Title: A new perspective on holographic entanglement

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URL: http://pirsa.org/16070011

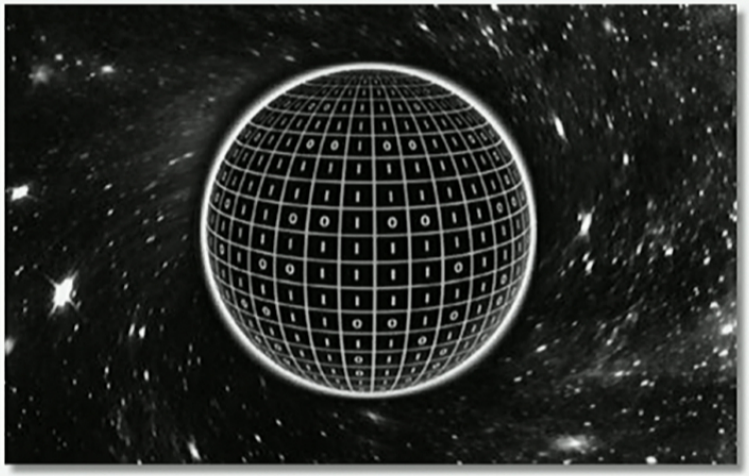
Abstract: The by-now classic Ryu-Takayanagi formula associates the entanglement entropy of a spatial region in a holographic field theory with the area of a certain minimal surface in the bulk. Despite its simplicity and beauty, this formula raises a number of stubborn conceptual problems. I will present a reformulation which does not involve the areas of surfaces. This reformulation leads to a picture of entanglement in the field theory being carried by Planck-thickness "bit threads" in the bulk. I will argue that this picture helps to resolve a number of the conceptual difficulties surrounding the RT formula

Entropy and area

In semiclassical gravity, entropies are related to surface areas

Bekenstein-Hawking ['74]:
For black hole (or other horizon)

$$S = \frac{\text{area(horizon)}}{4G_\text{N}}$$



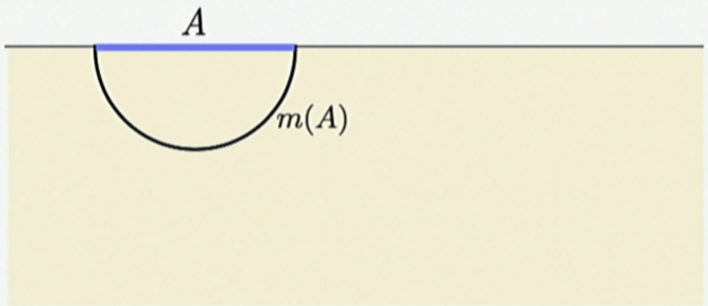
Why?

Naive answer: Microstate bits are located on horizon, density = one bit per 4 Planck areas

Ryu-Takayanagi ['06]:
For region in holographic field theory (static)

$$S(A) = \frac{1}{4G_N} \text{area}(m(A))$$

$m(A) =$ bulk minimal surface homologous to A



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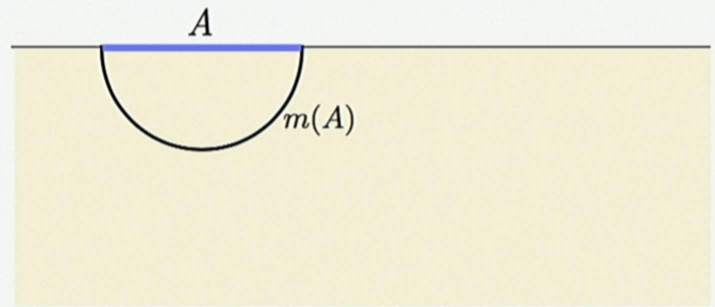
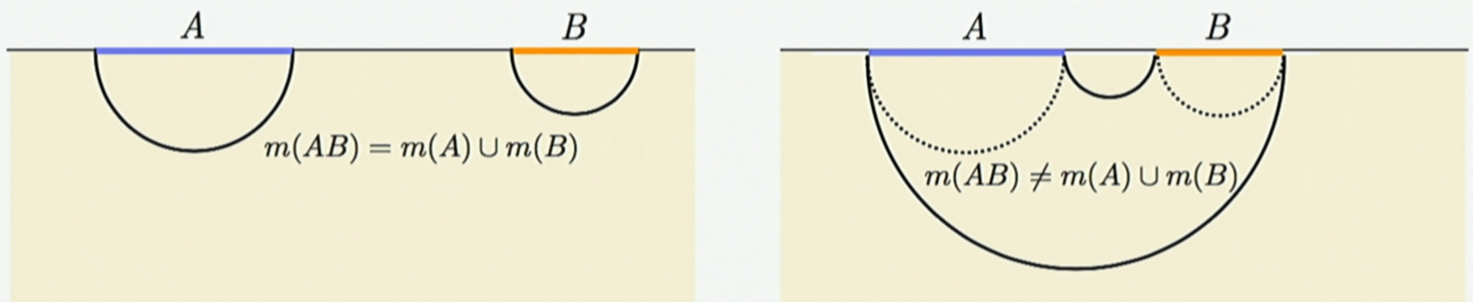


Do microstate bits of ρ_A live on $m(A)$?

Unlike horizon, $m(A)$ is not a special place; by choosing A, we can move it around at will

Leads to confusions:

- Under continuous changes in boundary region, minimal surface can jump, for example:



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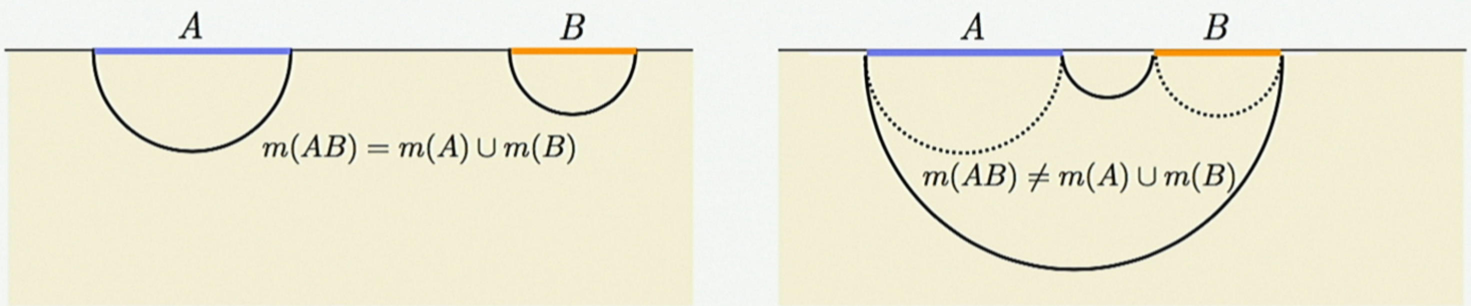

(Note: ρ_{AB} does not jump; not a conventional exchange-of-dominance phase transition [Headrick '13])

- Important quantities, like conditional entropy

$$H(A|B) = S(AB) - S(B)\,,$$

mutual information

$$I(A:B) = S(A) + S(B) - S(AB)\,,$$

conditional mutual information

$$I(A:B|C) = S(AB) + S(BC) - S(ABC) - S(C)\,,$$

are given by differences of areas of surfaces passing through different regions of bulk

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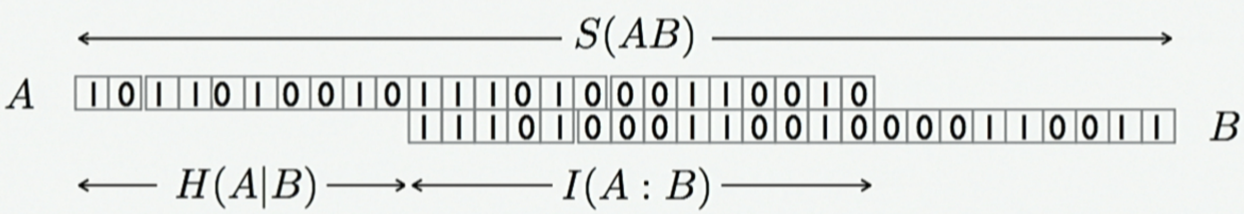
Let's recall their information-theoretic meaning

Classical:

$$\begin{aligned} H(A|B) &= \text{\# of (independent) bits belonging purely to } A \\ I(A:B) &= \text{\# shared with } B \end{aligned}$$



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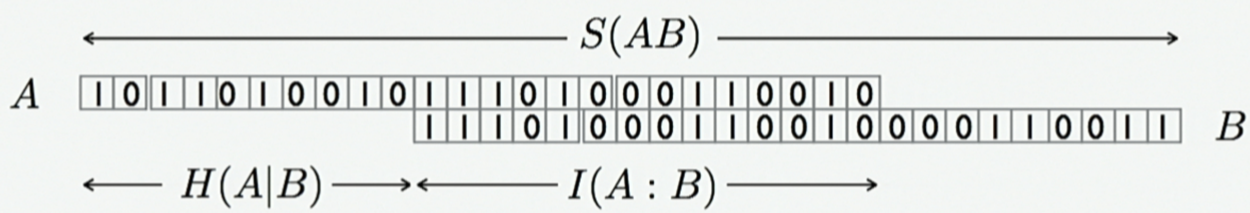
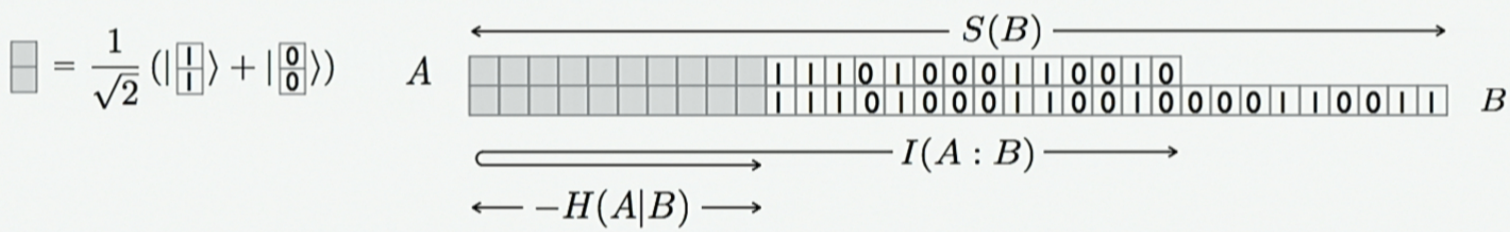
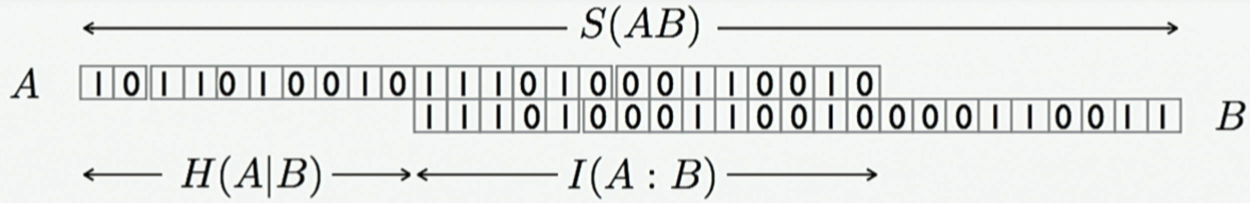


Quantum: Entangled pair of bits contributes 2 to $I(A:B)$, -1 to $H(A|B)$
Can lead to $H(A|B) < 0$





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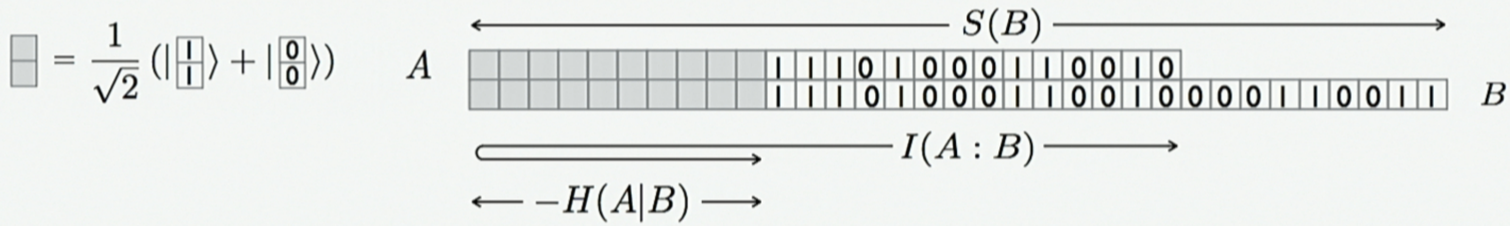
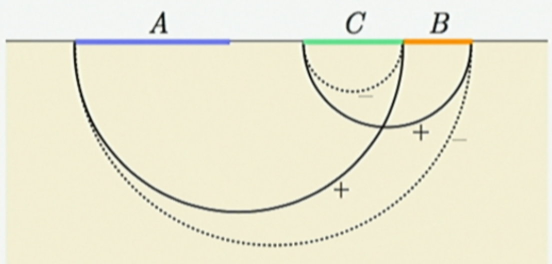


What do differences between areas of surfaces, passing through different parts of bulk, have to do with redundancy, entanglement, etc. between bits of A and B?

What does holographic proof of strong subadditivity have to do with monotonicity of correlations?



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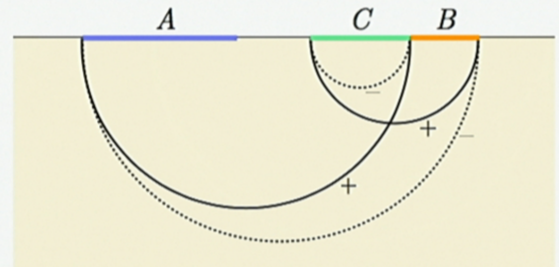

To answer these questions, I will present a new formulation of RT

- Does not refer to minimal surfaces; these are demoted to a calculational device
- Suggests a new way to think about the connection between spacetime geometry and information

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Max flow-min cut

(Originally on graphs, in context of network theory. Riemannian version [Federer '74, Strang '83, Nozawa '90])

Consider a Riemannian manifold with boundary

Define a *flow* as a vector field v s.t. $\nabla \cdot v = 0$, $|v| \leq 1$
Can be thought of as a set of oriented threads (flow lines) with transverse density $= |v| \leq 1$

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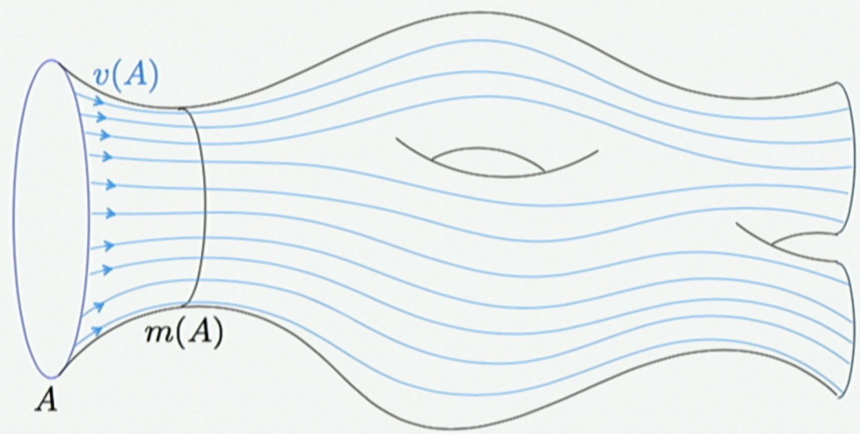
Let A be a subset of boundary
For any surface m homologous to A,

$$\int_A v = \int_m v \leq \text{area}(m)$$

Strongest bound is minimal area:

$$\max_v \int_A v \leq \min_{m \sim A} \text{area}(m)$$



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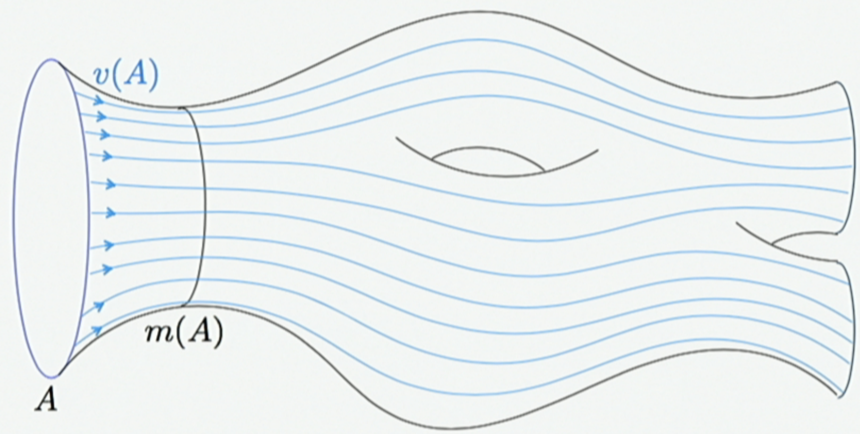

Max flow-min cut theorem: There are no other obstructions to increasing flux:

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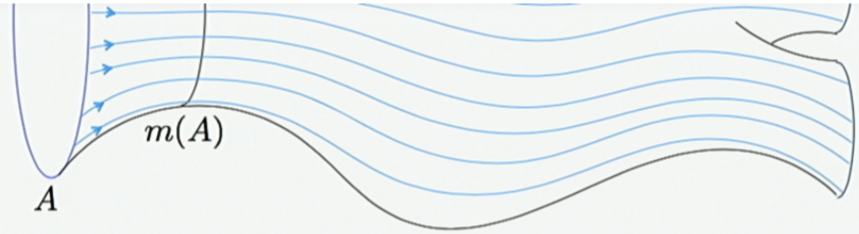

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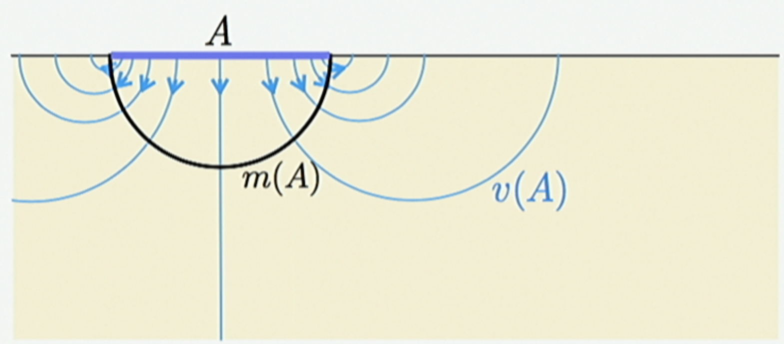
Note:

- Max flow is highly non-unique (except on $m(A)$, where $v =$ unit normal vector)
  Let $v(A)$ denote *any* max flow
- Finding max flow is a linear programming problem

RT version 2.0:

$$\begin{aligned} S(A) &= \max_v \int_A v \qquad (4G_{\mathrm{N}} = 1) \\ &= \text{max \# of threads coming out of } A \end{aligned}$$



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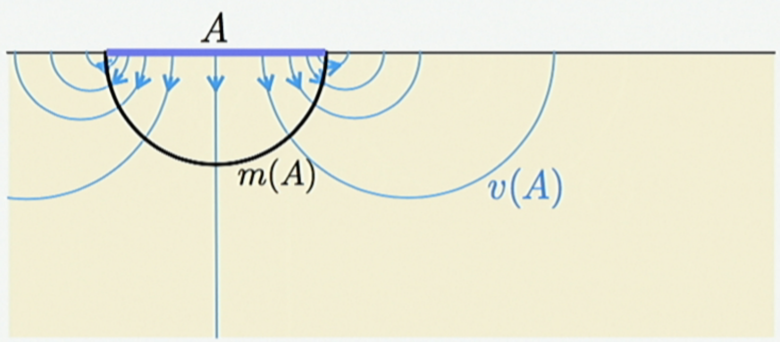
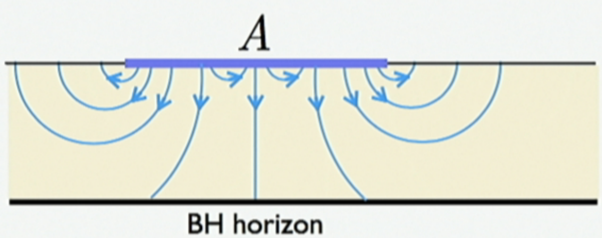


Automatically incorporates homology condition & global minimization



Threads can end on A^c or horizon

Each thread carries one independent bit of ρ_A, either entangled with A^c or in a mixed state

Threads are "floppy": lots of freedom to move them around in bulk & move where they end on A

Also lots of room near boundary to add extra threads that begin & end on A (don't contribute to $S(A)$)

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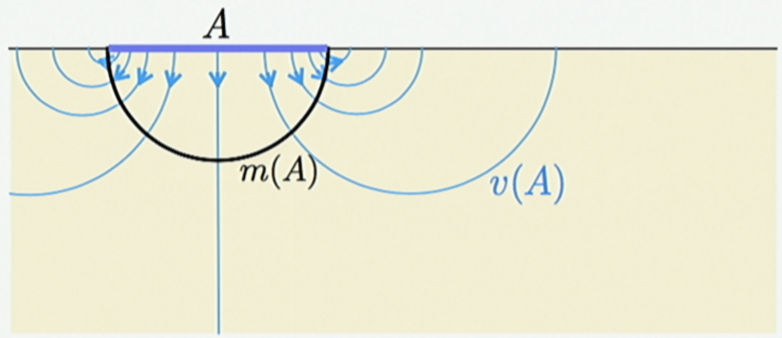

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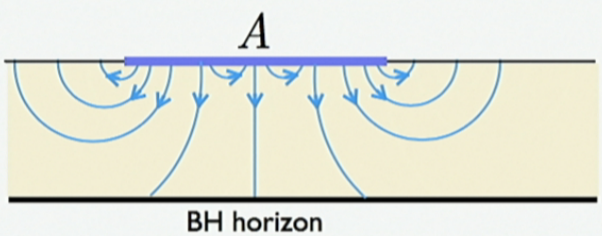

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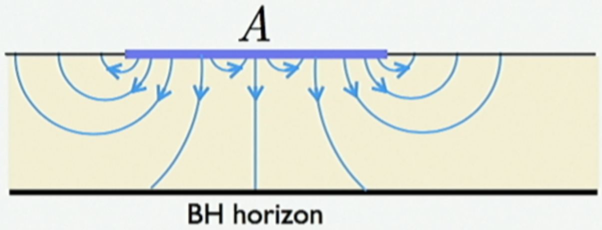

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Threads & information

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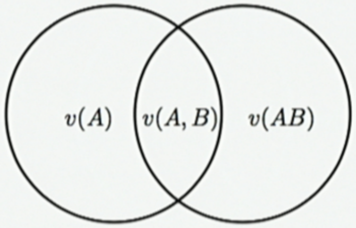
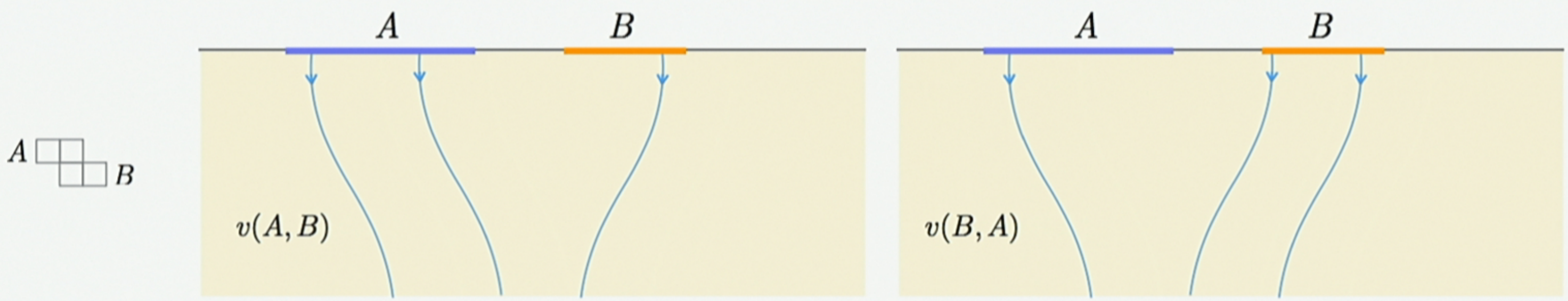
Now consider two regions A, B

We can maximize flux through A *or* B
If $S(AB) < S(A) + S(B)$, then we cannot simultaneously maximize through both
But we *can* always maximize through A *and* AB (nesting property)
Call such a flow $v(A, B)$



Example 1: $S(A) = S(B) = 2$, $S(AB) = 3 \Rightarrow I(A:B) = 1$, $H(A|B) = 1$

Maximizing on AB, we can also maximize on *either* A *or* B



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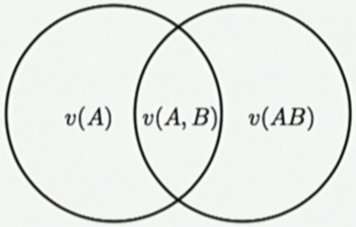

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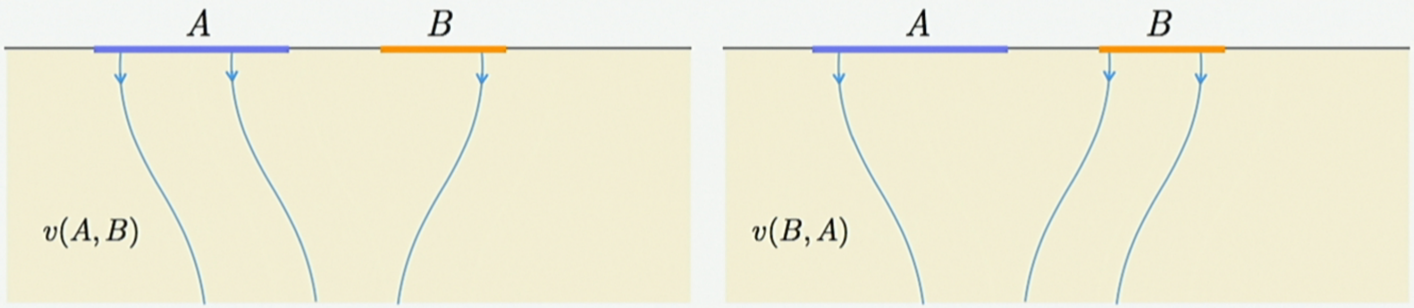
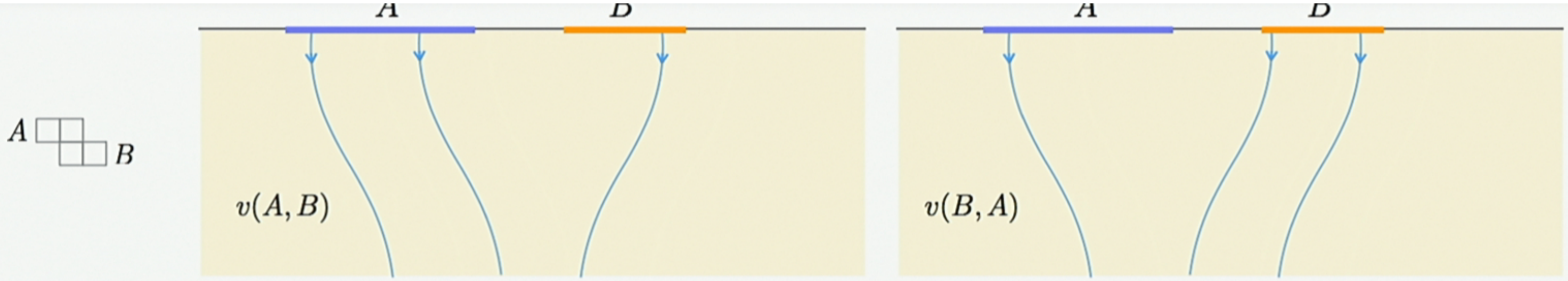


Lesson 1:

- Threads that are stuck on A represent bits unique to A
- Threads that can be moved between A & B represent correlated pairs of bits



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Example 2: $S(A) = S(B) = 2$, $S(AB) = 1 \Rightarrow I(A:B) = 3$, $H(A|B) = -1 \Rightarrow$ entanglement!

One thread leaving A *must* go to B, and vice versa

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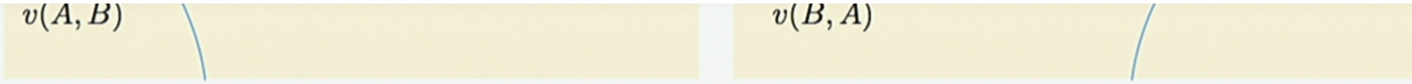
Apply lessons to single region:

- freedom to move thread endpoints around on A reflects correlations within A
- freedom to add loops that begin & end on A reflects entanglement within A



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In equations:

Conditional entropy:

$$\begin{aligned} H(A|B) &= S(AB) - S(B) \\ &= \int_{AB} v(AB) - \int_B v(B) \\ &= \int_{AB} v(B,A) - \int_B v(B,A) \\ &= \int_A v(B,A) \\ &= \text{min \# of threads on } A \text{ (maximizing on } AB\text{)} \end{aligned}$$

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Mutual information:

$$\begin{aligned} I(A:B) &= S(A) - H(A|B) \\ &= \int_A v(A,B) - \int_A v(B,A) \\ &= \text{max} - \text{min flux on } A \text{ (maximizing on } AB) \\ &= \text{flux movable between } A \text{ and } B \text{ (maximizing on } AB) \end{aligned}$$

Subadditivity is clear

Max flow can be defined even when flux is infinite: flow that cannot be augmented
Implies regulator-free definition of mutual information:

$$= \int_A v(B,A)$$

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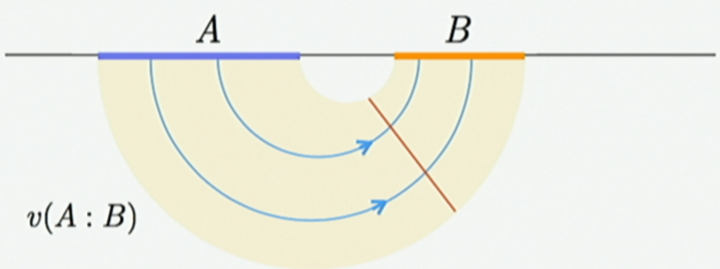
$$I(A:B) = \int_A (v(A,B) - v(B,A))$$

Define flow

$$v(A:B) = \frac{1}{2}(v(A,B) - v(B,A))$$



which goes from A to B through entanglement wedge $r(AB)$
Implies

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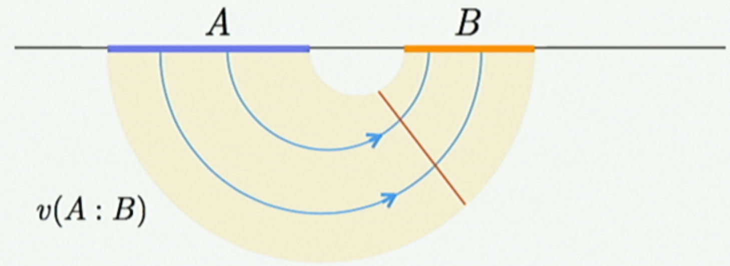

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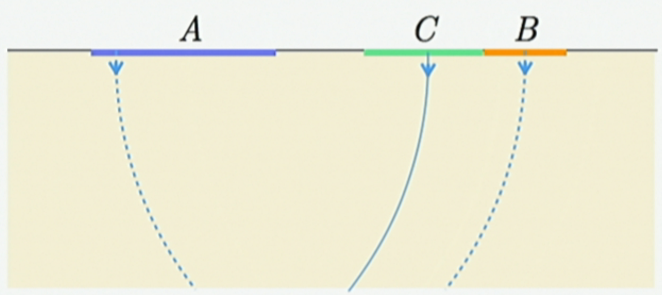
$$\frac{1}{2}I(A:B) \leq \mathrm{area}\,(\text{bottleneck of } r(AB))$$

Conditional mutual information:



$$\begin{aligned}
I(A:B|C) &= H(A|C) - H(A|BC) \\
&= \int_A v(C,A,B) - \int_A v(C,B,A) \\
&= \text{max} - \text{min flux on } A \text{ (maximizing on } C \text{ \& } ABC) \\
&= \text{flux movable between } A \text{ \& } B \text{ (maximizing on } C \text{ \& } ABC) \\
&= (\text{flux movable between } A \text{ \& } BC) - (\text{movable between } A \text{ \& } C) \\
&= I(A:BC) - I(A:C)
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Strong subadditivity is clear

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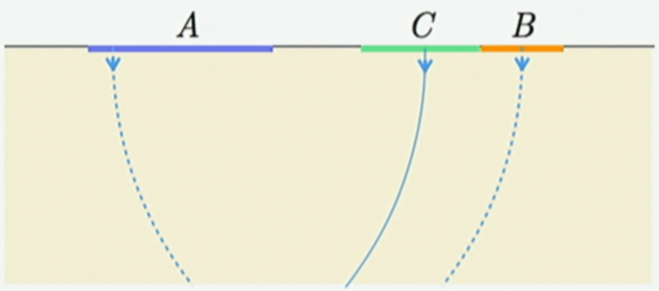

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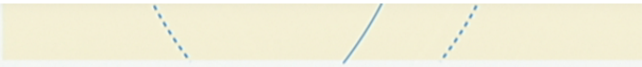
Strong subadditivity is clear

Exercise for reader: Find flow interpretation of other properties: Araki-Lieb, $S(A) = S(A^c)$ for pure states, ...

Exercise for reader (harder): Using flows, prove "monogamy of mutual information" inequality [Hayden-Headrick-Maloney '11],

$$I(A:BC) \geq I(A:B) + I(A:C)$$

and higher inequalities [Bao et al. '15]
Cannot be proved using just nesting property: 4-party GHZ obeys nesting but violates MMI
Indicates some new property of flows



$= \text{(flux movable between } A \text{ \& } BC) - \text{(movable between } A \text{ \& } C)$

$= I(A:BC) - I(A:C)$

Strong subadditivity is clear

Exercise for reader: Find flow interpretation of other properties: Araki-Lieb, $S(A) = S(A^c)$ for pure states, ...

Exercise for reader (harder): Using flows, prove "monogamy of mutual information" inequality [Hayden-Headrick-Maloney '11],

$$I(A:BC) \geq I(A:B) + I(A:C)$$

and higher inequalities [Bao et al. '15]
Cannot be proved using just nesting property: 4-party GHZ obeys nesting but violates MMI
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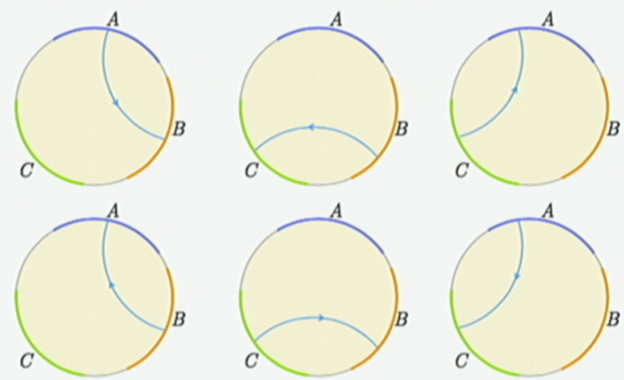
Do bit threads indicate that bipartite entanglement is privileged?
Possible to represent 3-party GHZ state



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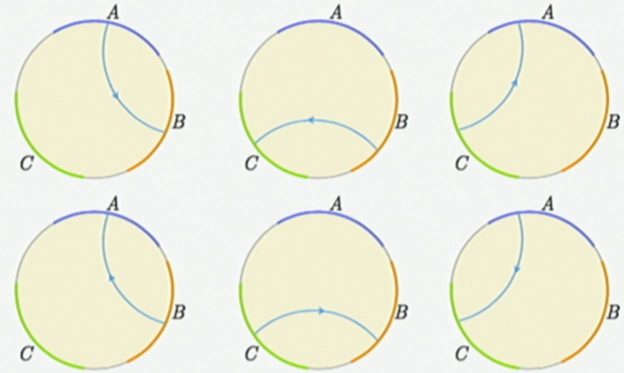
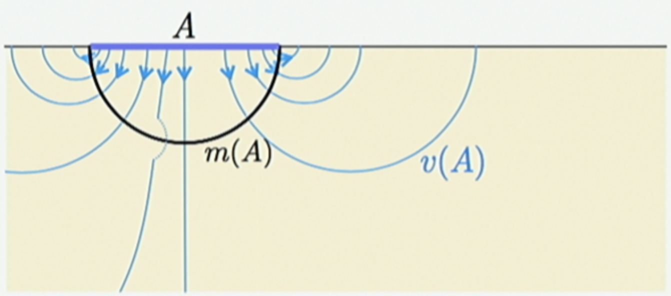


Quantum corrections



Faulkner-Lewkowycz-Maldacena ['13]: Quantum (order G_N^0) corrections to RT come from entanglement of bulk fields

This correction may be reproduced by allowing threads to jump (or tunnel through microscopic wormholes, à la ER = EPR)

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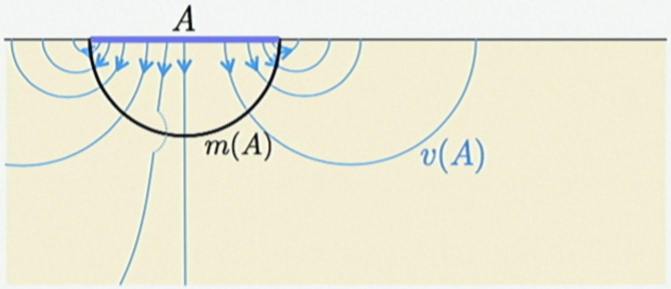

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Covariant flows

To appear (with Veronika Hubeny)

Hubeny-Rangamani-Takayanagi ['07] covariant entanglement entropy formula:

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Generalization of max flow-min cut theorem to Lorentzian setting:

Define a *flow* as a vector field v (in the full Lorentzian spacetime) obeying

- $\nabla \cdot v = 0$
- no flux into or out of singularities
- integrated norm bound: $\forall$ timelike curve C and unit normal vector field u on C,

$$\int_C ds\, u \cdot v \leq 1$$

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Then (assuming NEC)

$$\max_v \int_{D(A)} v = \text{area}(m(A))$$

Linearizes problem of finding extremal surface area

HRT version 2.0:

$$S(A) = \max_v \int_{D(A)} v$$

To maximize flux, threads seek out $m(A)$, automatically confining themselves to entanglement wedge

Threads can lie on common Cauchy slice (equivalent to Wall's ['12] maximin by standard max flow-min cut) or spread out in time