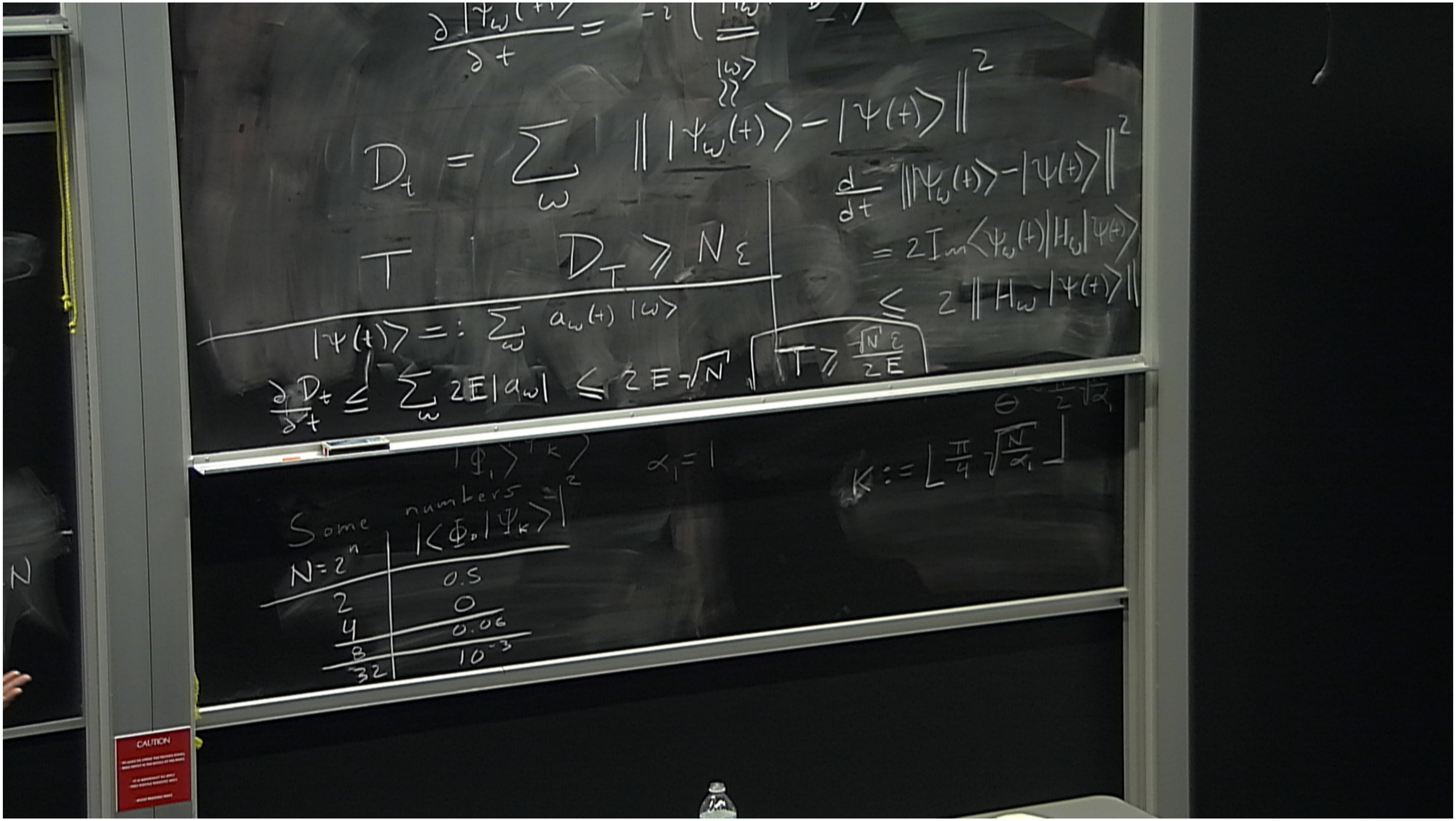

Hamiltonian Formulation
$H_f = -E P_{A_1} = -E \sum_{x \in A_1} |x\rangle\langle x|$
$Z_f = e^{-i \frac{\pi}{E} H_f}$
$T = \frac{\pi}{E} \lfloor \frac{\pi}{4} \sqrt{N} \rfloor$


$Z_f = e^{-i \frac{\pi}{E} H_f}$
$T = \frac{\pi}{E} \lfloor \frac{\pi}{4} \sqrt{N} \rfloor$
$f(x) = \delta_{\omega,x}$
$H_\omega(t) = H_\omega + H_D(t)$
$\sum_j P_j \leq 1$
$\sum_j P_j = 1$

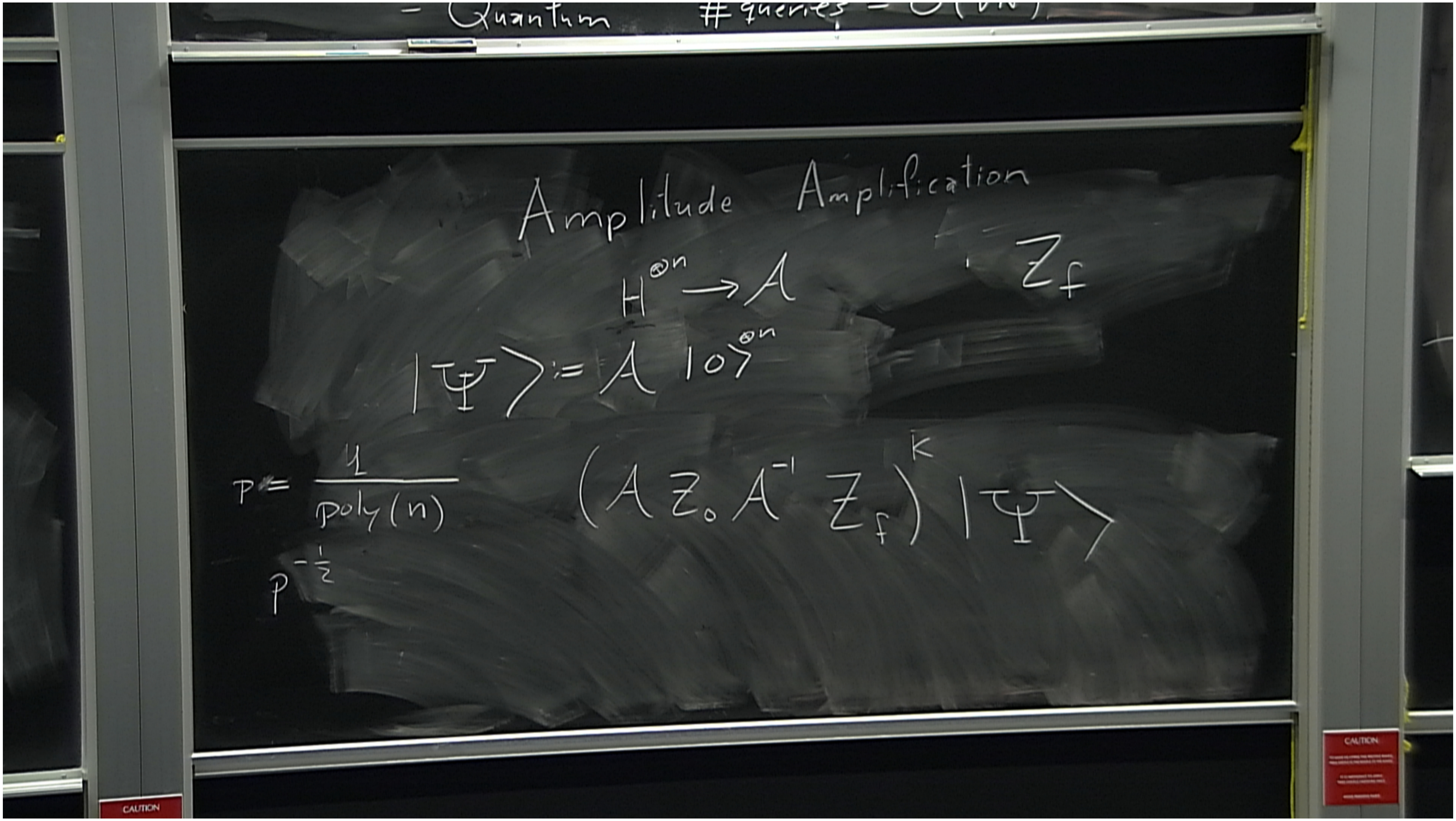

|Φ₀⟩
|Ψ⟩
θ
θ/2
H_Ψ
|Φ₁⟩
H_Ψ = − ⊗_{i=1}^{n} |+⟩⟨+|
H = H_Ψ + H_ω
||H|| ≈ 2 sin(θ/2) ≈ θ ≈ √(α₁/N)



|ψ_ω(0)⟩ = |ψ(0)⟩
∂|ψ_ω(t)⟩/∂t = −i (H_ω + H_D(t)) |ψ_ω(t)⟩
D_t = Σ_ω || |ψ_ω(t)⟩ − |ψ(t)⟩ ||²
d/dt || |ψ_ω(t)⟩ − |ψ(t)⟩ ||²
D_T ≥ N ε
4 0.06
8 10⁻³
32



$D_t = \sum_\omega \| |\psi_\omega(t)\rangle - |\psi(t)\rangle \|^2$
$T : D_T \geq N\varepsilon$
$|\psi(t)\rangle =: \sum_\omega a_\omega(t) |\omega\rangle$
$\frac{\partial D_t}{\partial t} \leq \sum_\omega 2E|a_\omega| \leq 2E\sqrt{N}$
$T \geq \frac{\sqrt{N}\varepsilon}{2E}$
$\frac{d}{dt} \| |\psi_\omega(t)\rangle - |\psi(t)\rangle \|^2$
$= 2\,\mathrm{Im}\langle\psi_\omega(t)|H_\omega|\psi(t)\rangle$
$\leq 2 \| H_\omega |\psi(t)\rangle \|$
$\alpha_1 = 1$
$K := \lfloor \frac{\pi}{4}\sqrt{\frac{N}{\alpha_1}} \rfloor$
Some numbers
$N = 2^n$
$|\langle\Phi_0|\Psi_K\rangle|^2$
2 0.5
4 0
8 0.06
32 10^{-3}


Quantum # queries
Amplitude Amplification
$H^{\otimes n} \to A$
Z_f
$|\Psi\rangle := A|0\rangle^{\otimes n}$
$p = \frac{1}{poly(n)}$
$(A Z_0 A^{-1} Z_f)^k |\Psi\rangle$
$p^{-\frac{1}{2}}$