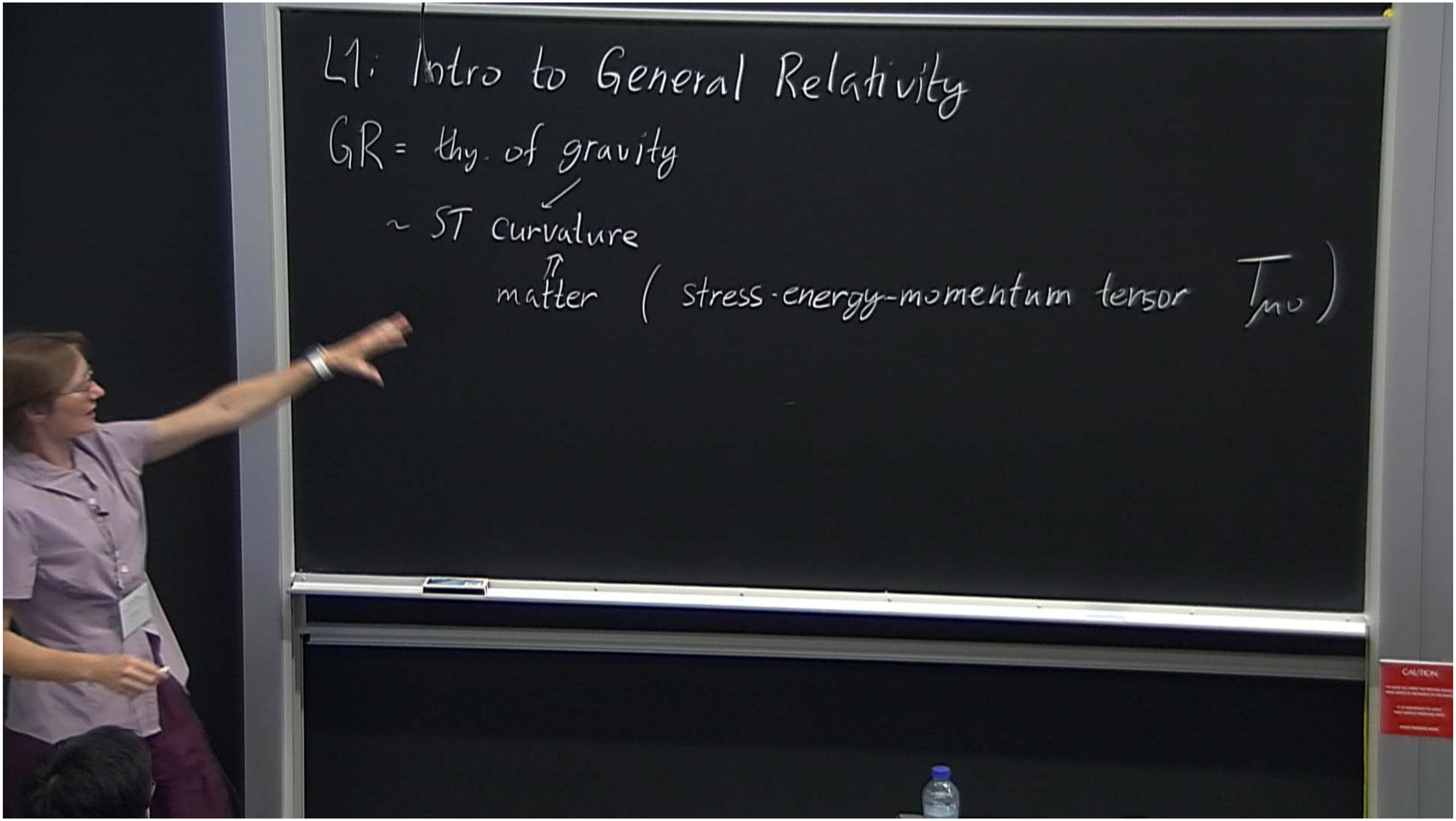
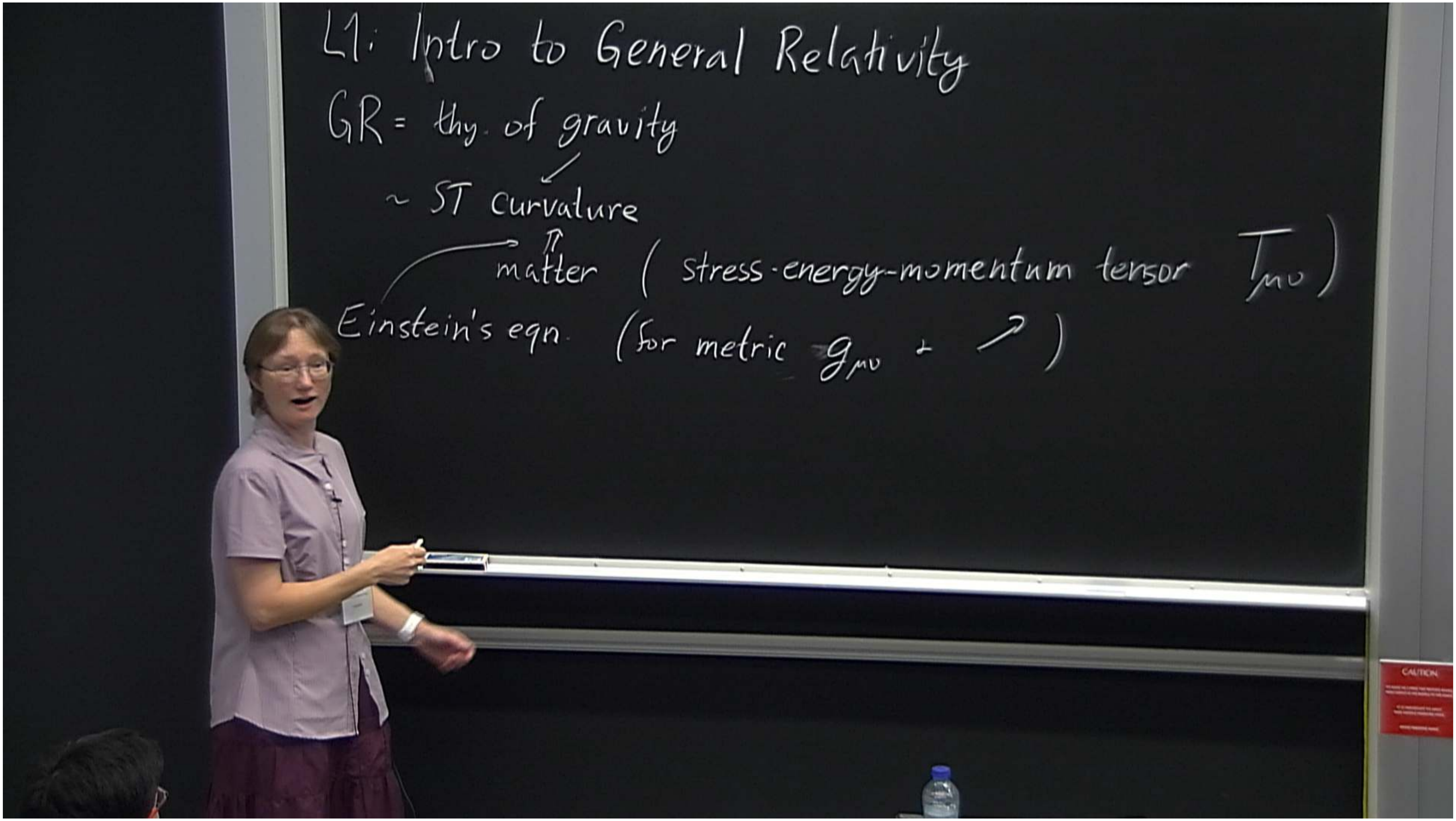
Title: Gravity Basics

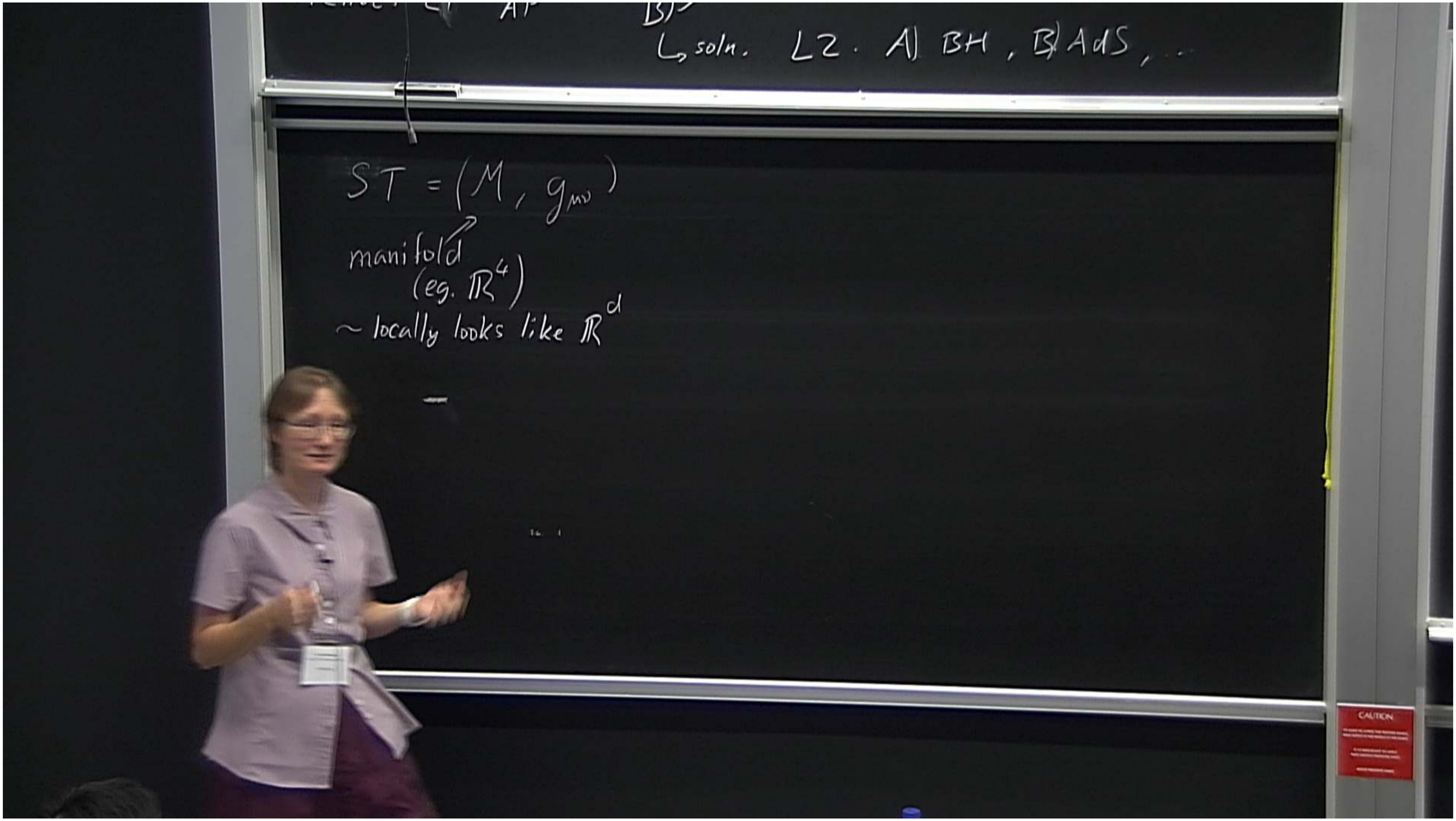
Date: Jul 18, 2016 09:10 AM

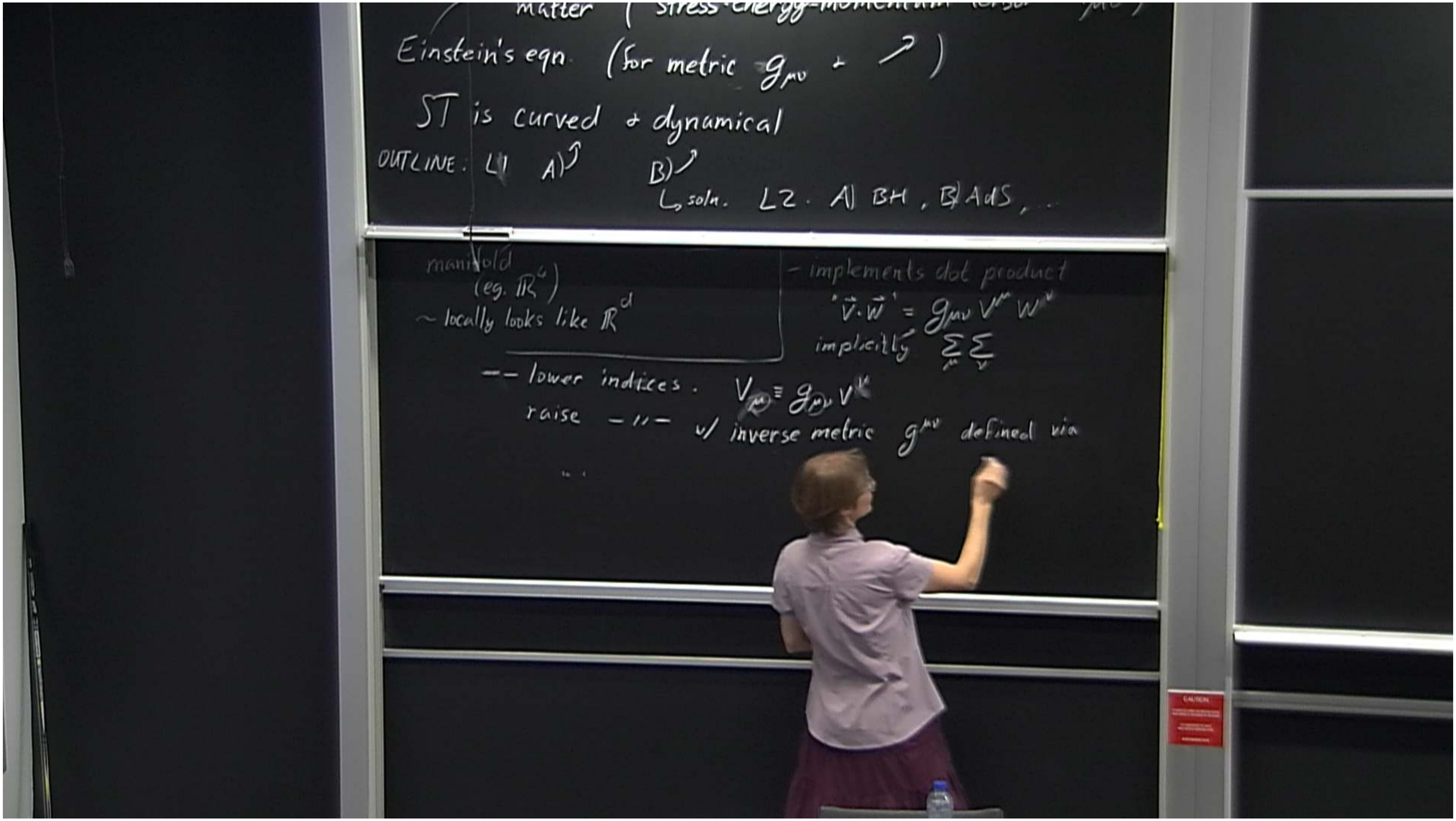
URL: http://pirsa.org/16070007

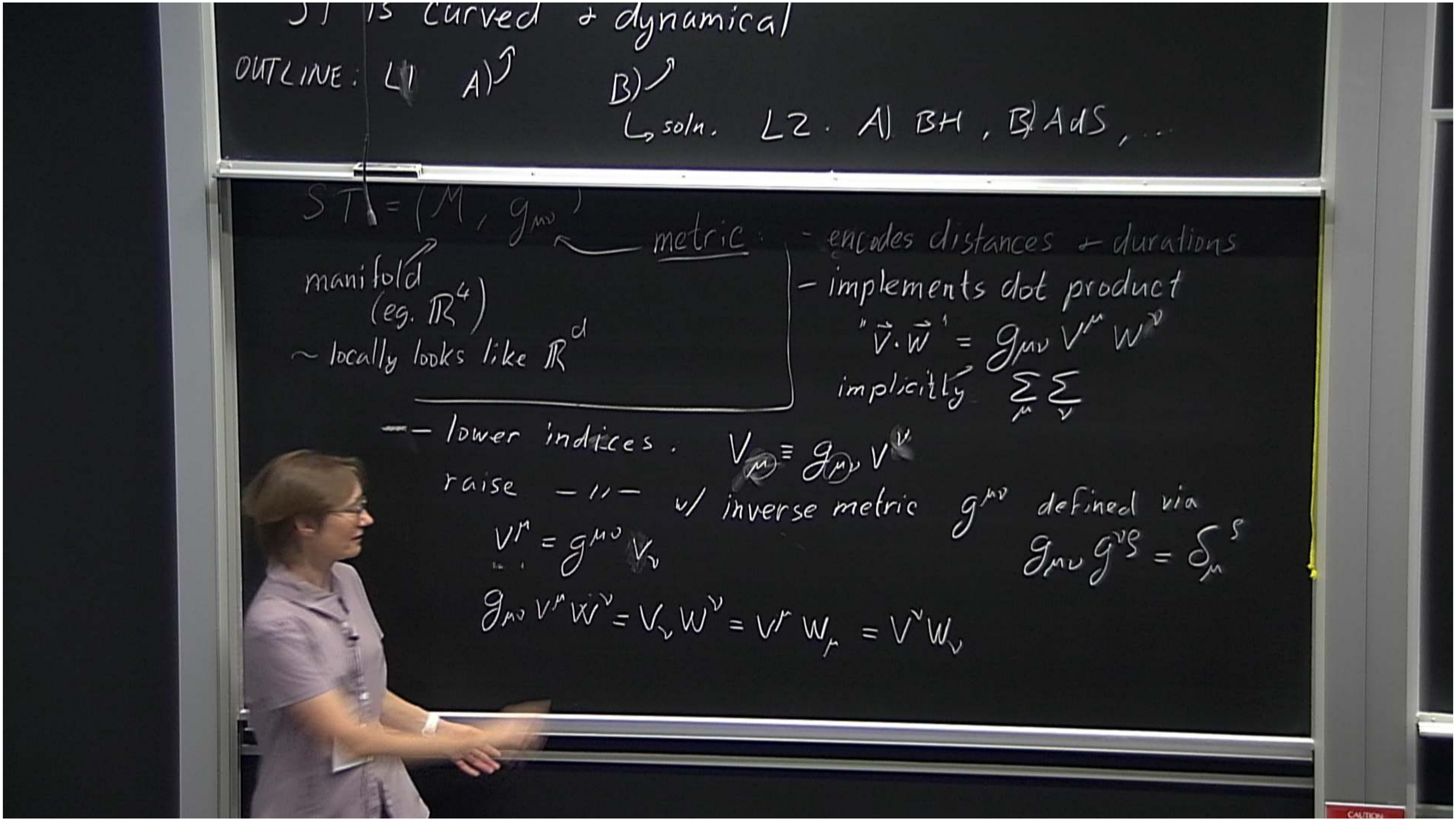
Abstract:

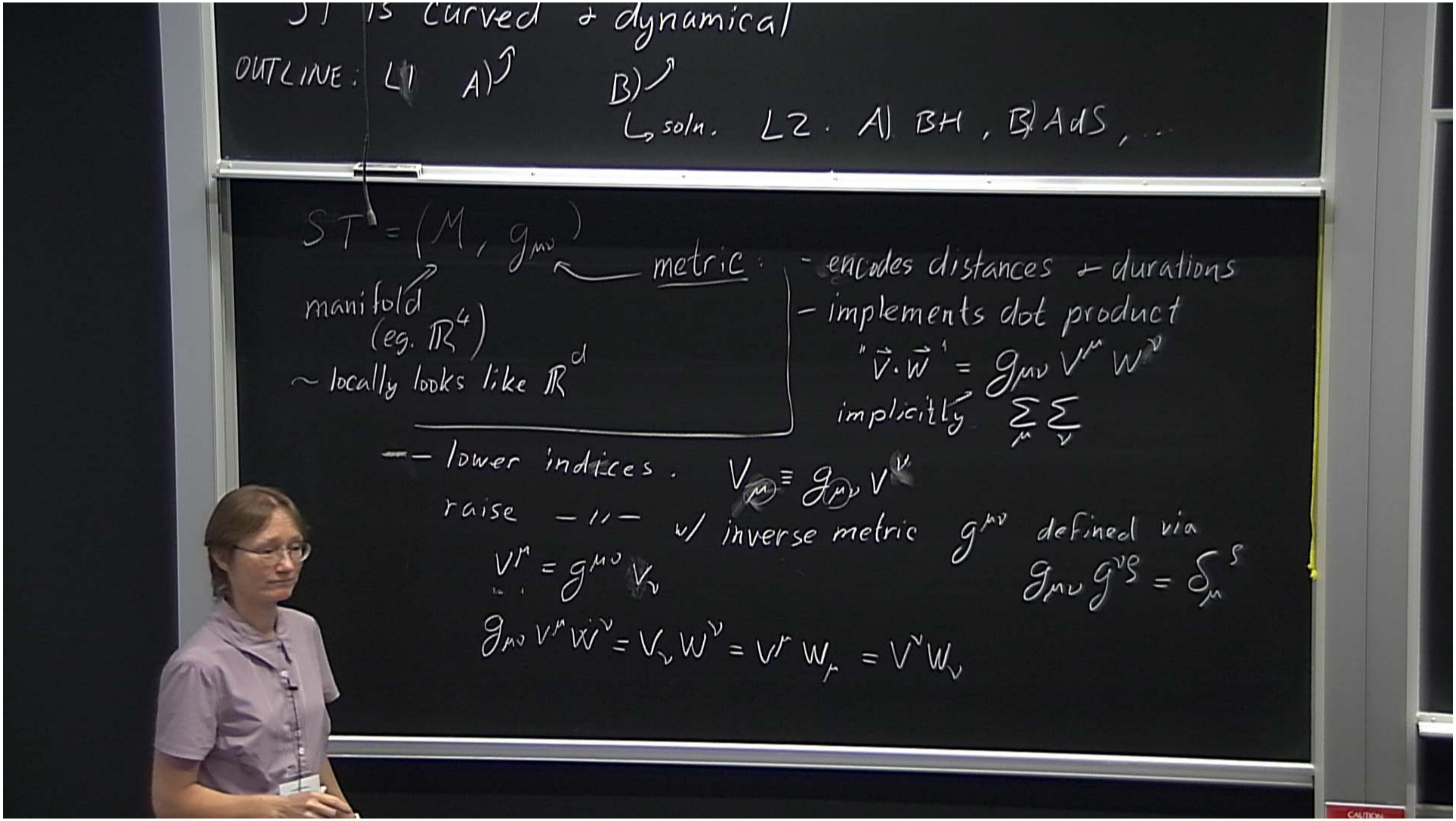

L1: Intro to General Relativity
GR = thy. of gravity
~ ST curvature
matter (stress-energy-momentum tensor $T_{\mu\nu}$)

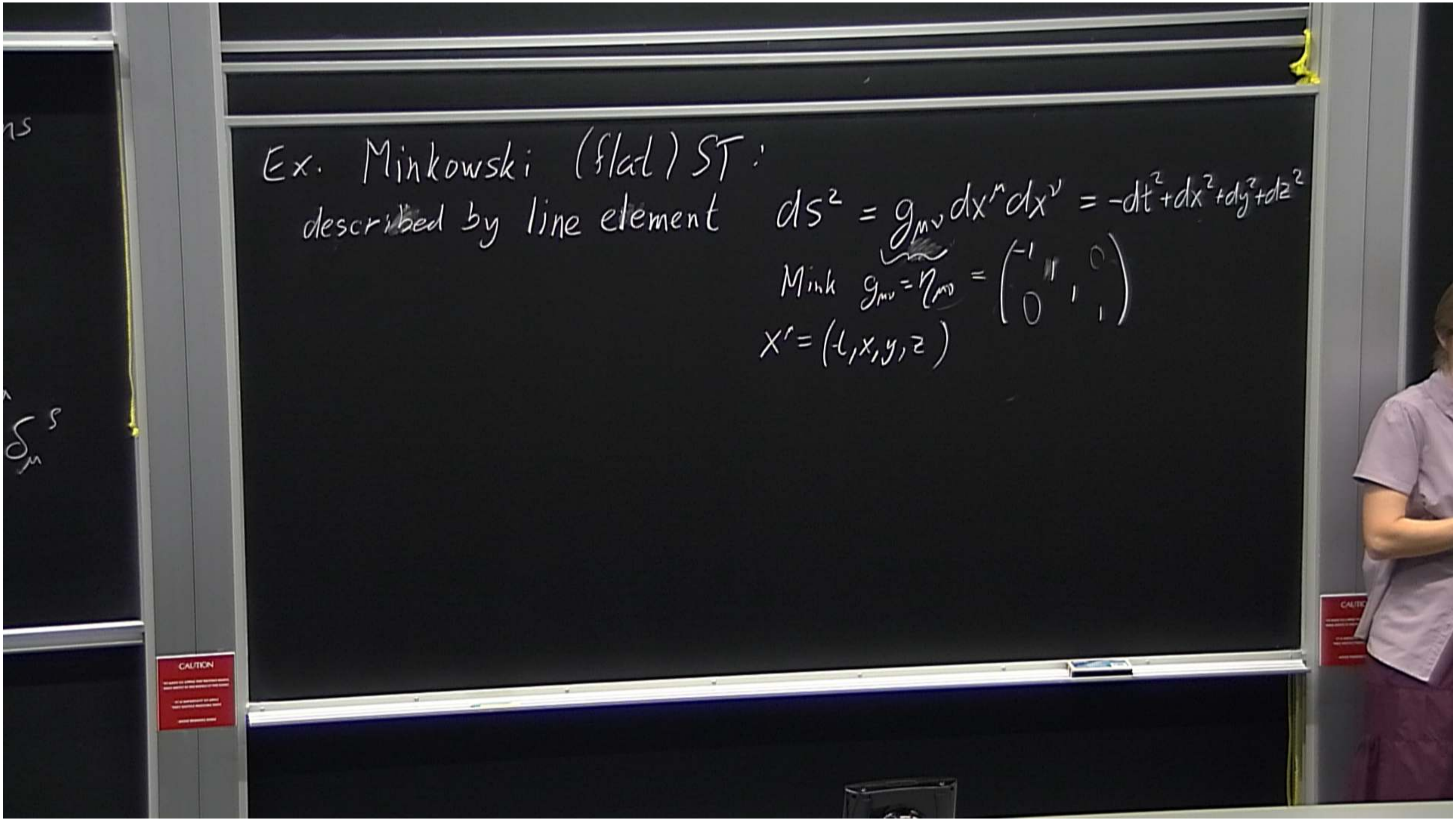

L1: Intro to General Relativity
GR = thy. of gravity
~ ST curvature
matter (stress-energy-momentum tensor $T_{\mu\nu}$)
Einstein's eqn. (for metric $g_{\mu\nu}$ +   )
CAUTION

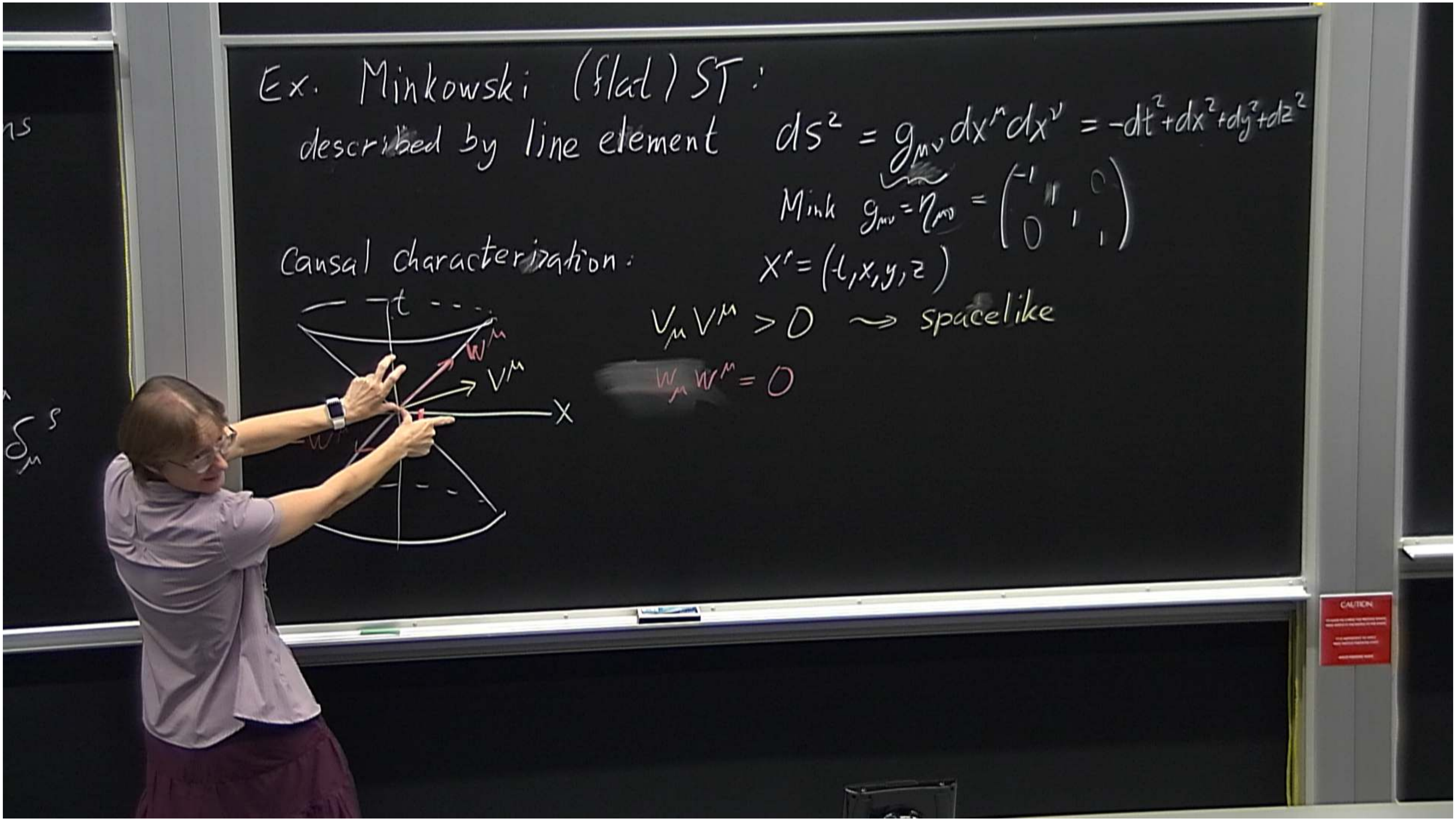

↳ soln. L2. A) BH, B) AdS, ..
ST = (M, g_μν)
manifold
(eg. ℝ^4)
~ locally looks like ℝ^d
CAUTION

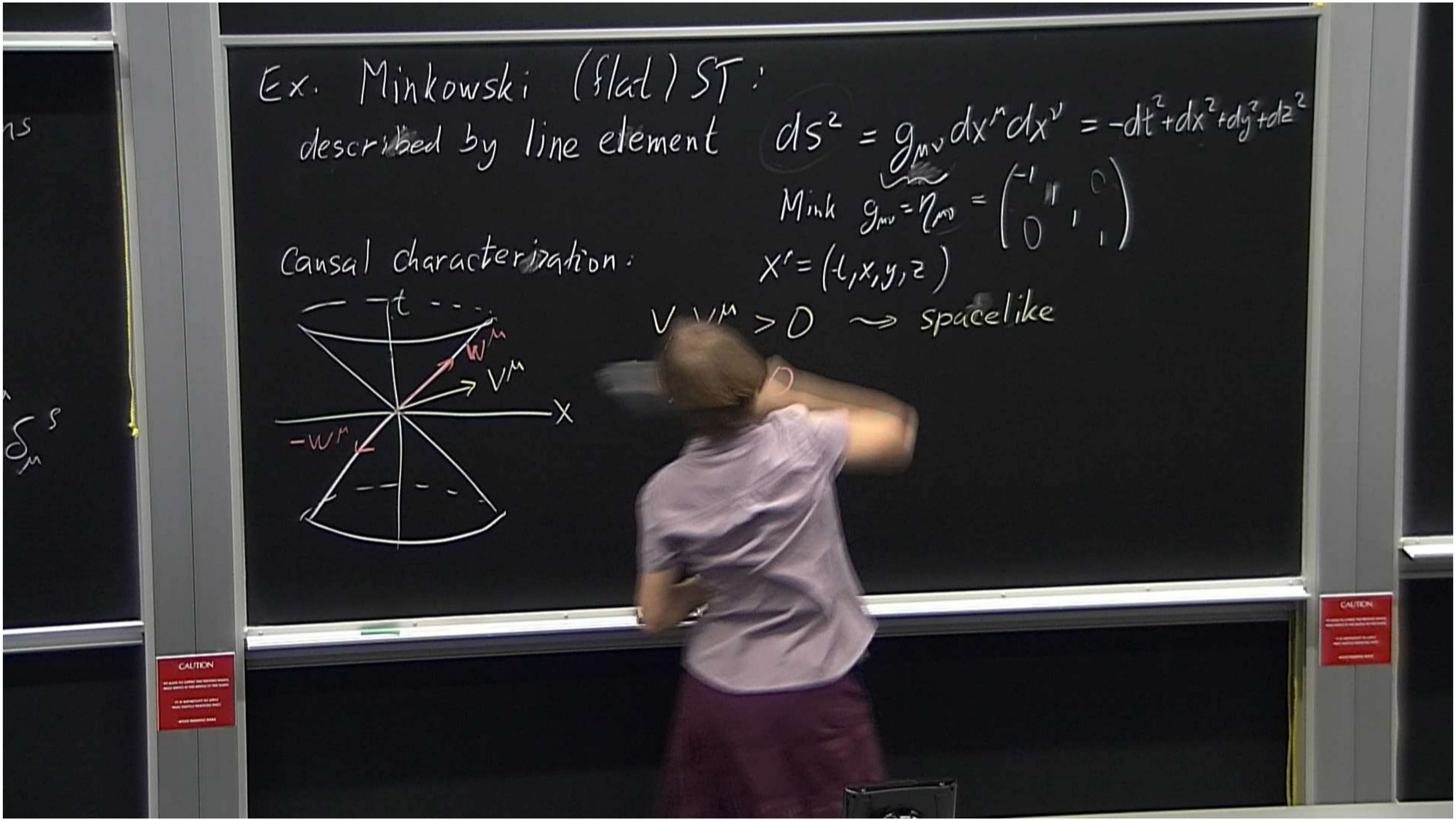

Einstein's eqn. (for metric $g_{\mu\nu}$ + ?)
ST is curved + dynamical
OUTLINE: L1 A) B)
soln. L2. A) BH, B) AdS, ..
manifold (eg. $\mathbb{R}^d$)
~ locally looks like $\mathbb{R}^d$
- implements dot product
" $\vec{V} \cdot \vec{W}$ " = $g_{\mu\nu} V^\mu W^\nu$
implicitly $\sum_\mu \sum_\nu$
-- lower indices. $V_\mu \equiv g_{\mu\nu} V^\nu$
raise -//- w/ inverse metric $g^{\mu\nu}$ defined via

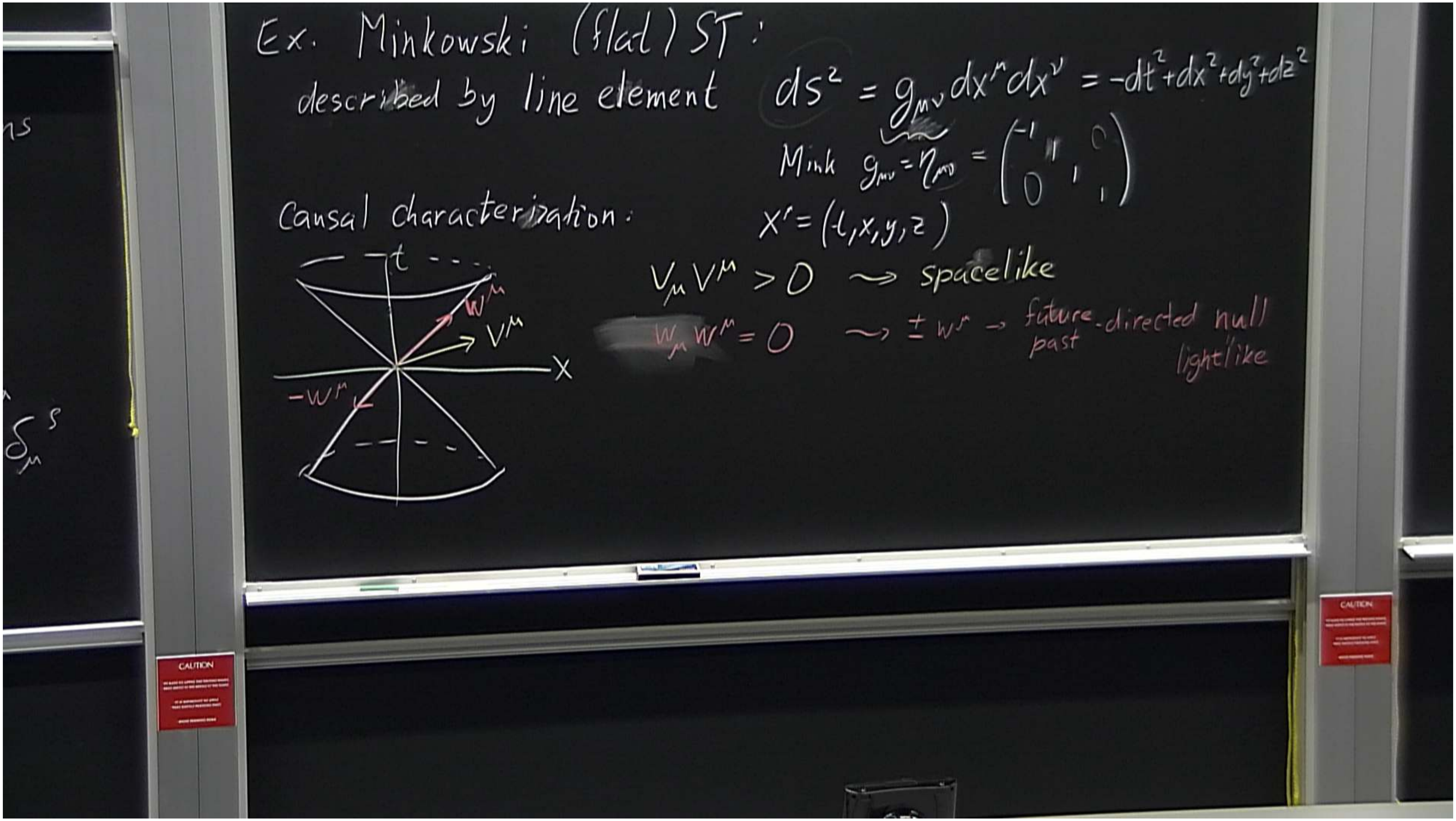

ST is curved & dynamical
OUTLINE: L1 A) B)
soln. L2. A) BH, B) AdS, ...
ST = (M, gμν)
manifold (eg. ℝ⁴)
~ locally looks like ℝ^d
metric: - encodes distances & durations
- implements dot product
"V⃗ · W⃗" = gμν V^μ W^ν
implicitly Σμ Σν
- lower indices. Vμ ≡ gμν V^ν
raise -"- w/ inverse metric g^μν defined via
gμν g^νρ = δμ^ρ
V^μ = g^μν Vν
gμν V^μ W^ν = Vν W^ν = V^μ Wμ = V^ν Wν

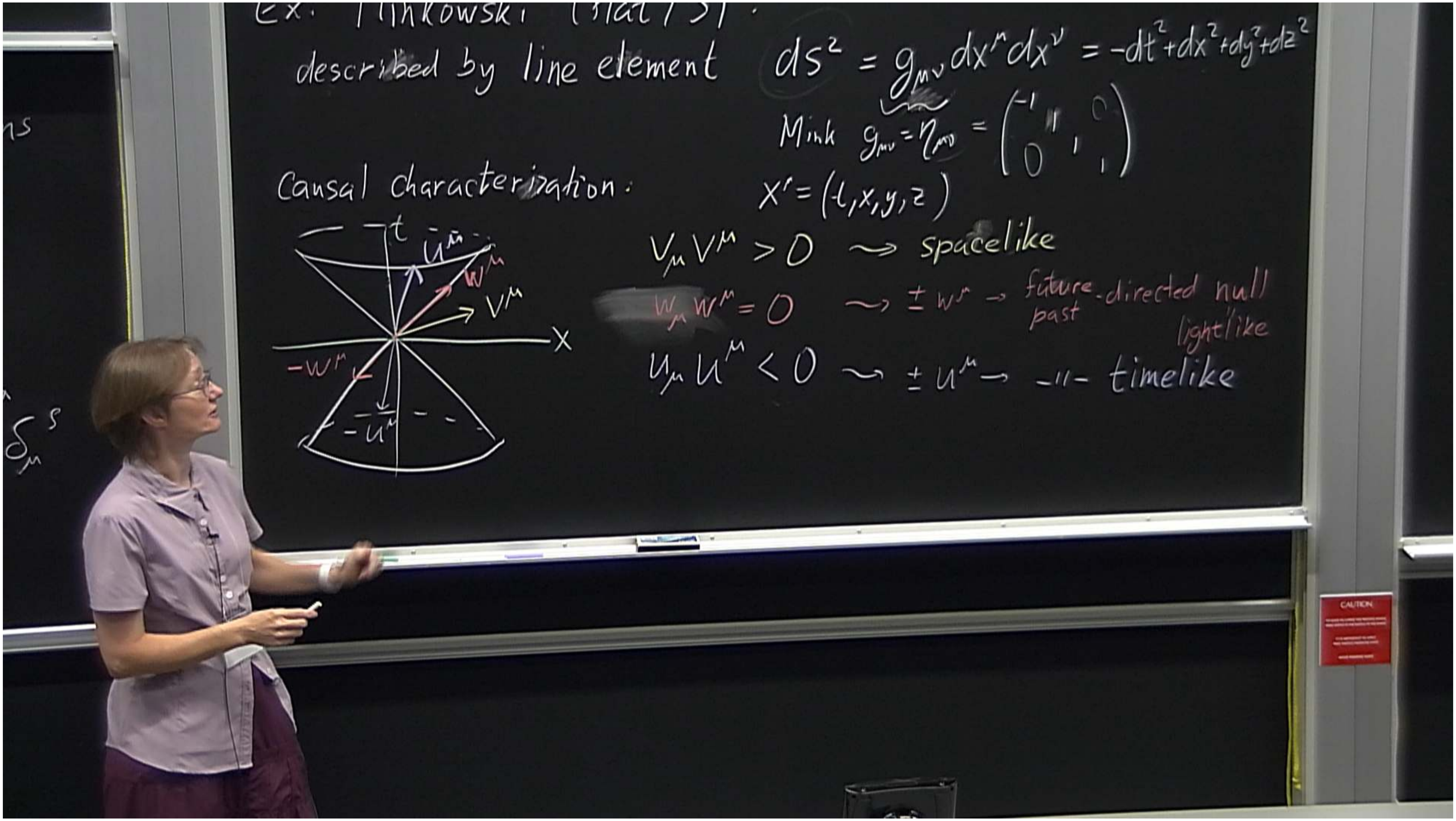

ST is curved & dynamical
OUTLINE: L1 A) B)
↳ soln. L2. A) BH, B) AdS, ..
ST = (M, $g_{\mu\nu}$)
manifold
(eg. $\mathbb{R}^4$)
~ locally looks like $\mathbb{R}^d$
metric:
- encodes distances & durations
- implements dot product
"$\vec{V} \cdot \vec{W}$" = $g_{\mu\nu} V^\mu W^\nu$
implicitly $\sum_\mu \sum_\nu$
- lower indices. $V_\mu \equiv g_{\mu\nu} V^\nu$
raise -//- w/ inverse metric $g^{\mu\nu}$ defined via
$V^\mu = g^{\mu\nu} V_\nu$
$g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho$
$g_{\mu\nu} V^\mu W^\nu = V_\nu W^\nu = V^\mu W_\mu = V^\nu W_\nu$

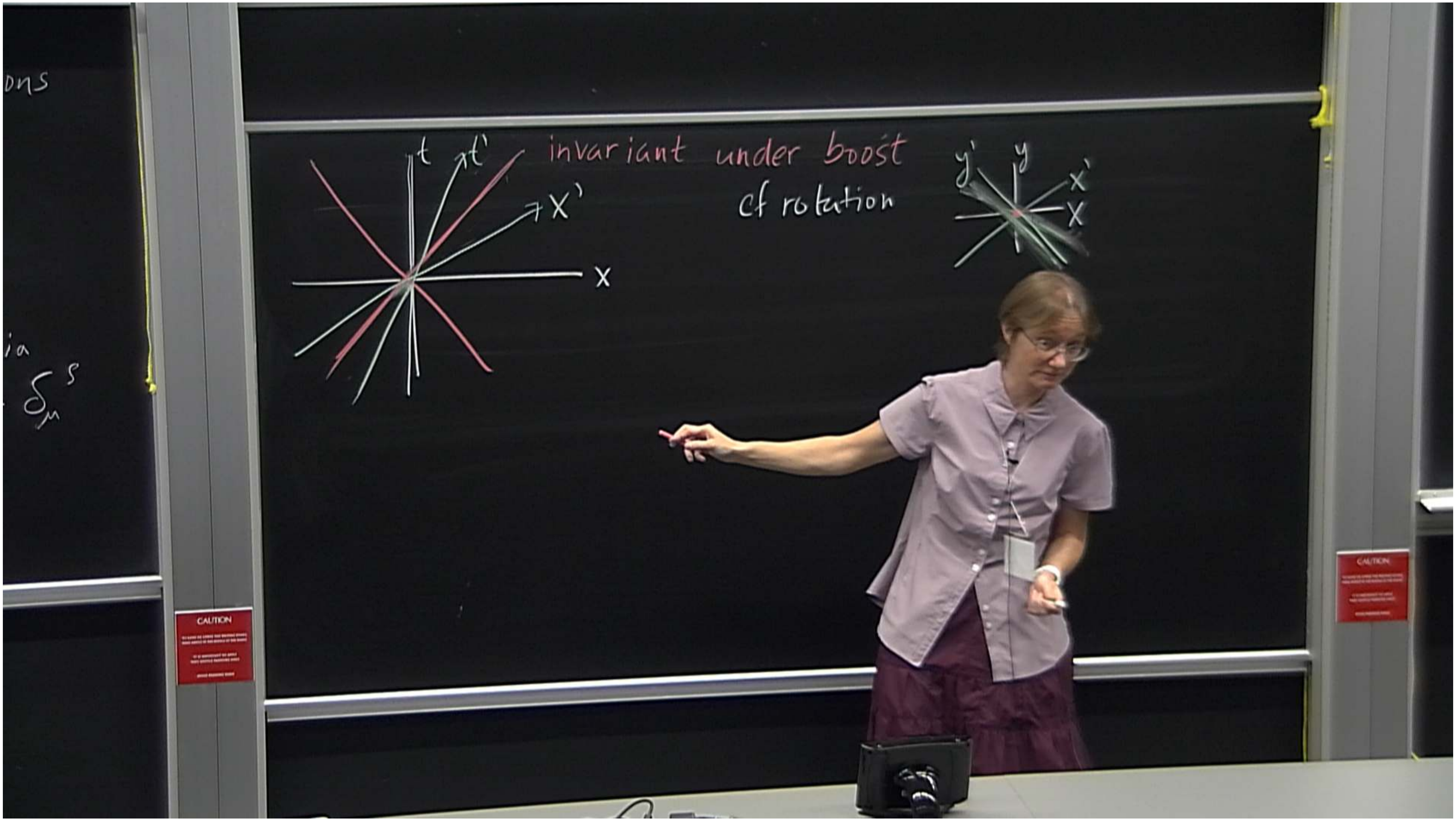

Ex. Minkowski (flat) ST:
described by line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$
Mink $g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$
$x^\mu = (t, x, y, z)$

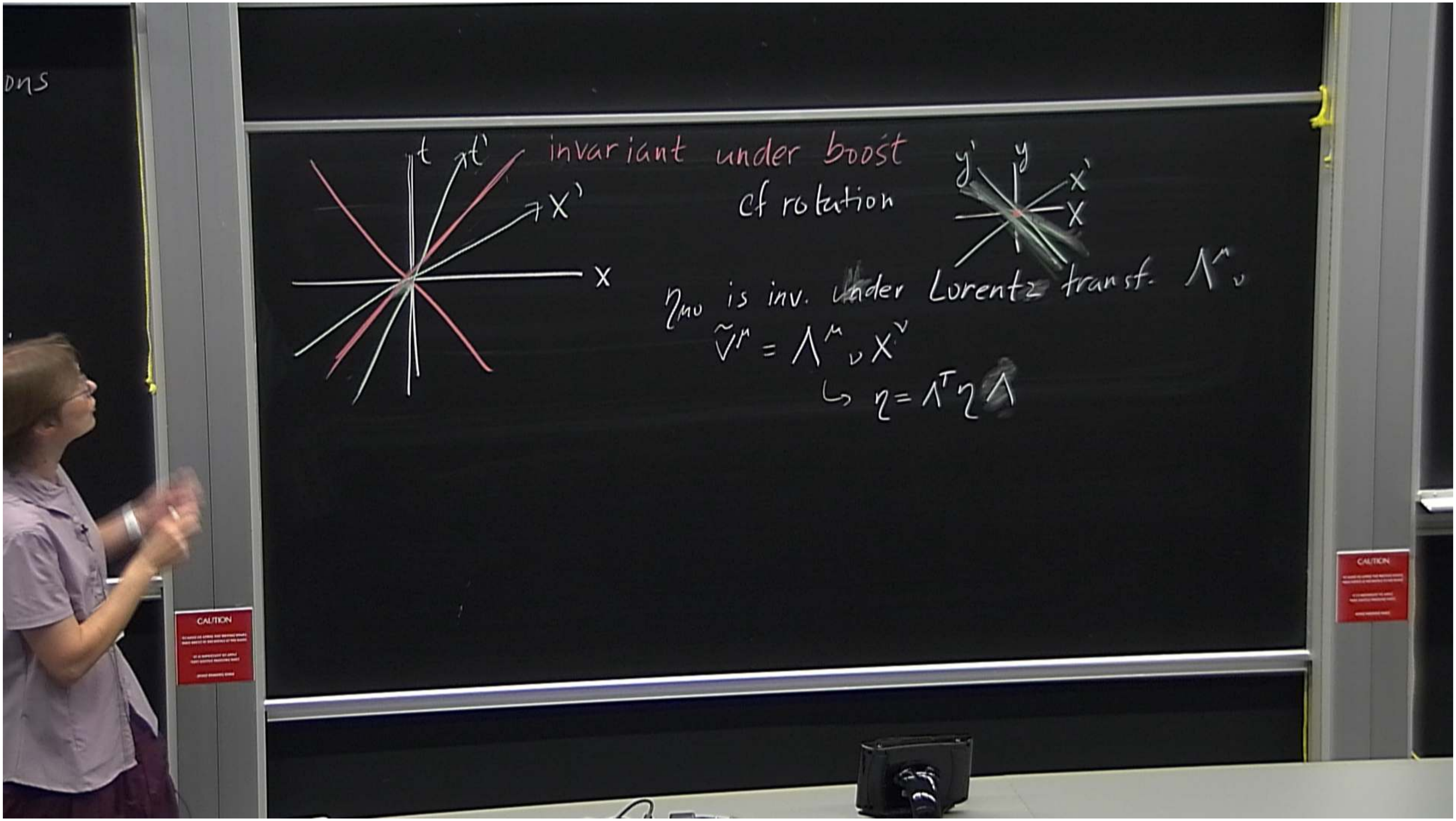

Ex. Minkowski (flat) ST:
described by line element ds² = gμν dx^μ dx^ν = -dt²+dx²+dy²+dz²
Mink gμν = ημν =
x^μ = (t,x,y,z)
Causal characterization:
t
W^μ
V^μ
x
VμV^μ > 0 ⇝ spacelike
WμW^μ = 0

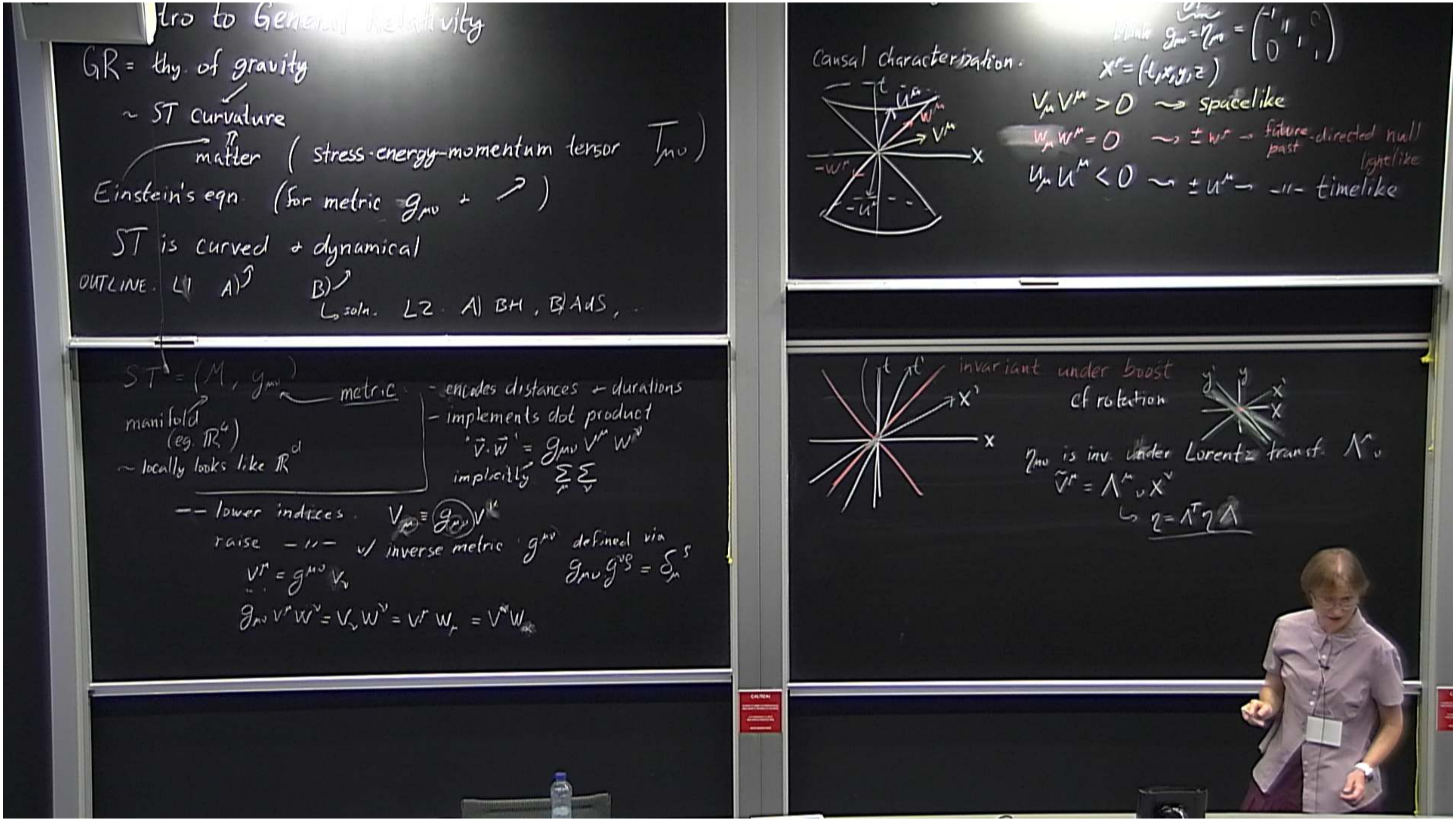

Ex. Minkowski (flat) ST:
described by line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2+dx^2+dy^2+dz^2$
Mink $g_{\mu\nu} = \eta_{\mu\nu}$
$x^\mu = (t,x,y,z)$
Causal characterization:
spacelike
t
x
W^μ
V^μ
$-W^\mu$
CAUTION

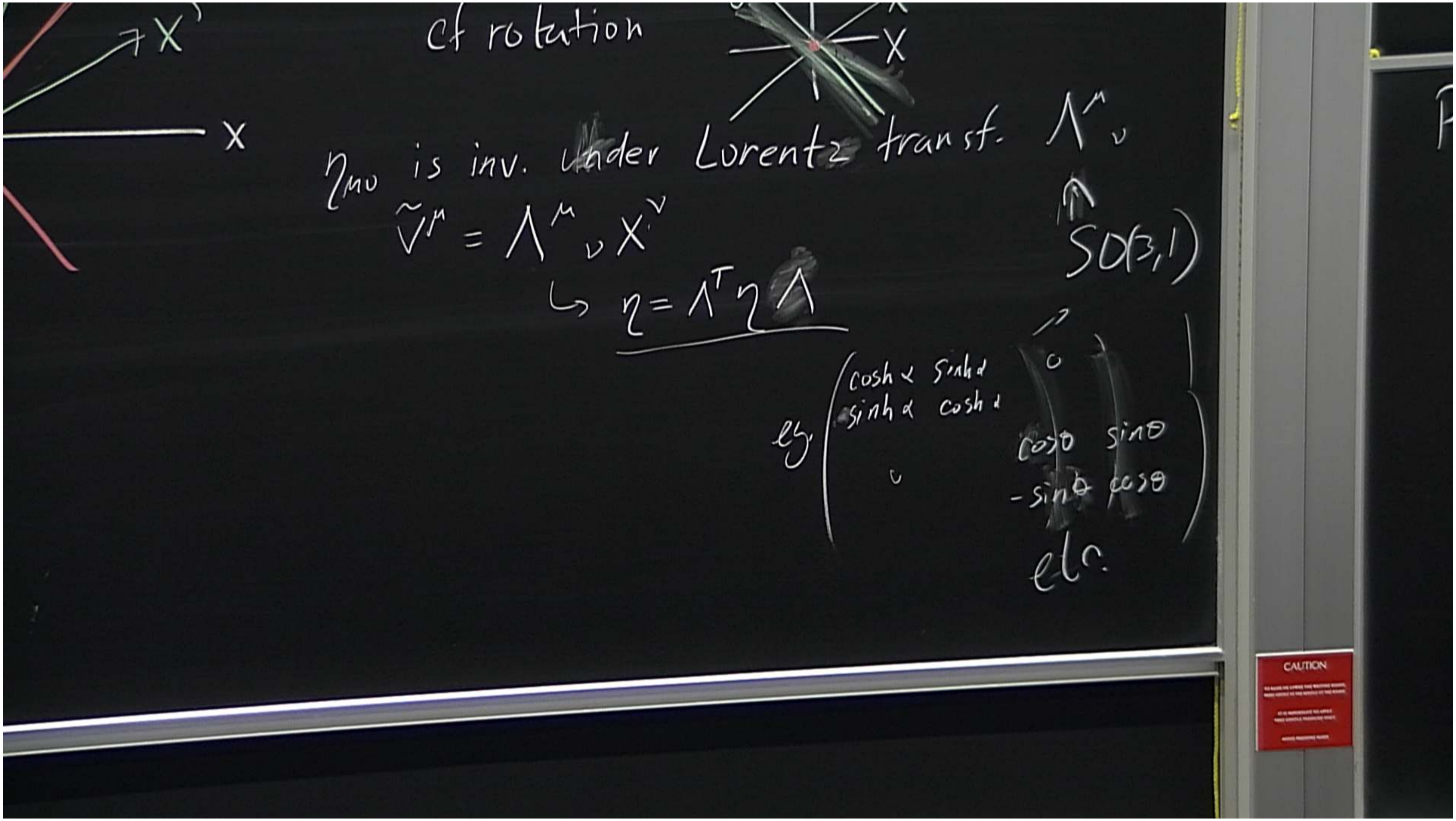

Ex. Minkowski (flat) ST:
described by line element ds² = g_μν dx^μ dx^ν = -dt² + dx² + dy² + dz²
Mink g_μν = η_μν =
x^μ = (t, x, y, z)
Causal characterization:
t
W^μ
V^μ
X
-W^μ
V_μ V^μ > 0 ⇝ spacelike
W_μ W^μ = 0 ⇝ ± W^μ → future-directed null
past
lightlike

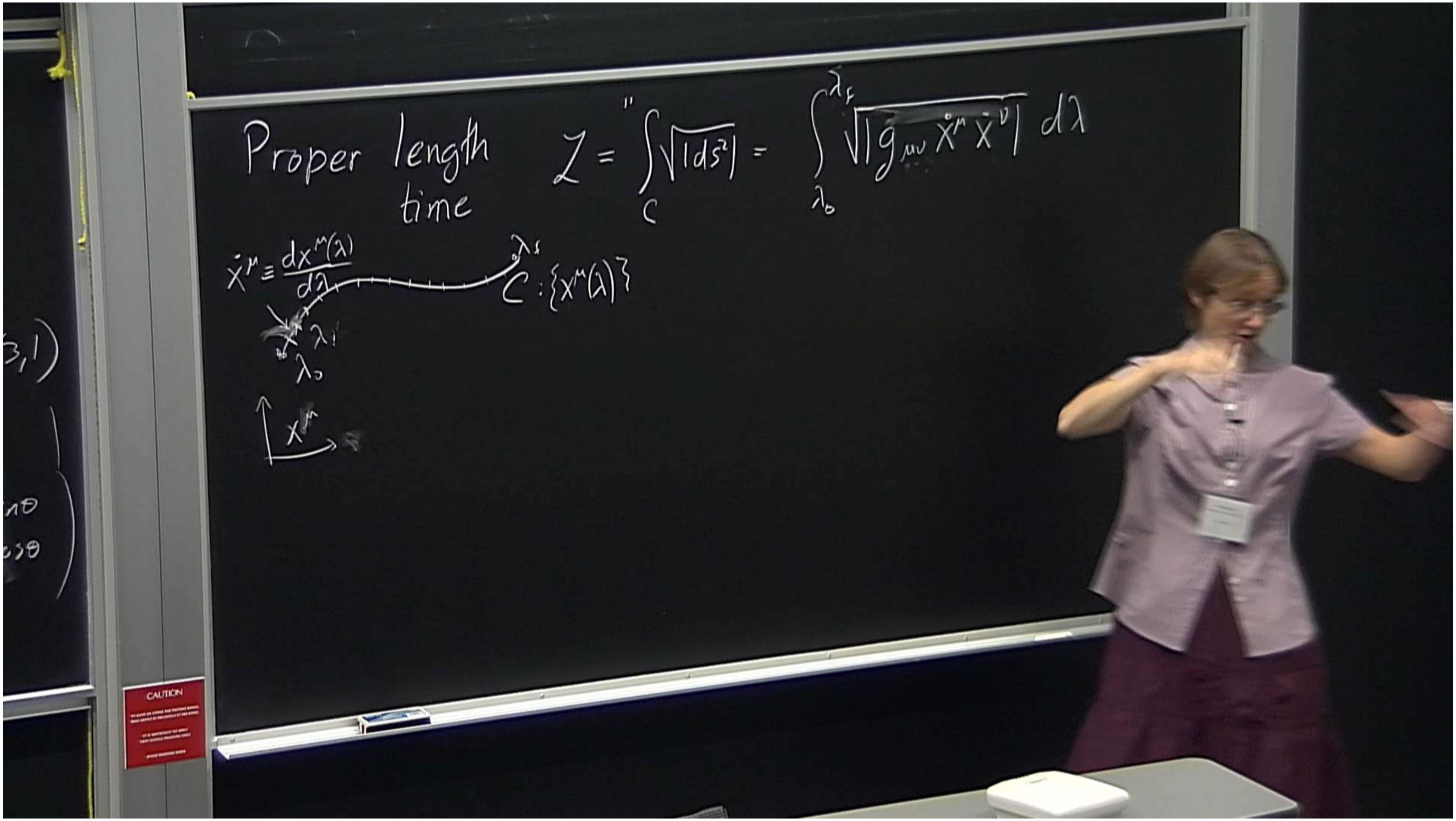

described by line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$
Mink $g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$
$x^\mu = (t, x, y, z)$
Causal characterization:
$V_\mu V^\mu > 0 \rightsquigarrow$ spacelike
$W_\mu W^\mu = 0 \rightsquigarrow \pm W^\mu \rightarrow$ future / past - directed null / lightlike
$U_\mu U^\mu < 0 \rightsquigarrow \pm U^\mu \rightarrow$ -"- timelike
t
x
U^μ
W^μ
V^μ
$-W^\mu$
$-U^\mu$

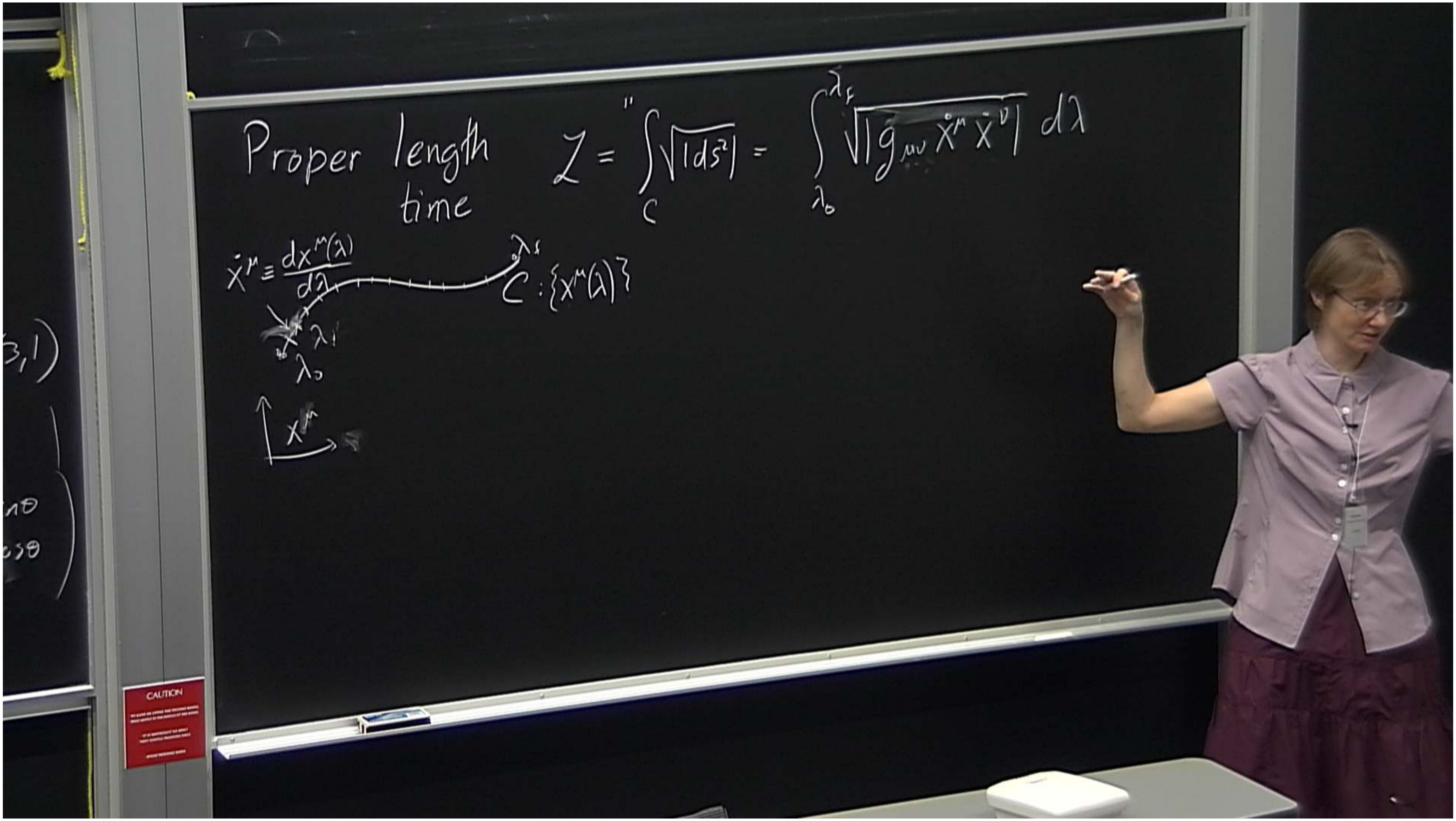

invariant under boost
cf rotation
t
t'
x'
x
y'
y
x'
x
CAUTION

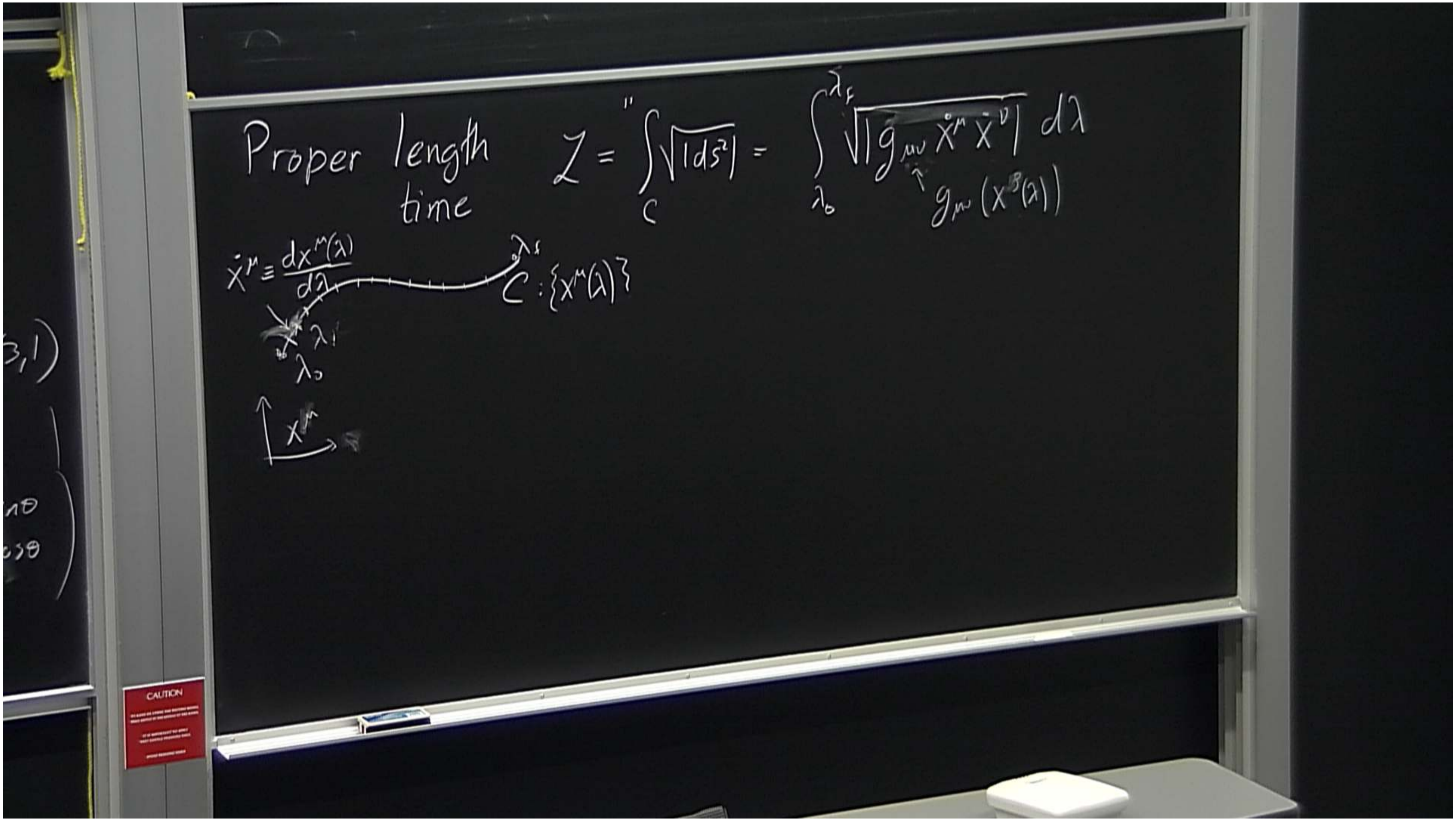

invariant under boost
cf rotation
t
t'
x'
x
y'
y
$\eta_{\mu\nu}$ is inv. under Lorentz transf. $\Lambda^\mu{}_\nu$
$\tilde{V}^\mu = \Lambda^\mu{}_\nu X^\nu$
$\hookrightarrow \eta = \Lambda^T \eta \Lambda$
CAUTION

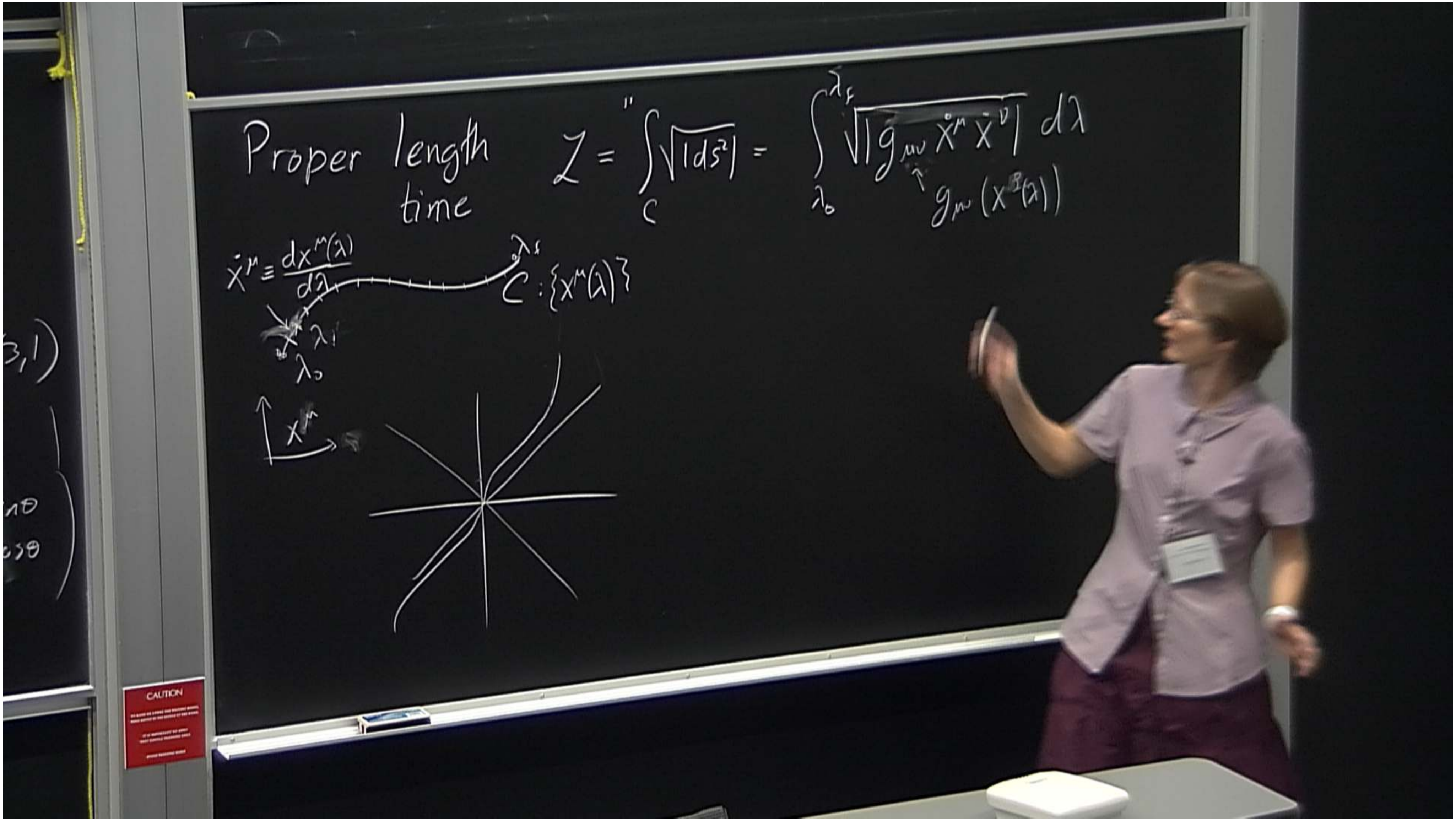

Intro to General Relativity
GR = thy. of gravity
~ ST curvature
matter (stress-energy-momentum tensor $T_{\mu\nu}$)
Einstein's eqn (for metric $g_{\mu\nu}$ + ↗)
ST is curved + dynamical
OUTLINE. L1 A) B)
soln. L2 A) BH , B) AdS , ..
ST = (M , $g_{\mu\nu}$)
manifold (eg. $\mathbb{R}^4$)
~ locally looks like $\mathbb{R}^d$
metric
- encodes distances + durations
- implements dot product
$\vec{V}\cdot\vec{W} = g_{\mu\nu} V^\mu W^\nu$
implicitly $\sum_\mu \sum_\nu$
-- lower indices $V_\mu \equiv g_{\mu\nu} V^\nu$
raise -- w/ inverse metric $g^{\mu\nu}$ defined via $g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho$
$V^\mu = g^{\mu\nu} V_\nu$
$g_{\mu\nu} V^\mu W^\nu = V_\nu W^\nu = V^\mu W_\mu = V^\alpha W_\alpha$
Causal characterization.
$g_{\mu\nu} = \eta_{\mu\nu}$
$x^\mu = (t, x, y, z)$
$V_\mu V^\mu > 0$ ⇝ spacelike
$W_\mu W^\mu = 0$ ⇝ $\pm W^\mu$ → future/past - directed null lightlike
$U_\mu U^\mu < 0$ ⇝ $\pm U^\mu$ → -"- timelike
t
x
U^μ
W^μ
V^μ
$-W^\mu$
$-U^\mu$
invariant under boost
cf rotation
t
t'
x'
x
y
y'
$\eta_{\mu\nu}$ is inv. under Lorentz transf. $\Lambda^\mu{}_\nu$
$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$
$\eta = \Lambda^T \eta \Lambda$

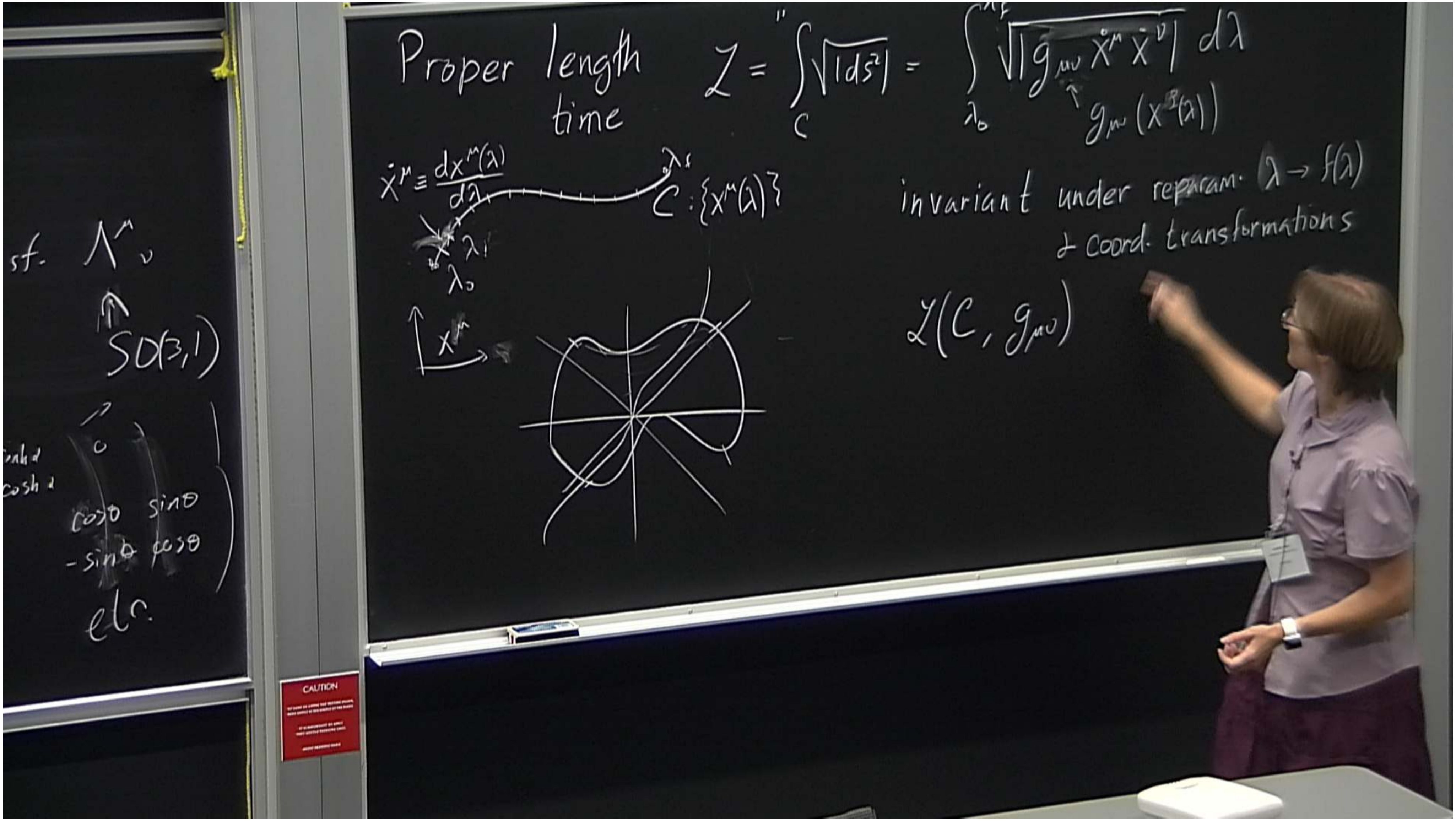

cf rotation
X
X
ημν is inv. under Lorentz transf. Λ^μ_ν
SO(3,1)
X̃^μ = Λ^μ_ν X^ν
↳ η = Λ^T η Λ
eg.
cosh α sinh α
sinh α cosh α
cos θ sin θ
-sin θ cos θ
etc.

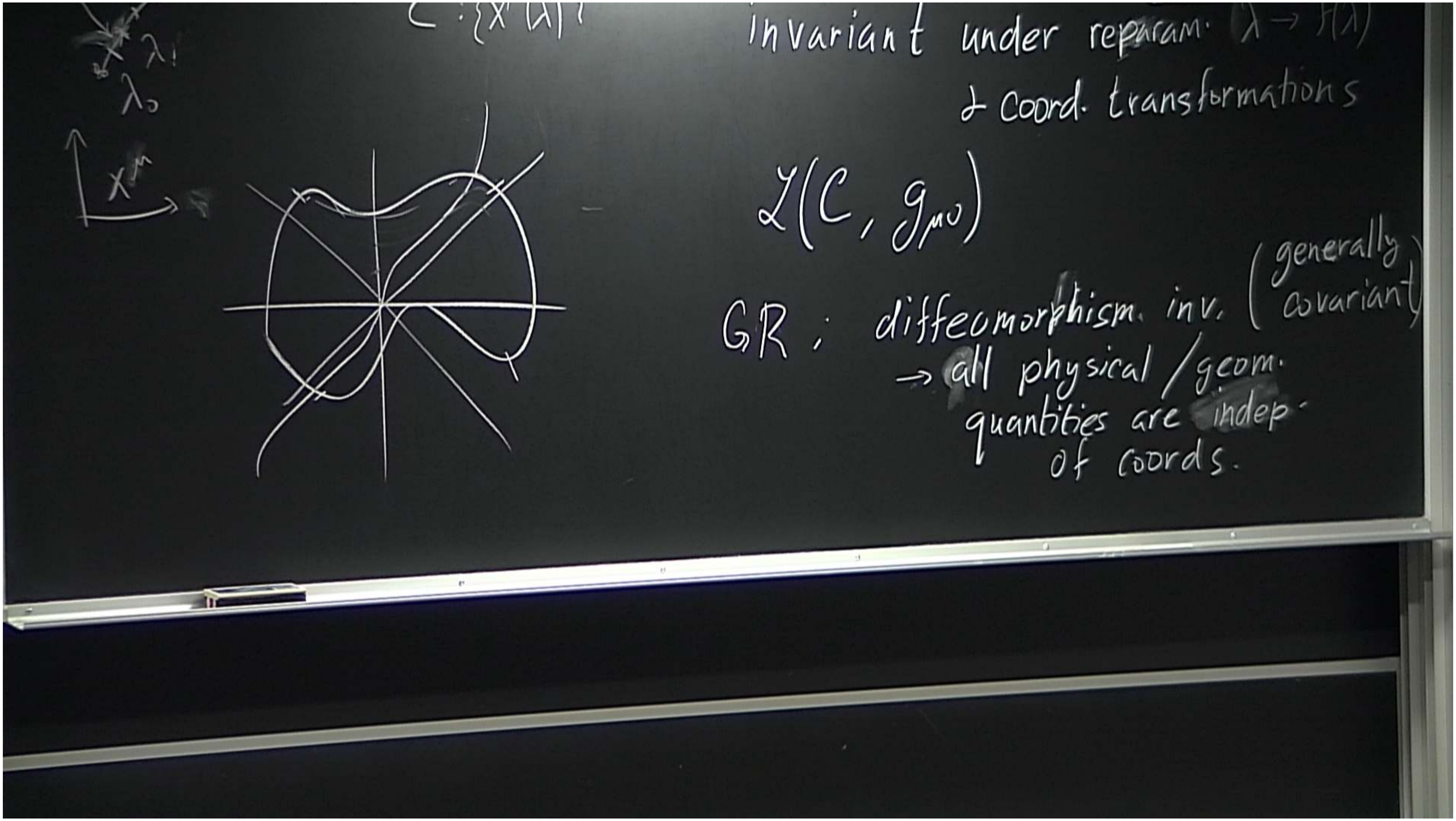

Proper length
time
$\mathcal{L} = \text{"}\int_C \sqrt{|ds^2|} = \int_{\lambda_0}^{\lambda_f} \sqrt{|g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu|}\, d\lambda$
$\dot{x}^\mu \equiv \frac{dx^\mu(\lambda)}{d\lambda}$
$C : \{x^\mu(\lambda)\}$
λ_0
λ_f
x^μ
CAUTION

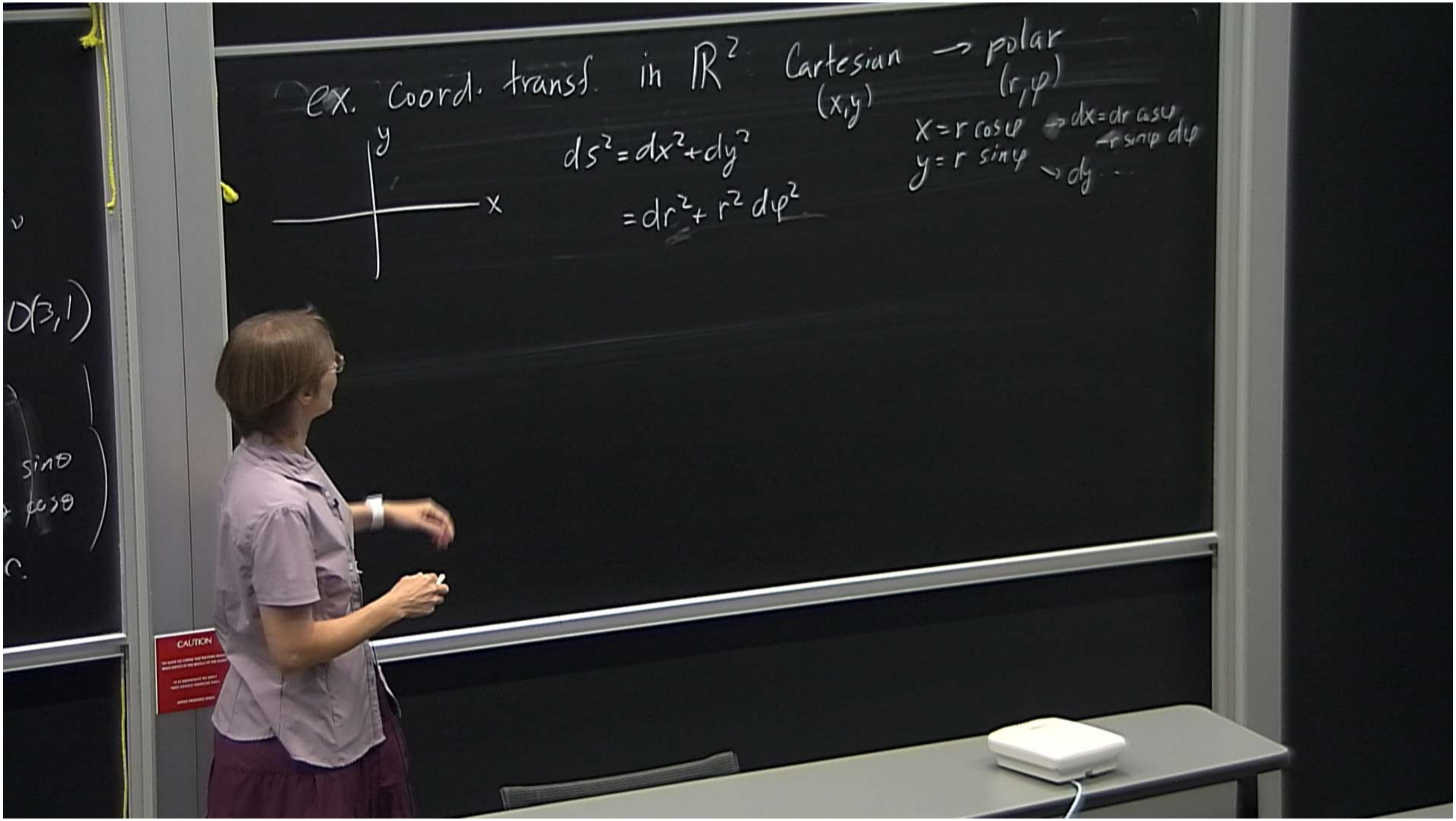

Proper length
time
$L = " \int_C \sqrt{|ds^2|} = \int_{\lambda_0}^{\lambda_f} \sqrt{|g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu|}\, d\lambda$
$\dot{x}^\mu \equiv \frac{dx^\mu(\lambda)}{d\lambda}$
$C : \{x^\mu(\lambda)\}$
λ_f
λ_1
λ_0
x^μ
CAUTION

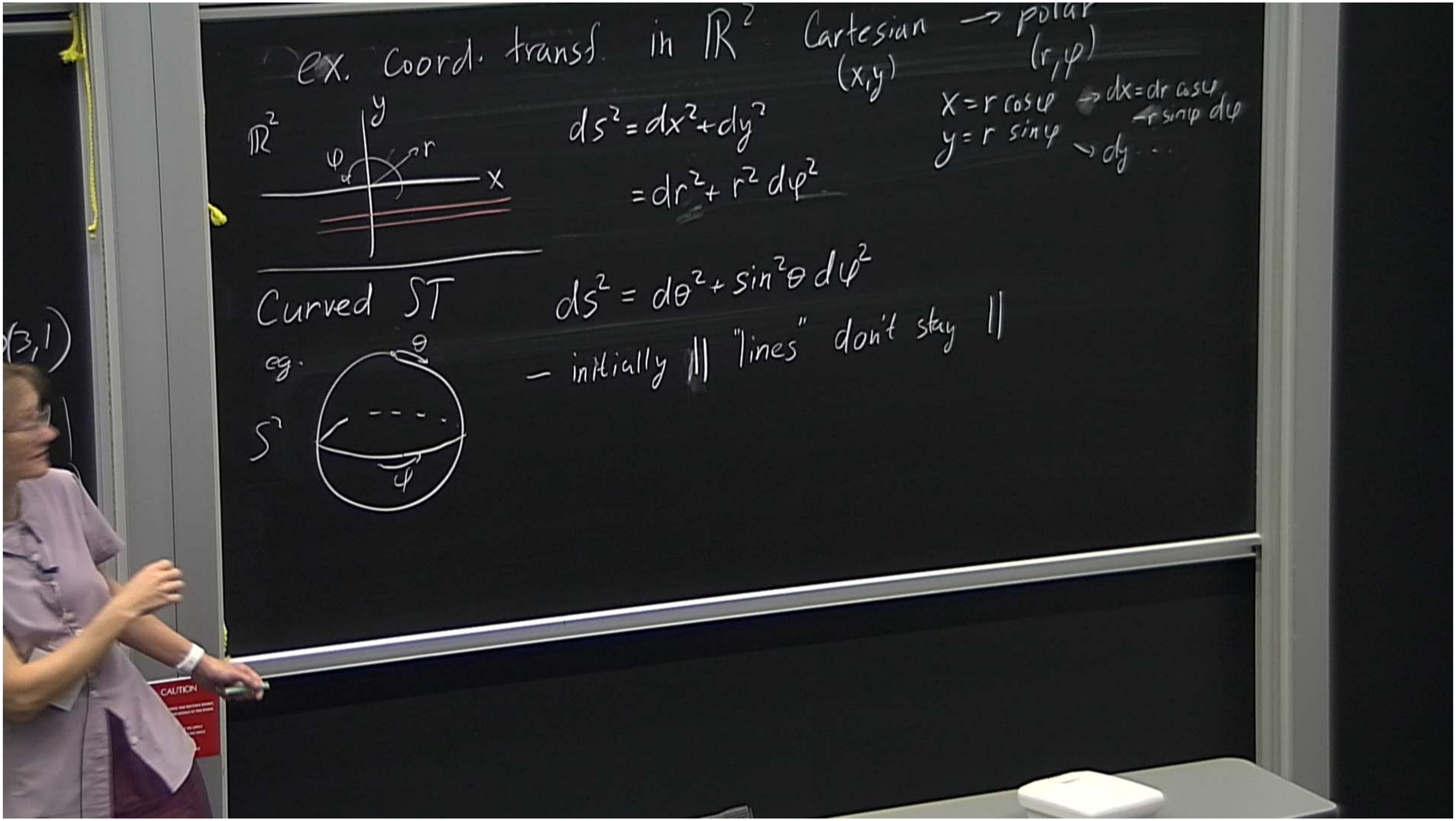

Proper length
time
$L = \int_C \sqrt{|ds^2|} = \int_{\lambda_0}^{\lambda_f} \sqrt{|g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu|}\, d\lambda$
$g_{\mu\nu}(x^\beta(\lambda))$
$\dot{x}^\mu \equiv \frac{dx^\mu(\lambda)}{d\lambda}$
$C : \{x^\mu(\lambda)\}$
λ_f
λ_0
x^μ

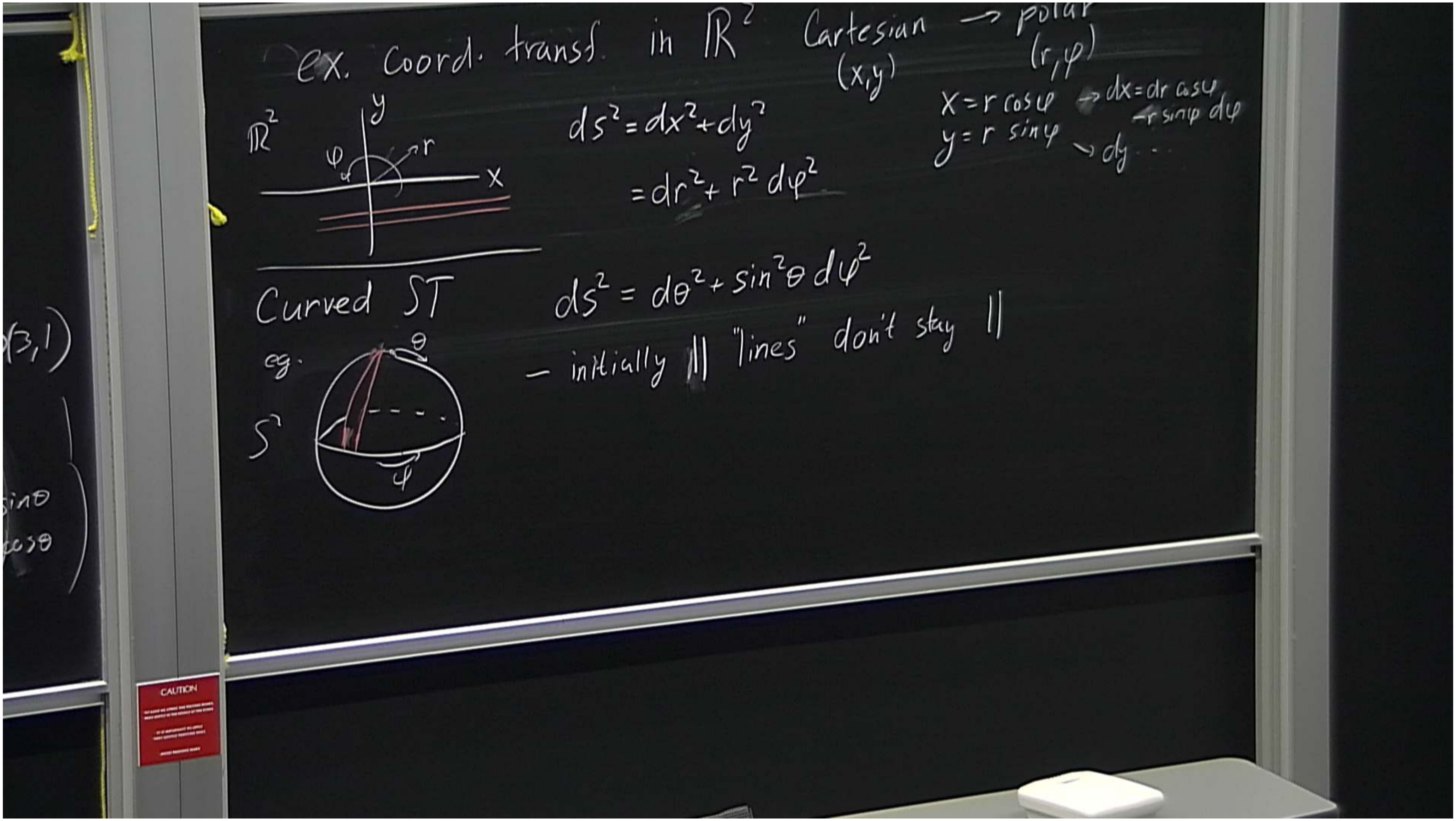

Proper length
time
$\mathcal{L} = " \int_C \sqrt{|ds^2|} = \int_{\lambda_0}^{\lambda_f} \sqrt{|g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}|}\, d\lambda$
$g_{\mu\nu}(x^{\beta}(\lambda))$
$\dot{x}^{\mu} \equiv \frac{dx^{\mu}(\lambda)}{d\lambda}$
$C : \{x^{\mu}(\lambda)\}$
λ_f
λ_1
λ_0
x^{μ}
CAUTION

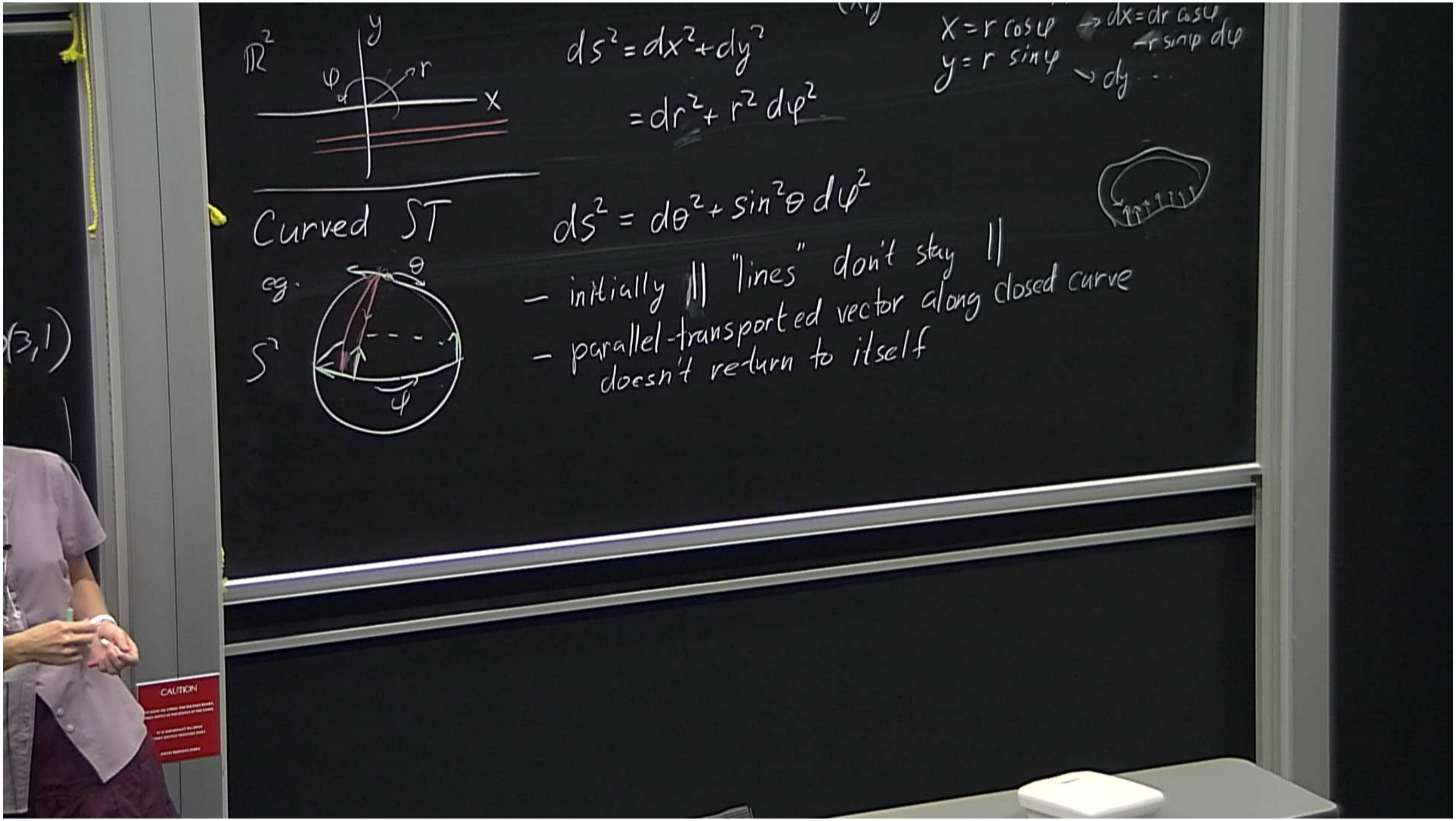

Proper length
time
invariant under reparam. λ → f(λ)
& coord. transformations
SO(3,1)
cos θ sin θ
- sin θ cos θ
etc.
CAUTION


λ₁
λ₀
x^μ
invariant under reparam. λ → f(λ)
& coord. transformations
L(C, g_μν)
GR: diffeomorphism inv. (generally covariant)
→ all physical / geom. quantities are indep. of coords.


ex. Coord. transf. in ℝ² Cartesian → polar
(x,y)
(r,φ)
ds² = dx² + dy²
= dr² + r² dφ²
x = r cos φ → dx = dr cos φ
− r sin φ dφ
y = r sin φ → dy . .
y
x
sin θ
cos θ
CAUTION


ex. Coord. transf. in ℝ² Cartesian → polar
(x,y) (r,φ)
x = r cos φ → dx = dr cos φ − r sin φ dφ
y = r sin φ → dy ...
ℝ²
ds² = dx² + dy²
= dr² + r² dφ²
Curved ST
ds² = dθ² + sin²θ dφ²
eg.
− initially || "lines" don't stay ||
S²
CAUTION

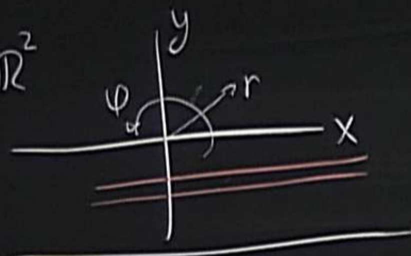

ex. Coord. transf. in ℝ²
Cartesian → polar
(x,y)
(r,φ)
x = r cos φ → dx = dr cos φ − r sin φ dφ
y = r sin φ → dy ...
ℝ²
y
r
φ
x
ds² = dx² + dy²
= dr² + r² dφ²
Curved ST
ds² = dθ² + sin²θ dφ²
eg.
θ
S²
φ
− initially ‖ "lines" don't stay ‖
CAUTION



$\mathbb{R}^2$
$ds^2 = dx^2 + dy^2$
$= dr^2 + r^2 d\varphi^2$
$x = r\cos\varphi \rightarrow dx = dr\cos\varphi - r\sin\varphi\, d\varphi$
$y = r\sin\varphi \rightarrow dy \ldots$
Curved ST
$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$
eg.
S^2
- initially || "lines" don't stay ||
- parallel-transported vector along closed curve doesn't return to itself
CAUTION

ex. Coord. transf in $\mathbb{R}^2$ Cartesian (x,y) $\to$ polar (r,φ)

$\mathbb{R}^2$



$$ds^2 = dx^2 + dy^2$$
$$= dr^2 + r^2 d\varphi^2$$

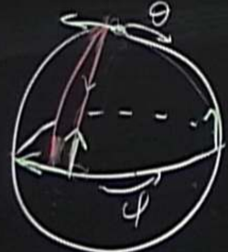
$x = r\cos\varphi \to dx = dr\cos\varphi - r\sin\varphi\, d\varphi$

$y = r\sin\varphi \to dy \ldots$

Curved ST

$$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

eg. S^2

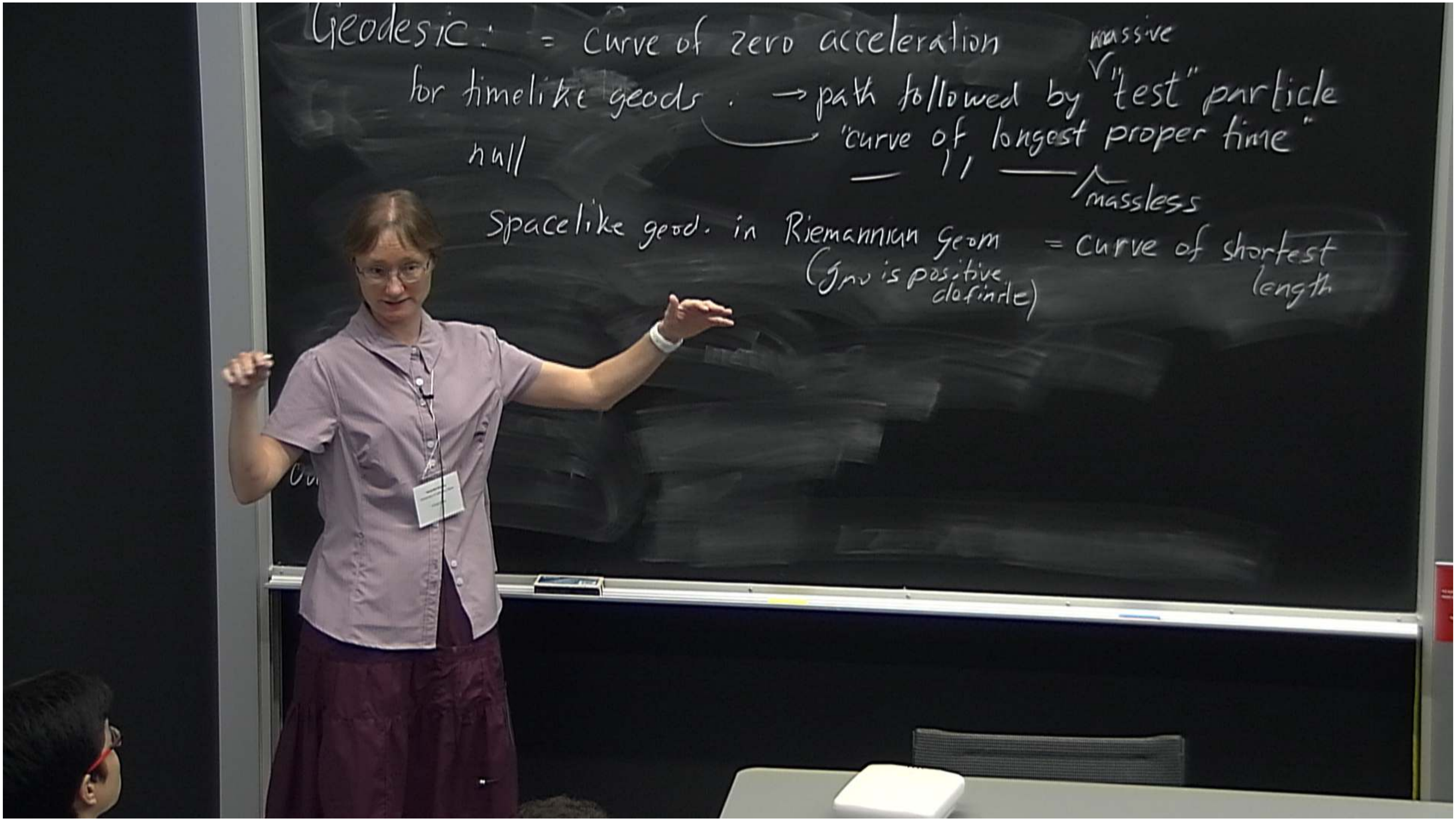


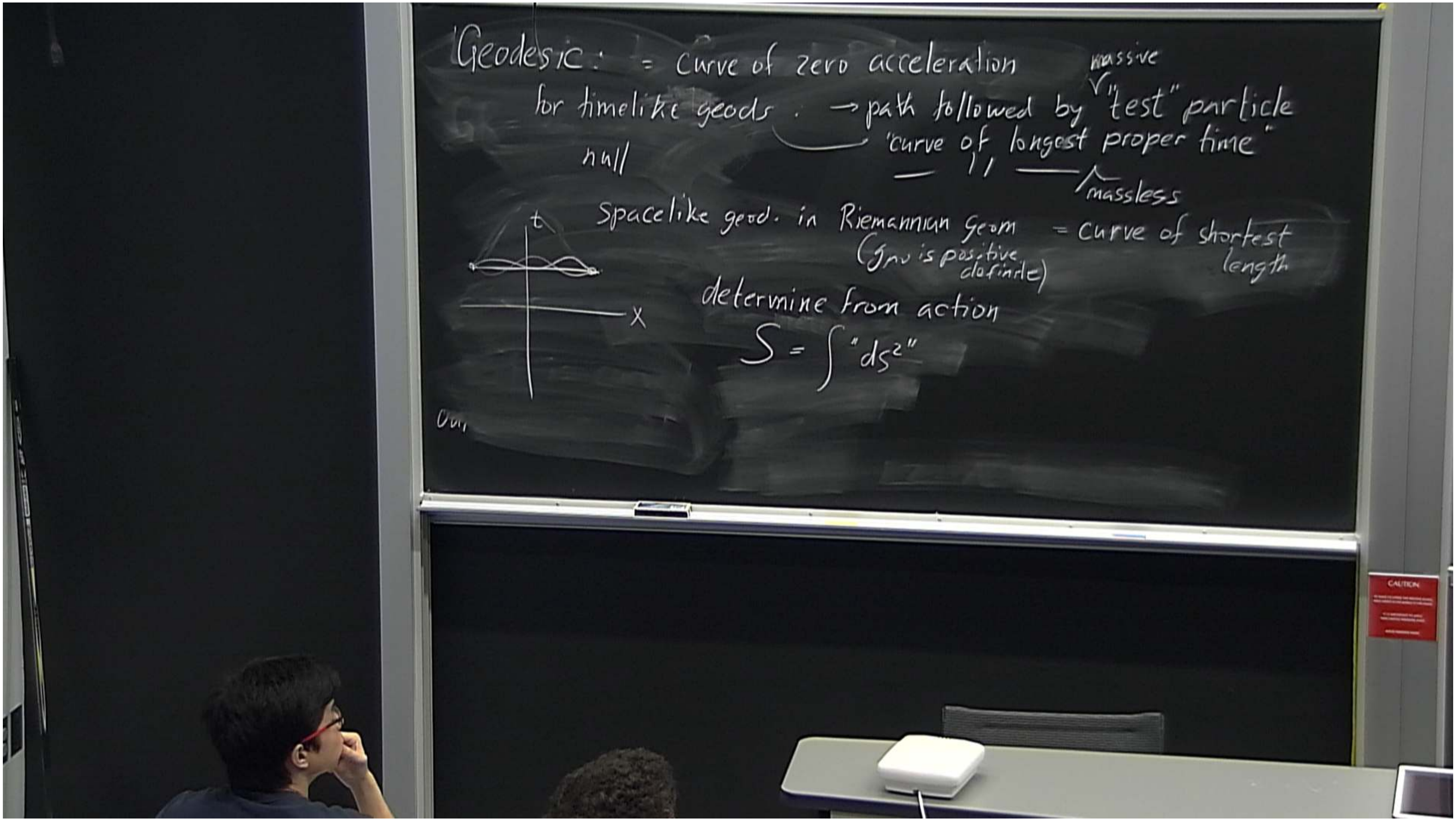
- initially $\parallel$ "lines" don't stay $\parallel$
- parallel-transported vector along closed curve doesn't return to itself
- curvature

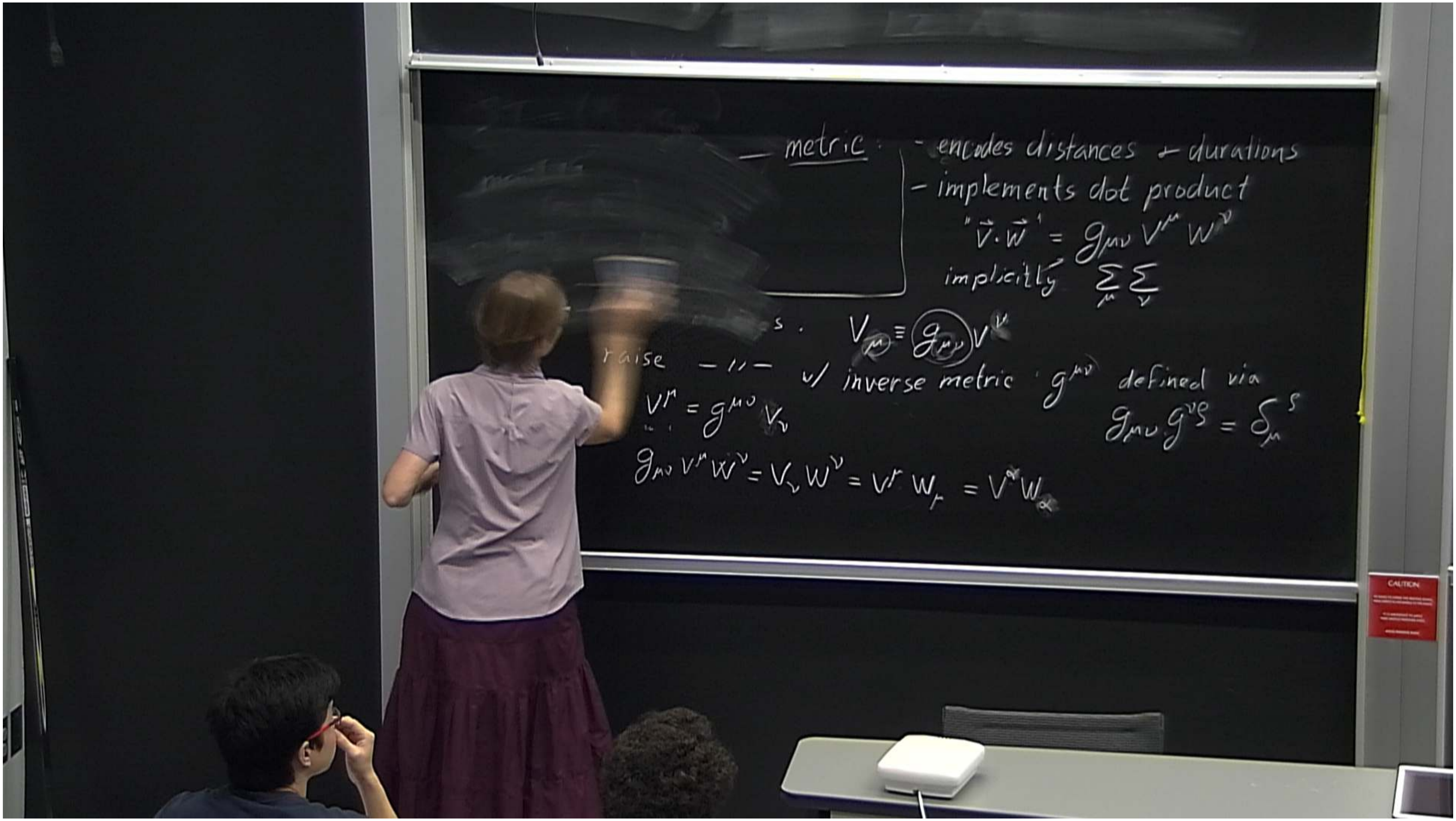
(3,1)

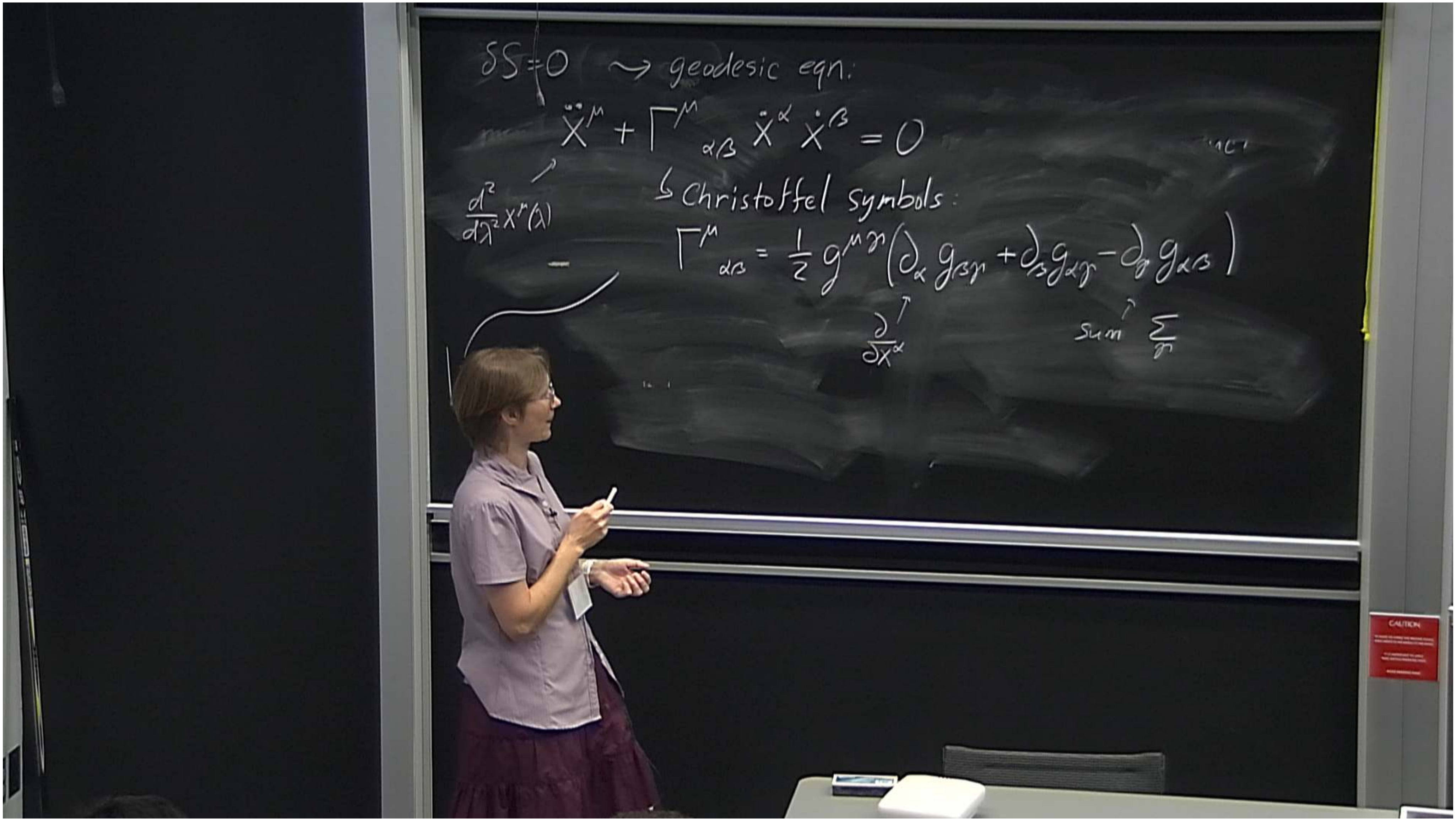
nθ

osθ

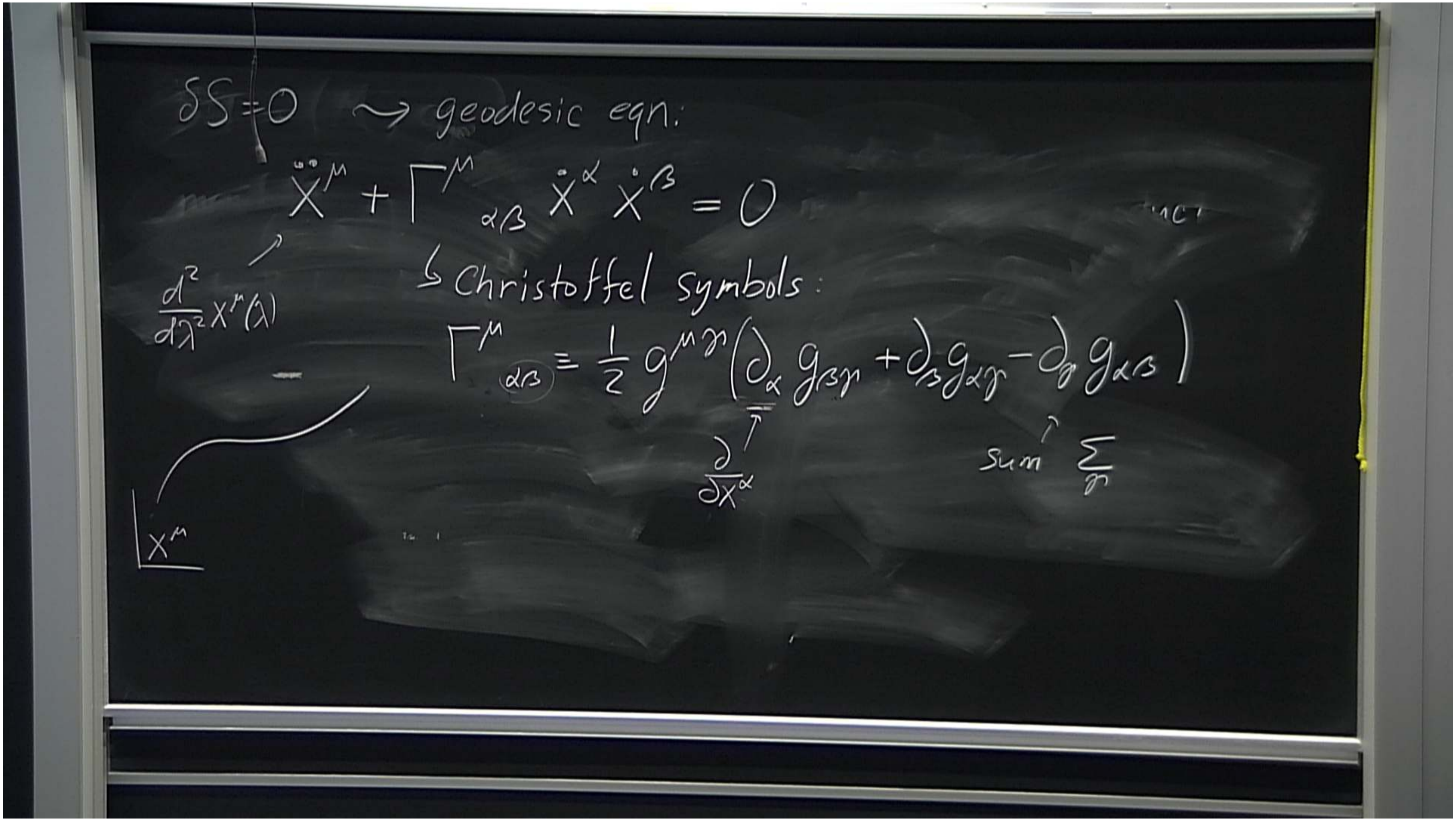

Geodesic: = Curve of zero acceleration
massive
for timelike geods. → path followed by "test" particle
"curve of longest proper time"
null
massless
Spacelike geod. in Riemannian geom = Curve of shortest length
(gμν is positive definite)

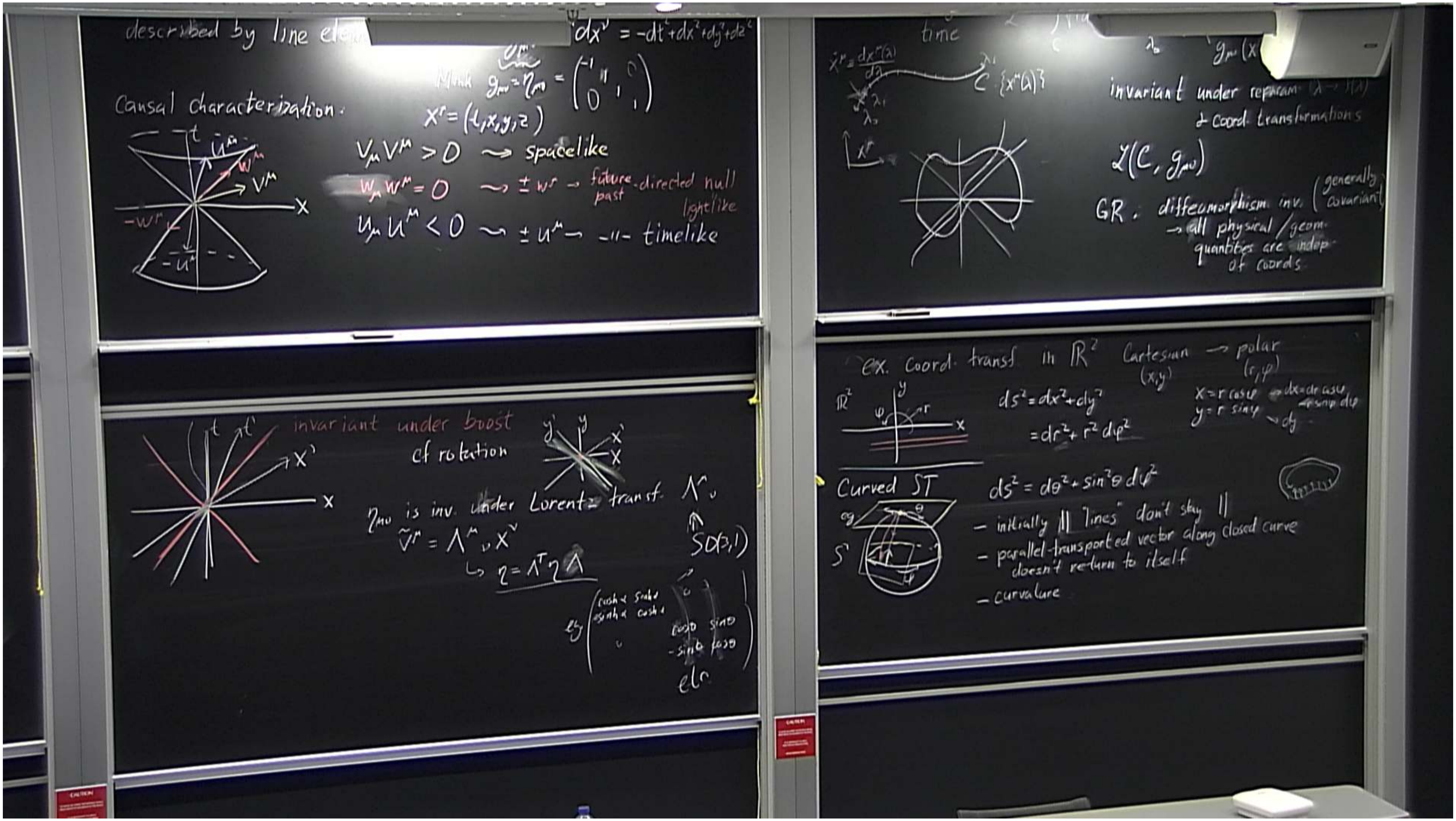

Geodesic: = curve of zero acceleration
massive
for timelike geods → path followed by "test" particle
"curve of longest proper time"
null
massless
spacelike geod. in Riemannian geom = curve of shortest length
($g_{\mu\nu}$ is positive definite)
t
x
determine from action
$S = \int$ "ds^2"

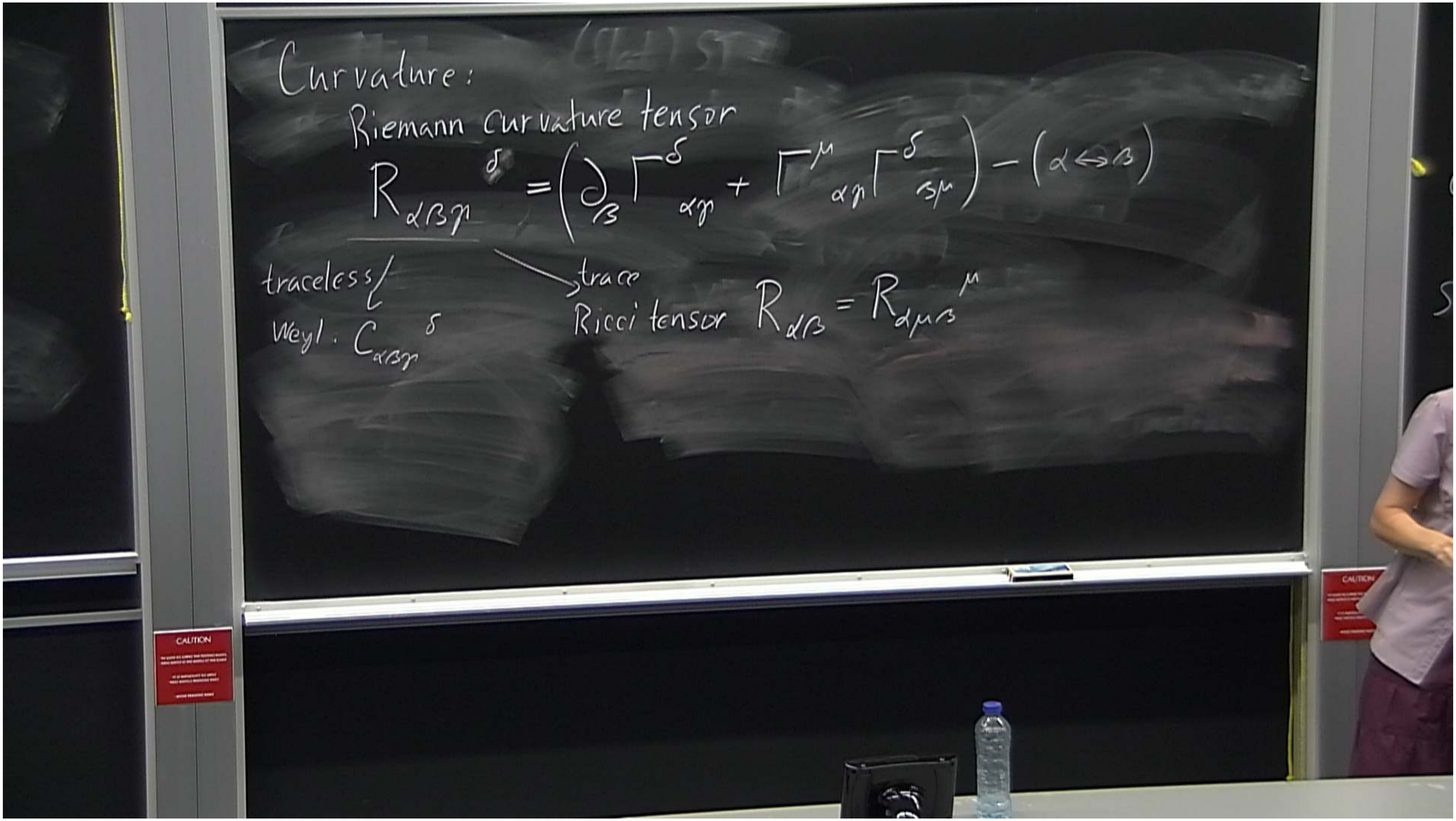

metric: - encodes distances & durations
- implements dot product
"V·W" = g_μν V^μ W^ν
implicitly Σ_μ Σ_ν
V_μ ≡ g_μν V^ν
raise -//- w/ inverse metric g^μν defined via
g_μν g^νρ = δ_μ^ρ
V^μ = g^μν V_ν
g_μν V^μ W^ν = V_ν W^ν = V^μ W_μ = V^α W_α

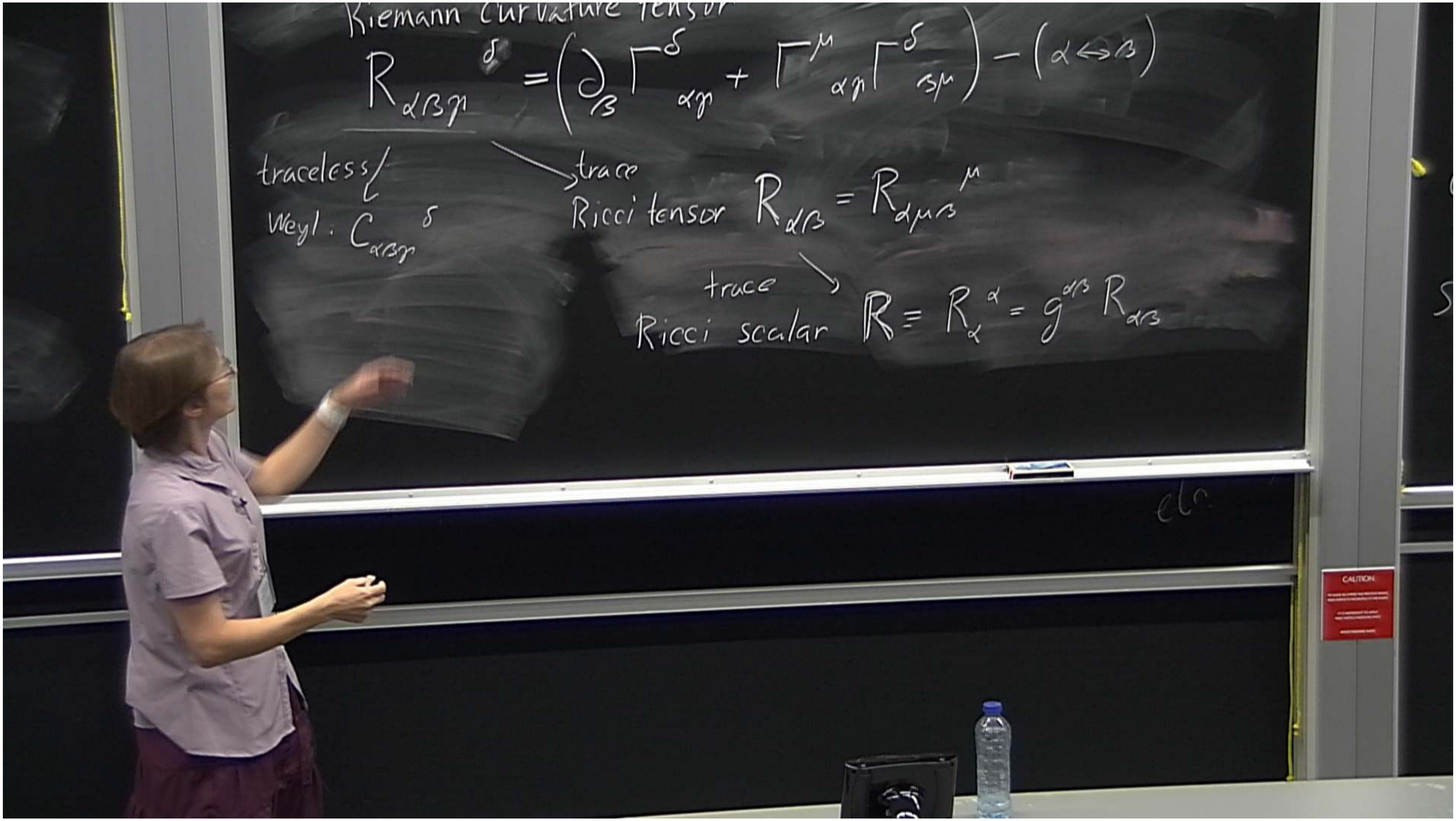


$\delta S = 0 \rightsquigarrow$ geodesic eqn:
$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta = 0$
$\frac{d^2}{d\lambda^2} x^\mu(\lambda)$
↳ Christoffel symbols:
$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}\right)$
$\frac{\partial}{\partial x^\alpha}$
sum $\sum_\gamma$
CAUTION

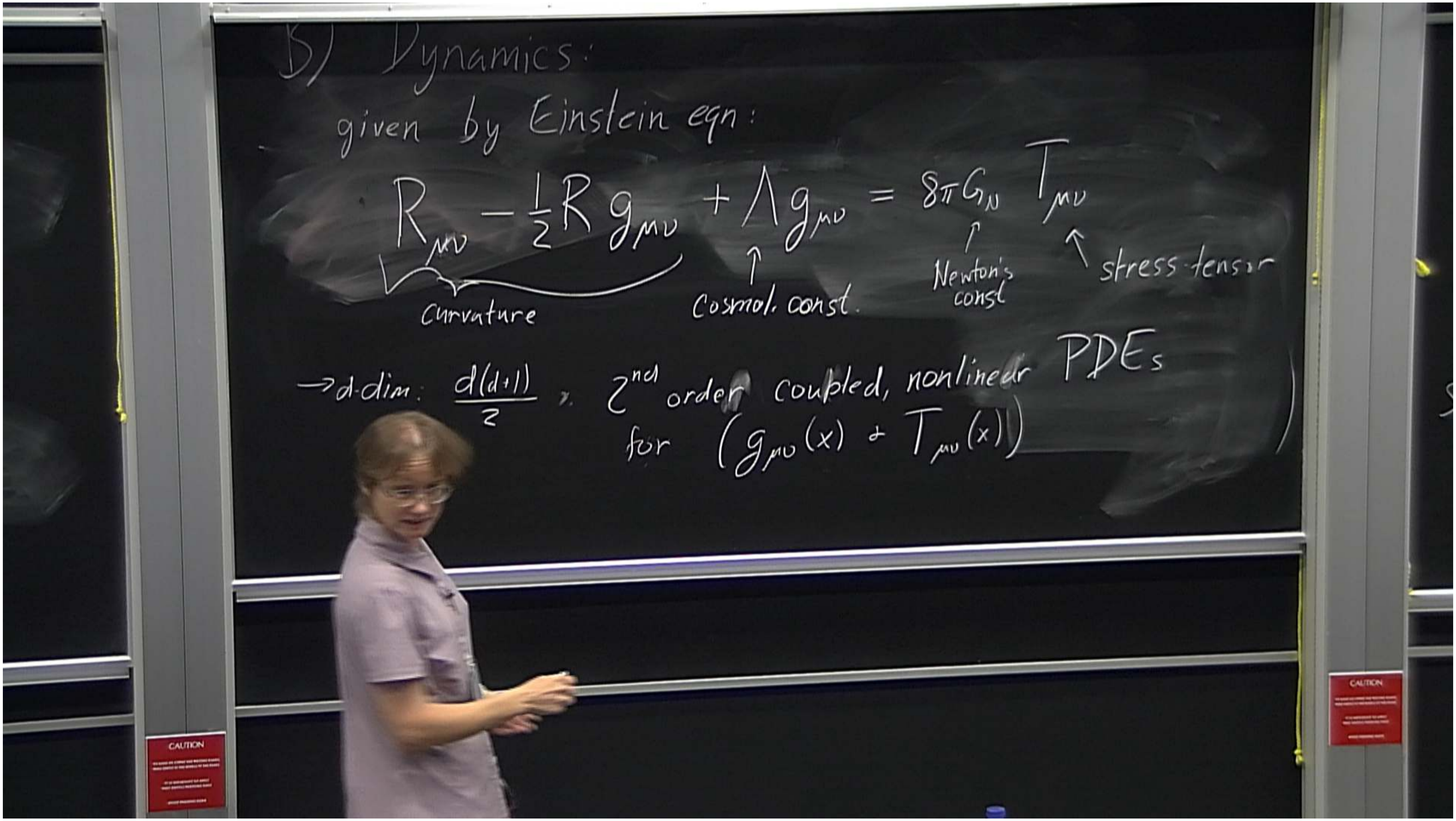

δS = 0 ⇝ geodesic eqn:
$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta = 0$
$\frac{d^2}{d\lambda^2} x^\mu(\lambda)$
↳ Christoffel symbols:
$\Gamma^\mu{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\gamma}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}\right)$
$\frac{\partial}{\partial x^\alpha}$
sum $\sum_\gamma$
x^μ

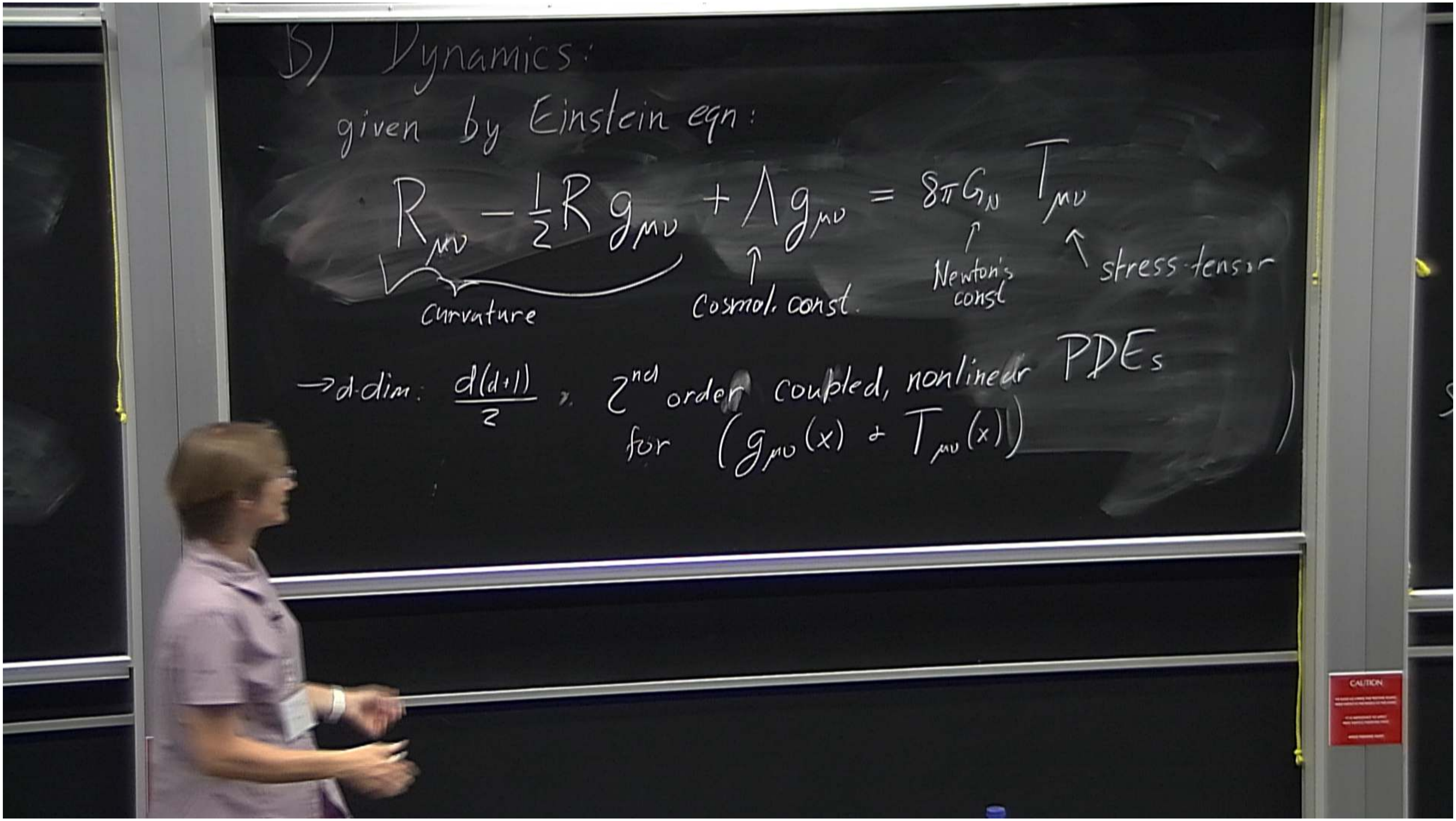

described by line ele
$dx^2 = -dt^2+dx^2+dy^2+dz^2$
Mink $g_{\mu\nu} = \eta_{\mu\nu}$
$x^\mu = (t,x,y,z)$
Causal characterization:
$V_\mu V^\mu > 0$ ⇝ spacelike
$W_\mu W^\mu = 0$ ⇝ $\pm W^\mu$ → future / past -directed null lightlike
$U_\mu U^\mu < 0$ ⇝ $\pm U^\mu$ → -"- timelike
time
$\mathcal{C}\ \{x^\mu(\lambda)\}$
invariant under reparam
& coord. transformations
$\mathcal{L}(\mathcal{C}, g_{\mu\nu})$
GR: diffeomorphism inv. (generally covariant)
→ all physical / geom. quantities are indep. of coords
invariant under boost
cf rotation
$\eta_{\mu\nu}$ is inv. under Lorentz transf. $\Lambda^\mu{}_\nu$
$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$
$\eta = \Lambda^T \eta \Lambda$
SO(3,1)
ex. Coord. transf. in $\mathbb{R}^2$ Cartesian (x,y) → polar (r,φ)
$ds^2 = dx^2+dy^2$
$= dr^2 + r^2 d\varphi^2$
$x = r\cos\varphi$
$y = r\sin\varphi$
Curved ST
$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$
– initially ‖ lines don't stay ‖
– parallel-transported vector along closed curve doesn't return to itself
– curvature

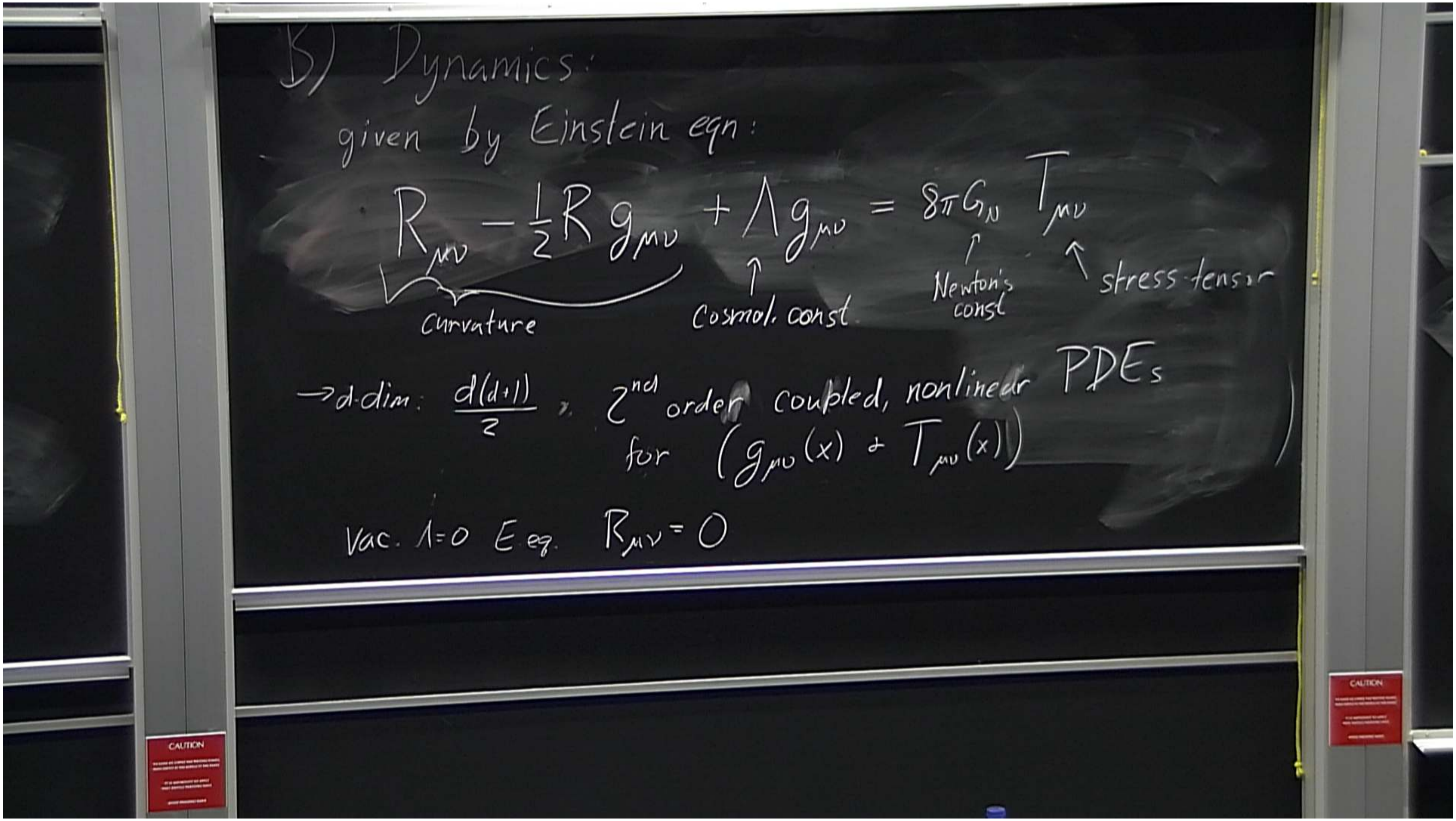

Curvature:
Riemann curvature tensor
$R_{\alpha\beta\gamma}{}^{\delta} = \left(\partial_\beta \Gamma^{\delta}{}_{\alpha\gamma} + \Gamma^{\mu}{}_{\alpha\gamma}\Gamma^{\delta}{}_{\beta\mu}\right) - (\alpha \leftrightarrow \beta)$
traceless
trace
Weyl: $C_{\alpha\beta\gamma}{}^{\delta}$
Ricci tensor $R_{\alpha\beta} = R_{\alpha\mu\beta}{}^{\mu}$
CAUTION
CAUTION

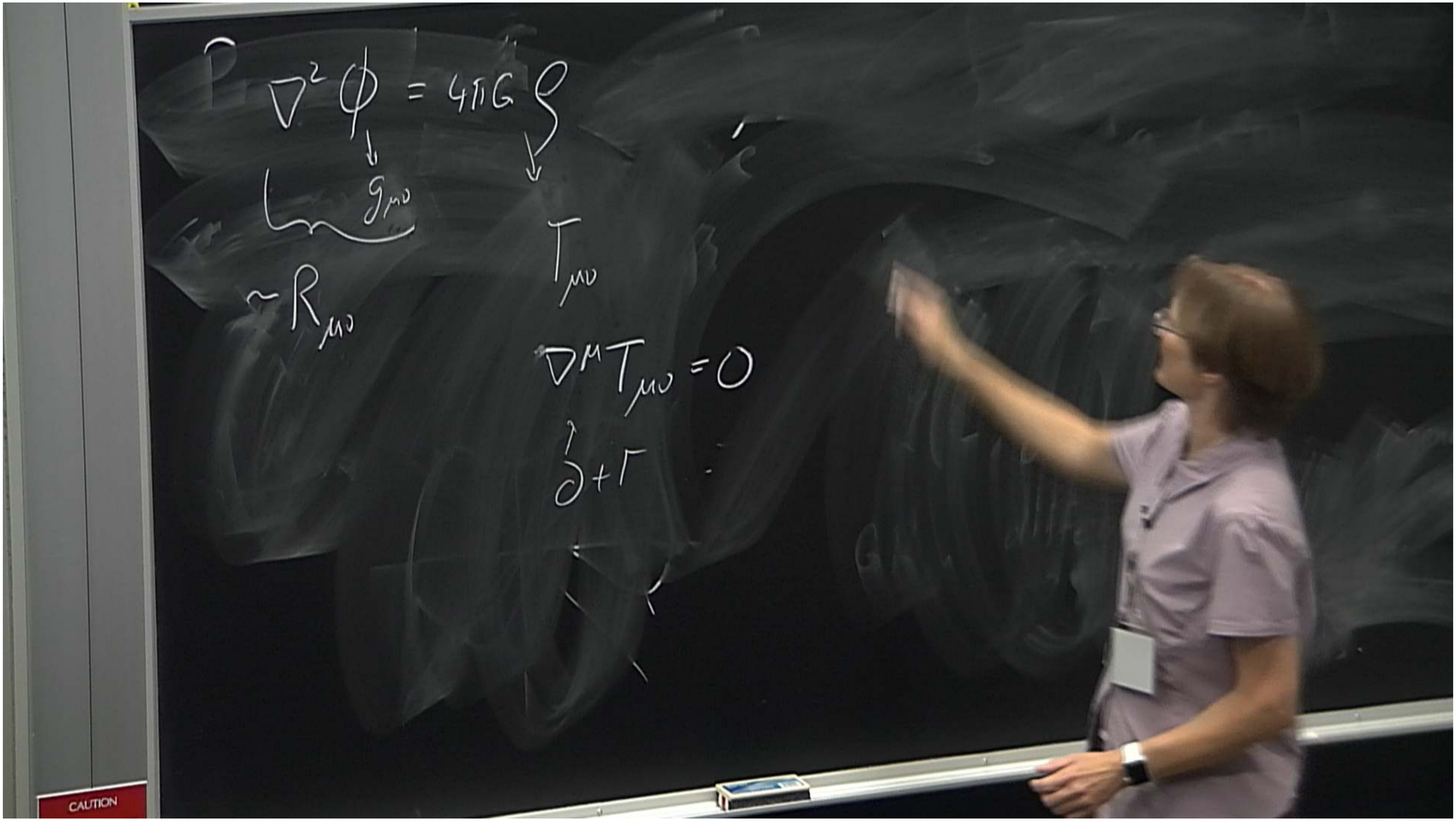

Riemann curvature tensor
$R_{\alpha\beta\gamma}{}^{\delta} = \left(\partial_\beta \Gamma^{\delta}{}_{\alpha\gamma} + \Gamma^{\mu}{}_{\alpha\gamma}\Gamma^{\delta}{}_{\beta\mu}\right) - (\alpha \leftrightarrow \beta)$
traceless
Weyl: $C_{\alpha\beta\gamma}{}^{\delta}$
trace
Ricci tensor $R_{\alpha\beta} = R_{\alpha\mu\beta}{}^{\mu}$
trace
Ricci scalar $R = R_{\alpha}{}^{\alpha} = g^{\alpha\beta} R_{\alpha\beta}$
CAUTION

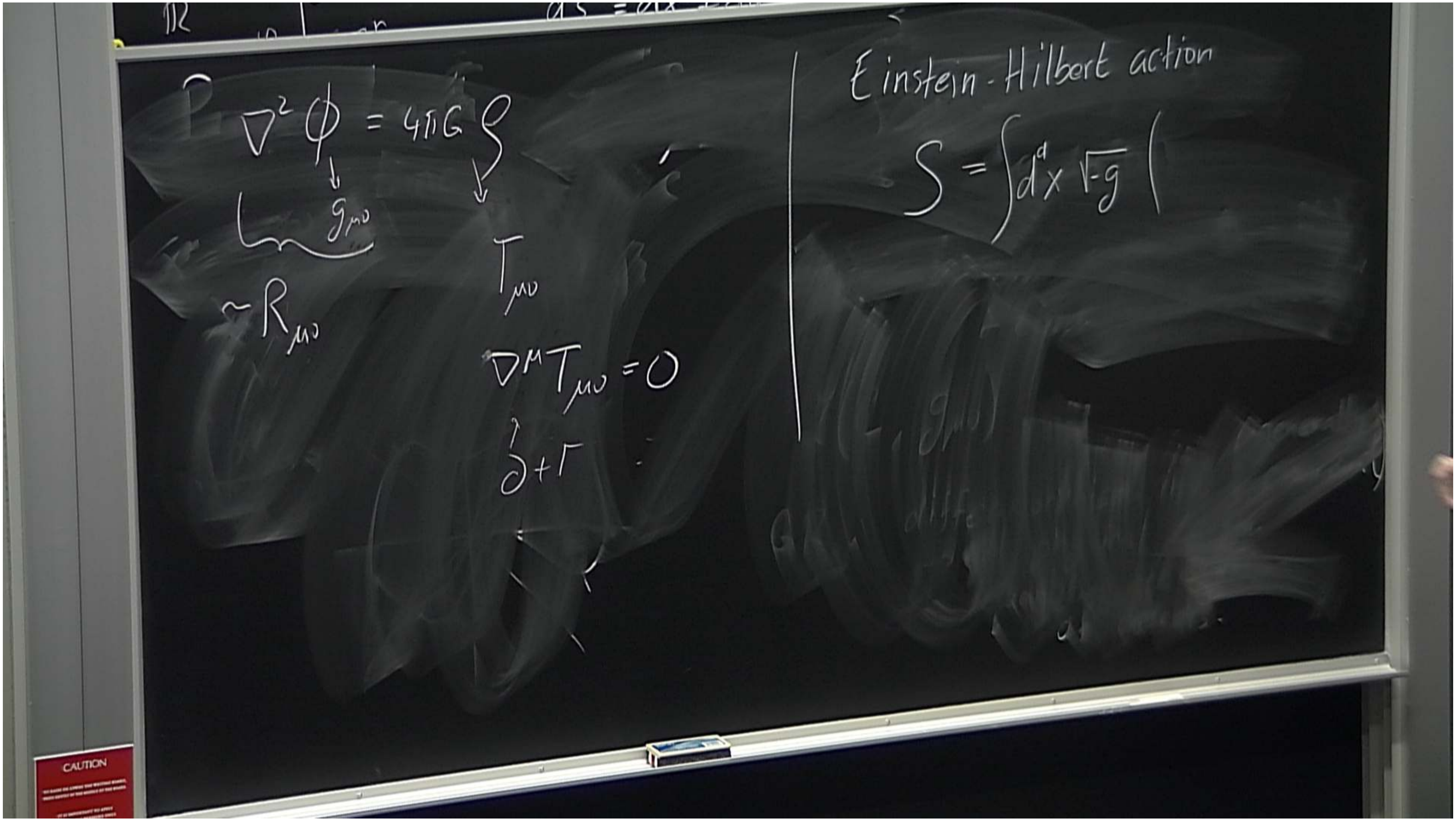

B) Dynamics:
given by Einstein eqn:
$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$
curvature
cosmol. const.
Newton's const
stress tensor

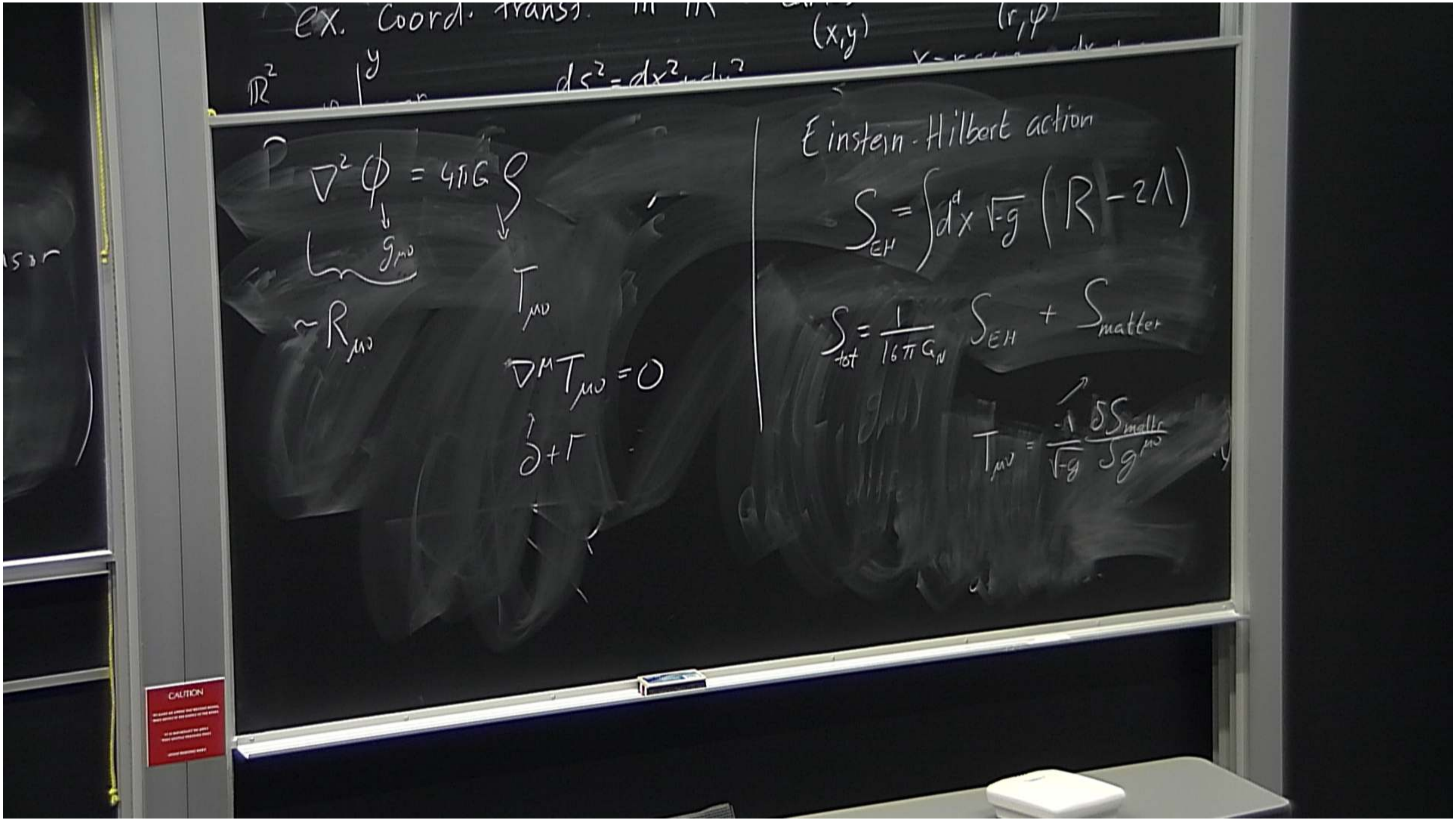

B) Dynamics:
given by Einstein eqn:
$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$
curvature
cosmol. const.
Newton's const
stress-tensor
→ d-dim: $\frac{d(d+1)}{2}$ × 2nd order coupled, nonlinear PDEs
for $(g_{\mu\nu}(x)$ & $T_{\mu\nu}(x))$
CAUTION
CAUTION


B) Dynamics:
given by Einstein eqn:
$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$
curvature
cosmol. const.
Newton's const
stress-tensor
→ d-dim: $\frac{d(d+1)}{2}$ × 2nd order coupled, nonlinear PDEs
for $(g_{\mu\nu}(x)$ + $T_{\mu\nu}(x))$


B) Dynamics:
given by Einstein eqn:
$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$
curvature
cosmol. const
Newton's const
stress-tensor
→ d-dim: $\frac{d(d+1)}{2}$ 2nd order coupled, nonlinear PDEs
for $(g_{\mu\nu}(x)$ & $T_{\mu\nu}(x))$
Vac. $\Lambda = 0$ E eq. $R_{\mu\nu} = 0$


$\nabla^2 \phi = 4\pi G \rho$
$g_{\mu\nu}$
$T_{\mu\nu}$
$\sim R_{\mu\nu}$
$\nabla^\mu T_{\mu\nu} = 0$
$\partial + \Gamma$
CAUTION


Einstein-Hilbert action
S = ∫d⁴x √-g (
∇²φ = 4πG ρ
g_μν
T_μν
~R_μν
∇^μ T_μν = 0
∂ + Γ
CAUTION


ex. Coord. transf.
(x,y)
(r,φ)
$\nabla^2 \phi = 4\pi G \rho$
$g_{\mu\nu}$
$\sim R_{\mu\nu}$
$T_{\mu\nu}$
$\nabla^\mu T_{\mu\nu} = 0$
$\partial + \Gamma$
Einstein-Hilbert action
$S_{EH} = \int d^dx \sqrt{-g} \left(R - 2\Lambda \right)$
$S_{tot} = \frac{1}{16\pi G_N} S_{EH} + S_{matter}$
$T_{\mu\nu} = \frac{-1}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}$