Title: Delayed choice qubit Lorentz rotations

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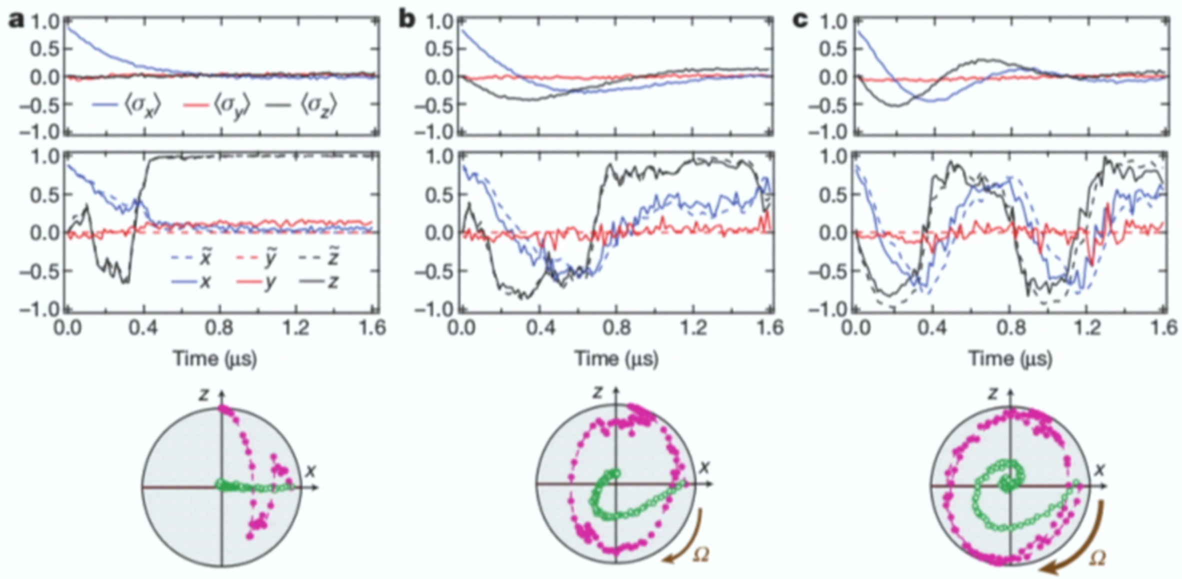
Abstract: <p>For a spin 1/2 (a qubit), Hamiltonian evolution is equivalent to an elliptic rotation of the (Bloch) spin vector in 3D space. In contrast, measurement alters the state norm, so may not be described as such a rotation. Nevertheless, extending the 3D spin vector to a 4D "spacetime" representation allows weak measurements to be interpreted as hyperbolic (boost) rotations. The combined Hamiltonian and measurement dynamics in continuous weak measurement trajectories are then equivalent to (stochastic) Lorentz transformations. Notably, in superconducting circuit QED implementations, the choice between which type of stochastic rotation occurs may be made long after the qubit and measurement field interact.</p>

Example: Monitored Rabi Oscillations



UCB, Nature **511**, 570 (2014)

A monitored quantum system (here a transmon qubit) has both **smooth unitary** dynamics and **random measurement** backaction

The composite dynamics has interesting theoretical structure

3

Qubits are 4-dimensional

- Qubits (e.g., spin-1/2) are 2-level quantum systems

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- But, qubit state space is generally 4-dimensional

$$\hat{\rho} = \frac{1}{2}\left(p\hat{1} + x\hat{\sigma}_1 + y\hat{\sigma}_2 + z\hat{\sigma}_3\right)$$

State norm
(preparation success probability)
(max radius of Bloch sphere)

(Conditional) Expectation values
(Bloch sphere coordinates)

$$x^2 + y^2 + z^2 \equiv R^2 \leq p^2$$

Operator basis:
Pauli operators

$$\hat{1} = |1\rangle\langle 1| + |0\rangle\langle 0|$$
$$\hat{\sigma}_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$$
$$\hat{\sigma}_2 = -i(|0\rangle\langle 1| - |1\rangle\langle 0|)$$
$$\hat{\sigma}_3 = |1\rangle\langle 1| - |0\rangle\langle 0|$$

Orthonormal under
operator inner product:

$$\hat{A}\cdot\hat{B} \equiv \frac{1}{2}\mathrm{Tr}\left[\frac{\hat{A}\hat{B}+\hat{B}\hat{A}}{2}\right]$$

4

Pauli Operators form a 3D Clifford Algebra (for spin)

- The Clifford algebra of 3-space has 8 elements
- The 4 "grades" of the algebra correspond geometrically to oriented subspaces of differing dimensions

Grade		Geometric meaning:
3	$\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z$	Volume Segment
2	$\hat{\sigma}_y\hat{\sigma}_z \quad \hat{\sigma}_z\hat{\sigma}_x \quad \hat{\sigma}_x\hat{\sigma}_y$	Plane Segment
1	$\hat{\sigma}_x \quad \hat{\sigma}_y \quad \hat{\sigma}_z$	Line Segment
0	$\hat{1}$	Point

5

Matrix Representation of 3D Clifford Algebra

- Simplification: $\hat{\sigma}_x \hat{\sigma}_y \hat{\sigma}_z = i\hat{1}$

(Representation-independent definition of "imaginary unit")

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0	$\hat{1}$	Point

6

Dot and Wedge Products

The usual Pauli operators are a faithful matrix representation, so the matrix product is precisely the Clifford product

$$\hat{A}\hat{B} = \frac{\hat{A}\hat{B}+\hat{B}\hat{A}}{2} + \frac{\hat{A}\hat{B}-\hat{B}\hat{A}}{2}$$

Symmetric part is **dot product**

$$\frac{\hat{A}\hat{B}+\hat{B}\hat{A}}{2} = (\hat{A}\cdot\hat{B})\hat{1}$$

Unit operator is an artifact of the matrix representation

Scalar projection removes representation:

$$\langle\hat{A}\rangle_0 \equiv \frac{1}{2}\mathrm{Tr}[\hat{A}]$$

Antisymmetric (noncommutative) part is **wedge product**

$$\frac{\hat{A}\hat{B}-\hat{B}\hat{A}}{2} = \hat{A}\wedge\hat{B}$$

Hodge Star operation flips grade k to (3-k):

$$\star\hat{A} \equiv -i\hat{A}$$

Cross Product is closed for grade 1 (vectors):

$$\hat{\sigma}_i \times \hat{\sigma}_j \equiv -i(\hat{\sigma}_i \wedge \hat{\sigma}_j) = \epsilon_{ijk}\hat{\sigma}_k$$

(Same "noncommutativity" in classical mechanics)

Why do we care?

For a physical spin, we expect rotations in 3D space to be relevant.

Identifying the implicit Clifford algebra makes geometry explicit.

$$\vec{s} \equiv \sum_j s_j \, \hat{\sigma}_j$$

A spin vector can be interpreted as a true vector in 3D space with this mapping.

Classical spin precession becomes obviously the same as the commutator evolution generated by a Hamiltonian operator:

$$\dot{\vec{s}} = \vec{\Omega} \times \vec{s} = -i(\vec{\Omega} \wedge \vec{s}) = \tfrac{-i}{2}\left[\vec{\Omega}\vec{s} - \vec{s}\vec{\Omega}\right] \equiv \tfrac{-i}{2}[\vec{\Omega}, \vec{s}]$$

The physical correspondence of the state to spin orientations becomes transparent:

$$\hat{\rho} = \tfrac{1}{2}[p\hat{1} + \vec{s}] \qquad \mathrm{Tr}[\hat{\rho}\,\vec{\ell}\,] = \langle (p\hat{1} + \vec{s})\,\vec{\ell}\,\rangle_0 = \vec{s} \cdot \vec{\ell}$$

8

What about Measurement?

- The Clifford algebra of 3D space works for Hamiltonian evolution, since the norm of the state never changes
- **Measurement changes the state norm**
- We either need nonlinear evolution (renormalization), or we need to consider the 4th state component.

Example: ground-state projection

$$\hat{\rho} \mapsto |0\rangle\langle 0|\hat{\rho}|0\rangle\langle 0| = \tfrac{p-z}{2}|0\rangle\langle 0| = \tfrac{p-z}{4}(\hat{1} - \hat{\sigma}_z)$$

Equivalent to: $p \mapsto (p-z)/2 \qquad z \mapsto (z-p)/2$

The state **norm** and spin **components** become **intertwined** by **measurement** (before renormalization)

9

Gaussian Measurements

- Weak measurements provide much more intuition about how to handle the change of the state norm

Example: Gaussian pointer r of variance V centered on z eigenvalues +/- 1

$$P(r) = \frac{\mathrm{Tr}[\hat{M}_r^\dagger \hat{M}_r \hat{\rho}]}{\mathrm{Tr}[\hat{\rho}]} = P_{z=1}\frac{\exp(-(r-1)^2/2V)}{\sqrt{2\pi V}} + P_{z=-1}\frac{\exp(-(r+1)^2/2V)}{\sqrt{2\pi V}}$$

Measurement (Kraus) operator:

$$\hat{M}_r = \frac{\exp[-(r-\hat{\sigma}_z)^2/4V]}{(2\pi V)^{1/4}} = C(r)\, e^{r\hat{\sigma}_z/2V} \qquad C^2(r) = \frac{\exp(-(r^2+1)/2V)}{\sqrt{2\pi V}}$$

State update:

$$\hat{\rho} \mapsto \hat{M}_r \hat{\rho} \hat{M}_r^\dagger = C^2(r)[\tfrac{p-z}{2}e^{-r/V}|0\rangle\langle 0| + \tfrac{p+z}{2}e^{r/V}|1\rangle\langle 1| + \tfrac{x}{2}\hat{\sigma}_x + \tfrac{y}{2}\hat{\sigma}_y]$$

(prefactor cancels in state renormalization: neglect)

$$p \mapsto p\cosh(r/V) + z\sinh(r/V) \qquad z \mapsto z\cosh(r/V) + p\sinh(r/V)$$

Hyperbolic rotation of state components in p-z plane by "rapidity" angle r/V

10

4D Clifford Algebra

Rotations in planes involving the state norm are hyperbolic.
Rotations in planes not involving the state norm are elliptic.
This is equivalent to the structure of **spacetime**.

Note: it is simple to "upgrade" the qubit representation to this 4D "spacetime"
3D Clifford algebra is a subalgebra of the 4D Clifford algebra of spacetime
Removing matrix representation changes no physics, but clarifies correspondence

Euclidean 3D		Minkowski 4D (+,-,-,-)
$\sigma_1,\ \sigma_2,\ \sigma_3$	$\mapsto$	$\gamma_0,\ \gamma_1,\ \gamma_2,\ \gamma_3$
$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$		$\gamma_0^2 = 1,\ \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1$

Apparent 3D vectors are timelike planes in 4D

$$\sigma_1 \equiv \gamma_1\gamma_0 \quad \sigma_2 \equiv \gamma_2\gamma_0 \quad \sigma_3 \equiv \gamma_3\gamma_0$$

Could represent 4D basis as Dirac "gamma matrices" if desired
(note vague connection to relativistic spin 1/2)

4D Clifford Algebra

- Simplification: $i \equiv \sigma_1\sigma_2\sigma_3 = \gamma_0\gamma_1\gamma_2\gamma_3$

(Representation-independent definition of "imaginary unit")

(but, **commutes** with **even** grade, *anti-commutes* with *odd* grade)

Grade

4	i	$2^4 = 16$ elements
3	$i\gamma_0$ | $i\gamma_1 \quad i\gamma_2 \quad i\gamma_3$	
2	$\sigma_1 \quad \sigma_2 \quad \sigma_3$ | $i\sigma_1 \quad i\sigma_2 \quad i\sigma_3$	← Planes (of rotation): 3 hyperbolic | 3 elliptic
1	γ_0 | $\gamma_1 \quad \gamma_2 \quad \gamma_3$	
0	1	Relative 3D space embedded as **even-graded** subspace

13

4D "Minkowski" Qubit

Drop 2D matrix representation, preserving physics in Clifford algebra:

$$\hat{\rho} = \frac{1}{2}(p\hat{1} + x\hat{\sigma}_1 + y\hat{\sigma}_2 + z\hat{\sigma}_3) \mapsto \rho \equiv p + x\sigma_1 + y\sigma_2 + z\sigma_3$$

Reinterpret algebra as embedded in 4D "spacetime":

$$\rho = (p\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3)\gamma_0 = \not{\rho}\gamma_0$$

Proper 4-vector

Proper time direction (of agent)
(defines relative space, and probability)

Rotations are then obvious **spinor transformations** of a proper 4-vector:

Hamiltonian: $$e^{dt(\Omega/2)(\star\sigma_1)}\rho e^{-dt(\Omega/2)(\star\sigma_1)} = e^{dt(\Omega/2)(\star\sigma_1)}\not{\rho}e^{-dt(\Omega/2)(\star\sigma_1)}\gamma_0$$

Elliptic rotation in spatial (spin) plane $\star\sigma_1 = -i\sigma_1 = \gamma_2\gamma_3$ with spin axis σ_1

Measurement: $$e^{dt(r/2\tau)(\sigma_3)}\rho e^{dt(r/2\tau)(\sigma_3)} = e^{dt(r/2\tau)(\sigma_3)}\not{\rho}e^{-dt(r/2\tau)(\sigma_3)}\gamma_0$$

Hyperbolic rotation (boost) in temporal (measurement) plane $\sigma_3 = \gamma_3\gamma_0$

14

4D Matrix Representation

- While a spinor representation of 4D rotations is efficient, a 4x4 real matrix representation is also computationally useful

$$\not{\phi} \mapsto \begin{bmatrix} p \\ x \\ y \\ z \end{bmatrix}$$

$$F = E + iB$$

$$E = (r_x/\tau)\sigma_1 + (r_y/\tau)\sigma_2 + (r_z/\tau)\sigma_3$$

Effective "electric field" polarizes spin

$$iB = (\Omega_x)\star\sigma_1 + (\Omega_y)\star\sigma_2 + (\Omega_z)\star\sigma_3$$

Effective "magnetic field" rotates spin

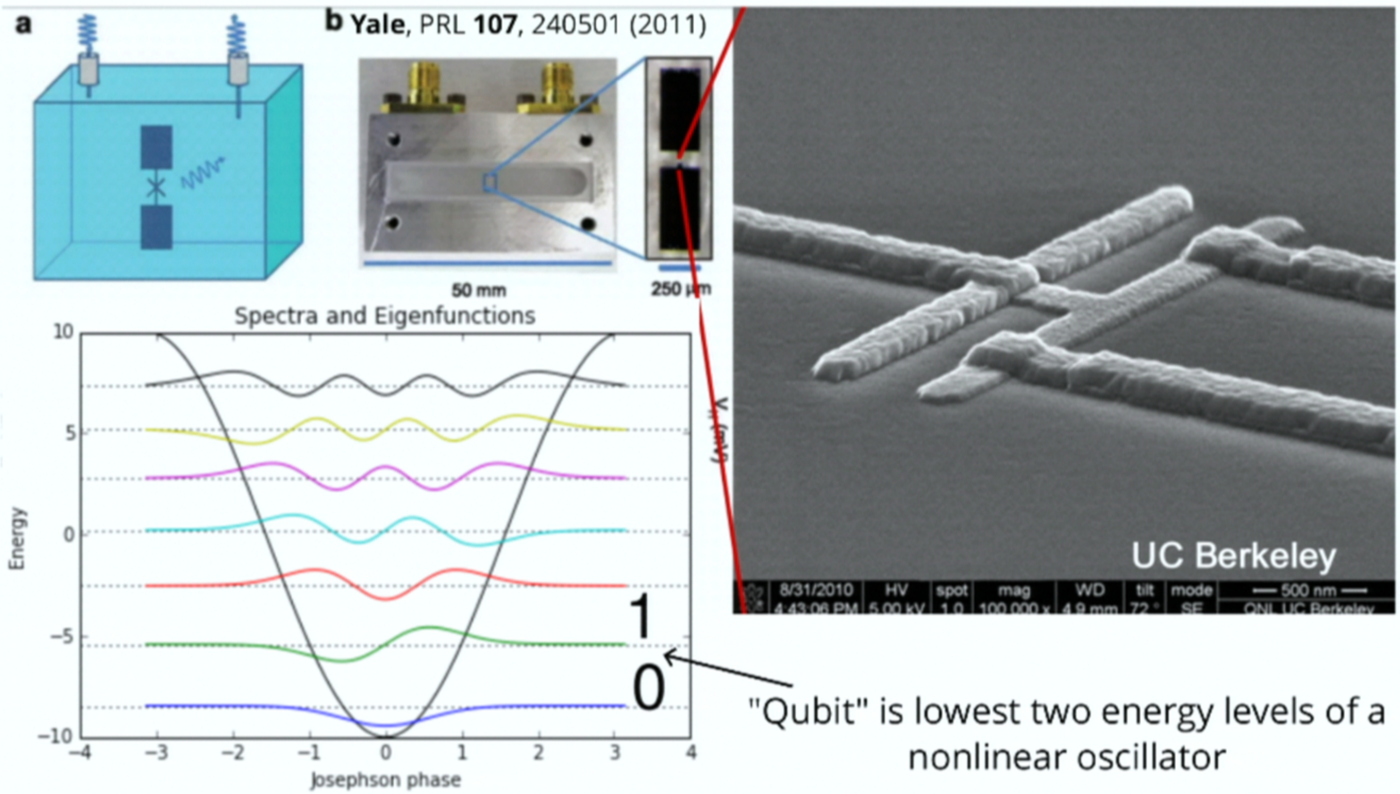
$$e^{dt\,F/2}\not{\phi}e^{-dt\,F/2} \mapsto \exp\left(dt \begin{bmatrix} 0 & r_x/\tau & r_y/\tau & r_z/\tau \\ r_x/\tau & 0 & -\Omega_z & \Omega_y \\ r_y/\tau & \Omega_z & 0 & -\Omega_x \\ r_z/\tau & -\Omega_y & \Omega_x & 0 \end{bmatrix} \right) \begin{bmatrix} p \\ x \\ y \\ z \end{bmatrix}$$

Lorentz rotation generator equivalent to "electromagnetic field tensor"

Describes continuous monitoring of all three qubit measurement axes: easily modified to add experimental inefficiencies (T1, T2, eta, etc.)

arXiv:1606.01407

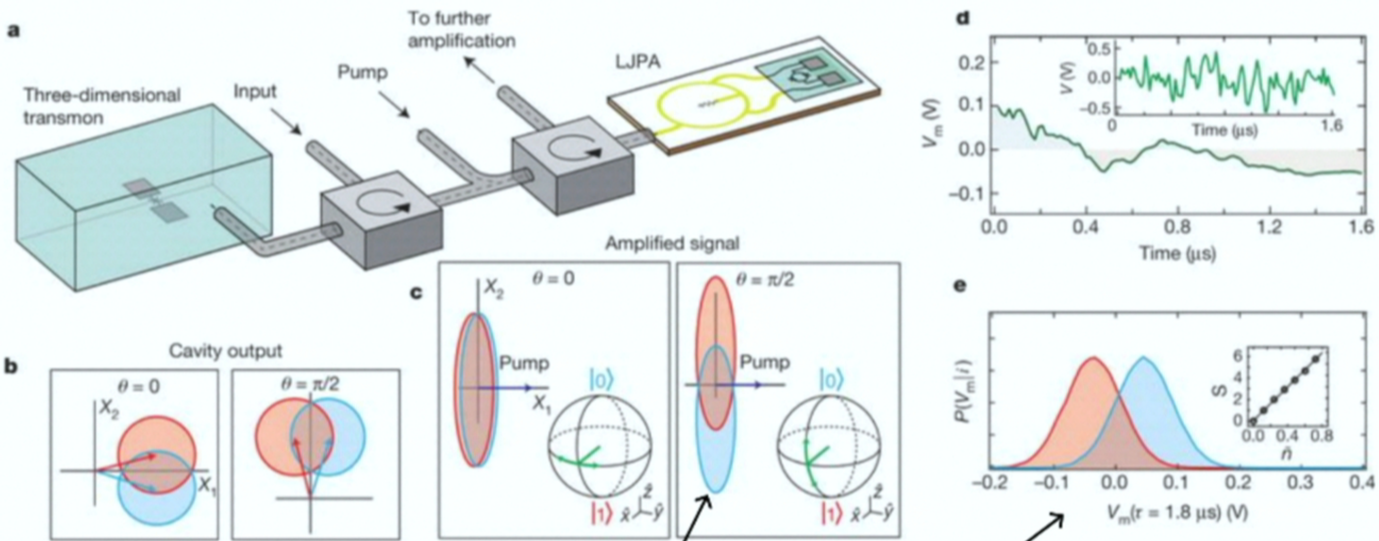
Superconducting Qubit (3D Transmon)



"Qubit" is lowest two energy levels of a nonlinear oscillator

Not spin-1/2, but formalism still useful.

Continuous Monitoring with Microwaves



UCB, Nature **502**, 211 (2013)

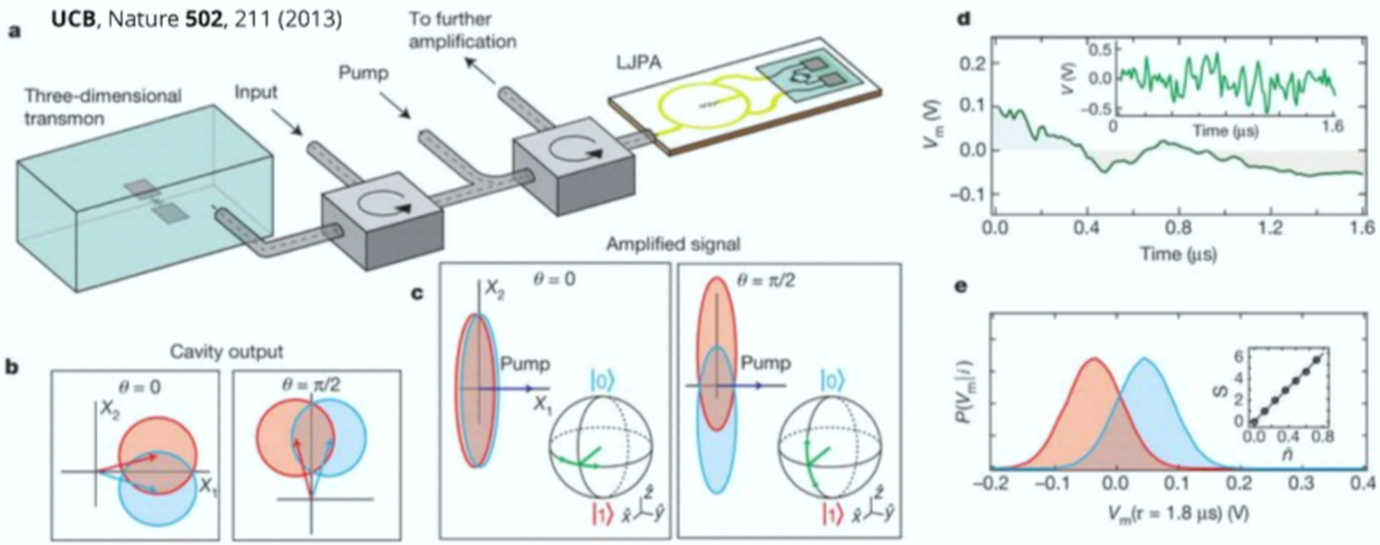
Phase-sensitive amplifier (LJPA) **squeezes** field along information-carrying quadrature

Gaussian measurement per dt
Distinguishable qubit states

17

Delayed Choice Rotations!

UCB, Nature **502**, 211 (2013)



Note the temporal progression:

1. Field interacts with qubit in cavity
2. Field leaks from cavity (unsqueezed)
3. **Field propagates away for a delay**
4. Amplifier phase chooses squeezed quadrature
5. Homodyne measures squeezed quadrature

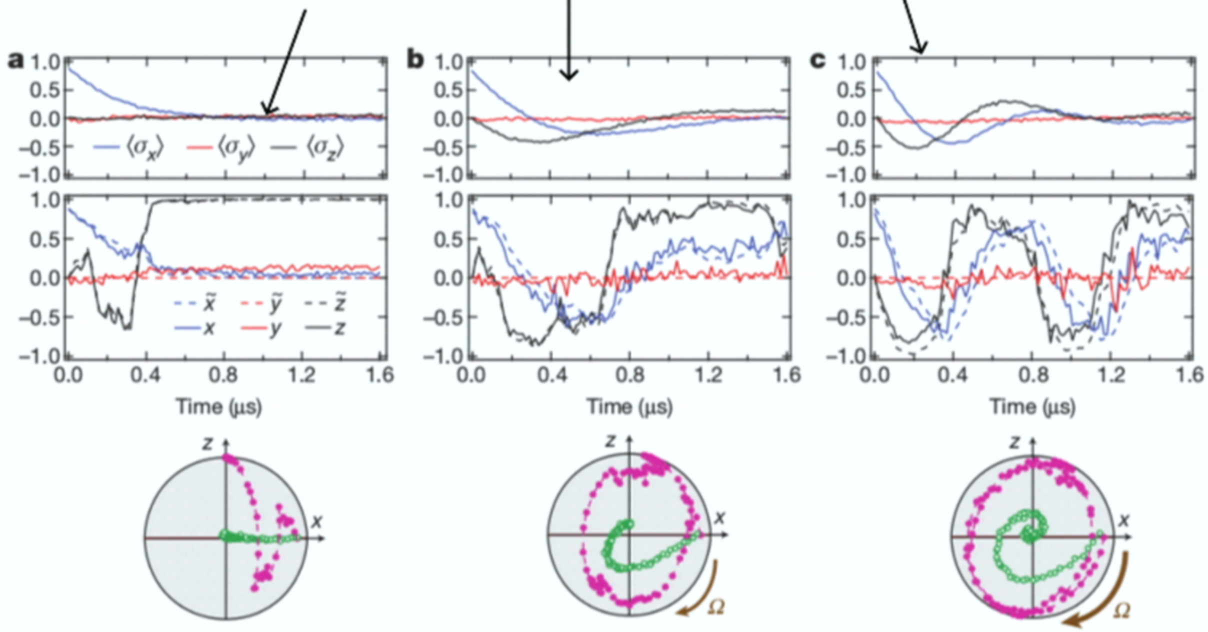
$$F = e^{i\phi}(r/\tau)\sigma_3$$

Phase of squeezing axis chosen long after field escapes cavity: type of qubit Lorentz rotation depends on this phase!

18

Causality still Preserved

Same **ensemble-averaged** (Lindblad) dynamics must occur regardless of (later) choice of measurement (and collected information)

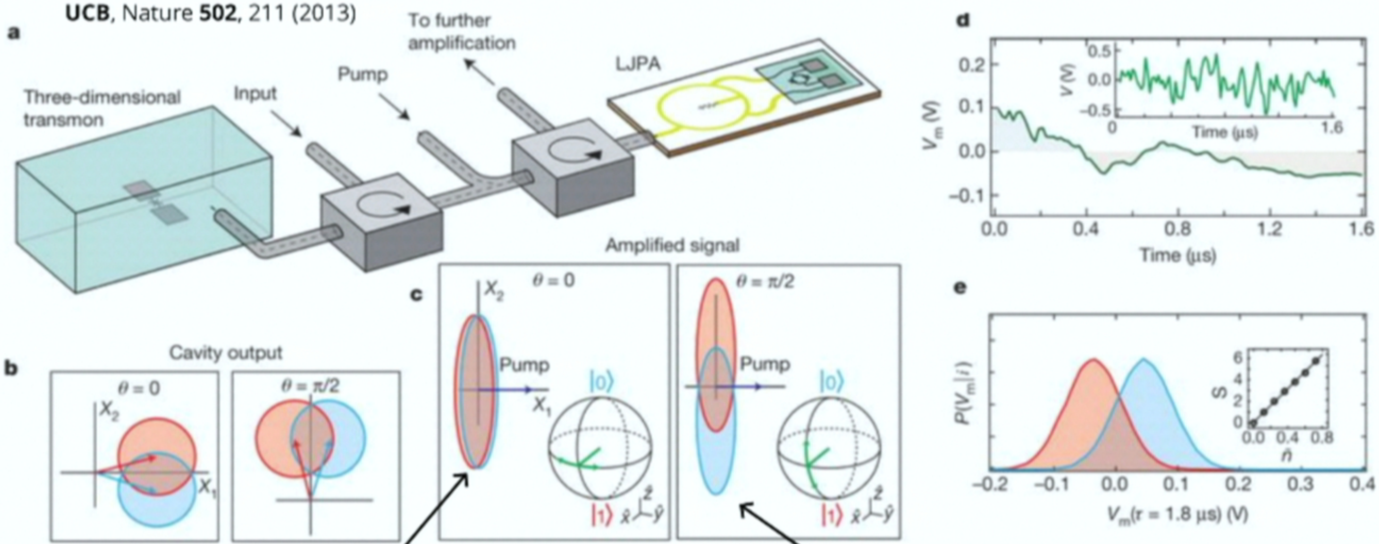


However, the **physical story** told by the observed readout will be very different

Korotkov, arXiv:1111.4016 (2011)

Fluctuations vs. Collapse

UCB, Nature **502**, 211 (2013)



Story #1:

The squeezing eliminates distinguishability of qubit states, but amplifies the intrinsic uncertainty of the cavity field photon number.

The **fluctuating photon number** made the qubit energy fluctuate, creating **random phase drifts** that **dephase** the qubit in the ensemble average (**purely elliptic rotations**).

Story #2:

The squeezing suppresses the intrinsic photon number uncertainty, but amplifies the field separation between distinct qubit states..

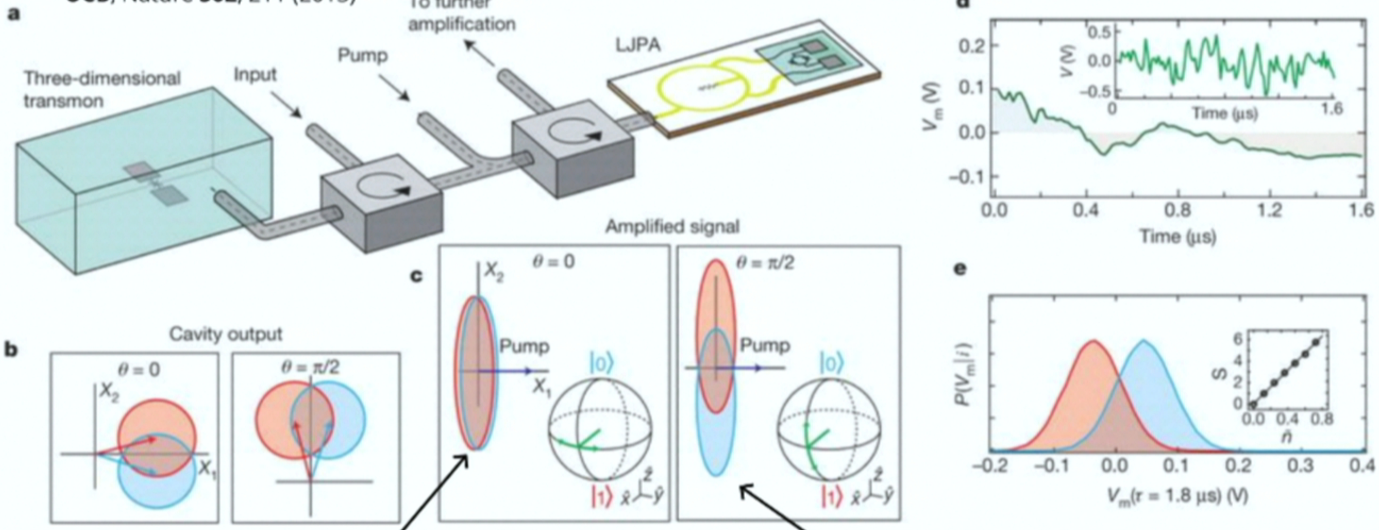
The cavity **photon number does not fluctuate**. Instead, continuous weak monitoring of z creates **partial collapses** that **decohere** the qubit in the ensemble average (**purely hyperbolic rotations**).

The **later** choice of squeezing axis completely changes the physical picture.

20

Fluctuations vs. Collapse

UCB, Nature **502**, 211 (2013)



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20

Conclusions

- **Two-level** systems undergoing **Gaussian continuous measurement** have a natural mapping to **stochastic Lorentz transformations** of a 4-vector

- **Circuit QED** measurements have a naturally **delayed choice** for which **Lorentz transformations** have occurred

Thank you!

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21