Title: QI Basics

Date: Jul 18, 2016 09:10 AM

URL: http://pirsa.org/16070006

Abstract:

ILITY

ELS

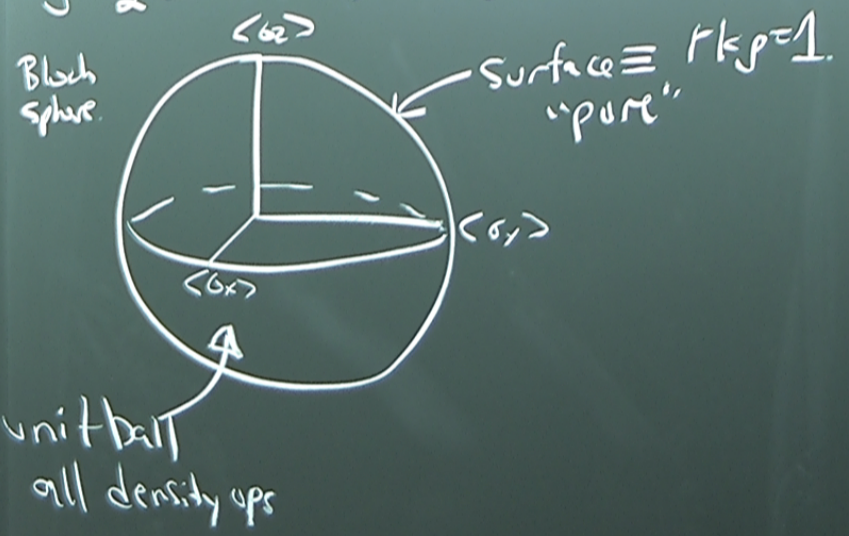
1) $\varphi_A = \mathrm{tr}_B |\varphi\rangle\langle\varphi|_{AB}$

$$\langle O_A \rangle_\varphi = {}_{AB}\langle\varphi| O_A \otimes I_B |\varphi\rangle_{AB}$$

$$= \mathrm{tr}\, O_A \varphi_A$$

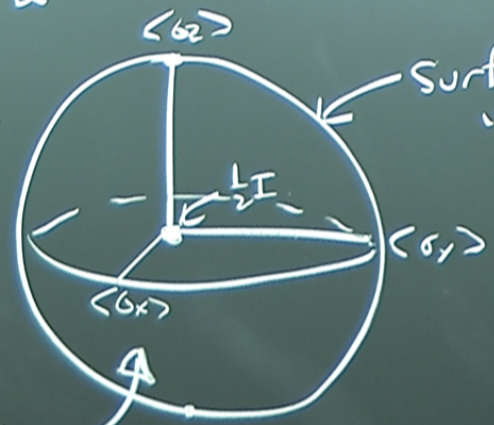
$$\varphi_A \geq 0 \qquad \mathrm{tr}\, \varphi_A = 1$$

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}) \qquad r_j = \mathrm{tr}(\rho\sigma_j)$$



$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}) \qquad r_j = \mathrm{tr}(\rho\,\sigma_j)$$

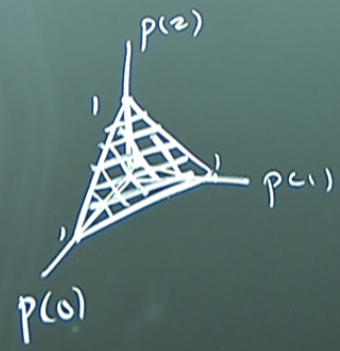
Bloch sphere.



Surface $\equiv$ "pure" $\quad r^2 = \rho^2 = 1.$

unit ball
all density ops

CONTRAST CLASSICAL.



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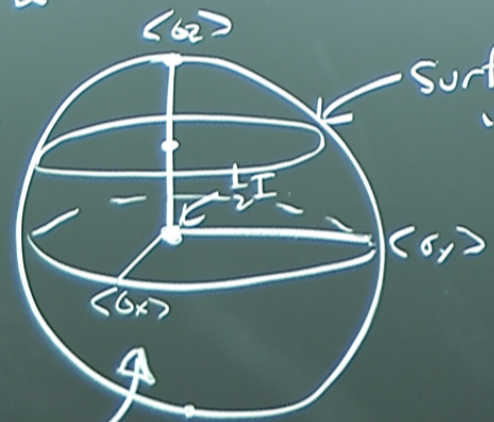

Surface $\equiv$ "pure" $\mathrm{rk}\,\rho = 1$.

unit ball
all density ops

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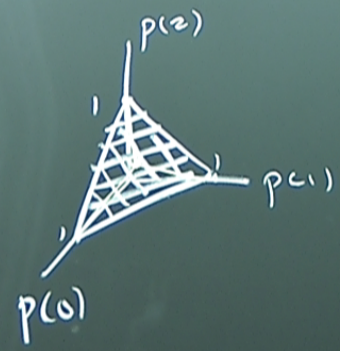
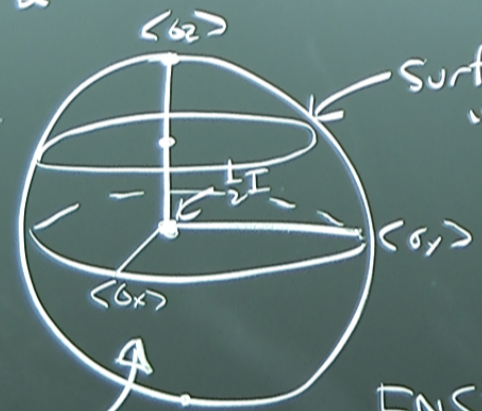


$\otimes I_B |\varphi\rangle_{AB}$

$= 1.$

$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$ $\quad r_j = tr(\rho\sigma_j)$

Bloch sphere

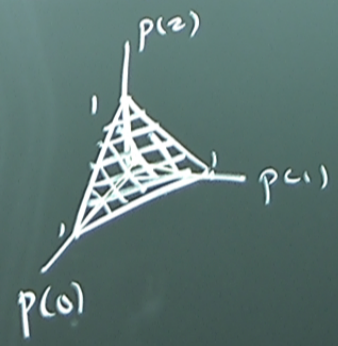


surface $\equiv$ "pure" $rk\,\rho = 1.$

unit ball
all density ops

ENSEMBLE AMBIGUITY.

CONTRAST CLASSICAL.



FIDELITY

Generalize $|\langle\varphi|\psi\rangle|$ to mixed states.

- $0 \leq F(\rho,\sigma) \leq 1$

iff $\rho \perp \sigma$, $\mathrm{tr}\,\rho\sigma = 0$

iff $\rho = \sigma$

- $F(\rho,\sigma)$

$$= \max\left\{ |\langle\varphi|\psi\rangle| ;\ |\varphi\rangle, |\psi\rangle \in A \otimes B,\ \varphi_A = \rho,\ \psi_A = \sigma \right\}$$

$$= \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}.$$

- CONCAVE.

TRACE DISTANCE

FIDELITY

Generalize $|\langle\varphi|\psi\rangle|$ to mixed states.

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$=$ iff $\rho\perp\sigma$, $\mathrm{tr}\,\rho\sigma=0$

$=$ iff $\rho=\sigma$

- $F(\rho,\sigma)$

$$= \max\left\{ |\langle\varphi|\psi\rangle| ;\ |\varphi\rangle, |\psi\rangle \in A\otimes B,\ \varphi_A=\rho,\ \psi_A=\sigma \right\}$$

$$= \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}.$$

- CONCAVE.

TRACE DISTANCE

Recall Any matrix X:

$$X = UDV \quad \text{SVD}$$

U, V — UNITARY

D — $\mathrm{DIAG}\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \geq 0$

1) $\varphi_A =$

$\langle O_A \rangle$

$\varphi_A \geq$

all Any matrix X:

$= UDV$ SVD

U — UNITARY

D — DIAG $\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \geq 0$

- $0 \leq T(\rho, \sigma) \leq 2$

  0: iff $\rho = \sigma$

  2: iff $\rho \perp \sigma$

- CONVEX

- $1 - F \leq \frac{T}{2} \leq \sqrt{1 - F^2}$

all Any matrix X:

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- CONVEX

- $1 - F \leq \frac{T}{2} \leq \sqrt{1 - F^2}$

  NO DIMENSION FACTORS $T \equiv F$

$\mathrm{tr}\sqrt{X^\dagger X}$

$\rho \perp \sigma$

- $T(\rho,\gamma) = \max \mathrm{tr}\, \sigma(\rho-\gamma)$.

$\|\sigma\|_\alpha \le 1$

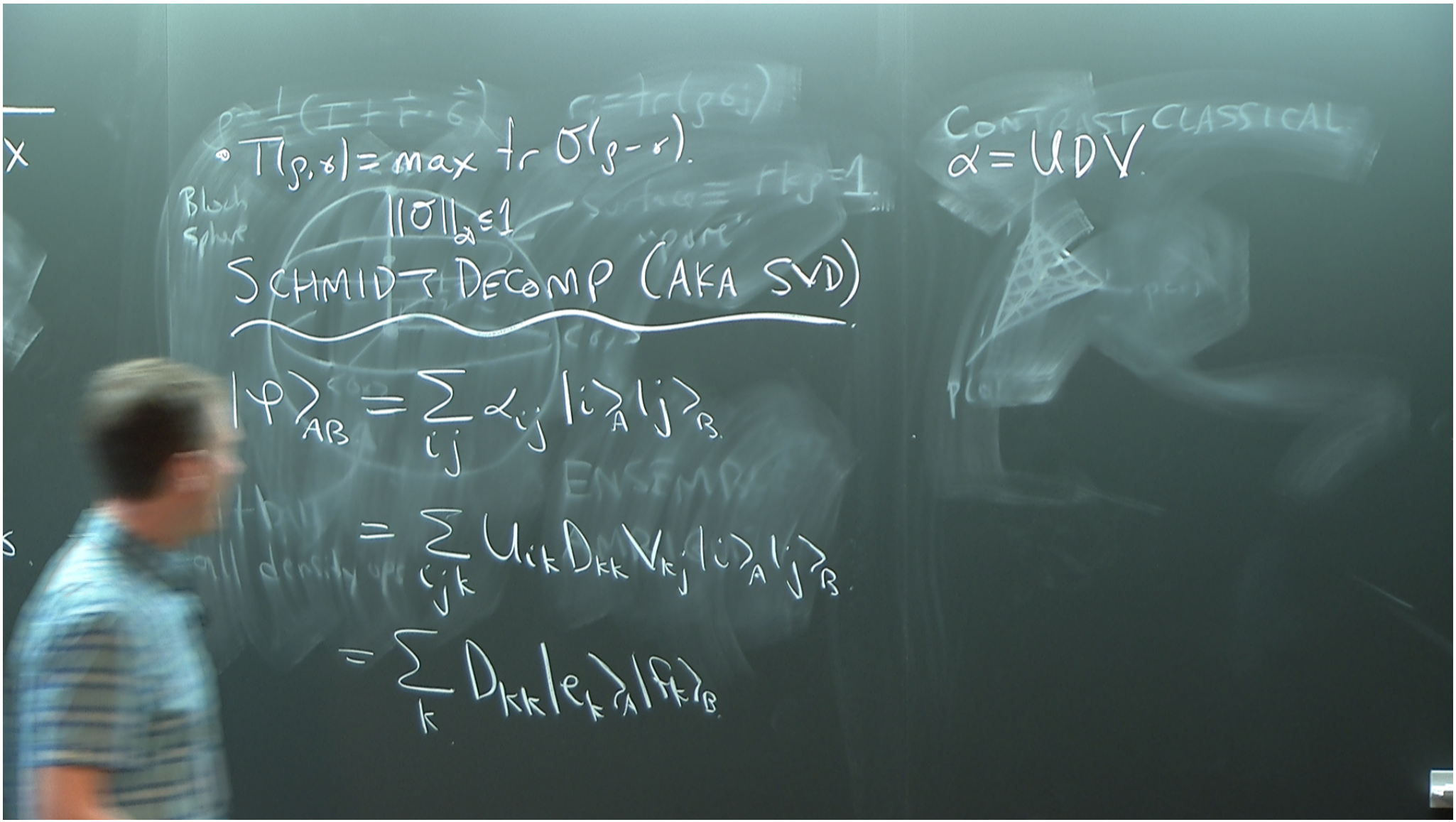
Bloch sphere

SCHMIDT DECOMP (AKA SVD)

all density ops

ENSEMBLE AMBIGUITY

$p(0)$


$T(\rho,\sigma) = \max_{\|\mathcal{O}\|_\alpha \le 1} \mathrm{tr}\, \mathcal{O}(\rho-\sigma)$
SCHMIDT DECOMP (AKA SVD)
$|\varphi\rangle_{AB} = \sum_{ij} \alpha_{ij} |i\rangle_A |j\rangle_B$
$= \sum_{ijk} U_{ik} D_{kk} V_{kj} |i\rangle_A |j\rangle_B$
$= \sum_k D_{kk} |e_k\rangle_A |f_k\rangle_B$
CONTRAST CLASSICAL
$\alpha = UDV$

$$\cdot\ T(\rho,\sigma) = \max_{\|\mathcal{O}\|_\infty \le 1} \operatorname{tr} \mathcal{O}(\rho-\sigma).$$

SCHMIDT DECOMP (AKA SVD)

$$|\varphi\rangle_{AB} = \sum_{ij} \alpha_{ij} |i\rangle_A |j\rangle_B$$

$$= \sum_{ijk} U_{ik} D_{kk} V_{kj} |i\rangle_A |j\rangle_B$$

$$= \sum_k D_{kk} |e_k\rangle_A |f_k\rangle_B$$

$$\Rightarrow \varphi_A = \sum_k D_{kk}^2 |e_k\rangle\langle e_k|_A$$

$$\alpha = UDV$$

$$\langle e_i | e_j \rangle = \delta_{ij} = \langle f_i | f_j \rangle$$

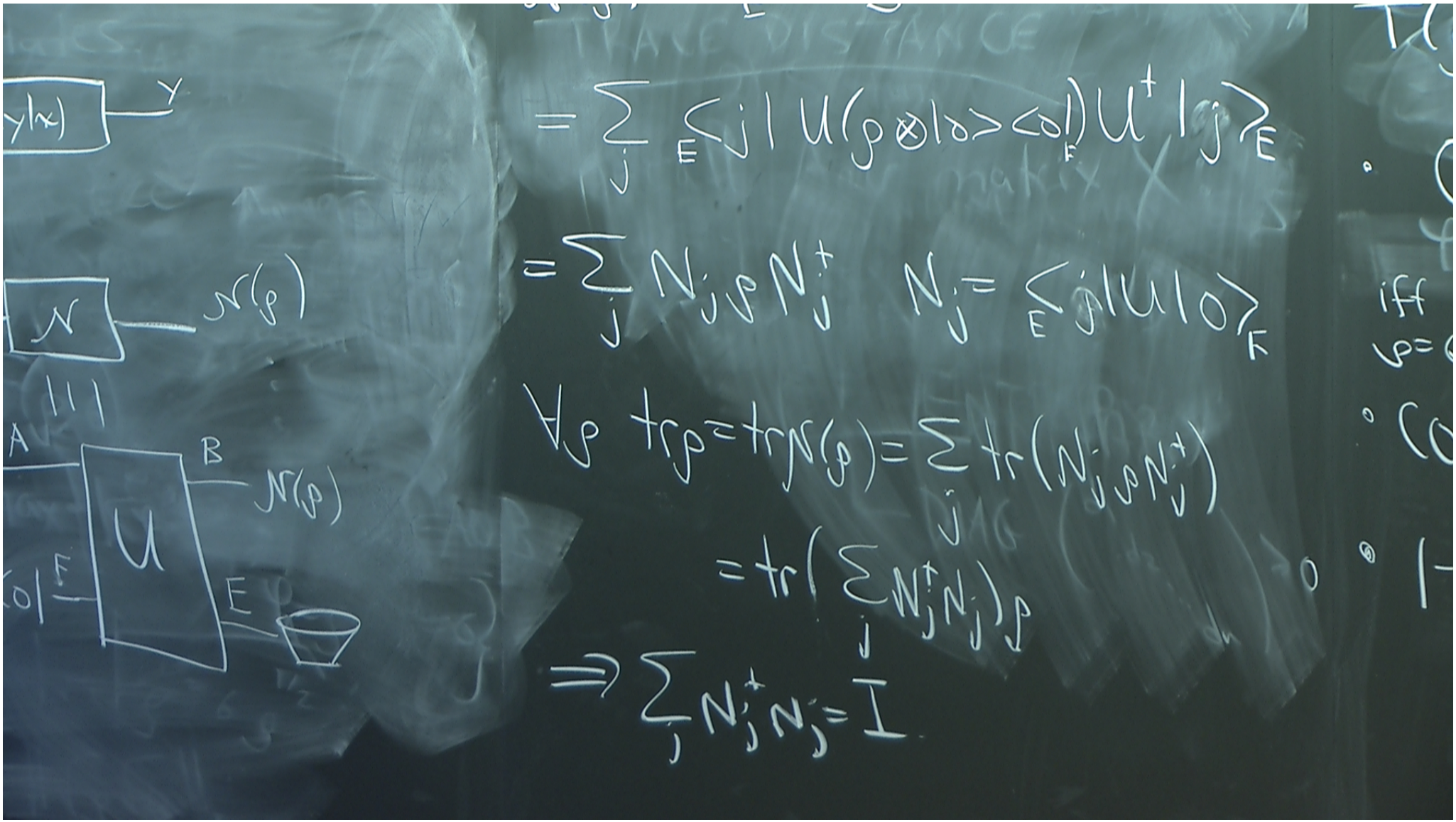
$$|\varphi\rangle_{AB} = \sum_{ij} \alpha_{ij} |i\rangle_A |j\rangle_B$$

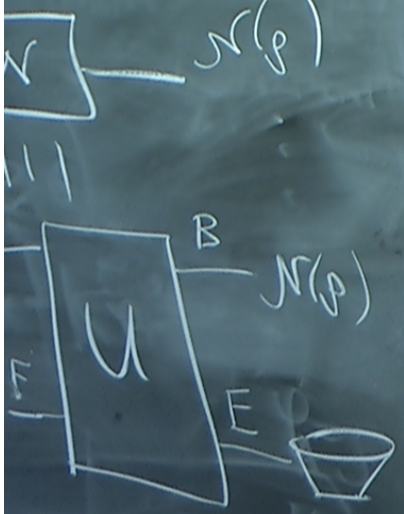
$$= \sum_{ijk} U_{ik} D_{kk} V_{kj} |i\rangle_A |j\rangle_B$$

$$= \sum_k D_{kk} |e_k\rangle_A |f_k\rangle_B$$

$$\Rightarrow \varphi_A = \sum_k D_{kk}^2 |e_k\rangle\langle e_k|_A$$

$$\Rightarrow \mathrm{Eig}_{\neq 0}(\varphi_A) = \mathrm{Eig}_{\neq 0}(\varphi_B)$$


y|x)
Y
$\mathcal{N}$
$\mathcal{N}(\rho)$
A
B
$\mathcal{N}(\rho)$
U
F
E
$= \sum_j {}_E\langle j| U(\rho \otimes |0\rangle\langle 0|_F) U^\dagger |j\rangle_E$
$= \sum_j N_j \rho N_j^\dagger \quad N_j = {}_E\langle j|U|0\rangle_F$
$\forall \rho \ \mathrm{tr}\,\rho = \mathrm{tr}\,\mathcal{N}(\rho) = \sum_j \mathrm{tr}(N_j \rho N_j^\dagger)$
$= \mathrm{tr}\left(\sum_j N_j^\dagger N_j\right)\rho$
$\Rightarrow \sum_j N_j^\dagger N_j = I$
iff



$$= \sum_j N_j \rho N_j^\dagger \qquad N_j = {}_E\langle j | U | 0 \rangle_F$$

$$\forall \rho \quad \mathrm{tr}\,\rho = \mathrm{tr}\,\mathcal{N}(\rho) = \sum_j \mathrm{tr}\left(N_j \rho N_j^\dagger\right)$$

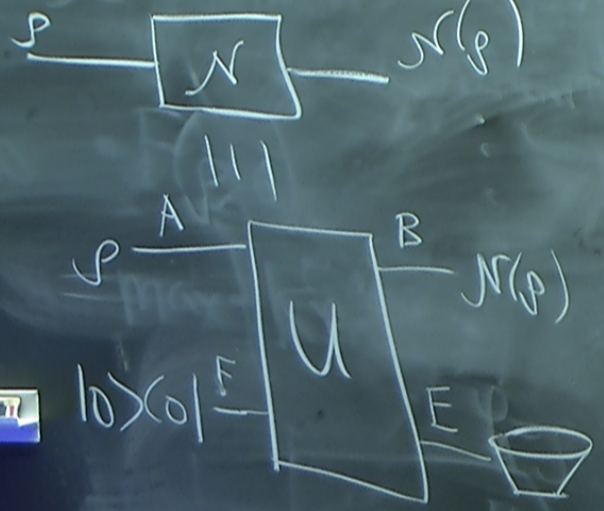
$$= \mathrm{tr}\left(\sum_j N_j^\dagger N_j\right)\rho$$

$$\Rightarrow \sum_j N_j^\dagger N_j = I$$

ALT DEFN OF Q. CHANNEL

iff $\rho = \sigma$.

- CONVEX
- $1 - F \leq \frac{T}{2}$



$$= \sum_j N_j \rho N_j^\dagger \qquad N_j = {}_E\langle j|U|0\rangle_E$$

$$\forall \rho \quad \mathrm{tr}\,\rho = \mathrm{tr}\,\mathcal{N}(\rho) = \sum_j \mathrm{tr}\left(N_j \rho N_j^\dagger\right)$$

$$= \mathrm{tr}\left(\sum_j N_j^\dagger N_j\right)\rho$$

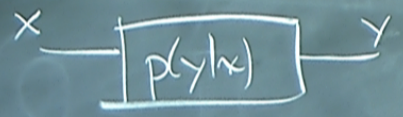
$$\Rightarrow \sum_j N_j^\dagger N_j = I$$

ALT DEFN OF Q. CHANNEL

$\equiv$ CPTP

CHANNELS

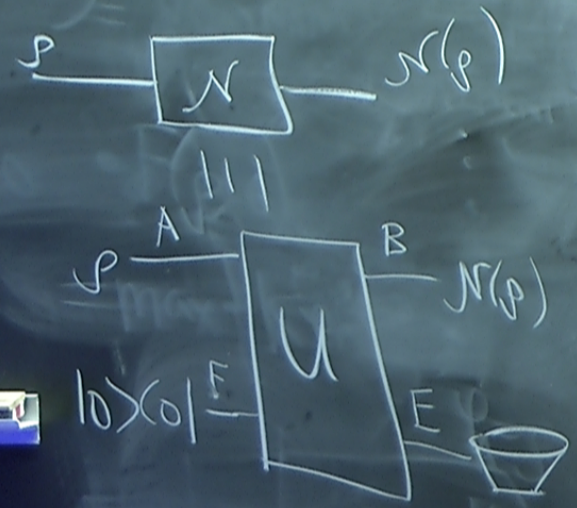
CLASSICAL.



$p(y|x) \geq 0$

$\sum_y p(y|x) = 1.$

QUANTUM



MONOTONICITY

- $F(\rho,\sigma) \leq F(\mathcal{N}(\rho), \mathcal{N}(\sigma))$
- $T(\rho,\sigma) \geq T(\mathcal{N}(\rho), \mathcal{N}(\sigma))$

$N(\rho), N(\sigma))$

$N(\rho), N(\sigma))$

NTUM INFO"?

END SOME ENTANG.

$(N \otimes I_R)(\Phi)) > 1-\epsilon.$

$$\mathcal{D}(\rho_R) = \sum_j |j\rangle\langle j| \rho_R |j\rangle\langle j|$$

$$F(\Phi_{AR}, (N \otimes \mathcal{D})(\Phi))$$

$$\geq F(\Phi_{AR}, (N \otimes I_R)(\Phi)) \quad \text{(MONO)}$$

$$\geq 1-\epsilon.$$

Write $|\Phi\rangle_{AR} = \sum_j \sqrt{p_j}\, |\varphi_j\rangle_A |j\rangle_B$

$$\mathcal{D}(\rho_R) = \sum_j |j\rangle\langle j|\rho_R|j\rangle\langle j|$$

$$F(\Phi_{AR}, (\mathcal{N}\otimes\mathcal{D})(\Phi))$$

$$\geq F(\Phi_{AR}, (\mathcal{N}\otimes I_R)(\Phi)) \quad \text{(MONO)}$$

$$\geq 1-\epsilon.$$

Write $|\Phi\rangle_{AR} = \sum_j \sqrt{p_j}\,|\varphi_j\rangle_A |j\rangle_R$

not $\perp$ in general

$$(\mathcal{N}\otimes\mathcal{D})(\Phi_{AR}) = \sum_j p_j\, \mathcal{N}(\varphi_j)\otimes|j\rangle\langle j|_R.$$

ASIDE: $F(|\xi\rangle, \rho) = \sqrt{\langle\xi|\rho|\xi\rangle}$

$$\mathcal{D}(\rho_R) = \sum_j |j\rangle\langle j|\rho_R|j\rangle\langle j|$$

$$F(\Phi_{AR}, (\mathcal{N}\otimes\mathcal{D})(\Phi))$$

$$\geq F(\Phi_{AR}, (\mathcal{N}\otimes I_R)(\Phi)) \quad \text{(MONO)}$$

$$\geq 1-\epsilon$$

Write $|\Phi\rangle_{AR} = \sum_j \sqrt{p_j}\,|\varphi_j\rangle_A |j\rangle_R$

not $\perp$ in general

$$(\mathcal{N}\otimes\mathcal{D})(\Phi_{AR}) = \sum_j p_j\, \mathcal{N}(\varphi_j)\otimes |j\rangle\langle j|_R$$

ASIDE: $F(|\xi\rangle, \rho) = \sqrt{\langle\xi|\rho|\xi\rangle}$

$$(1-\epsilon)^2 \leq F(\Phi, (\mathcal{N}\otimes\mathcal{D})(\Phi))^2$$

$$= \sum_j p_j \langle\varphi_j|\mathcal{N}(\varphi_j)|\varphi_j\rangle$$

$$\mathcal{D}(\rho_R) = \sum_j |j\rangle\langle j|\, \rho_R\, |j\rangle\langle j|$$

$$F(\Phi_{AR}, (\mathcal{N}\otimes\mathcal{D})(\Phi))$$

$$\geq F(\Phi_{AR}, (\mathcal{N}\otimes I_R)(\Phi)) \quad \text{(MONO)}$$

$$\geq 1-\epsilon$$

Write $|\Phi\rangle_{AR} = \sum_j \sqrt{p_j}\, |\varphi_j\rangle_A |j\rangle_R$

not $\perp$ in general

$$(\mathcal{N}\otimes\mathcal{D})(\Phi_{AR}) = \sum_j p_j\, \mathcal{N}(\varphi_j)\otimes |j\rangle\langle j|_R$$

ASIDE: $F(|\xi\rangle, \rho) = \sqrt{\langle\xi|\rho|\xi\rangle}$

$$(1-\epsilon)^2 \leq F(\Phi, (\mathcal{N}\otimes\mathcal{D})(\Phi))^2$$

$$= \sum_j \underset{\leq 1}{p_j^2}\, \underbrace{\langle\varphi_j|\mathcal{N}(\varphi_j)|\varphi_j\rangle}_{\leq 1}$$

$$\leq \sum_j p_j\, F(\varphi_j, \mathcal{N}(\varphi_j))$$

$$(\mathcal{N}\otimes\mathcal{D})(\Phi_{AR}) = \sum_j p_j\, \mathcal{N}(\varphi_j)\otimes|j\rangle\langle j|_R$$

ASIDE: $F(|\xi\rangle, \rho) = \sqrt{\langle\xi|\rho|\xi\rangle}$

$$1-2\epsilon \le (1-\epsilon)^2 \le F(\Phi, (\mathcal{N}\otimes\mathcal{D})(\Phi))^2$$

(MONO)

$$= \sum_j p_j^2 \underbrace{\langle\varphi_j|\mathcal{N}(\varphi_j)|\varphi_j\rangle}_{\le 1}$$

$$\le \sum_j p_j F(\varphi_j, \mathcal{N}(\varphi_j))$$

CONTRAST CLASSICAL

GIVEN $F(|\Phi\rangle_{AR}, (\mathcal{N}\otimes I)(\Phi_{AR})) > 1-\epsilon$

$\forall$ decomp $\Phi_A = \sum_j q_j |\xi_j\rangle\langle\xi_j|$

$$\sum_j q_j F(\xi_j, \mathcal{N}(\xi_j)) > 1-2\epsilon$$

VN ENTROPY

$$S(\rho) = -\mathrm{tr}\,\rho \log \rho \qquad S(A)_\rho = S(\rho_A)$$

$$0 \leq S(\rho) \leq \log \dim \mathcal{H}$$

$= \text{iff } \mathrm{rk}\,\rho = 1$

$= \text{iff } \rho = \frac{I}{\dim \mathcal{H}}$

CONCAVITY

$$S\left(\sum_j p_j \sigma_j\right) \geq \sum_j p_j S(\sigma_j)$$

RELATIVE ENTROPY

$$S(\rho \| \sigma) = \mathrm{tr}\,\rho \log \rho - \mathrm{tr}\,\rho \log \sigma$$

$$0 \leq S(\rho \| \sigma)$$

$= \text{iff } \rho = \sigma$

$$S(\rho) = -\mathrm{tr}\,\rho \log \rho \qquad S(A)_\rho = S(\rho_A)$$

$$0 \leq S(\rho) \leq \log \dim \mathcal{H}$$

$=$ iff $\mathrm{rk}\,\rho = 1$

$=$ iff $\rho = \frac{I}{\dim \mathcal{H}}$

$$S\left(\sum_j p_j \sigma_j\right) \geq \sum_j p_j S(\sigma_j)$$

RELATIVE ENTROPY

$$S(\rho \| \sigma) = \mathrm{tr}\,\rho \log \rho - \mathrm{tr}\,\rho \log \sigma$$

$$0 \leq S(\rho \| \sigma) \leq \infty$$

$=$ iff $\rho = \sigma$

if $\mathrm{Supp}(\rho) \not\subseteq \mathrm{Supp}(\sigma)$

$$S(\rho \| \sigma) \geq S(\mathcal{N}(\rho) \| \mathcal{N}(\sigma))$$

$p_j S(\sigma_j)$

Y

$\mathrm{tr}\,\rho \log \sigma$

$\mathrm{Supp}(\rho) \not\subseteq \mathrm{Supp}(\sigma)$

$N(\sigma)$

RE IS EVERYWHERE!

- $\sigma_\beta = \dfrac{e^{-\beta H}}{Z}$

$$S(\rho \| \sigma_\beta) = \mathrm{tr}\,\rho \log \rho - \mathrm{tr}\,\rho \log \frac{e^{-\beta H}}{Z}$$

$$= -S(\rho) + \beta \langle H \rangle_\rho + \log Z$$

$$= \beta\big(F(\rho) - F(\sigma_\beta) \big)$$

$F = U - TS$

$(\mathcal{N} \otimes \mathcal{D})(\Phi_{AR}) = \sum_j p$

ASIDE: $F(|\xi\rangle, \rho) =$

$(1-\epsilon)^2 \leq F(\Phi, (\mathcal{N}$

$$= \sum_j p_j^2 \langle \varphi_j$$

$$\leq 1$$

$$\leq \sum_j p_j F(\varphi_j$$

INFORMATION.

$= S(\rho_{AB} \| \rho_A \otimes \rho_B)$

$)_\rho + S(B)_\rho - S(AB)_\rho$

$I(A' : B')_{(\mathcal{N}_1 \otimes \mathcal{N}_2)(\rho)}$

SPECIAL CASE:

$I(A : BC)_\rho \geq I(A : B)_\rho$

$S(AB)_\rho + S(BC)_\rho \geq S(ABC)_\rho + S(B)_\rho$

$\underline{\underline{SSA}}$ $\quad S(AB) \geq S(A)$