

Title: Entanglement

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Abstract:

Entanglement Theory

Part II

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Entanglement theory for bipartite pure states

$$|\Psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$$

Deterministic state conversion

Nielsen's theorem

$$\psi \xrightarrow{LOCC} \phi \quad \Leftrightarrow \quad \lambda(\rho_\phi) \succ \lambda(\rho_\psi)$$

$\lambda(\rho)$ = vector of eigenvalues of ρ

where $x \succ y$ Is the majorization relation

Deterministic state conversion

Nielsen's theorem

$$\psi \xrightarrow{LOCC} \phi \quad \Leftrightarrow \quad \lambda(\rho_\phi) \succ \lambda(\rho_\psi)$$

$\lambda(\rho)$ = vector of eigenvalues of ρ

where $x \succ y$ Is the majorization relation
for $x := (x_1, x_2, \dots, x_d) \in \mathbb{R}_+^d$

define $x_1^\downarrow \geq x_2^\downarrow \cdots \geq x_d^\downarrow$

Similarly for y

$$\begin{aligned} x_1^\downarrow &\geq y_1^\downarrow \\ x_1^\downarrow + x_2^\downarrow &\geq y_1^\downarrow + y_2^\downarrow \\ &\vdots \\ x_1^\downarrow + x_2^\downarrow + \cdots + x_{d-1}^\downarrow &\geq y_1^\downarrow + y_2^\downarrow + \cdots + y_{d-1}^\downarrow \end{aligned}$$

LOCC equivalence

$$\begin{array}{c} \psi \xrightarrow{\text{LOCC}} \phi \\ \psi \xleftarrow{\text{LOCC}} \phi \end{array} \iff \lambda(\rho_\phi) = \lambda(\rho_\psi)$$

i.e., equivalent up to local unitaries

Measures of entanglement

Def'n: A function E from states to the reals is a measure of entanglement if

$$\rho \xrightarrow{LOCC} \sigma \Rightarrow E(\rho) \geq E(\sigma)$$

Measures of entanglement

$$E(\psi) = -\text{Tr}(\rho_\psi^2)$$

Purity-based entanglement measure

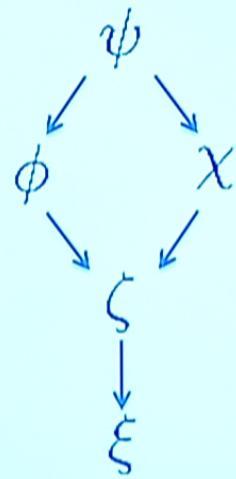
$$E(\psi) = -\text{Tr}(\rho_\psi \log \rho_\psi)$$

Entanglement entropy

$$E(\psi) = \pm \frac{1}{\alpha-1} \log \text{Tr}(\rho_\psi^\alpha)$$

Order- α Renyi entanglement

Bipartite pure states
under LOCC form a
partial order



$$\psi \xrightarrow{LOCC} \phi \iff \begin{aligned} E^{(1)}(\psi) &\geq E^{(1)}(\phi) \\ E^{(2)}(\psi) &\geq E^{(2)}(\phi) \\ &\vdots \end{aligned}$$

Complete set of entanglement measures

Recall definition of majorization $x \succ y$

$$\begin{aligned}x_1^\downarrow &\geq y_1^\downarrow \\x_1^\downarrow + x_2^\downarrow &\geq y_1^\downarrow + y_2^\downarrow \\&\vdots \\x_1^\downarrow + x_2^\downarrow + \cdots + x_{d-1}^\downarrow &\geq y_1^\downarrow + y_2^\downarrow + \cdots + y_{d-1}^\downarrow\end{aligned}$$

Recall definition of Schur-concavity

$$x \succ y \Rightarrow G(x) \leq G(y).$$

$$k \in \{1, \dots, d-1\}$$

Recall definition of majorization $x \succ y$

$$\begin{aligned}x_1^\downarrow &\geq y_1^\downarrow \\x_1^\downarrow + x_2^\downarrow &\geq y_1^\downarrow + y_2^\downarrow \\&\vdots \\x_1^\downarrow + x_2^\downarrow + \cdots + x_{d-1}^\downarrow &\geq y_1^\downarrow + y_2^\downarrow + \cdots + y_{d-1}^\downarrow\end{aligned}$$

Recall definition of Schur-concavity
 $x \succ y \Rightarrow G(x) \leq G(y)$.

$$G^{(k)}(x) := - \sum_{i=1}^k x_i^\downarrow \quad \text{is Schur-concave}$$

The Ky Fan k-norm

$$\tilde{E}^{(k)}(\psi) := - \sum_{i=1}^k \lambda_i^\downarrow(\rho_\psi) \quad k \in \{1, \dots, d-1\}$$

The set of such measures forms a complete set

$$\psi \xrightarrow{LOCC} \phi \iff \begin{aligned} E^{(1)}(\psi) &\geq E^{(1)}(\phi) \\ E^{(2)}(\psi) &\geq E^{(2)}(\phi) \\ &\vdots \\ E^{(d-1)}(\psi) &\geq E^{(d-1)}(\phi) \end{aligned}$$

$$E^{(k)}(\psi) := \sum_{i=k+1}^d \lambda_i^\downarrow(\rho_\psi)$$

Complete set of entanglement measures

Stochastic state conversion

$$\psi \xrightarrow{LOCC} \phi \quad \iff \quad p \leq \min_k \frac{E^{(k)}(\psi)}{E^{(k)}(\phi)}$$

with probability p

$$E^{(k)}(\psi) := \sum_{i=k+1}^d \lambda_i^\downarrow(\rho_\psi)$$

Vidal's theorem

Note: $p = 1 \Rightarrow \forall k : E^{(k)}(\psi) \geq E^{(k)}(\phi)$

Stochastic-LOCC equivalence

$$\begin{array}{ccc} \psi \xrightarrow{\text{LOCC}} \phi & \rightleftharpoons & \text{rank}(\rho_\psi) = \text{rank}(\rho_\phi) \\ \text{with nonzero probability} & & \text{"Schmidt number" is the same} \\ \psi \xleftarrow{\text{LOCC}} \phi & & \\ \text{with nonzero probability} & & \end{array}$$

Catalytic state conversion

$$\begin{array}{ccc} \psi \xrightarrow[\exists \text{ entangled state } \eta :]{\cancel{LOCC}} \phi & \rightleftharpoons & \lambda(\rho_\phi) \neq \lambda(\rho_\psi) \\ \psi \otimes \eta \xrightarrow{LOCC} \phi \otimes \eta & & \exists \text{ probability distribution } z \\ & & \lambda(\rho_\phi) \otimes z \succ \lambda(\rho_\psi) \otimes z \end{array}$$

Note: additive measures of entanglement cannot distinguish deterministic order from catalytic order

$$\begin{aligned} E(\psi \otimes \eta) &\geq E(\phi \otimes \eta) \\ E(\psi) + E(\eta) &\geq E(\phi) + E(\eta) \\ E(\psi) &\geq E(\phi) \end{aligned}$$

Asymptotic state conversion

$$\psi^{\otimes n} \xrightarrow{LOCC} \phi^{\otimes m}$$

at asymptotic rate

$$R = \lim_{n \rightarrow \infty} \frac{m}{n}$$



$$R(\psi \rightarrow \phi) = \frac{E(\psi)}{E(\phi)}$$

$$E(\psi) = -\text{Tr}(\rho_\psi \log \rho_\psi)$$

$$R(\psi \rightarrow \phi) = \sup_{\mathcal{E} \in \text{LOCC}} \left\{ \frac{m}{n} : \lim_{n \rightarrow \infty} \|\mathcal{E}(\psi^{\otimes n}) - \phi^{\otimes m}\|_1 = 0 \right\}$$

The rate in one direction is the inverse of the rate in the other:
States are reversibly interconvertible asymptotically

Heuristic argument

$$\begin{aligned}
 \psi^{\otimes n} \xrightarrow{LOCC} \phi^{\otimes m} &\iff \lambda(\rho_{\phi}^{\otimes m}) \succ \lambda(\rho_{\psi}^{\otimes n}) \\
 &\iff \lambda(\rho_{\phi})^{\otimes m} \succ \lambda(\rho_{\psi})^{\otimes n} \\
 &\iff \lambda(\rho_{\phi})^{\otimes m} \longrightarrow \lambda(\rho_{\psi})^{\otimes n} \\
 &\qquad\qquad\qquad \text{by classical data processing}
 \end{aligned}$$

$x^{\otimes m} \longrightarrow y^{\otimes n}$ by classical data processing

Shannon's noiseless channel coding theorem implies

$$R(x \rightarrow y) = \lim_{m \rightarrow \infty} \frac{n}{m} = \frac{H(x)}{H(y)}$$

$$R(\lambda(\rho_{\phi}) \rightarrow \lambda(\rho_{\psi})) = \lim_{m \rightarrow \infty} \frac{n}{m} = \frac{H(\lambda(\rho_{\phi}))}{H(\lambda(\rho_{\psi}))}$$

$$R(\psi \rightarrow \phi) = \lim_{n \rightarrow \infty} \frac{m}{n} = \frac{H(\lambda(\rho_{\psi}))}{H(\lambda(\rho_{\phi}))} = \frac{E(\psi)}{E(\phi)}$$

Noting that $E(\Phi^+) = 1$ “one e-bit”

copies Φ^+ per copy of ψ Distillable entanglement
 $R(\psi \rightarrow \Phi^+) = E(\psi)$ $E(\psi)$

copies ψ per copy of Φ^+ Entanglement cost
 $R(\Phi^+ \rightarrow \psi) = \frac{1}{E(\psi)}$ $E(\psi)$

Mixed state entanglement

Questions one could try to address

- Deterministic state conversion
 - LOCC-equivalence
- Stochastic state conversion
- Stochastic-LOCC equivalence
 - Catalytic state conversion
- Asymptotic state conversion
- Measures of entanglement
 - ...

Entanglement measures based on distance from separable states

Recall that D is a distinguishability measure if

$$\forall \mathcal{E} \in \text{CPTP} : D(\mathcal{E}(\rho_1), \mathcal{E}(\rho_2)) \leq D(\rho_1, \rho_2)$$

i.e. quantum data processing cannot increase distinguishability

$$E(\rho) = \inf_{\tau \in \text{Sep}} D(\rho, \tau) \quad \text{is an entanglement measure}$$

Proof: we need to show that $\forall \mathcal{E} \in \text{LOCC} : E(\mathcal{E}(\rho)) \leq E(\rho)$

$$D(\rho, \tau^*) = \inf_{\tau \in \text{Sep}} D(\rho, \tau)$$

$$D(\mathcal{E}(\rho), \mathcal{E}(\tau^*)) \leq D(\rho, \tau^*)$$

$$\inf_{\tau \in \text{Sep}} D(\mathcal{E}(\rho), \tau) \leq D(\mathcal{E}(\rho), \mathcal{E}(\tau^*)) \quad \text{because} \quad \mathcal{E}(\tau^*) \in \text{Sep}$$

Use relative entropy as distinguishability measure

$$S_{\text{Rel}}(\rho_1 \parallel \rho_2) = \text{Tr} (\rho_1 (\log \rho_1 - \log \rho_2))$$

$$E_{\text{Rel}}(\rho) = \inf_{\tau \in \text{Sep}} \text{Tr} (\rho (\log \rho - \log \tau))$$

Relative entropy of entanglement

$$D_{\text{Bures}}(\rho_1, \rho_2) = 2(1 - \sqrt{F(\rho_1, \rho_2)})$$

$$\text{where } F(\rho_1, \rho_2) = (\text{Tr} \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}})^2$$

$$E_{\text{Bures}}(\rho) = \inf_{\tau \in \text{Sep}} 2(1 - \sqrt{F(\rho, \tau)})$$

Entanglement distance based on Bures metric

Distillable entanglement

Largest number of e-bits one can distill per copy of ρ

$$E_D(\rho) = \sup_{\mathcal{E} \in \text{LOCC}} \left\{ \frac{m}{n} : \lim_{n \rightarrow \infty} \|\mathcal{E}(\rho^{\otimes n}) - (\Phi^+)^{\otimes m}\|_1 = 0 \right\}$$

Entanglement cost

Smallest number of e-bits one requires per copy of ρ one prepares

$$E_C(\rho) = \inf_{\mathcal{E} \in \text{LOCC}} \left\{ \frac{m}{n} : \lim_{n \rightarrow \infty} \|\rho^{\otimes n} - \mathcal{E}[(\Phi^+)^{\otimes m}]\|_1 = 0 \right\}$$

Lower bounds for various classes of states are known

Peres criterion for entanglement

$$\rho_{AB} \text{ separable} \Rightarrow \rho_{AB}^{T_B} \geq 0$$

“Positive partial transpose” (PPT)

$$\begin{aligned} & \langle \eta|_A \langle \mu|_B \rho_{AB}^{T_B} |\tau\rangle_A |\nu\rangle_B \\ &= \langle \eta|_A \langle \nu|_B \rho_{AB} |\tau\rangle_A |\mu\rangle_B \end{aligned}$$

In general

$$\rho_{AB} \text{ NPT} \quad \cancel{\rightleftarrows} \quad \rho_{AB} \text{ entangled}$$

i.e. there are PPT states that are entangled

Some PPT states have nonzero entanglement cost

Note that:

$$\begin{aligned} \rho \in \text{PPT} &\Rightarrow \forall \mathcal{E} \in \text{LOCC} : \mathcal{E}(\rho) \in \text{PPT} \\ &\Phi^+ \in \text{NPT} \end{aligned}$$

All PPT states have zero distillable entanglement

This is called
bound entanglement

Multi-partite entanglement

Stochastic-LOCC equivalence for 3 qubits

$\psi \xrightarrow{LOCC} \phi$
with nonzero probability \rightleftharpoons
 $\psi \xleftarrow{LOCC} \phi$
with nonzero probability

Among true tripartite entangled states,
There are two equivalence classes

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

Unlike bipartite case, not just about the
ranks of the reductions

Monogamy of entanglement

(Quantum entanglement cannot be shared)

A and B maximally entangled \rightarrow A not entangled with C

Proof:

A and B are maximally entangled \rightarrow AB state must be pure

$$\rho_{ABC} = |\psi\rangle_{AB}\langle\psi| \otimes \sigma_C \rightarrow \text{AB not entangled with C}$$

$$\rho_{AC} = \rho_A \otimes \sigma_C \rightarrow \text{A not entangled with C}$$

More generally: for some measures one can derive a tradeoff between AB entanglement and AC entanglement

Monogamy is a useful property for proving security of key distribution

More applications

- Beating classical communication complexity bounds
- Possible role in computational speed-up
- Quantum metrology

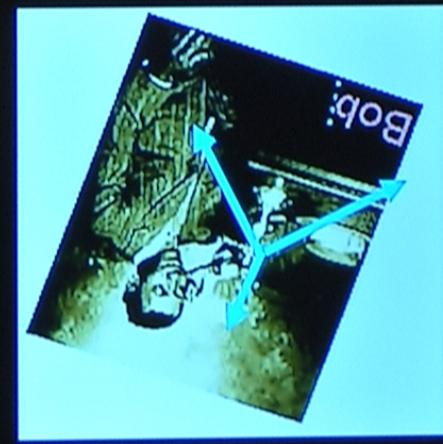
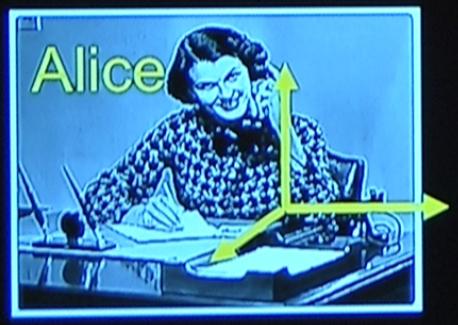
Even if entanglement is necessary, is it sufficient?
If not, what is the property that is sufficient for each of these tasks?

Quantum resource theories

	Entanglement theory	Asymmetry theory	Athermality theory
Free operations	LOCC	Symmetric operations	Thermal operations
Resources	Entangled states	Asymmetric states	Athermal states

Also: coherence, nongaussianity, incompatibility, ...

The resource theory of asymmetry





Frame
alignment

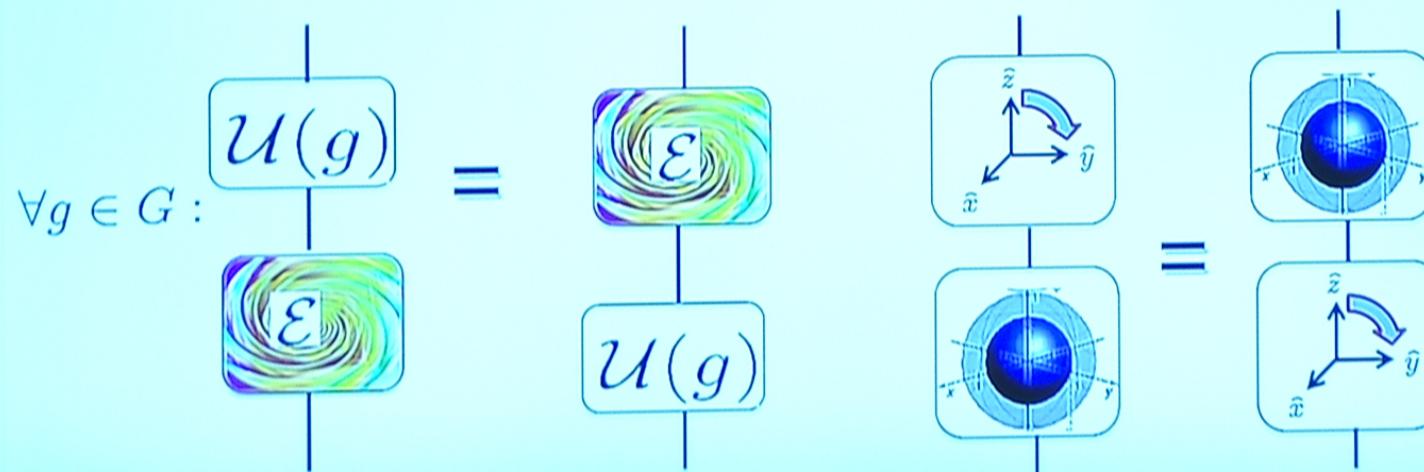


Symmetric operations

A symmetry is defined by a group G
and a unitary representation $g \in G \rightarrow U(g)$

A symmetric operation is any completely-positive trace-preserving map \mathcal{E} that commutes with the action of the group

$$\forall g \in G : \mathcal{E}[U(g)\rho U^\dagger(g)] = U(g)\mathcal{E}[\rho]U^\dagger(g)$$

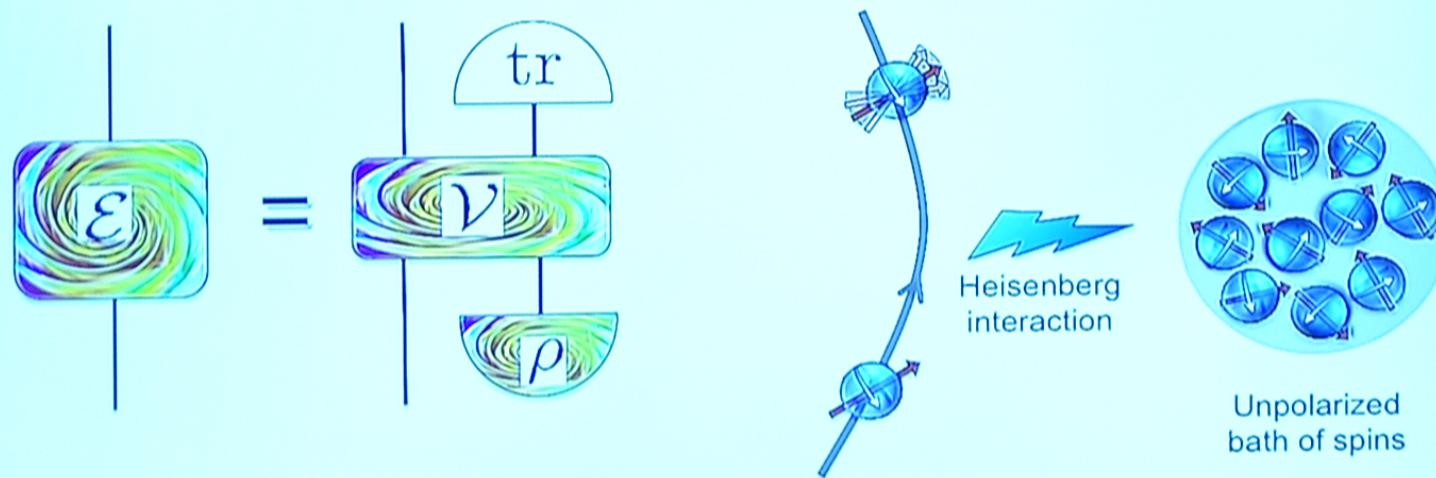


Every such operation can be achieved as follows:

$$\mathcal{E}(\rho_S) = \text{tr}_{A'} \left(V_{SA} (\rho_S \otimes \rho_{A'}) V_{SA}^\dagger \right)$$

$\rho_{A'}$ a symmetric state

V_{SA} a symmetric unitary



Questions to address

- Deterministic state conversion
 - Deterministic equivalence
 - Stochastic state conversion
 - Stochastic equivalence
 - Catalytic state conversion
 - Asymptotic state conversion
 - Measures of asymmetry

Deterministic state conversion

$$\rho \xrightarrow{sym} \sigma \iff ? \text{ Conditions on } \rho \text{ and } \sigma ?$$

\exists symmetric \mathcal{E} :
 $\mathcal{E}(\rho) = \sigma$

What are the constraints on state transitions arising from symmetric dynamics?



Noether's theorem

Suppose the symmetry transformation has L as a generator $U(g) = e^{i\theta L}$

$$[V, U(g)] = 0 \quad \forall g \in G \Rightarrow [V, L] = 0$$

$$\rho \xleftrightarrow{\text{sym}} \sigma \quad \Rightarrow \quad \begin{aligned} \text{tr}(\rho L^k) &= \text{tr}(\sigma L^k) \\ \forall k \in \mathbb{N} \end{aligned}$$

Symmetry	Conservation law
Rotational	Angular momentum
Spatial Translation	Linear momentum
Time translation	Energy
Phase shift	Number

Two deficiencies:

- It is silent about open-system dynamics
- Even for closed-system dynamics, it does not capture all of the consequences of symmetry

Even for closed-system dynamics, Noether conservation laws do not capture **all** the constraints implied by the symmetries

$$\rho = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| \otimes |0\rangle\langle 0| + |\downarrow\rangle\langle\downarrow| \otimes |1\rangle\langle 1|)$$

$$\sigma = \frac{1}{2}(|\leftarrow\rangle\langle\leftarrow| \otimes |0\rangle\langle 0| + |\rightarrow\rangle\langle\rightarrow| \otimes |1\rangle\langle 1|)$$

This transition is forbidden by virtue of symmetry but Noether's theorem does not prohibit it

$$\forall L, k : \text{tr}(\rho L^k) = \text{tr}(\sigma L^k)$$

Measures of asymmetry

Def'n: A function A from states to the reals is a measure of asymmetry if

$$\rho \xrightarrow{sym} \sigma \Rightarrow A(\rho) \geq A(\sigma)$$

Open-system dynamics
(irreversible):
every measure yields a
monotonicity constraint

Closed-system dynamics
(reversible):
every measure yields a
conservation law

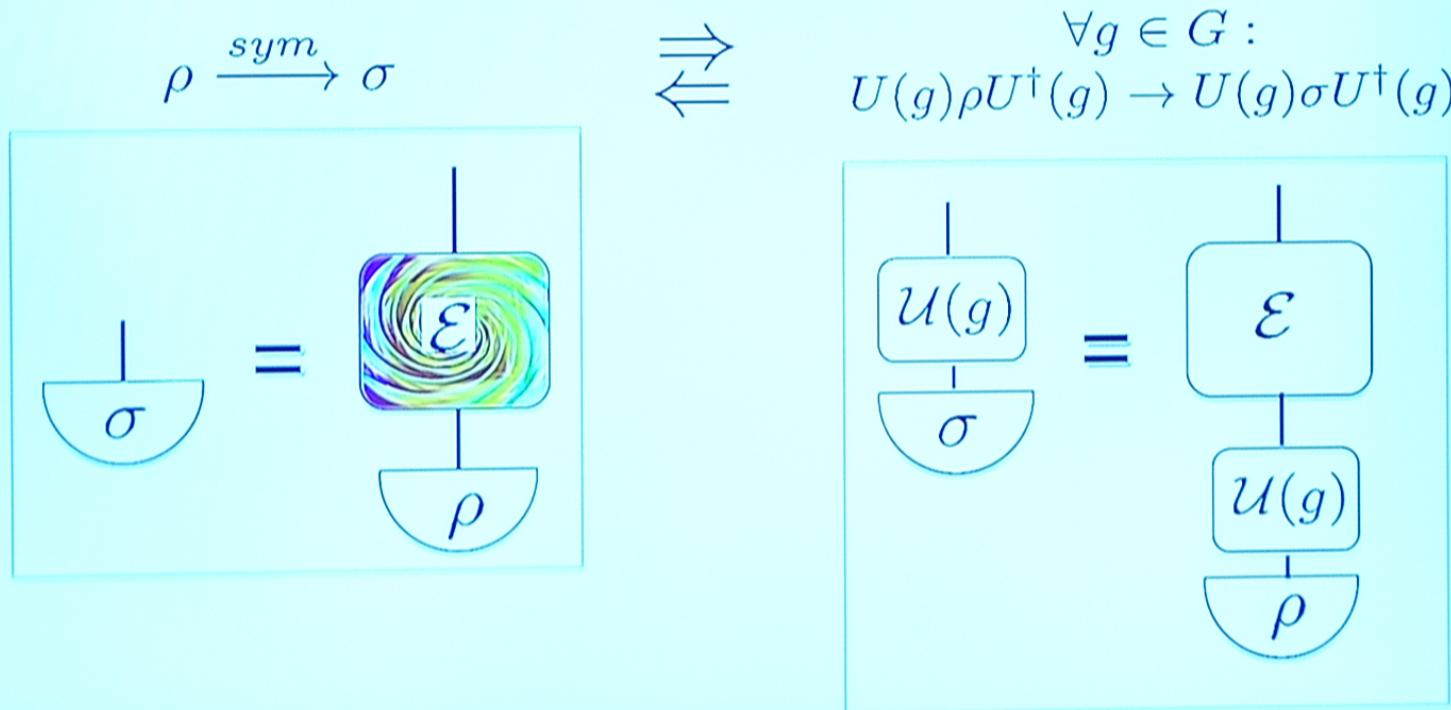
Are the Noether conserved quantities $\text{tr}(\rho L^k)$ nontrivial measures of asymmetry? NO!

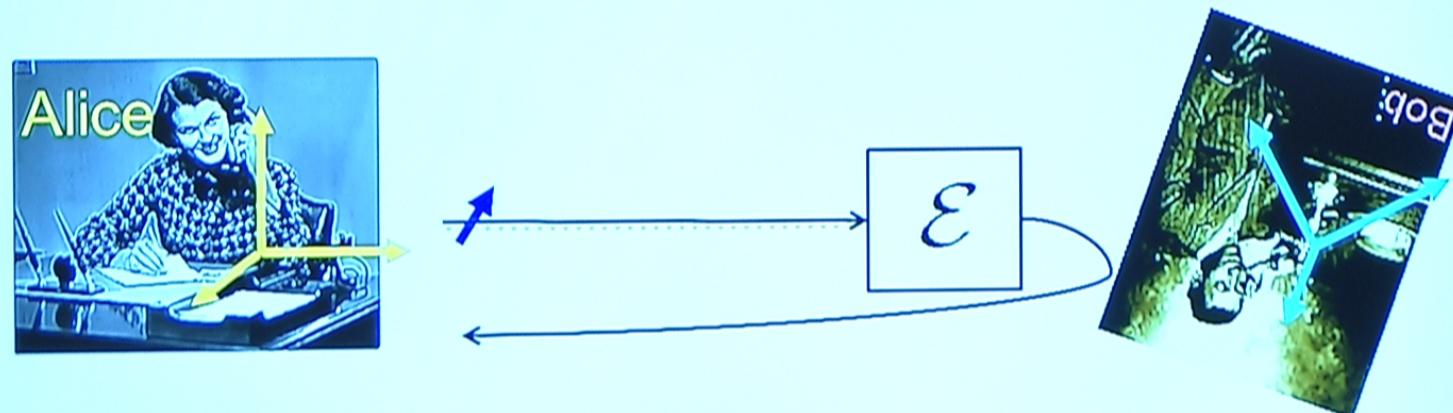
$$U(\theta) = e^{i\theta L} \quad \rho = \sum_l |c_l|^2 |l\rangle\langle l| \quad \sigma = \sum_{l,l'} c_l c_{l'}^* |l\rangle\langle l'|$$

The difference is not seen by any fn' of the Noether quantities
 $\forall k \in \mathbb{N} : \text{tr}(\rho L^k) = \text{tr}(\sigma L^k)$

How can we find measures of asymmetry?

Bridge lemma:

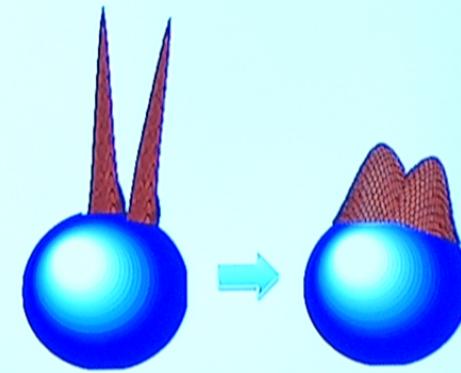
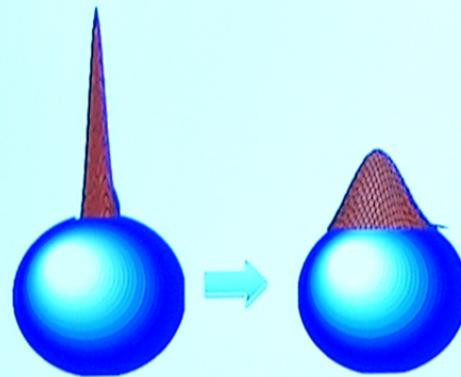




Measure of asymmetry
of the state ρ

=

Measure of information about G
encoded in the orbit of ρ
 $\{U(g)\rho U^\dagger(g) : g \in G\}$



Take the measure of information to be
the Holevo quantity

$$H_p(\{\rho_i\}) \equiv S(\sum_i p_i \rho_i) - \sum_i p_i S(\rho_i)$$

$$S(\rho) \equiv -\text{tr}(\rho \log \rho)$$

$$H_p(\{U(g)\rho U(g)^\dagger\}) \equiv S(\int dg p(g) U(g)\rho U^\dagger(g)) - \int dg p(g) S(U(g)\rho U^\dagger(g))$$

$$A_p^{\text{Hol}}(\rho) \equiv S(\mathcal{G}_p(\rho)) - S(\rho)$$

$$\mathcal{G}_p(\rho) \equiv \int dg p(g) U(g)\rho U^\dagger(g)$$

Holevo asymmetry

Opt'l int'n: provides an upper bound on accessible information about g

Even for closed-system dynamics, Noether conservation laws do not capture **all** the constraints implied by the symmetries

$$\rho = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| \otimes |0\rangle\langle 0| + |\downarrow\rangle\langle\downarrow| \otimes |1\rangle\langle 1|)$$

$$\sigma = \frac{1}{2}(|\leftarrow\rangle\langle\leftarrow| \otimes |0\rangle\langle 0| + |\rightarrow\rangle\langle\rightarrow| \otimes |1\rangle\langle 1|)$$

This transition is forbidden by virtue of symmetry but Noether's theorem does not prohibit it

$$\forall L, k : \text{tr}(\rho L^k) = \text{tr}(\sigma L^k)$$

Holevo asymmetry prohibits the transition

$p(g)$ = uniform measure over rotations around z

$$0 = S(\mathcal{G}_{p(g)}[\rho]) - S(\rho) \not\geq S(\mathcal{G}_{p(g)}[\sigma]) - S(\sigma) = 2 - 1$$

Measure of information based on relative Renyi entropy
measure of distinguishability

$$D_s(\rho_1, \rho_2) = \frac{1}{s-1} \log \left(\text{tr}(\rho_1^s \rho_2^{1-s}) \right)$$
$$s \in (0, 1) \cup (1, \infty)$$

$$A^{(g)}(\rho) = D_s(\rho, U(g)\rho U^\dagger(g))$$

$$U(g) = e^{i\theta L}$$

For element infinitessimally close to identity

$$A_{L,s}^{\text{skew}}(\rho) \equiv \text{tr}(\rho L^2) - \text{tr}(\rho^s L \rho^{1-s} L)$$
$$s \in (0, 1) \cup (1, \infty)$$

Wigner-Yanase-Dyson skew information

Asymmetry measures based on distance from symmetric set

Recall that D is a distinguishability measure if

$$\forall \mathcal{E} \in \text{CPTP} : D(\mathcal{E}(\rho_1), \mathcal{E}(\rho_2)) \leq D(\rho_1, \rho_2)$$

Use relative entropy as distinguishability measure

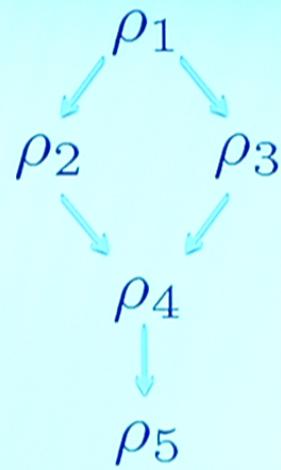
$$S_{\text{Rel}}(\rho_1 \parallel \rho_2) = \text{Tr} (\rho_1 (\log \rho_1 - \log \rho_2))$$

$$A_{\text{Rel}}(\rho) = \inf_{\tau \in \text{Sym}} \text{Tr} (\rho (\log \rho - \log \tau))$$

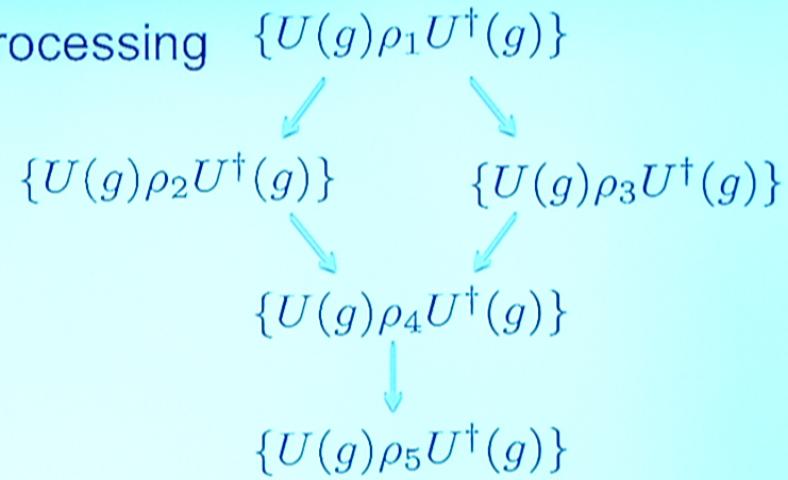
Relative entropy of asymmetry

$$\begin{aligned} \inf_{\tau \in \text{sym}} S_{\text{Rel}}(\rho \parallel \tau) &= S_{\text{Rel}}(\rho \parallel \mathcal{G}(\rho)) \\ &= S(\mathcal{G}(\rho)) - S(\rho) \\ \mathcal{G}(\rho) &\equiv \int dg U(g)\rho U^\dagger(g) \end{aligned}$$

Symmetric
operations



Data processing



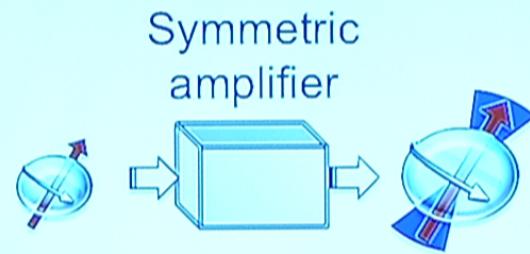
Rehabilitated version of Noether's theorem

$$\rho \xrightarrow{sym} \sigma \quad \rightleftharpoons \quad \begin{matrix} \text{conditions on} \\ \rho \text{ and } \sigma \end{matrix}$$

So far, only solved for the case of pure states

$$|\psi\rangle \xrightarrow{sym} |\phi\rangle \quad \rightleftharpoons \quad \begin{matrix} \forall g \in G : \\ \langle\psi|U(g)|\psi\rangle = e^{i\theta(g)}\langle\phi|U(g)|\phi\rangle \end{matrix}$$

Fundamental noise limits in quantum amplifiers



$$\rho \xrightarrow{sym} \sigma \quad \Rightarrow \quad S(\mathcal{G}_p[\rho]) - S(\rho) \geq S(\mathcal{G}_p[\sigma]) - S(\sigma)$$

$$S(\sigma) - S(\rho) \geq S(\mathcal{G}_p[\sigma]) - S(\mathcal{G}_p[\rho])$$

Quantum Metrology



$$\text{Variance in unbiased estimator of a phase } \theta \leq \frac{1}{4(\text{tr}(\rho L^2) - \text{tr}(\rho^{1/2} L \rho^{1/2} L))}$$

Quantum speed limits

Mandelstam-Tamm bound

$$\tau_{\perp}(\rho) \geq \frac{\pi}{2\Delta E(\rho)}$$

$$\Delta E(\rho) = \text{tr}(\rho H^2) - \text{tr}^2(\rho H)$$

Margolus-Levitin bound

$$\tau_{\perp}(\rho) \geq \frac{\pi}{2[E(\rho) - E_{\min}(\rho)]}$$

$$E(\rho) = \text{tr}(\rho H)$$

$$E_{\min}(\rho) = \lambda_{\min}(H) \text{ on } \text{supp}(\rho)$$

Define the speed of evolution as a measure of time-translation asymmetry:

Minimum time t such that $\rho(t)$ is ϵ -distinguishable from $\rho(0)$ relative to measure of distinguishability D

$$\tau_\epsilon^D(\rho) \equiv \begin{cases} \infty , & \text{if } \forall t \in \mathbb{R}^+ : D(\rho, \rho(t)) < \epsilon \\ \min\{t : t \in \mathbb{R}^+, D(\rho, \rho(t)) \geq \epsilon\} , & \text{otherwise.} \end{cases}$$

The resource theory of athermality

Free energy excess

$$\Delta F(\rho) \equiv F(\rho) - F(\gamma)$$

where $F(\rho) \equiv \text{tr}(\rho H) - S(\rho)$

l_1 -norm-based athermality

$$\Delta F^{l_1}(\rho) \equiv \|\rho - \gamma\|_1$$

Generalized free energy excess

$$\Delta F_s(\rho) \equiv kT S_s(\rho || \gamma)$$

where $S_s(\rho || \sigma) \equiv \frac{1}{1-s} \log \text{tr} [\rho^s \sigma^{1-s}]$
 $s \in (0, 1) \cup (1, \infty)$

The appeal of resource theories

Constraints are implied by the laws of quantum physics,
not technological limitations or choices of model

Questions are well-posed and unambiguous, and their
answers help one to develop a better conceptual
understanding of the property being studied

The study of many different resources helps to identify which
resources are relevant to which tasks

“It from qubit”
=

“Relevance of information theory to modern physics”

No surprise that entanglement theory uses information theory
given that it is defined in terms of classical communication

The role that information theory plays in
other resources theories is more significant

