Title: How to Characterize the Quantum Correlations of a Generic Causal Structure

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Abstract: <p>The ideas of no-signalling, nonlocality, Bell inequalities, and quantum correlations can all be understood as implications of a presumed causal structure. In particular, the causal structure of the Bell scenario implies the Bell inequalities whenever the shared resource is presumed to act like a classical hidden random variable. If the shared resource in the scenario is a quantum system, however, then the quantum causal structure can give rise to a larger set of correlations, including probability distributions which violate Bell inequalities up to Tsirelson's bound. It is hard to generically distinguish between the classical and quantum correlations, though, because the standard method for computing Bell inequalities cannot be generalized to general causal scenarios. We therefore introduce a method (the "Inflation DAG" technique) to heuristically constrain the set of correlations compatible with a given classical causal structure, and we demonstrate how it may be used to derive explicit inequalities in terms of probabilities. We also discuss deriving physics-independent constraints, i.e. non-trivial inequalities which are nevertheless satisfied even by all quantum correlations, thereby quantifying some absolute limits as to what the universe allows.</p>

<p>[Unpublished results of E.W., Rob Spekkens, Tobias Fritz]</p>

Causal Structures
&
Classical vs. Quantum Correlations

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June 16 2016

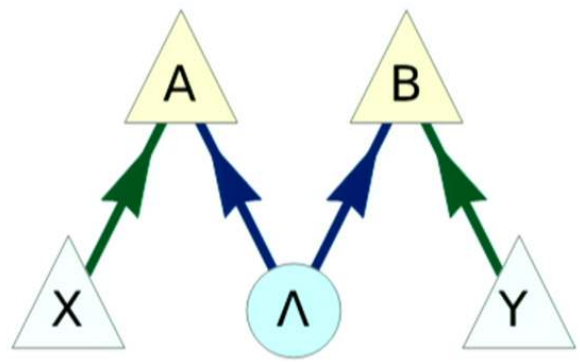
Nonlocality = Causal Structure Compatibility

Bell's Theorem states that quantum correlations contradict a set of reasonable assumptions (local realism)

- **Locality / No-Signalling:** Alice's choice of measurement setting cannot influence Bob's measurement outcome
- **Realism / Ontic State:** The state shared by Alice and Bob should be described as a hidden probability distribution
- **Measurement independence / Free Will:** The choice of the parties' measurement settings are not related to the shared resource

Bell "local"

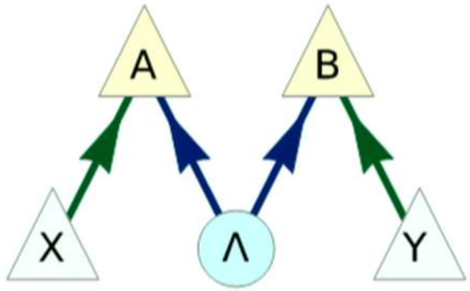
compatible with



See also *The lesson of causal discovery algorithms for quantum correlations: causal explanations of Bell-inequality violations require fine-tuning* by Wood & Spekkens NJP 2015, arXiv:1208.4119.

Structure → Inequalities → Distributions

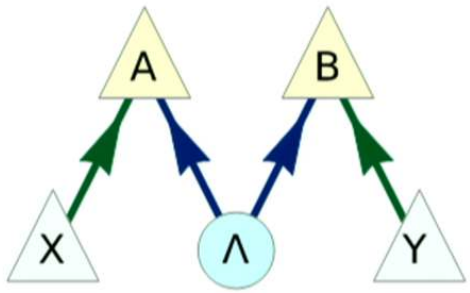
Bipartite Bell Scenario



Nonlocality = violating causal infeasibility criteria, see Wood & Spekkens NJP 2015, arXiv:1208.4119.

Structure → Inequalities → Distributions

Bipartite Bell Scenario

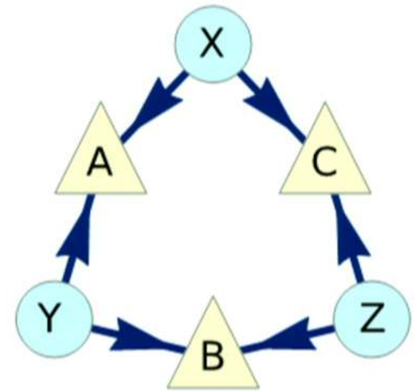


$$\langle A_0B_0\rangle + \langle A_1B_0\rangle + \langle A_0B_1\rangle - \langle A_1B_1\rangle \le 2$$

$$p(ab \mid xy) \le p(a\bar{b} \mid x\bar{y}) + p(\bar{a}b \mid \bar{x}y) + p(ab \mid \overline{xy})$$

PR-box $\quad p(ab \mid xy) = \dfrac{\delta(a \oplus b = x \cdot y)}{2}$

Triangle Scenario

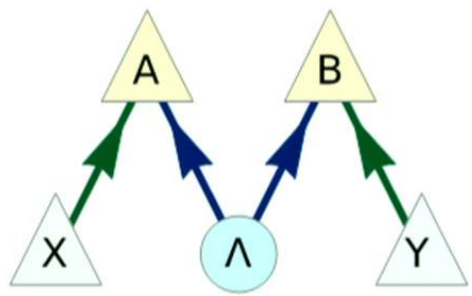


$$p(a)p(b)p(c) \le p(\bar{a}\bar{b}\bar{c}) + p(c)p(ab) + p(b)p(ac) + p(a)p(bc)$$

Nonlocality = violating causal infeasibility criteria, see Wood & Spekkens NJP 2015, arXiv:1208.4119.

Structure → Inequalities → Distributions

Bipartite Bell Scenario

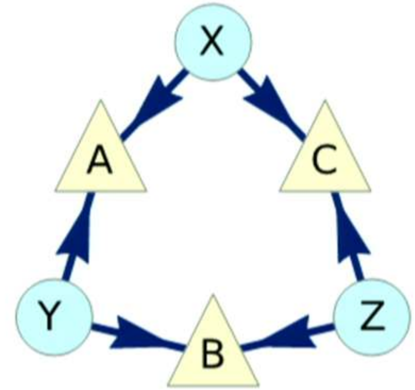


$$\langle A_0B_0\rangle+\langle A_1B_0\rangle+\langle A_0B_1\rangle-\langle A_1B_1\rangle\leq 2$$

$$p(ab\mid xy)\leq p(a\bar{b}\mid x\bar{y})+p(\bar{a}b\mid\bar{x}y)+p(ab\mid\bar{x}\bar{y})$$

PR-box $\quad p(ab\mid xy)=\frac{\delta(a\oplus b=x\cdot y)}{2}$

Triangle Scenario



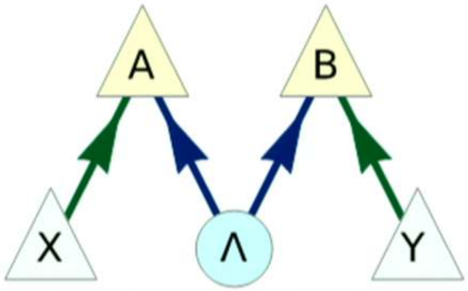
$$p(a)p(b)p(c)\leq p(\bar{a}\bar{b}\bar{c})+p(c)p(ab)+p(b)p(ac)+p(a)p(bc)$$

W-distribution $\quad p(abc)=\frac{[100]+[010]+[001]}{3}$

Nonlocality = violating causal infeasibility criteria, see Wood & Spekkens NJP 2015, arXiv:1208.4119.

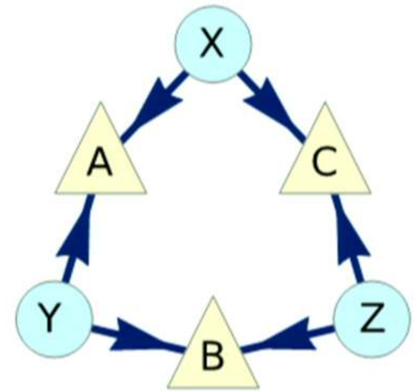
Conditional Independence Relations

Bipartite Bell Scenario



$(X \perp Y)$ $(A \perp Y | B)$ $(B \perp X | A)$

Triangle Scenario

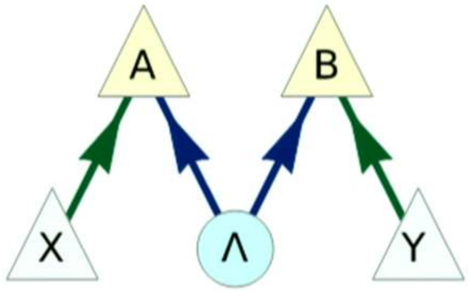


- There are no observable conditional independence relations in the Triangle Scenario.

- All C.I. relations are **satisfied by all quantum distributions** as well.

CI useless for distinguishing quantum from classical: J. Henson, R. Lal, & M. Pusey NJP 2014, arXiv:1405.2572

Bell Scenarios "Easy" (Convex, Polytope)



$$p(ab\,|\,xy) \leq \int p(a\,|\,x\lambda)\,p(b\,|\,y\lambda)\,p(\lambda)\,\mathrm{d}\lambda$$

- Any combination of product distributions is itself Bell scenario compatible
- There are 16 **extremal** (deterministic) product distributions
- The **convex span** of those 16 distributions is the compatible set

Bell Type Scenarios	Generic Causal Scenarios
Compatible distributions = **convex polytope**	Compatible distributions = **nonconvex** set
Easy to define & **linear inequalities** suffice	Polynomial **inequalities** required

For example, see *Polynomial Bell Inequalities* by R. Chaves PRL 2016, arXiv:1506.04325.

The Inflation DAG Method

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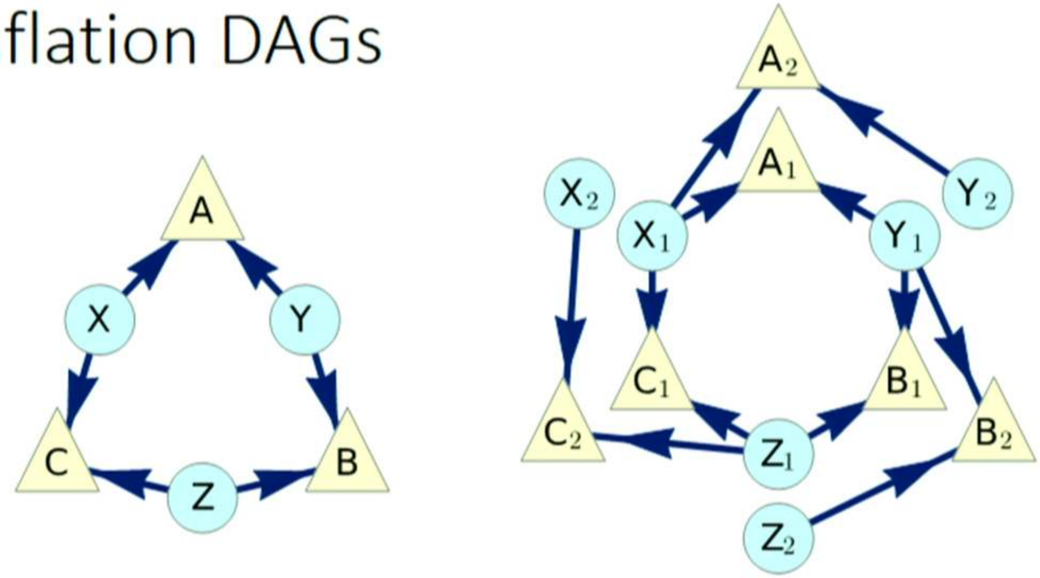
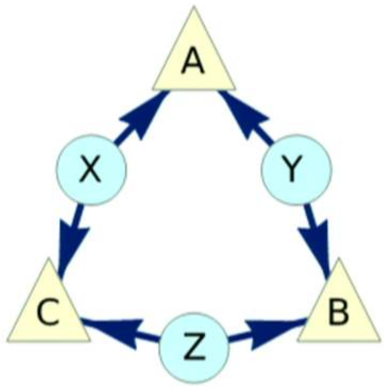
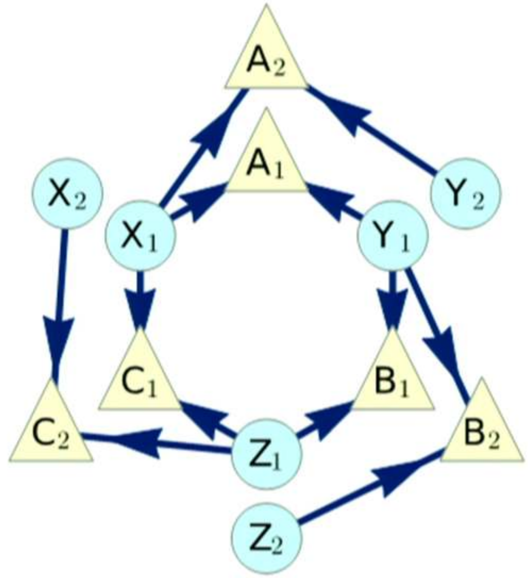
Inflation DAGs



Every variable in the inflation DAG has the **same parents** as in the original DAG (up to dummy indices)

Inflation DAGs



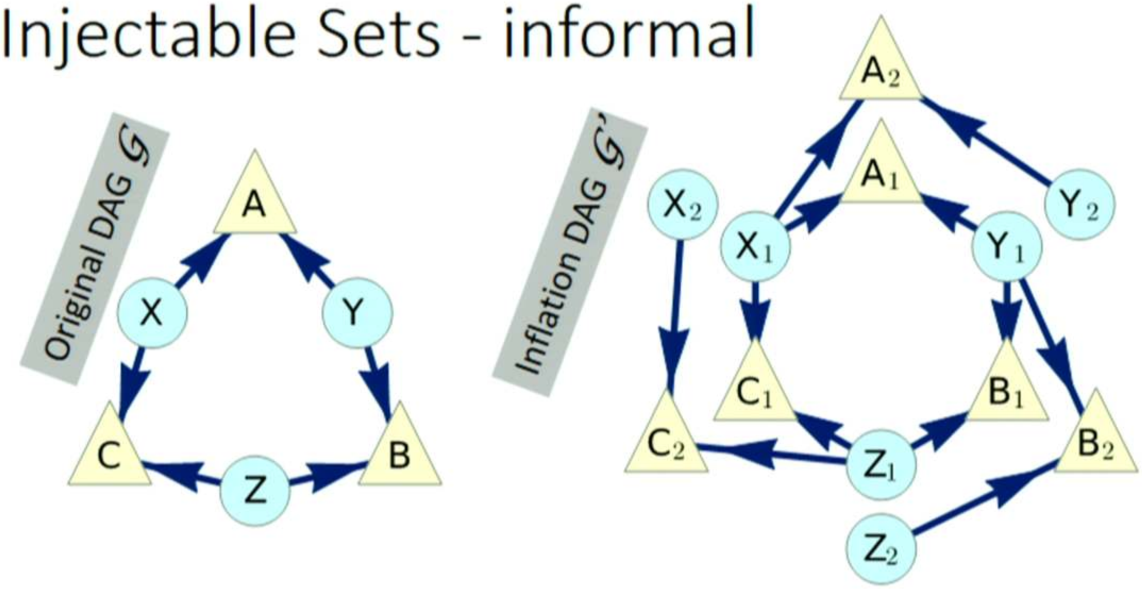


Every variable in the inflation DAG has the **same parents** as in the original DAG (up to dummy indices)

Therefore every variable in the inflation DAG can have the **same distribution** as in the original DAG

$$P(A_1) = P(A_2) = P(A) \qquad P(B_1) = P(B_2) = P(B) \qquad \text{etc.}$$

Injectable Sets - informal



$$P_{A_1}(a) = P_A(a) \qquad P_{A_2}(a) = P_A(a) \qquad P_{B_1}(b) = P_B(b) \quad \ldots$$

$$P_{A_1B_2}(ab) = P_{AB}(ab) \qquad P_{A_1B_1C_1}(abc) = P_{ABC}(abc) \quad \ldots$$

$$P_{A_1B_1C_2}(abc) \neq P_{ABC}(abc)$$

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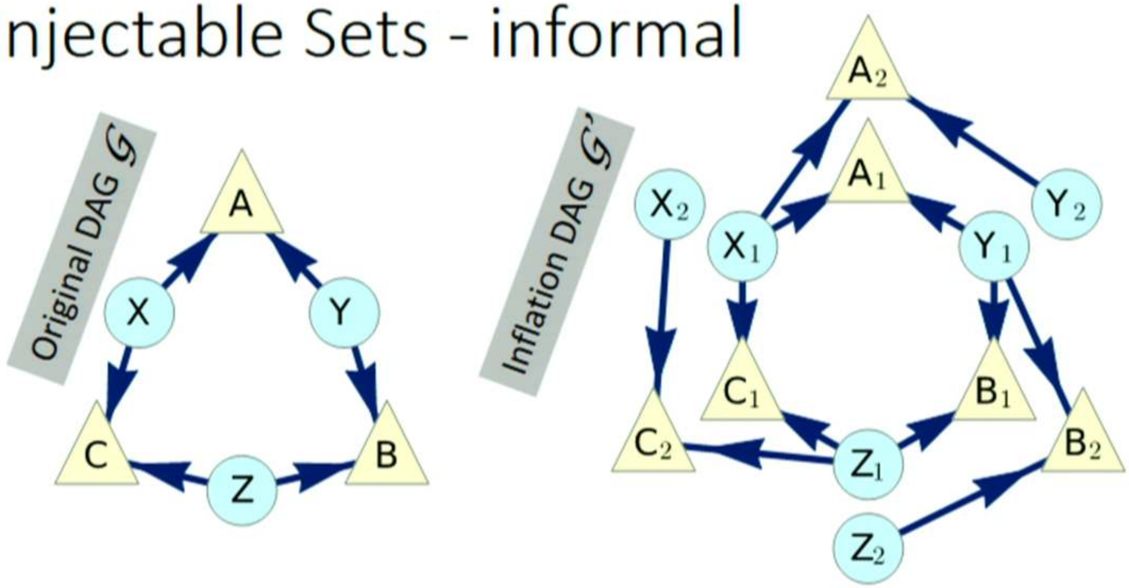
Injectable Sets - informal



$$P_{A_1}(a) = P_A(a) \qquad P_{A_2}(a) = P_A(a) \qquad P_{B_1}(b) = P_B(b) \ \ldots$$

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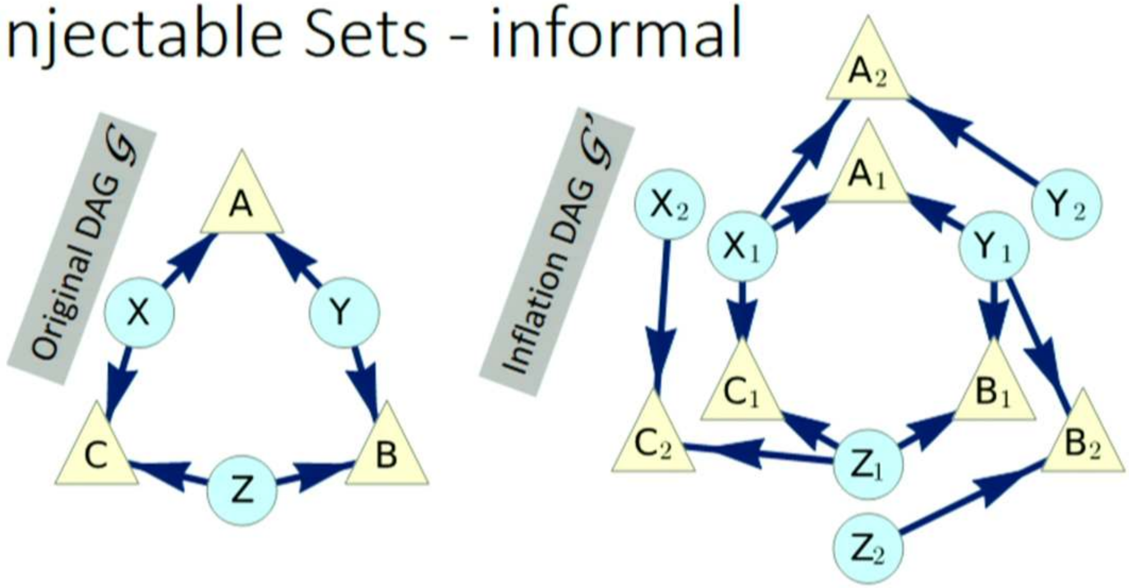
Injectable Sets - informal



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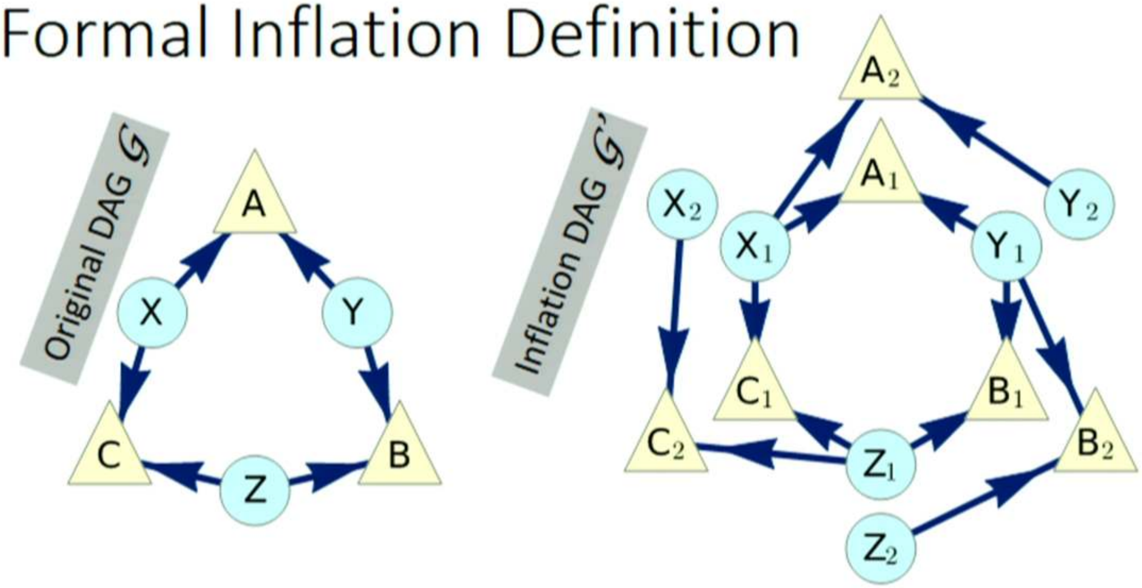
Formal Inflation Definition



The Inflation Assumption:

All functional dependencies *could be* preserved

$$\forall_V \forall_i:\ \text{AncestralSubgraph}_{G'}\left[V_i\right] \sim \text{AncestralSubgraph}_G\left[V\right]$$

Injectable Sets - Formal

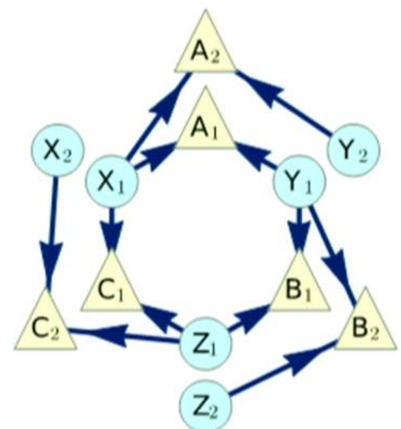
$\mathbf{U}` \in \text{InjectableSets}[G']$

- iff $\text{AncestralSubgraph}_{G'}[\mathbf{U}`] \sim \text{AncestralSubgraph}_{G}[\mathbf{U}]$
- iff $\text{Ancestors}[\mathbf{U}`]$ is *irredundant*

  (no two nodes which differ only by subscipt)
- iff every pair of nodes in $\mathbf{U}`$ is injectable

The spiral-shaped inflation DAG has **four** *maximal* injectable sets

A_1B_2 B_1C_2 C_1A_2 $A_1B_1C_1$



The Inflation DAG Logic

$P_{specific}(U)$ is compatible with G

$\{P_{specific}(U\text{\`{}}\#1)\ldots P_{specific}(U\text{\`{}}\#n)\}$

is compatible with G`

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The Inflation DAG Logic

$P_{specific}(U)$ is compatible with ***G***

&

Inequality[$P_{abstract}(U`_{\#1})$…$P_{abstract}(U`_{\#n})$]

is necessary for compatibility with ***G`***

Inequality[$P_{specific}(U`_{\#1})$…$P_{specific}(U`_{\#n})$]

is satisfied

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The Inflation DAG Logic

$Inequality[P_{specific}(U`_{\#1})...P_{specific}(U`_{\#n})]$

is violated

&

$Inequality[P_{abstract}(U`_{\#1})...P_{abstract}(U`_{\#n})]$

is necessary for compatibility with ***G`***

$P_{specific}(U)$ is NOT compatible with ***G***

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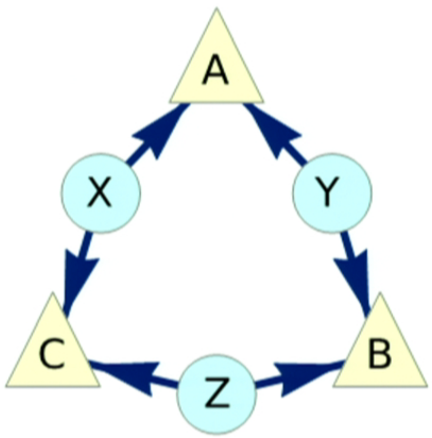
A simple question – perfect correlation?



GHZ-distribution

$$p(abc) = \frac{[000]+[111]}{2}$$

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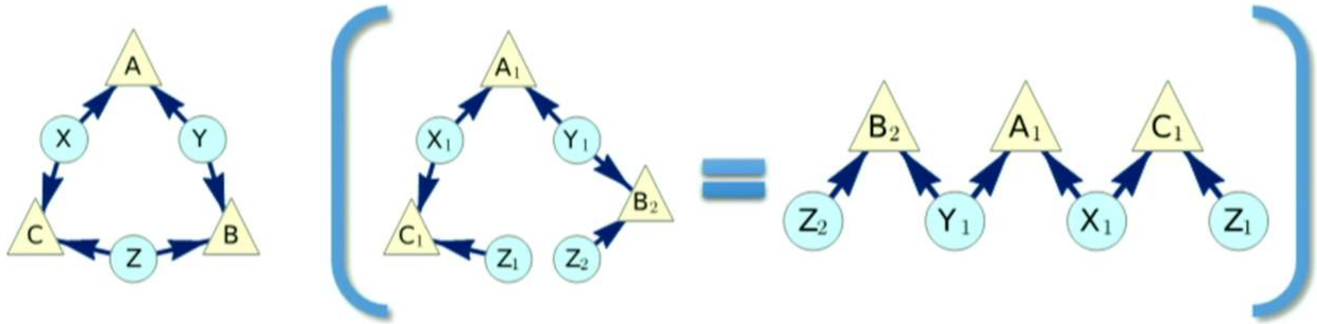
Weak constraints on inflated DAG…



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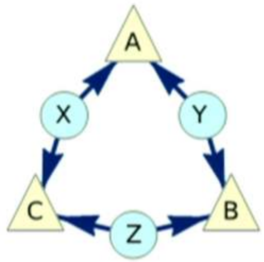
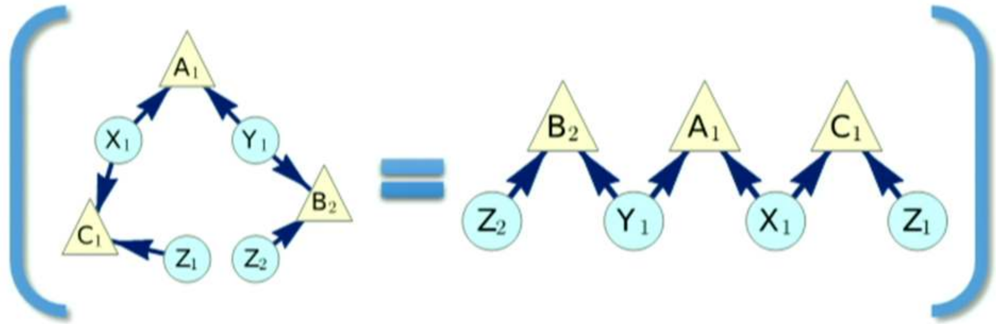




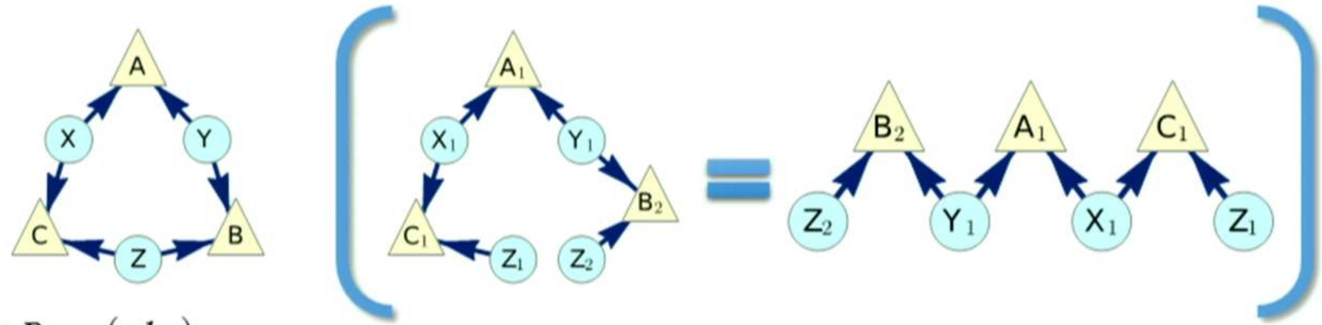
$$0 \le P_{A_1B_2C_1}(abc)$$

$$0 \le P_{A_1B_2C_1}(\bar{a}\bar{b}\bar{c}) = 1 - P_{A_1}(a) - P_{B_2}(b) - P_{C_1}(c) + P_{A_1B_2}(ab) + P_{A_1C_1}(ac) + P_{B_2C_1}(bc) - P_{A_1B_2C_1}(abc)$$

$$P_{B_2C_1}(bc) = P_{B_2}(b)\,P_{C_1}(c)$$

$$\therefore \boxed{0 \le 1 - P_{A_1}(a) - P_{B_2}(b) - P_{C_1}(c) + P_{A_1B_2}(ab) + P_{A_1C_1}(ac) + P_{B_2}(b)\,P_{C_1}(c)}$$

Weak constraints on inflated DAG...



$$0 \le P_{A_1B_2C_1}(abc)$$

$$0 \le P_{A_1B_2C_1}(\bar{a}\bar{b}\bar{c}) = 1 - P_{A_1}(a) - P_{B_2}(b) - P_{C_1}(c) + P_{A_1B_2}(ab) + P_{A_1C_1}(ac) + P_{B_2C_1}(bc) - P_{A_1B_2C_1}(abc)$$

$$P_{B_2C_1}(bc) = P_{B_2}(b)P_{C_1}(c)$$

$$\therefore \boxed{0 \le 1 - P_{A_1}(1) - P_{B_2}(0) - P_{C_1}(0) + P_{A_1B_2}(10) + P_{A_1C_1}(10) + P_{B_2}(0)P_{C_1}(0)}$$

GHZ-distribution

$$p(abc) = \frac{[000]+[111]}{2}$$

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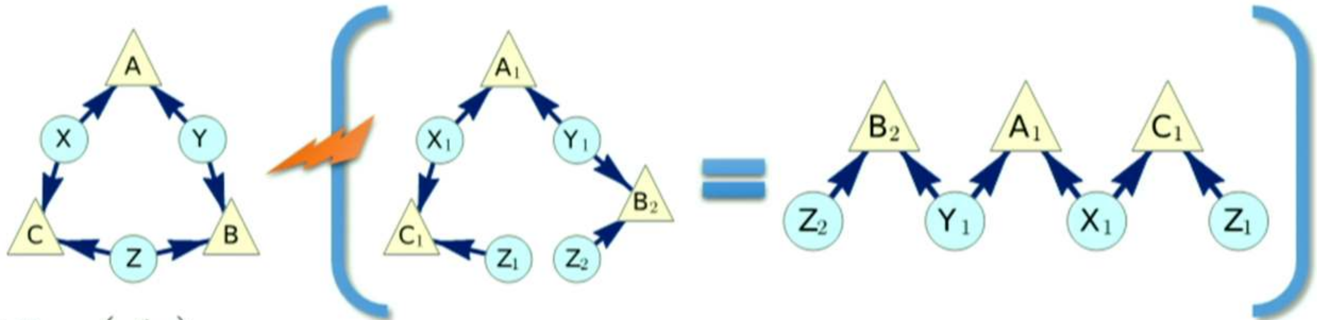
Weak constraints on inflated DAG...



$$0 \le P_{A_1B_2C_1}(abc)$$

$$0 \le P_{A_1B_2C_1}(\bar{a}\bar{b}\bar{c}) = 1 - P_{A_1}(a) - P_{B_2}(b) - P_{C_1}(c) + P_{A_1B_2}(ab) + P_{A_1C_1}(ac) + P_{B_2C_1}(bc) - P_{A_1B_2C_1}(abc)$$

$$P_{B_2C_1}(bc) = P_{B_2}(b)P_{C_1}(c)$$

$$\therefore \boxed{0 \le 1 - P_{A_1}(1) - P_{B_2}(0) - P_{C_1}(0) + P_{A_1B_2}(10) + P_{A_1C_1}(10) + P_{B_2}(0)P_{C_1}(0)}$$

...can be mapped into stronger constraints on the original DAG!

$$\boxed{0 \le 1 - P_A(a) - P_B(b) - P_C(c) + P_{AB}(ab) + P_{AC}(ac) + P_B(b)P_C(c)}$$

GHZ-distribution

$$p(abc) = \frac{[000]+[111]}{2}$$

This recovers a result of
B. Steudel & N. Ay arXiv:1010.5720.

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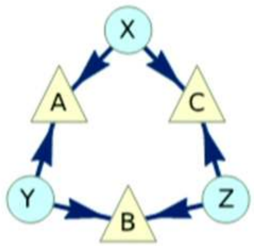
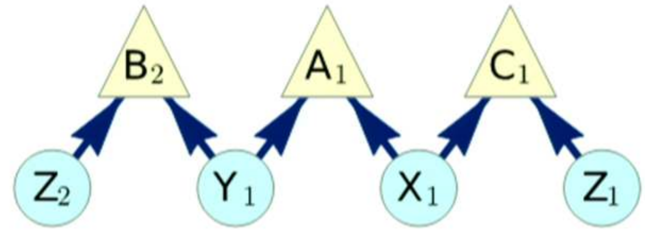
Weak constraints on inflated DAG…





$$I(A_1 : B_2) + I(A_1 : C_1) - I(B_2 : C_1) \leq H(A_1)$$

$$I(B_2 : C_1) \to 0$$

$$\therefore \boxed{I(A_1 : B_2) + I(A_1 : C_1) \leq H(A_1)}$$

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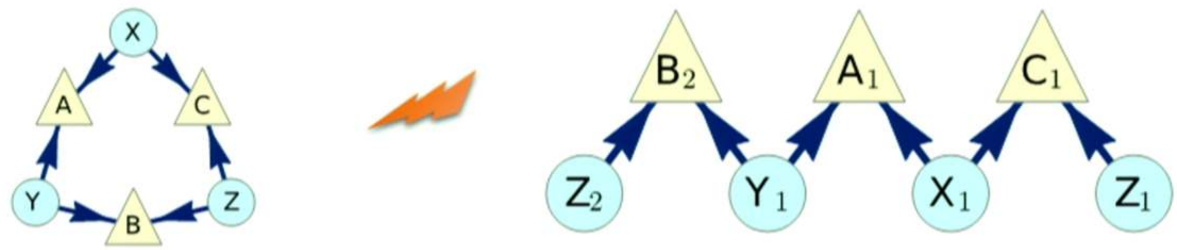
Weak constraints on inflated DAG…



$$I(A_1 : B_2) + I(A_1 : C_1) - I(B_2 : C_1) \le H(A_1)$$

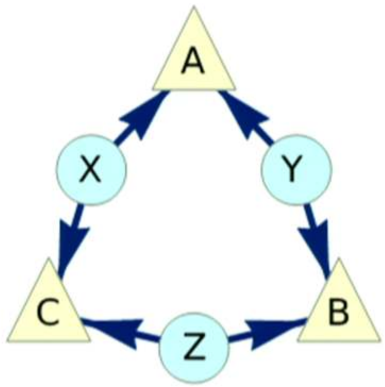
$$I(B_2 : C_1) \to 0$$

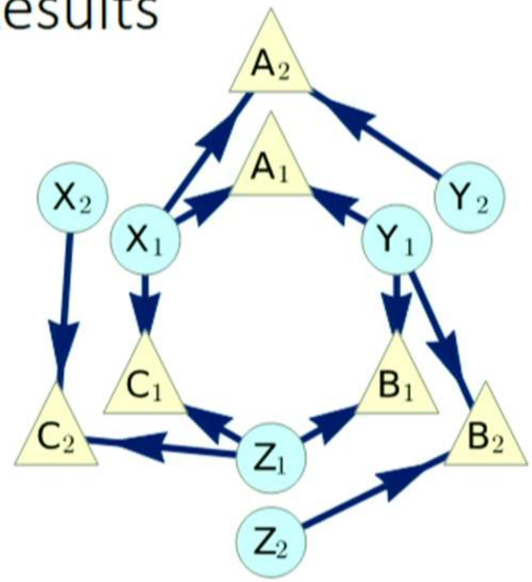
$$\therefore \boxed{I(A_1 : B_2) + I(A_1 : C_1) \le H(A_1)}$$

$$\boxed{I(A : B) + I(A : C) \le H(A)}$$

This recovers a result of T. Fritz NJP 2012, arXiv:1206.5115.

Some Triangle Scenario Results

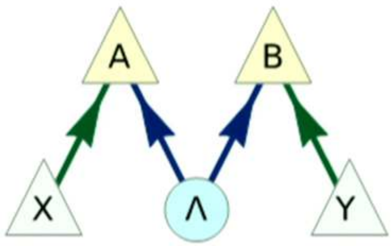


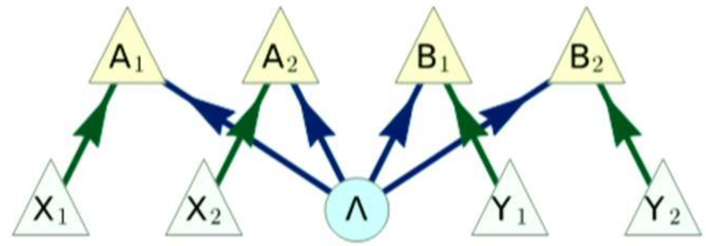


$$0 \le 3+\langle A\rangle+\langle B\rangle+\langle C\rangle-\langle ABC\rangle+2\left(\langle AB\rangle+\langle AC\rangle+\langle BC\rangle\right)$$
$$+\langle A\rangle\langle B\rangle+\langle B\rangle\langle C\rangle+\langle A\rangle\langle C\rangle+\langle A\rangle\langle BC\rangle+\langle B\rangle\langle AC\rangle+\langle C\rangle\langle AB\rangle-\langle A\rangle\langle B\rangle\langle C\rangle$$

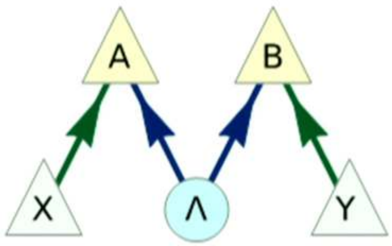
$$0 \le 5+\langle A\rangle+\langle B\rangle+\langle C\rangle-\langle AB\rangle-\langle AC\rangle-\langle BC\rangle-\langle ABC\rangle$$
$$+2\left(\langle A\rangle\langle B\rangle+\langle B\rangle\langle C\rangle+\langle A\rangle\langle C\rangle+\langle B\rangle\langle AC\rangle+\langle BC\rangle\langle A\rangle+\langle C\rangle\langle AB\rangle\right)$$

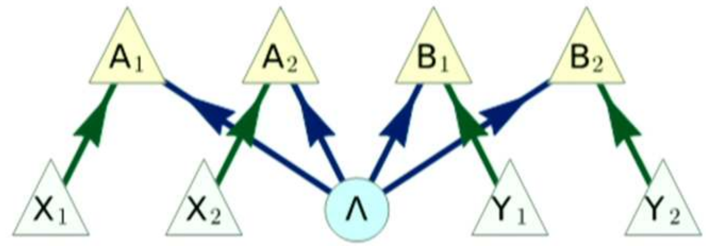
Weak constraints on inflated DAG…





Weak constraints on inflated DAG…





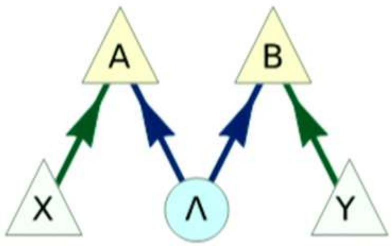
$$P_{A_1B_1A_2B_2X_1Y_1X_2Y_2}(ab{*}{*}xy\bar{x}\bar{y}) \le P_{A_1B_2A_2B_1X_1Y_1X_2Y_2}(a{*}{*}\bar{b}xy\bar{x}\bar{y}) + P_{A_1B_1A_2B_2X_1Y_1X_2Y_2}({*}b\bar{a}{*}xy\bar{x}\bar{y}) + P_{A_1B_1A_2B_2X_1Y_1X_2Y_2}({*}{*}abxy\bar{x}\bar{y})$$

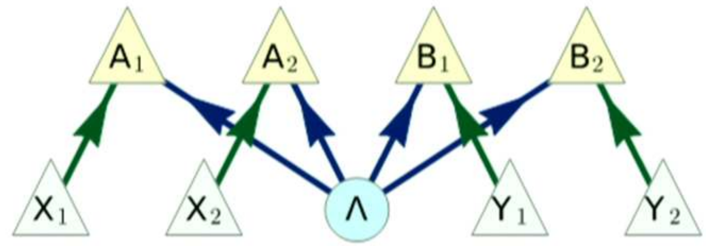
$$P_{A_1B_1X_1Y_1}(abxy)P_{X_2}(\bar{x})P_{Y_2}(\bar{y}) \le P_{A_1B_2X_1Y_2}(a\bar{b}x\bar{y})P_{X_2}(\bar{x})P_{Y_1}(y) + P_{A_2B_1X_2Y_1}(\bar{a}b\bar{x}y)P_{X_1}(x)P_{Y_2}(\bar{y}) + P_{A_2B_2X_2Y_2}(ab\bar{x}\bar{y})P_{X_1}(x)P_{Y_1}(y)$$

$$\frac{P_{A_1B_1X_1Y_1}(abxy)}{P_{X_1}(x)P_{Y_1}(y)} \le \frac{P_{A_1B_2X_1Y_2}(a\bar{b}x\bar{y})}{P_{X_1}(x)P_{Y_2}(\bar{y})} + \frac{P_{A_2B_1X_2Y_1}(\bar{a}b\bar{x}y)}{P_{X_2}(\bar{x})P_{Y_1}(y)} + \frac{P_{A_2B_2X_2Y_2}(ab\bar{x}\bar{y})}{P_{X_2}(\bar{x})P_{Y_2}(\bar{y})}$$

$$\therefore \boxed{P_{A_1B_1|X_1Y_1}(ab|xy) \le P_{A_1B_2|X_1Y_2}(a\bar{b}|x\bar{y}) + P_{A_2B_1|X_2Y_1}(\bar{a}b|\bar{x}y) + P_{A_2B_2|X_2Y_2}(ab|\bar{x}\bar{y})}$$

Recovers Bell Scenario Results





The spider-shaped inflation DAG has **four** *maximal* injectable sets

$A_1B_1X_1Y_1$ $A_1B_2X_1Y_2$

$A_2B_1X_2Y_1$ $A_2B_2X_2Y_2$

$$\langle A_0B_0\rangle+\langle A_1B_0\rangle+\langle A_0B_1\rangle-\langle A_1B_1\rangle\le 2$$

…the inflation DAG technique recovers **ALL** Bell inequalities!

(Any number of parties, measurements settings, possible outcomes…)

The Inflation DAG technique

- Generate a set of (weak) constraints pertaining to the **inflation DAG**'s variables

  (e.g. marginal independences – *YOU CHOOSE*)

- Identify the **injectable subsets** (of the inflation observable variables)

  (ancestral subgraph comparison)

- Perform Quantifier Elimination to obtain **purely injectable** inequalities

  (e.g. Fourier-Motzkin / Convex Hull Enumeration)

Many polytope projection algorithms are available, see Jones *et. al.* DOI:10.1007/s10957-008-9384-4

GPT / Quantum Constraints

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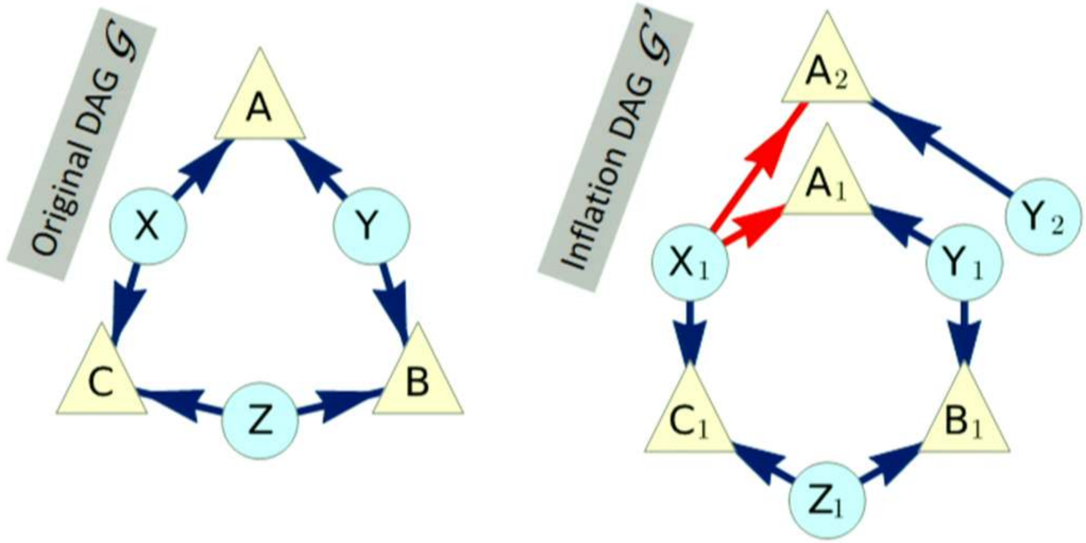
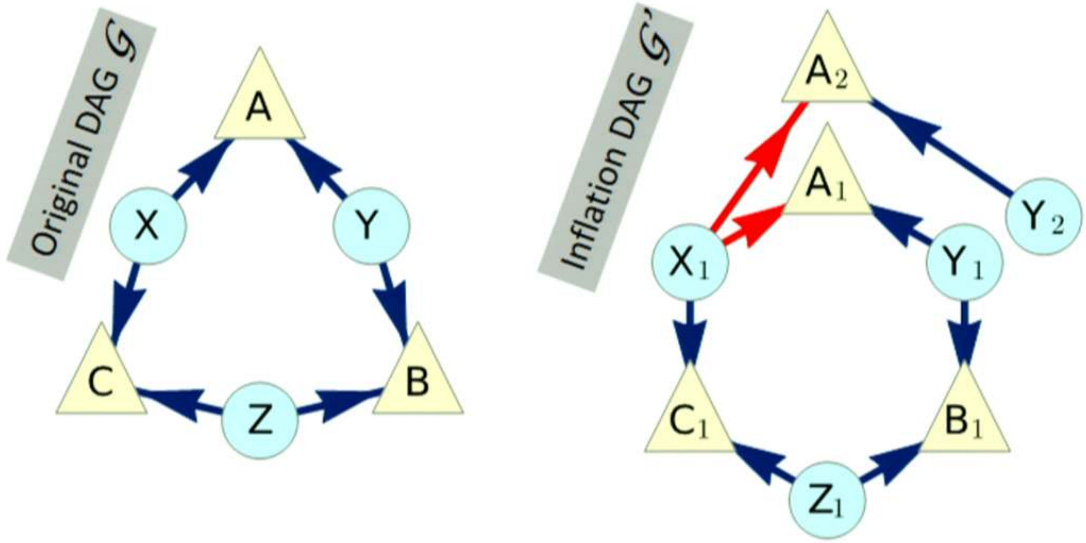
Classical or Quantum constraints? (1 of 3)



- The variable **Y** gets prepared twice, but the information **X→A** is **broadcast**
- If **X** is **classical** then physically we just clone/retransmit the ontological state
- If **X** is **quantum** then *such* an inflation cannot physically be constructed

Classical or Quantum constraints? (1 of 3)



- The variable **Y** gets prepared twice, but the information **X→A** is **broadcast**
- If **X** is **classical** then physically we just clone/retransmit the ontological state
- If **X** is **quantum** then *such* an inflation cannot physically be constructed

∴ we obtain **classical** causal constraints whenever we assume **broadcasting**

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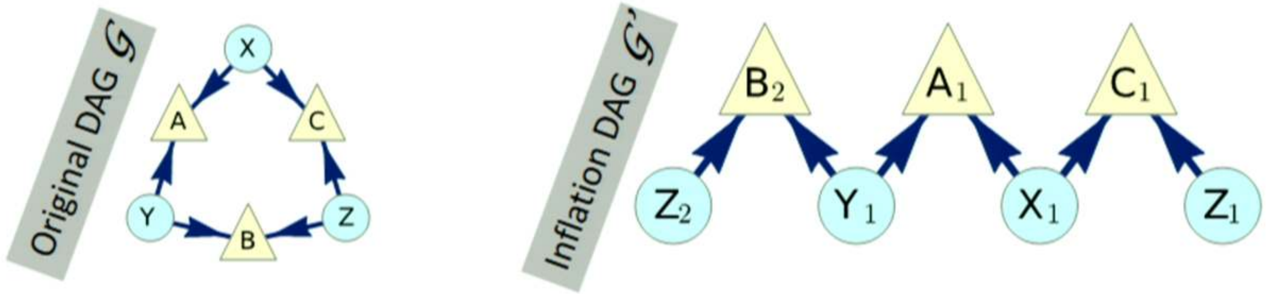
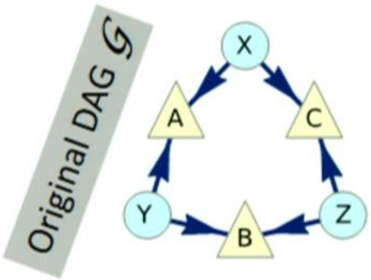
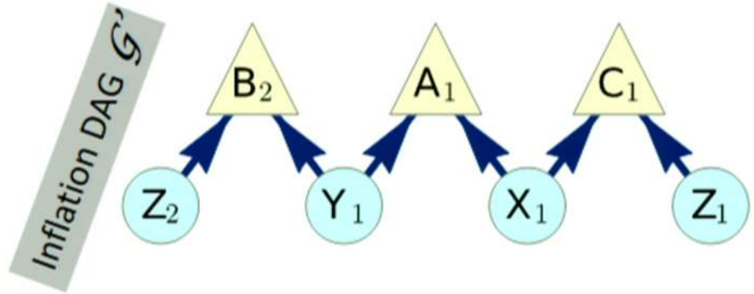
Classical or Quantum constraints? (2 of 3)



This simple inflation DAG does **not** make use of any broadcasting.

Classical or Quantum constraints? (2 of 3)





This simple inflation DAG does **not** make use of any broadcasting.

..so constraints derived from it hold **even in the GPT paradigm**

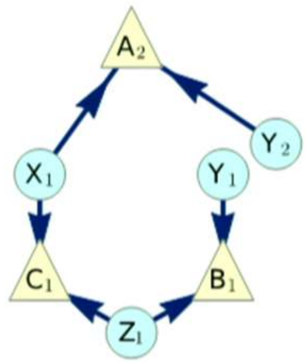
...so even for GPT $0 \le 1 - P_A(a) - P_B(b) - P_C(c) + P_{AB}(ab) + P_{AC}(ac) + P_B(b)P_C(c)$

Classical or Quantum constraints? (3 of 3)

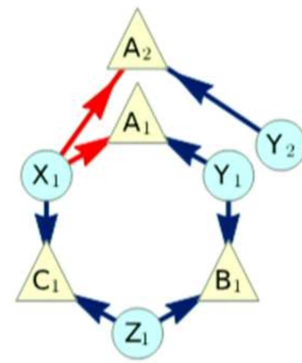
Non-broadcasting Inflation DAG ***G`***:

The children of each latent note in ***G`*** are *irredundant*

Classical or Quantum constraints? (3 of 3)



VS



Non-broadcasting Inflation DAG ***G`***:

The children of each latent note in ***G`*** are *irredundant*

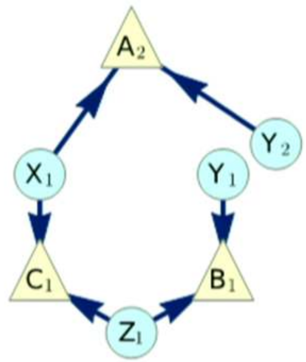
Non-broadcasting node subset ***U`***:

The children of each latent note in Ancestors[***U`***] are *irredundant*

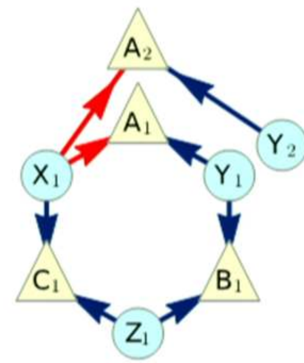
Broadcasting sets ⊂ Non-injective sets

If an inequality is derived from purely non-broadcasting sets then that inequality remains a valid necessary compatibility criterion for all GPTs

Classical or Quantum constraints? (3 of 3)



VS



Non-broadcasting Inflation DAG ***G`***:

The children of each latent note in ***G`*** are *irredundant*

Non-broadcasting node subset ***U`***:

The children of each latent note in Ancestors[***U`***] are *irredundant*

Broadcasting sets $\subset$ Non-injective sets

If an inequality is derived from purely non-broadcasting sets then that inequality remains a valid necessary compatibility criterion for all GPTs

Summary

- The inflation DAG technique allows one to infer **strong constraints on the original DAG** from **weak constraints on the inflation DAG** **+** quantifier existential closure

- The **initial constraints** can be **any causal criteria**.

- Options: **Linear** or **nonlinear** existential quantifiers**?** Quantifier **elimination** or **satisfiability** checking**?** Derive **all implications** or just **some constraints?**

- We derived novel constraints for the Triangle scenario & rejected the W-distribution.

$$p(a)p(b)p(c) \le p(\overline{a}\overline{b}\overline{c}) + p(c)p(ab) + p(b)p(ac) + p(a)p(bc)$$

$$p(abc) = \frac{[100] + [010] + [001]}{3}$$

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Thank You

Status: **unpublished** (draft available upon request)

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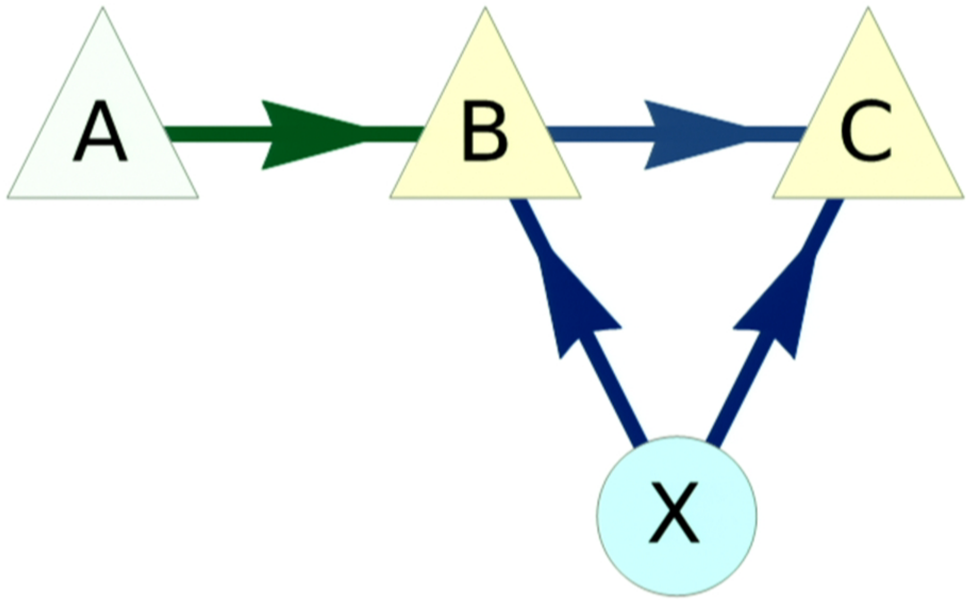
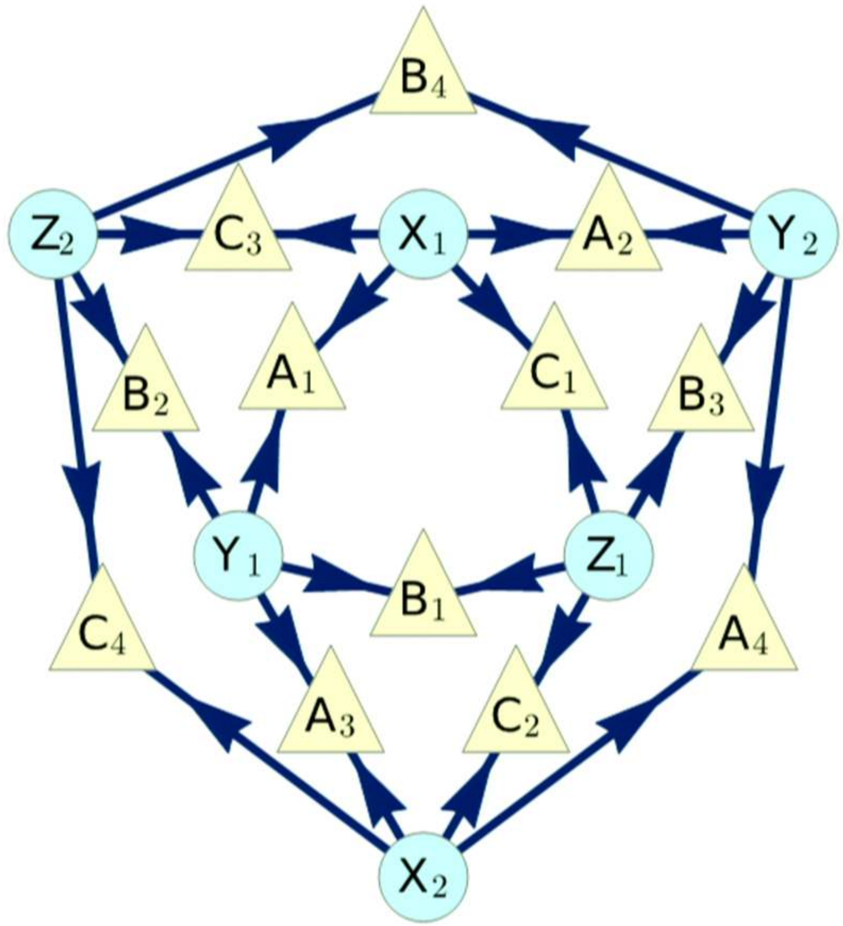
Instrumental Scenario



Larger Triangle Scenario Inflation DAG



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